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Paul RUET

LIENS - 97 - 14

Département de Mathématiques et Informatique

CNRS URA 1327

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Laboratoire d'Informatique de l'Ecole Normale Supérieure
45 rue d'Ulm 75230 PARIS Cedex 05

Tel : (33)(1) 44 32 30 00

Adresse électronique : ruet@dmi.ens.fr

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Paul Ruet
LIENS-CNRS, Ecole Normale Supérieure
45 rue d'Ulm, 75005 Paris, France
Phone: +33 1 44 32 20 83 Fax: +33 1 44 32 20 80
Email: ruet@dmi.ens.fr

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Abstract

We define a mixed version of classical propositional linear logic, which combines both commutative and non-commutative connectives, with the basic features of a proof theory: sequent calculus and phase semantics. The multiplicative fragment of this logic extends commutative MLL on the one hand, and cyclic MLL on the other hand.

A motivation for this mixed logic is the semantical study of synchronization mechanisms in concurrent programming.

1 Introduction

This paper presents a mixed version of classical propositional linear logic (LL [4]), which combines both commutative and non-commutative connectives. The multiplicative fragment of this logic extends commutative MLL on the one hand, and cyclic MLL [5, 12] on the other hand. We give a sequent calculus (where the information on the way formulas should be combined in a sequent is represented by a series-parallel order), and a phase semantics.

The present logic is based on two previous works: the intuitionistic multiplicative version of de Groote (with phase semantics) and a proposal of the author (a classical system, with modalities, but not extending commutative LL) [3, 10]. The present work extends the version of de Groote to the classical case, with all the connectives (and thus Lambek's calculus [6]). It differs from other proposals made by Retoré [8, 9] to combine both kinds of connectives. This mixed logic admits a syntax in terms of proof nets as well (briefly sketched here) with a sequentialization criterium, but their detailed presentation goes beyond the scope of the present paper.

A motivation for this mixed logic has been the logical characterization of synchronization mechanisms in concurrent programming, specifically in concurrent constraint programming [11]. Namely this mixed logic enables the characterization of finer observations (than the ones provided by intuitionistic or commutative linear logic): the suspensions of a process. Another motivation is a possible approach to the so-called "logical dilemma" in the proof theory of the classical sequent calculus [2].

Further research topics are: the development of semantics of proofs (coherent, categorical), and the application to the semantics of concurrency (phase semantics and simulations of processes, proof nets as parallel executions...).

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2 Sequent calculus

We concentrate our attention on the propositional case (the extension to first-order is not a problem), and make the choice of a single negation (which is more natural for usual commutative LL to be a particular case of the present one). The formulas are defined by:

Definition 1 (Formulas) *The* formulas *are built from atomic propositions* $p, q, \ldots, p^\perp, q^\perp, \ldots$ *with:*
- *non-commutative connectives: the conjunction* $\bullet$ (then) *and the disjunction* $\lhd$ (sequential)*,*
- *multiplicative commutative connectives:* $\otimes$ (tensor) *and* $⅋$ (par)*,*
- *additive connectives:* & (with) *and* $\oplus$ (plus)*,*
- *exponential connectives:* ! (of course) *and* ? (why not)*,*
- *constants: multiplicative* $\mathbf{1}$ *and* $\perp$*, and additive* $\top$ *and* $\mathbf{0}$*.*

The set of formulas is denoted Φ*.*

As usual *negation* is a connective defined by De Morgan laws:

$$\begin{array}{ll}
(p)^\perp = p^\perp & (p^\perp)^\perp = p \\
(A \bullet B)^\perp = B^\perp \lhd A^\perp & (A \lhd B)^\perp = B^\perp \bullet A^\perp \\
(A \otimes B)^\perp = B^\perp ⅋ A^\perp & (A ⅋ B)^\perp = B^\perp \otimes A^\perp \\
(A \& B)^\perp = B^\perp \oplus A^\perp & (A \oplus B)^\perp = B^\perp \& A^\perp \\
(!A)^\perp = ?A^\perp & (?A)^\perp = !A^\perp \\
\mathbf{1}^\perp = \perp & \perp^\perp = \mathbf{1} \\
\top^\perp = \mathbf{0} & \mathbf{0}^\perp = \top
\end{array}$$

Negation is then an involution: for any formula A, $A^{\perp\perp} = A$.

Defining a sequent calculus for a linear logic mixing both commutative and non-commutative multiplicatives raises the problem of representing the information on the way the formulas in the sequent must be combined (either by $\otimes$ or by $\bullet$). In the purely non-commutative case (Abrusci [1]) sequents are lists of formulas. In Retoré's Pomset logic [8, 9] this information is represented by orders attached to the sequents. Among these orders, the well-known series-parallel orders (see, e.g., [7] for a survey on series-parallel orders) play an important rôle, as the orders for which the sequent may be interpreted as a single formula. Here this information is represented by a series-parallel order as well.

