ECOLE NORMALE SUPERIEURE

Voronoi L-decomposition of PSD_n and the Hypermetric Correlation Cone

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LIENS - 97 - 5

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CNRS URA 1327

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April 1997

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Abstract

We describe connections of Voronoi's results with hypermetrics.

Denote by PSD_n and $HYCO_n$ the cones in $\mathbf{R}^{\binom{n+1}{2}}$ consisting of real symmetric matrices $(a_{ij})_{i,j=1}^n$ such that

$$PSD_n = \{a \in \mathbf{R}^{\binom{n+1}{2}} : \sum_{1 \le i,j \le n} a_{ij}x_ix_j \ge 0 \text{ for all } \textit{integral } x_i\},$$

$$HYCO_n = \{a \in \mathbf{R}^{\binom{n+1}{2}} : \sum_{1 \le i,j \le n} a_{ij}x_ix_j - \sum_{1 \le i \le n} a_{ii}x_i \ge 0 \text{ for all } \textit{integral } x_i\}.$$

The first cone PSD_n is non-polyhedral cone of all positive semidefinite matrices. Note that the inequalities defining PSD_n are satisfied also by all rational x, and, by continuity, by all real x. The second cone $HYCO_n$ (called the *hypermetric correlation* cone) is polyhedral. There is the third cone COR_n tightly related to these cones and called the *correlation* cone. This cone is defined not by inequalities but by its extreme rays which are all $2^n - 1$ binary matrices $(a_i(S)a_j(S))_{i,j=1}^n$ for all nonzero indicator vectors $a(S) \in \{0,1\}^n$, $S \subseteq \{1, 2, ..., n\}$. Hence

$$COR_n = \{a \in \mathbf{R}^{\binom{n+1}{2}} : a_{ij} = \sum_{S \subseteq \{1,\dots,n\}} x_S a_i(S)a_j(S) \text{ for all } \textit{nonnegative } x_S\}.$$

COR_n and $HYCO_n$ are very interesting because images of them under some linear mapping of $\mathbf{R}^{\binom{n+1}{2}}$ correspond to semimetrics on $n+1$ points. The semimetrics of the map of COR_n are metric subspaces of some l_1^m (i.e. of $\mathbf{R}^m$ with the metric $d_{pq} = ||p - q||_{l_1}$). The semimetrics of the map of $HYCO_n$ are *hypermetrics* $(d_{ij})_{i,j=1}^{n+1}$, i.e. they satisfy the following *hypemetric inequalities*:

$$\sum_{1 \le i < j \le n+1} b_ib_jd_{ij} \le 0 \text{ for all } b \in \mathbf{Z}^{n+1} \text{ with } \sum_{1 \le i \le n+1} b_i = 1.$$

See book [DL97] for detailed study of those semimetrics; see, especially, Chapters 13–17 there.

The purpose of this note is to present the cone $HYCO_n$ (especially its polyhedrality) briefly and only in the context of the remarkable decomposition of PSD_n into polyhedral L-domains and the finiteness of the number of types of L-polytopes given in the famous Deuxième mémoire of Voronoi published in 1908 and (the second part) in 1909 by Crelle Journal.

First, we mention that $HYCO_n = COR_n$ iff $n \leq 5$ (see [De60] for $n = 4$ and [Ba71] for $n = 5$). The set of all facets of $HYCO_6$ is given in [Ba95]. The number $O(n)$ of orbits of facets under the permutation group S_n, and the number of all facets of $HYCO_n$ are as follows

n	2	3	4	5	6
$O(n)$	1	1	2	4	14
$N(n)$	3	12	40	210	3773

Now we will to connect the polyhedrality of $HYCO_n$ with the following two theorems of Voronoi; their proof is, actually, the main content of Deuxéme mémoire [Vor08], [Vor09].

Theorem A (Voronoi) *The action of the group $GL(n, \mathbf{Z})$ induces a partition of the cone PSD_n into disjoint relatively open convex subcones, called the* L-type domains*, of dimension $1, 2, ..., \binom{n+1}{2}$, and having the following properties:*

(i) On each of these subcones the affine structure of the L- decomposition of corresponding lattices is constant.

(ii) Subcones of dimension $\binom{n+1}{2}$ correspond to general lattices, i.e. having simplicial L-decompositions. These L-type domain are polyhedral.

(iii) A subcone of dimension less than $\binom{n+1}{2}$ is a relatively open face of two or more L-type domains. If such a cone makes contact with the boundary of an L-type domain, then it is necessarily a face of that domain. The lattice corresponding to a quadratic form on such a face is special, i.e. it has among its Delaunay polytopes some that are not simplices.

Theorem B (Voronoi) *The numbers of distinct L-types (i.e. the numbers of stars of L-polytopes) of lattices in $\mathbf{R}^n$ is finite for each dimension n.*

Our main theorem is that $HYCO_n$ is polyhedral. The first proof ([DGL93]) used Theorem B. Since Voronoi's proof of Theorem B was very involved, we gave also in [DGL93] much shorter (2 pages) proof of Theorem B. Another proof of the polyhedrality of $HYCO_n$ follows from the Theorem A of Voronoi. In fact, let $v_1, ..., v_n \in \mathbf{R}^k$, where $1 \leq k \leq n$. We set $L =: \mathbf{Z}(v_1, ..., v_n)$ and define $p \in PSD_n$ by setting $p_{ij} = v_i^T v_j$ for $1 \leq i, j \leq n$. Then

(*) $p \in HYCO_n$ if and only if L is a lattice and $v_1, ..., v_n$ are all vertices of the same Delaunay polytope in the star of L at the origin.

