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A Generalization of Strongly Regular Graphs

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Abstract: Motivated from an example of ridge graphs of metric polytopes, we consider a class of connected regular graphs such that the squares of their adjacency matrices lies in some symmetric Bose-Mesner algebras of dimension 3, as a generalization of strongly regular graphs. In addition to a detailed analysis of this prototype example defined over $(MetP_5)^*$, some general properties of these graphs are studied from the combinatorial view point.

1. Introduction

The notion of ridge graphs is given in [3] for studying metric polytopes $MetP_n$ and their relatives. The complement of one of those is interesting to us in this paper. Indeed, after some modifications, a connected regular graph is found such

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that the square of its adjacency matrix lies in a symmetric Bose-Mesner algebra of dimension 3, though itself is not strongly regular. It therefore leads to another interesting generalization of strongly regular graphs besides the notion of distance regular graphs.

A connected simple graph G is called *strongly regular* with parameters v, k, λ, μ if it consists of v vertices such that

$$|G(x) \cap G(y)| = \begin{cases} k & \text{if } x = y, \\ \lambda & \text{if } x, y \text{ are adjacent}, \\ \mu & \text{otherwise}, \end{cases}$$

where $G(x) = \{z | z \in V(G) \text{ is adjacent to } x\}$. This condition can be restated in terms of its adjacency matrix A as $A^2 = kI + \lambda A + \mu(J - I - A)$. As a matter of fact, its adjacency algebra is a symmetric Bose-Mesner algebra of dimension 3. A subspace $\mathcal{A} \subseteq M_n(\mathbf{C})$ of symmetric matrices is called a *symmetric Bose-Mesner algebra* if $I, J \in \mathcal{A}$, and $\mathcal{A}$ is closed under both the ordinary product and Hadamard product of matrices, and under the conjugate transposing as well. It is known that each symmetric Bose-Mesner algebra has a base $\{A_i \mid 0 \leq i \leq d\}$ consisting of (0,1)-matrices such that $A_0 = I$, $\sum_{k=0}^{d} A_i = J$, and $A_i A_j = \sum_{k=0}^{d} p_{ij}^k A_k$ for suitable parameters p_{ij}^k, $0 \leq i, j, k \leq d$. This leads to combinational structures such as *symmetric association schemes*, and in particular P-polynomial association schemes, i.e, *distance-regular graphs*, refer to [1, 2] for more details.

Some classical combinatorial structures were defined in terms of their incidence matrices through Bose-Mesner algebras. Among many others, recall that a $2-(v, k, \lambda)$ design $\Pi = (X, \mathcal{B})$ is called *quasi-symmetric* with sizes of intersections α, β if $M^t M = kI + \alpha A + \beta(J - I - A)$ where A is the square (0,1)-matrix indexed by $\mathcal{B} \times \mathcal{B}$ such that $A(B_1, B_2) = 1$ if and only if $|B_1 \cap B_2| = \alpha$. It is worth noting that $M^t M$ lies in the Bose-Mesner algebra of dimension 3 generated by A. One of the purposes of this paper is to study those (0,1)-matrices M such that $M^t M$ lie in symmetric Bose-Mesner algebras of dimension 3, in particular for those symmetric ones. Necessary background regarding ridge graphs is given in section 2. As a prototype example, the complement Γ_5 of the ridge graph of metric polytopes is given in section 3 in details. Among other things, we show that its adjacency matrix lies in a symmetric Bose-Mesner algebra of dimension 5. The observations made in section 3 leads to a generalization of strongly regular graphs, called quasi strongly regular graphs (QSRG), in section 4.

2. The complement of ridge graphs

We will first recall the notion of ridge graphs introduced in [3] for completeness, and then we will correct some misprints found in [3].