Recall that the class of series-parallel orders is the smallest class of orders containing singletons and closed by series ($x <_{\mathrm{i;j}} y$ iff $x <_{\mathrm{i}} y$ or $x <_{\mathrm{j}} y$ or $(x, y) \in \mathrm{i} \times \mathrm{j}$) and parallel ($x \leq_{\mathrm{i,j}} y$ iff $x \leq_{\mathrm{i}} y$ or $x \leq_{\mathrm{j}} y$) compositions. Like [3], we adopt the ",-;" notation for series-parallel orders.

Definition 2 (Sequents) *The* sequents *are of the form* $\vdash \Gamma$, *where* $\Gamma \in \mathcal{H}$. $\mathcal{H}$ *and* $\mathcal{H}_0$ *are respectively the sets of* blocks *and* non-empty blocks *and are defined by the following grammar:*

- $\mathcal{H} ::= () \mid \mathcal{H}_0$
- $\mathcal{H}_0 ::= \Phi \mid (\mathcal{H}_0, \mathcal{H}_0) \mid (\mathcal{H}_0; \mathcal{H}_0)$

$\Gamma, \Delta \ldots$ will denote (possibly empty) blocks. We use the notation $\Gamma[\ ]$ to denote a *context*, i.e. a block with a hole (a leaf of the binary tree Γ), and $\Gamma[\Delta]$ is the block obtained by "filling" the hole with Δ. We use the notation $?\Gamma$ for any block whose formulas are all under a ?.

In the following presentation, we choose to have explicit rules for associativity (of ";" and ","). Another choice would have been to omit these rules, considering sequents "up to associativity", but this could have led to some confusion, since of course, for instance $\Gamma; (\Delta, \Sigma) \neq (\Gamma; \Delta), \Sigma$.

The rules of the *sequent calculus* are:

Axiom - Cut

$$\vdash A^{\perp}, A \qquad\qquad \frac{\vdash \Gamma, A \qquad \vdash A^{\perp}, \Delta}{\vdash \Gamma, \Delta}$$

Structural rules

$$\frac{\vdash \Gamma[(\Delta, \Sigma), \Pi]}{\vdash \Gamma[\Delta, (\Sigma, \Pi)]} \qquad \frac{\vdash \Gamma[\Delta, (\Sigma, \Pi)]}{\vdash \Gamma[(\Delta, \Sigma), \Pi]} \qquad \frac{\vdash \Gamma[(\Delta; \Sigma); \Pi]}{\vdash \Gamma[\Delta; (\Sigma; \Pi)]} \qquad \frac{\vdash \Gamma[\Delta; (\Sigma; \Pi)]}{\vdash \Gamma[(\Delta; \Sigma); \Pi]}$$

$$\frac{\vdash \Gamma[\Delta, \Sigma]}{\vdash \Gamma[\Sigma, \Delta]}\,\text{exchange} \qquad \frac{\vdash \Gamma[\Delta; \Sigma]}{\vdash \Gamma[\Delta, \Sigma]}\,\text{entropy} \qquad \frac{\vdash \Gamma, \Delta}{\vdash \Gamma; \Delta}$$

Logical rules

$$\frac{\vdash \Gamma, A \qquad \vdash \Delta, B}{\vdash (\Delta;\Gamma), A \bullet B} \qquad \frac{\vdash \Gamma, A \qquad \vdash \Delta, B}{\vdash (\Delta,\Gamma), A \otimes B} \qquad \frac{\vdash \Gamma, (A;B)}{\vdash \Gamma, A \lhd B} \qquad \frac{\vdash \Gamma, (A,B)}{\vdash \Gamma, A \parr B}$$

$$\frac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \& B} \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B} \qquad \frac{\vdash \Gamma, B}{\vdash \Gamma, A \oplus B}$$

$$\vdash \mathbf{1} \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \bot} \qquad \text{(no rule for } \mathbf{0}\text{)} \qquad \vdash \Gamma, \top$$

$$\frac{\vdash \Gamma, A}{\vdash \Gamma, ?A} \qquad \frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A} \qquad \frac{\vdash \Gamma, (?A, ?A)}{\vdash \Gamma, ?A} \qquad \frac{\vdash \Gamma}{\vdash \Gamma, ?A}$$

$$\frac{\vdash \Gamma, (?\Delta, ?\Sigma)}{\vdash \Gamma, (?\Delta; ?\Sigma)}$$

Remarks:

$\Im$ The series-parallel order associated to a sequent enables to express commutativity / non-commutativity constraints on formulas. Commas stand for $\parr$'s and semicolumns stand for $\lhd$'s, so the exchange rule is restricted in general to blocks separated by a comma, and the entropy rule says exactly that $A \otimes B \vdash A \bullet B$.