Moreover, if $p \in HYCO_n$, then the whole L-type domain containing p is entirely contained in $HYCO_n$. Hence $HYCO_n$ is a union of L-type domains. In fact, this union is *finite*. Indeed, $HYCO_n$ contains only finitely many L-type domains whose associated lattices have the same L-type (by (*) and since, in a given lattice L, there are only finitely many ways of choosing a set of n vectors $v_1, ..., v_n$ that all are vertices of the same Delaunay polytope in the star of L). Finally, the number of distinct L-types of lattices of given dimension is finite. Therefore, $HYCO_n$ is a finite union of L-type domains. But, a fundamental property is that each L-type domain is a polyhedral cone. Hence $HYCO_n$ is a polyhedral cone.

The best known upper bound on the number of facets of $HYCO_n$ (using a result of [Lo94]) is $(1 + 2^{n+1}n!/\binom{2n}{n})^n$. So, the problem of testing membership in $HYCO_n$ is in *co-*NP, but it is not known whether this problem is NP-hard. In [AG93], more precise complexity results are given for integer-valued members of $HYCO_n$.

L-polytopes were introduced in the Deuxième mémoire of Voronoi; they also called Delaunay polytopes, since Delaunay and his school worked a lot on them, see the introduction of [NRDS92] for a historical account; see also [Er92], [RB79]. The link between hypermetrics and L-polytopes was established in [As82]. L-polytopes versus perfect lattices were considered in [Gr93], versus equiangular lines, in [DG95a], versus covering radius of a lattice, in [DG96]. Low-dimensional L-polytopes were considered in [ER87] and [DG95b]. Subclasses of members of $HYCO_n$ (coming from intger-valued graphic hypermetric) were considered in [TD87] and [DG93].

Besides the polyhedrality of $HYCO_n$, the main notion introduced in [DGL92], [DGL93], [DGL95] is the *rank* of an L- polytope, and we hope that it will be useful in the geometry of quadratic forms. In short, the rank of an L-polytope is the number of its degrees of freedom that it has when one deforms it in such a way that the deformed polytope remains an L-polytope. More precisely, the rank $rk(P)$ of a Delaunay polytope P is the dimension (in a topological sense) of the set of all affine bijections T of $\mathbf{R}^k$ (if P is k-dimensional) up to translations and orthogonal transformations, such that $T(P)$ is again an L-polytope, see [La96]. The main results about the rank given in [DGL92], [DGL93], [DGL95] are:

1) $rk(P)$ is equal to the rank of any generating subset of the set $V(P)$ of vertices of P;

2) $rk(P_1 \times P_2) = rk(P_1) + rk(P_2)$;

3) Suppose that an n-dimensional L-polytope P is *basic*, i.e. $V(P)$ contains a basis of the lattice generated by $V(P)$; all known L- polytopes are basic. Then

$$\binom{n+2}{2} - |V(P)| \leq rk(P) \leq \binom{n+1}{2}$$

and, moreover, $\binom{n+1}{2} - \frac{|V(P)|}{2} + 1 \leq rk(P)$ if P is centrally symmetric. For example, the simplex α_n realizes both the inequalities in the first bound; the cross-polytope β_n realizes the equality in the second bound on the rank of a symmetric Delaunay polytope.

4) The *repartitioning* polytopes, introduced in the Deuxème mémoire of Voronoi, but named and studied in [RB79], are exactly L-polytopes of *corank* 1, i.e. of rank $\binom{n}{2} - 1$, see [AG93].

We call L-polytopes of rank 1 *extreme*. An extreme L-polytope is rigid in the sense that a homotety is the only affine transformation of it which is still an L-polytope. In terms of the correspondence between L-polytopes and hypermetrics, the above two cases of Delaunay polytopes of maximal and minimal rank correspond to facets and extreme rays of $HYCO_n$. Examples of extreme L-polytopes, given in [DGL92], [DGL93], [DGL95], are:

the segment α_1, the Schläfli $2_{21} \in \mathbf{R}^6$ and the Gosset $3_{21} \in \mathbf{R}^7$ polytopes (constructed from sections of the root lattice E_8); two centrally symmetric polytopes in $\mathbf{R}^{16}$ and 3 polytopes in $\mathbf{R}^{15}$ (from sections of the Barnes-Wall lattice Λ_{16}); a centrally symmetric L-polytope in $\mathbf{R}^{23}$ with 552 vertices and an asymmetric L-polytope in $\mathbf{R}^{22}$ with 275 vertices (from the Leech lattice Λ_{24}).

It will be interesting to construct an infinite sequence of extreme L-polytopes.

For any n, $HYCO_n$ and COR_n have $2^n - 1$ common extreme rays $(a_i(S)a_j(S))_{i,j=1}^n$, which are preimages of the cut metrics. These are only extreme rays for $n \leq 5$. We know ([DGL92], [DGL93]) additionally 26 orbits of the group S_6 of extreme rays of $HYCO_6$ distinct from cut rays. We expect that there are no other extreme rays. In this case, the segment α_1 and the Schläfli polytope 2_{21} are only Delaunay polytopes of rank 1 having dimension at most 6.

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