Let us prepare some notation before we give the definition of these graphs. For each n, define vectors $u_{ijk}, v_{ijk} \in \mathbf{R}^{\binom{n}{2}}$, a vector space of dimension $\binom{n}{2}$ over $\mathbf{R}$ indexed by $\{(p,q) | 1 \le p < q \le n\}$, such that

i) the (p,q)-entry of u_{ijk} $(1 \le i < j < k \le n)$ is given by

$$u_{ijk}(p,q) = \begin{cases} 1 & \text{if } (p,q) = (i,j),\ (i,k),\ \text{or } (j,k), \\ 0 & \text{otherwise;} \end{cases}$$

ii) the (p,q)-entry of v_{ijk} $(1 \le i < j \le n,\ k \ne i,j)$ is given by

$$v_{ijk}(p,q) = \begin{cases} 1 & \text{if } (p,q) = (i,j), \\ -1 & \text{if } \{p,q\} = \{i,k\} \text{ or } \{j,k\}, \\ 0 & \text{otherwise.} \end{cases}$$

The graph Γ_n is defined over the vectors

$$\{u_{ijk} | 1 \le i < j < k \le n\} \cup \{v_{ijk} | 1 \le i < j \le n, k \ne i,j\}$$

such that two vectors a, b are adjacent if there is an index (p,q) such that the (p,q)-entries of a and b are nonzero and their sum $a_{(p,q)} + b_{(p,q)}$ is zero. Some small examples include:

1. Γ_3 is the complete graph K_4 of 4 vertices;
2. Γ_4 is a strongly regular graph $SRG(16,9,4,6)$, and its complement $\overline{\Gamma_4}$ is the (4×4)-grid $L(K_{4,4})$ which is a strongly regular graph $SRG(16,6,2,2)$.

Indeed, those *ridge graphs* in [3] are the complements of these graphs Γ_n $(n \geq 3)$. It is worth mentioning here that $\overline{\Gamma_n}$ is the skeleton of the dual metric polytope $(MetP_n)^*$, where $MetP_n$ denotes the full dimensional polytope in $\mathbf{R}^{\binom{n}{2}}$, defined by the inequalities

$$\langle x, v_{ijk} \rangle = x_{ij} - x_{ik} - x_{jk} \leq 0,$$
$$\langle x, u_{ijk} \rangle = x_{ij} + x_{ik} + x_{jk} \leq 2.$$

The first $3\binom{n}{3}$ homogeneous inequalities above define the cone $MetP_n$ of all semi-metrics on n points, and its dual cone is the cone of feasible multicommodity flows. The graph $\overline{\Gamma_n}$ is also the edge-graph of the dual metric polytope $(MetP_n)^*$, i.e., the *ridge* (subfacet, co-edge) graph of $MetP_n$. The local graph of $\overline{\Gamma_n}$ is the bouquet of $n-3$ copies of (3×3)-grids with a common K_3, i.e. any two above grids are disjoint except the triangle, one for all of them. Refer to [3, 4] for further details on this topic.

Corrections for some misprints found in [3] are given in the following:

1. (Theorem 2.2, [3]) For $n \geq 4$, Γ_n is locally the bouquet of $(n-3)$ copies of (3×3)-grids with a common K_3 having parameters $v = 4\binom{n}{3}$, $k = 3(2n - 5)$, $\lambda = 2(n-2)$ or 4, and

$$\mu = \begin{cases} 2(n-1),\ 4,\ \text{or } 0 & \text{if } n \geq 6, \\ 2(n-1),\ 4 & \text{if } n = 5. \end{cases}$$

2. (line +5, p.p. 362 [3]) Some parameters of $\overline{\Gamma_n}$ are the valency $\frac{2(n-3)(n^2-7)}{3}$, and $\mu = \frac{2(n-3)(n^2-13)}{3}$ or $\frac{2(n-3)(n^2-16)}{3} + 2$.

3. (Theorem 2.9 [3]) For $n \geq 4$, the ridge graph G'_n of the metric cone Met_n is locally the bouquet of $(n-3)$ hexagons with a common edge having parameters $v = 3\binom{n}{3}$, $k = 2(2n-5)$, $\lambda = n-2$ or 2, and

$$\mu = \begin{cases} 2n-4,\ n,\ n-1 \text{ or } 0 & \text{if } n \geq 5, \\ 2n-4,\ n,\ n-1 & \text{if } n = 4. \end{cases}$$

4. (line +10, p.p. 364 [3]) Some parameters of G'_n are the valency $\frac{(n-3)(n^2-6)}{2}$, and $\mu = \frac{(n-3)(n^2-12)}{2}$ or $\frac{(n-3)(n^2-14)}{2} + 1$.