The choice of a single negation forces cyclic exchange, as in cyclic LL, and indeed the rule

$$\frac{\vdash \Gamma, \Delta}{\vdash \Gamma; \Delta}$$

together with entropy, implies cyclic exchange. But it says a bit more: it enables to replace a comma by a semicolumn in case that comma is the outermost separator of the sequent. Of course this does not imply commutativity, and in fact this rule is almost forced if one wants both commutative LL and cyclic LL to be parts of a single (simple) classical calculus. Note that this hypothesis is very natural, and it relies on the intuition coming from proof nets that planar proof nets with at most 2 conclusions can freely pivot, so for such subproofs of a larger proof, commutative and non-commutative should be indistinguishable.

$\Im$ The rules for the additives and constants are quite evident.

Exponentials naturally enable commutativity, in such a way that $!A \otimes !B \cong !(A \& B) \cong !A \bullet !B \cong !B \bullet !A$. This is the reason of the last rule for exponentials.

$\Im$ At first sight the relatively short number of rules (especially logical ones) could be amazing, since one could have expected other rules for, say, $\bullet$, like:

$$\frac{\vdash \Gamma, A \qquad \vdash \Delta[B]}{\vdash \Delta[\Gamma; A \bullet B]}$$

a reminiscence of the intuitionistic rule:

$$\frac{\Gamma \vdash A \qquad \Delta[B] \vdash C}{\Delta[\Gamma; A -\!\!\bullet B] \vdash C}$$

As we shall see, such rules with nested blocks are useless. Indeed the ",-;" notation for series-parallel orders is convenient for writing and reading proofs, but because of permutations (specifically cyclic exchange) there will be several ways of writing equivalent forms of a single sequent. The important point is that, with the structural rules given here, any formula A in the sequent can be "taken off" from the other formulas (to get a sequent of the form $\vdash \Gamma, A$). This simplifies the calculus like in the pure cyclic case.

Proposition 1 *Let $\vdash \Gamma$ be a sequent and A a formula of Γ. There is a sequence of structural rules, the application of which leads from $\vdash \Gamma$ to a sequent of the form $\vdash \Delta, A$ with the same formulas.*

Proof. A is a leaf in the binary tree Γ representing a series-parallel order. The nodes of Γ are separators: either ";" or ",". We proceed by induction on the length of the path from the root of Γ to A.

– If $\Gamma = \Delta, A$ or A, Δ, we apply at most one exchange rule, and we get the result.

– If $\Gamma = \Delta, \Sigma$ and A is in Δ, then by the induction hypothesis, there is a sequence of structural rules mapping $\vdash \Delta$ to $\vdash \Pi, A$, hence from $\vdash \Delta, \Sigma$ to $\vdash (\Pi, A), \Sigma$, then apply an exchange to get $\vdash (A, \Pi), \Sigma$ and an associativity to get $\vdash A, (\Pi, \Sigma)$, qed.

– If $\Gamma = \Delta, \Sigma$ and A is in Σ: identical.

– If $\Gamma = \Delta; \Sigma$, the procedure is similar.

Note that the result is unique, up to associativity of ";" and "," and commutativity of ",". g

Examples:

ℑ In the following examples, we shall omit a minimum of (obvious) structural rules and parentheses. Here if a proof of $A \otimes B \vdash B \otimes A$:

$$\begin{array}{c}
\vdash A^\perp, A \qquad \vdash B^\perp, B \\ \hline
\vdash (A^\perp, B^\perp), B \otimes A \\ \hline
\vdash (B^\perp, A^\perp), B \otimes A \\ \hline
\vdash B^\perp \parr A^\perp, B \otimes A
\end{array}$$

ℑ Here is a proof of $A \otimes (A -\!\!\bullet B) \vdash B$:

$$\begin{array}{c}
\vdash A^\perp, A \qquad \vdash B, B^\perp \\ \hline
\vdash (A^\perp; B), B^\perp \bullet A \\ \hline
\vdash A^\perp; B; B^\perp \bullet A \\ \hline
\vdash B^\perp \bullet A; A^\perp; B \\ \hline
\vdash (B^\perp \bullet A; A^\perp); B \\ \hline
\vdash (B^\perp \bullet A, A^\perp); B \\ \hline
\vdash (B^\perp \bullet A) \parr A^\perp; B \\ \hline
\vdash (B^\perp \bullet A) \parr A^\perp, B
\end{array}$$

$\Im$ Here are proofs of $!A \bullet !B \cong !(A\&B)$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash A^\perp, A}{\vdash ?A^\perp, A}}{\vdash ?B^\perp, ?A^\perp, A} \qquad \dfrac{\dfrac{\vdash B^\perp, B}{\vdash ?B^\perp, B}}{\vdash ?B^\perp, ?A^\perp, B}}{\vdash ?B^\perp, ?A^\perp, A\&B}}{\vdash (?B^\perp; ?A^\perp), A\&B}}{\vdash (?B^\perp; ?A^\perp), !(A\&B)}}{\vdash ?B^\perp \lhd ?A^\perp, !(A\&B)}
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash A^\perp, A}{\vdash B^\perp \oplus A^\perp, A}}{\vdash ?(B^\perp \oplus A^\perp), A}}{\vdash ?(B^\perp \oplus A^\perp), !A} \qquad \dfrac{\dfrac{\dfrac{\vdash B^\perp, B}{\vdash B^\perp \oplus A^\perp, B}}{\vdash ?(B^\perp \oplus A^\perp), B}}{\vdash ?(B^\perp \oplus A^\perp), !B}}{\vdash (?(B^\perp \oplus A^\perp); ?(B^\perp \oplus A^\perp)), !A \bullet !B}}{\vdash ?(B^\perp \oplus A^\perp), ?(B^\perp \oplus A^\perp), !A \bullet !B}}{\vdash ?(B^\perp \oplus A^\perp), !A \bullet !B}$$

$\Im$ Here is a proof of $A \otimes \mathbf{1} \vdash A$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash A^\perp, A}{\vdash A^\perp; A}}{\vdash (A^\perp; A), \perp}}{\vdash (A^\perp; A); \perp}}{\vdash \perp; (A^\perp; A)}}{\vdash (\perp; A^\perp); A}}{\vdash (\perp, A^\perp), A}}{\vdash \perp \parr A^\perp, A}$$

We can already note that the cut-free sequent calculus enjoys the *subformula property.*

Besides it enjoys *cut-elimination*, but this is more easily proved in a proof-nets syntax (with a sequentialization criterium, which is the long trip condition of [4] plus the condition that for trips with $\bullet$ links switched on 'L', for any $\lhd$ link, the part of the trip from the left premisse to the right one does not go through a conclusion link): full details are beyond the scope of this paper.

3 Phase semantics

Definition 3 (Phase space) *A* phase space *consists in the following data:*
– a poset $(P, \leq)$, whose elements are called phases*,*
– a monoidal product $\cdot$ and a commutative monoidal product $\star$, both with unit 1, monotonic in both arguments ($x \leq x'$ and $y \leq y'$ imply $x \cdot y \leq x' \cdot y'$ and $x \star y \leq x' \star y'$); and such that $x \star y \leq x \cdot y$ for all $x, y \in P$,
– a set $\perp_P$, whose elements are called the antiphases*, downward closed ($x \in \perp_P$ and $y \leq x$ imply $y \in \perp_P$) and such that for all $x, y \in P$, $x \cdot y \in \perp_P$ iff $y \cdot x \in \perp_P$ iff $x \star y \in \perp_P$.*

The phase space will be simply denoted P.

Definition 4 *If G is a subset of P, its* dual *is defined by:*

$$G^\perp = \{p \in P \mid \forall q \in G, p \star q \in \perp_P\}$$

If G and H are subsets of P, then define:

$$G \cdot H = \{p \cdot q \mid p \in G, q \in H\} \quad G \star H = \{p \star q \mid p \in G, q \in H\}$$

Alternatively $G^\perp = \{p \in P \mid \forall q \in G, p \cdot q \in \perp_P\} = \{p \in P \mid \forall q \in G, q \cdot p \in \perp_P\}$

Definition 5 (Fact) *A* fact *is a subset A of P such that $A^{\perp\perp} = A$. A is* valid *when $1 \in A$.*

As for usual commutative and cyclic linear logic, we have the:

Properties 1 *(i) For any $G \subset P$, $G \subset G^{\perp\perp}$.*
(ii) For any $G, H \subset P$, $G \subset H \Rightarrow H^\perp \subset G^\perp$.
(iii) $G \subset P$ is a fact iff it is of the form $H^\perp$ for some $H \subset P$.
(iv) $\perp_P$ is a fact since $\perp_P = \{1\}^\perp$ (we shall sometimes simply call it $\perp$).
(v) Facts are downward closed.
(vi) Facts are closed under arbitrary intersections.

Proof. (i) to (iv) are immediate.

For (v), let G be a fact, and take $x \in G$ and $y \leq x$. If $z \in G^\perp$, then $x \cdot z \in \perp$ so $y \cdot z \in \perp$ (monotonicity of $\cdot$). Therefore $y \in G^{\perp\perp} = G$.

For (vi) it suffices to verify that if $(G_i)_{i \in I}$ is a family of facts, then $\bigcap G_i = (\bigcup G_i^\perp)^\perp$: if $x \in \bigcap G_i$ then for all $i \in I$, $x \in G_i$, now if $y \in \bigcup G_i^\perp$, then $y \in G_{i_0}^\perp$ for some $i_0 \in I$, so $x \cdot y \in \perp_P$; conversely if $x \in (\bigcup G_i^\perp)^\perp$, then for all $i \in I$ and all $y \in G_i^\perp$, $x \cdot y \in \perp_P$, so $x \in G_i^{\perp\perp} = G_i$, qed. ❡

Definition 6 *A few facts: the largest one $\top = \emptyset^\perp = P$ (w.r.t. inclusion), the smallest one $\mathbf{0} = \top^\perp$, and $\mathbf{1} = \perp^\perp$.*

Definition 7 *Define the following operations on facts A, B:*
- $A \bullet B = (A \cdot B)^{\perp\perp}$,
- $A \lhd B = (B^\perp \cdot A^\perp)^\perp$,
- $A \otimes B = (A \star B)^{\perp\perp}$,
- $A \parr B = (B^\perp \star A^\perp)^\perp$,
- $A \& B = A \cap B$,
- $A \oplus B = (A \cup B)^{\perp\perp}$.

Lemma 1 *For any subsets F and G of P, $F^{\perp\perp} \cdot G^{\perp\perp} \subset (F \cdot G)^{\perp\perp}$ and $F^{\perp\perp} \star G^{\perp\perp} \subset (F \star G)^{\perp\perp}$.*

Proof. The proof is the same as in [4], using the fact that $x \cdot y \in \perp_P$ iff $x \star y \in \perp_P$.

Let us consider the case of $\cdot$. Let $p \in F^{\perp\perp}$ and $q \in G^{\perp\perp}$. If $v \in (F \cdot G)^\perp$ then for all $f \in F$ and $g \in G$, $v \cdot (f \cdot g) = (v \cdot f) \cdot g \in \perp_P$, so for all $g \in G$, $v \cdot f \in G^\perp = G^{\perp\perp\perp}$, and $q \cdot (v \cdot f) = (q \cdot v) \cdot f \in \perp_P$, whence $q \cdot v \in F^\perp = F^{\perp\perp\perp}$. Therefore $p \cdot q \cdot v \in \perp_P$, qed. ❡

Lemma 2 *If G is any subset of P, then $G^{\perp\perp}$ is the smallest fact containing G.*

Properties 2 *(i) De Morgan laws hold for $\bullet$ and $\lhd$, $\otimes$ and $\parr$, $\&$ and $\oplus$. Moreover these 6 operations are associative, $\otimes$, $\parr$, $\&$ and $\oplus$ are commutative, $\mathbf{1}$ is neutral for $\bullet$ and $\otimes$, $\perp$ is neutral for $\lhd$ and $\parr$, $\top$ and $\mathbf{0}$ are respectively neutral for $\&$ and $\oplus$. Distributivity properties hold for $\bullet$ and $\oplus$, $\otimes$ and $\oplus$, $\lhd$ and $\&$, $\parr$ and $\&$.*

(ii) Let A and B be any facts:

$$A \otimes B \subset A \bullet B \ (\textit{dually } A \lhd B \subset A \parr B).$$

Proof. Only the following deserve attention:
– Associativity of the multiplicatives (by duality, we just consider the conjunctions): using Lemma 1, we have $(A \bullet B) \bullet C = ((A \cdot B)^{\perp\perp} \cdot C)^{\perp\perp} = ((A \cdot B)^{\perp\perp} \cdot C^{\perp\perp})^{\perp\perp} \subset (A \cdot B \cdot C)^{\perp\perp}$, and $(A \cdot B \cdot C)^{\perp\perp} \subset (A \bullet B) \bullet C$ is immediate. Hence $(A \bullet B) \bullet C = (A \cdot B \cdot C)^{\perp\perp}$, qed. The case of $\otimes$ is similar.
– Neutrality: these properties rely on the neutrality of $1 \in \mathbf{1}$ for both $\cdot$ and $\star$.
– Distributivities: the proofs are exactly the same as for commutative LL, using Lemma 2.
– $A \otimes B \subset A \bullet B$: it is enough to show that $A \star B \subset A \bullet B = (A \cdot B)^{\perp\perp}$. If $a \in A$ and $b \in B$, then $a \cdot b \in A \cdot B \subset (A \cdot B)^{\perp\perp}$. $(A \cdot B)^{\perp\perp}$ is a fact and $a \star b \leq a \cdot b$ so by Property 1 (v), $a \star b \in (A \cdot B)^{\perp\perp}$. $\square$

Definition 8 – $A \multimap B = \{x \in P \mid \forall a \in A, a \star x \in B\}$,
– $A {-\!\!\bullet} B = \{x \in P \mid \forall a \in A, a \cdot x \in B\}$,
– $B {\bullet\!\!-} A = \{x \in P \mid \forall a \in A, x \cdot a \in B\}$.

Properties 3 *Let A and B be any facts:*

$$A {-\!\!\bullet} B = A^\perp \lhd B, \quad B {\bullet\!\!-} A = B \lhd A^\perp, \quad A \multimap B = A^\perp \parr B,$$
$$A \multimap \perp = A {-\!\!\bullet} \perp = \perp {\bullet\!\!-} A = A^\perp.$$

Hence $A {-\!\!\bullet} B$, $B {\bullet\!\!-} A$ and $A \multimap B$ are facts.

Proof. Here we use again the fact that for all $x, y \in P$, $x \cdot y \in \perp_P$ iff $x \star y \in \perp_P$.

Let us just consider the case of ${-\!\!\bullet}$ (the others are similar). Assume $x \in A {-\!\!\bullet} B$. Let $a \in A$ and $y \in B^\perp$; $a \cdot x \in B$ so $y \cdot a \cdot x \in \perp_P$, thus $x \in A^\perp \lhd B = (B^\perp \cdot A)^\perp$. Conversely, assume $x \in A^\perp \lhd B$, and take $a \in A$. For all $y \in B^\perp$, $y \cdot a \cdot x \in \perp_P$, thus $a \cdot x \in B^{\perp\perp} = B$, whence $x \in A {-\!\!\bullet} B$. $\square$

Definition 9 *A* topolinear space *consists in a phase space P together with a set $\mathcal{F}_P$ of facts, called* closed facts*, such that:*
– *$\mathcal{F}_P$ is closed under arbitrary intersections (so $P \in \mathcal{F}_P$), under finite $\parr$'s and $\lhd$'s (so $\perp \in \mathcal{F}_P$), with smallest element $\perp$,*
– *$F \parr F = F = F \lhd F$ for all $F \in \mathcal{F}_P$.*

The topolinear space will be still denoted P.

The duals of closed facts are called open *facts.*

Definition 10 *Let A be a fact:*
– *$!A$ is defined as the greatest open fact included in A.*
– $?A = (!A^\perp)^\perp$.

Properties 4 *(i) De Morgan laws hold for* ! *and* ?.

(ii) For any fact A, $?A$ *is the smallest closed fact containing* A.

(iii) For any facts A *and* B, *we have* $!!A \ = !A \subset A$, $A \subset B \ \Rightarrow !A \subset !B$, *and*

$$!A \otimes !B \ = !(A \& B) \ = !A \bullet !B \ = !B \bullet !A.$$

Proof. De Morgan laws are immediate.

For the second assertion: if F is closed and $A \subset F \subset ?A$, then $F^{\perp}$ is open and $!A^{\perp} = (?A)^{\perp} \subset F^{\perp} \subset A^{\perp}$, so by definition of !, $!A^{\perp} = F^{\perp}$ whence $F = ?A$.

Among the other assertions only the last one deserves attention. $!(A\&B)$ is a fact, so $!(A\&B) \ = !(A\&B) \otimes !(A\&B) \ \subset !A \otimes !B$. On the other hand, $\mathbf{1}$ is the greatest open fact, so for any fact C, $!C \subset \mathbf{1}$. Thus $!A \otimes !B \subset !A \subset A$ and $!A \otimes !B \subset !B \subset B$ so $!(!A \otimes !B) \subset !(A\&B)$, but $!A \otimes !B$ is open so $!(!A \otimes !B) \ = \ !A \otimes !B$, qed. The case of $\bullet$ is similar, using the properties that $!A \bullet !B$ is open and $F \lhd F = F$ for all $F \in \mathcal{F}_P$. ɡ

Definition 11 (Phase structure, Validity) *A* phase structure (P, S) *is a topolinear space* P, *together with a* valuation *that assigns a fact* $S(p)$ *to any propositional symbol* p.

Given a phase structure, we inductively define the interpretation $S(A)$ *of a formula* A *in the obvious way. The interpretation of a block* Γ *is defined by:* $S(()) = \perp$, $S(\Gamma; \Delta) = S(\Gamma) \lhd S(\Delta)$ *and* $S(\Gamma, \Delta) = S(\Gamma) \parr S(\Delta)$.

Let A *be a formula:* A *is* valid in S *when* $1 \in S(A)$. A *is a* tautology *if it is valid in any phase structure. A sequent* $\vdash \Gamma$ *is valid when* $1 \in S(\Gamma)$ *for any phase structure* S.

Theorem 1 (Soundness) *If a sequent is provable in the sequent calculus, then it is valid.*

Proof. Let (P, S) be any phase structure, and $\vdash \Gamma$ a sequent provable in the sequent calculus. We proceed by induction on a proof of $\vdash \Gamma$:

– The proof is an axiom $\vdash A^{\perp}, A$: the interpretation is $S(A^{\perp}, A) = S(A^{\perp} \parr A) = S(A \ (\ A)$ by Property 3, so $S(A^{\perp}, A) = S(A) \ (\ S(A)$ and $1 \in S(A^{\perp}, A)$.

– The proof is an axiom $\vdash \mathbf{1}$: $1 \in \mathbf{1} = \perp^{\perp}$.

– The proof is an axiom $\vdash \Gamma, \top$: the interpretation is $S(\Gamma, \top) = S(\Gamma) \parr S(\top) = S(\top) \parr S(\Gamma) = S(\mathbf{0}) \ (\ S(\Gamma) = \mathbf{0} \ (\ S(\Gamma)$, and the smallest fact $\mathbf{0} \subset S(\Gamma)$, besides 1 is neutral for $\star$ so this means $1 \in \mathbf{0} \ (\ S(\Gamma)$.

– The proof ends with a cut rule: by induction hypothesis $1 \in S(\Gamma) \parr S(A)$ and $1 \in S(A^{\perp}) \parr S(\Delta)$, which means $S(\Gamma)^{\perp} \subset S(A)$ and $S(A^{\perp})^{\perp} = S(A) \subset S(\Delta)$, qed.

– The proof ends with an associativity rule or the exchange rule: the result follows immediately from the associativity of $\parr$ and $\lhd$, or the commutativity of $\parr$ (Property 2 (i)).

– The proof ends with an entropy rule: immediate from Property 2 (ii).

– The proof ends with the rule:

$$\frac{\vdash \Gamma, \Delta}{\vdash \Gamma; \Delta}$$

By induction hypothesis, $1 \in S(\Gamma)\invamp S(\Delta)$, i.e. $S(\Gamma)^\perp \subset S(\Delta)$, besides 1 is neutral for $\cdot$ so this is equivalent to $1 \in S(\Gamma)^\perp {-\!\bullet} S(\Delta) = S(\Gamma) \lhd S(\Delta)$.
– The proof ends with a & rule: immediate from $A\&B = A \cap B$.
– The proof ends with a $\oplus$ rule: immediate from $A \oplus B = (A \cup B)^{\perp\perp}$ and Lemma 2.
– The proof ends with a $\lhd$ or $\invamp$ rule: immediate.
– The proof ends with a $\bullet$ rule: by induction $S(\Gamma)^\perp \subset S(A)$ and $S(\Delta)^\perp \subset S(B)$, so $(S(\Delta) \lhd S(\Gamma))^\perp = S(\Gamma)^\perp \bullet S(\Delta)^\perp \subset S(A) \bullet S(B)$, i.e. $1 \in (S(\Delta) \lhd S(\Gamma)) \lhd S(A \bullet B) = S((\Delta;\Gamma); A \bullet B)$.
– The proof ends with a $\otimes$ rule: similar argument, using $G^\perp \subset H$ iff $1 \in G\invamp H$.
– The proof ends with a dereliction rule: by induction $S(\Gamma)^\perp \subset S(A)$, and by Property 4 (translated for ?'s) $S(A) \subset S(?A)$, qed.
– The proof ends with a promotion rule: immediate from monotonicity of ! (Property 4 (iii)).
– The proof ends with a contraction rule: immediate from $S(?A) = S(?A\invamp ?A)$, since $S(?A)$ is closed.
– The proof ends with a weakening rule: immediate from $\perp \subset S(?A)$ and $\perp$ neutral for $\invamp$ (Property 2 (i)).
– The proof ends with the rule:

$$\frac{\vdash \Gamma, (?A, ?B)}{\vdash \Gamma, (?A; ?B)}$$

Consequence of $S(?A \lhd ?A) = S(?A\invamp ?A)$, (Property 4 (iii)).
– The proof ends with an introduction of $\perp$: as for weakening.

g

Theorem 2 (Completeness) *If a sequent is valid, then it is provable in the sequent calculus.*

Proof. Let us consider the following phase space:
– P is the set of blocks, up to associativity of ";" and "," and commutativity of ",", $\Gamma \cdot \Delta$ is the sequential composition $(\Gamma;\Delta)$, $\Gamma \star \Delta$ is the parallel composition (Γ, Δ), unit is $()$,
– the ordering $\leq$ is the least ordering such that: $(\Gamma,\Delta) \leq (\Gamma;\Delta)$ for all $\Gamma,\Delta \in P$, and $\Gamma \leq \Gamma'$ and $\Delta \leq \Delta'$ imply $(\Gamma;\Delta) \leq (\Gamma';\Delta')$ and $(\Gamma,\Delta) \leq (\Gamma',\Delta')$,
– $\perp_P$ is the set of blocks Γ such that $\vdash \Gamma$ is provable in the sequent calculus. By the entropy rule and the rule

$$\frac{\vdash \Gamma, \Delta}{\vdash \Gamma; \Delta}$$

$\perp_P$ satisfies the conditions of Definition 3.

Let $Pr(A) = \{\Gamma \in P \mid \vdash \Gamma, A \text{ is provable in the sequent calculus}\}$.

First remark that sets $Pr(A)$ are facts, namely that $Pr(A) = Pr(A^\perp)^\perp$ (the proof is the same as in [4]). Define a phase structure S by $S(p) = Pr(p)$ for any propositional symbol p.

P is extended to a topolinear space by saying that the set $\mathcal{F}_P$ of closed facts is the set of arbitrary intersections of facts of the form $S(?A)$.

Let us verify that $\mathcal{F}_P$ satisfies the properties of topolinear spaces. We have already given proofs in sequent calculus that $!A \bullet !B \cong !(A \& B)$, or equivalently $?A \lhd ?B \cong ?(A \oplus B)$, and of course the same holds for $\parr$: $?A \parr ?B \cong ?(A \oplus B)$, so it is immediate that $\mathcal{F}_P$ is closed under finite $\parr$'s and $\lhd$'s, and the equivalence $A \cong A \oplus A$ implies $?A \cong ?A \parr ?A \cong ?A \lhd ?A$ whence $S(?A) = S(?A) \parr S(?A) = S(?A) \lhd S(?A)$. The equivalence $\perp \cong ?\mathbf{0}$ shows that $\perp \in \mathcal{F}_P$, and by the weakening rule, $\perp$ is the smallest closed fact. The distributivity of $\parr$ and $\lhd$ w.r.t. arbitrary intersections is enough to conclude.

It is then routine to prove by induction on A that $S(A) = Pr(A)$. g

Remarks:

ℑ The extension to first-order quantifiers does not raise any problem.

ℑ One can see easily that the phase semantics we have defined, when particularized to $\parr$ and $\otimes$ (resp. $\lhd$ and $\bullet$) is the phase semantics of commutative (resp. cyclic) LL (For the exponentials there is the slight difference with [12] that ! and ? are not central: the fundamental equality $?A \lhd ?B = ?(A \oplus B)$ of Property 4 just implies that two consecutive ?'s should commute.)

References

[1] M. Abrusci. Phase semantics and sequent calculus for pure noncommutative classical linear logic. *Journal of Symbolic Logic*, 56(4), 1991.

[2] V. Danos, J.B. Joinet, and H. Schellinx. A new deconstructive logic: linear logic. In *Preprint 936, Dept. of Mathematics, University of Utrecht, To appear in the J. of Symbolic Logic*, 1995.

[3] Ph. de Groote. Partially commutative linear logic: sequent calculus and phase semantics. In *Proofs and Linguistic Categories, Proceedings 1996 Roma Workshop*. In V. M. Abrusci and C. Casadio, eds. Cooperativa Libraria Universitaria Editrice Bologna, 1996.

[4] J.Y. Girard. Linear logic. *Theoretical Computer Science*, 50, 1987.

[5] J.Y. Girard. Towards a geometry of interaction. *Contemporary Mathematics*, 92:69–108, 1989.

[6] J. Lambek. The mathematics of sentence structure. *American Mathematical Monthly*, 65(3), 1958.

[7] R. Mohring. Computationally tractable classes of ordered sets. *NATO ASI Series (I.Rival, ed)*, 255, 1989.

[8] Ch. Retoré. *Réseaux et séquents ordonnés.* PhD thesis, Université Paris 7, 1993.

[9] Ch. Retoré. Pomset logic - A non-commutative extension of commutative linear logic. In *Computer Science Logic'95*, 1995.

[10] P. Ruet. Non-commutative linear logic with mobilities. *Abstract submitted to the Bulletin of Symbolic Logic*, Presented at the Logic Colloquium'96, San Sebastian, Spain (`http://www.dmi.ens.fr/~ruet/PAPIERS/nll4.ps`), 1996.

[11] P. Ruet and F. Fages. Concurrent constraint programming and mixed non-commutative linear logic. *Submitted to Computer Science Logic'97*, `http://www.dmi.ens.fr/~ruet/PAPIERS/ccnll97.ps`, 1997.

[12] D.N. Yetter. Quantales and (non-commutative) linear logic. *J. of Symbolic Logic*, 55(1), 1990.