Finally, lines $-5 \sim -8$ in p.p. 362 [3] should be deleted.

Among these graphs, Γ_5 is of particular interest to us, the details of the graph Γ_5 will be given in the next section as a prototype for a generalization of strongly regular graphs.

3. The prototype

Among the family $\{\Gamma_n | n \geq 4\}$ of graphs defined in the previous section, the graph Γ_5 is the focus in this section. A maximal clique partition of the vertex set of Γ_5 is given, it then leads to a symmetric Bose-Mesner algebra of dimension 5 containing its adjacency matrix.

Consider the following partition of the vertex set of Γ_5 into $\{X_i | 1 \leq i \leq 10\}$, where

$$\begin{aligned}
X_1 &= \{123, 12.3, 13.2, 23.1\}, & X_6 &= \{145, 14.5, 15.4, 45.1\},\\
X_2 &= \{124, 12.4, 14.2, 24.1\}, & X_7 &= \{234, 23.4, 24.3, 34.2\},\\
X_3 &= \{125, 12.5, 15.2, 25.1\}, & X_8 &= \{235, 23.5, 25.3, 35.2\},\\
X_4 &= \{134, 13.4, 14.3, 34.1\}, & X_9 &= \{245, 24.5, 25.4, 45.2\},\\
X_5 &= \{135, 13.5, 15.3, 35.1\}, & X_{10} &= \{345, 34.5, 35.4, 45.3\},
\end{aligned}$$

and ijk corresponds to u_{ijk}, $ij.k$ corresponds to v_{ijk} respectively for suitable i, j and $k \leq 5$. Note that each X_i gives a maximal clique of Γ_5 and there are no others. Let $A(i,j)$ be the adjacency matrix of Γ_5 with respect to X_i (in rows) and X_j (in columns) in the orders as given respectively. Then $A(i,j)$ is one of the following:

1. $\boxed{\begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}}$ for $(i,j) = (1,6), (1,9), (1,10), (2,5), (2,8), (2,10), (3,4), (3,7),$ $(3,10), (4,8), (4,9), (5,7), (5,9), (6,7), (6,8)$ and their transposes;

2. $\boxed{\begin{matrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{matrix}}$ for $1 \leq i = j \leq 10$;

3. $\boxed{\begin{matrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{matrix}}$ for $(i, j) = (1, 2), (1, 3), (2, 3), (4, 5), (7, 8)$ and their transposes;

4. $\boxed{\begin{matrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{matrix}}$ for $(i, j) = (2, 4), (3, 5), (3, 6), (5, 6), (8, 9)$ and their transposes;

5. $\boxed{\begin{matrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{matrix}}$ for $(i, j) = (4, 7), (5, 8), (6, 9), (6, 10), (9, 10)$ and their transposes;

6. $\boxed{\begin{matrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{matrix}}$ for $(i, j) = (1, 4), (1, 5), (2, 6), (4, 6), (7, 9)$;

7. $\boxed{\begin{matrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{matrix}}$ for $(i, j) = (1, 7), (1, 8), (2, 9), (4, 10), (7, 10)$;

8. $\boxed{\begin{matrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{matrix}}$ for $(i, j) = (2, 7), (3, 8), (3, 9), (5, 10), (8, 10)$;

Note also that $A(i, j)$ is the transpose of case 6) if $(i, j) = (4, 1), (5, 1), (6, 2), (6, 4), (9, 7)$; $A(i, j)$ is the transpose of case 7) if $(i, j) = (7, 1), (8, 1), (9, 2), (10, 4)$ and $(10, 7)$; finally $A(i, j)$ is the transpose of case 8) if $(i, j) = (7, 2), (8, 3), (9, 3), (10, 5)$ and $(10, 8)$. Indeed, $A = [A_{ij}]_{10\times 10}$ is an adjacency matrix of Γ_5.

Some information of the graph Γ_5 can be derived from the relative positions of the above eight 4×4 matrices in its adjacency matrix A. The positions for those 30 copies of 4×4 zero matrix can be kept record as the Petersen graph. This also gives the clique graph $\mathcal{C}$ of Γ_5.

Proposition 3.1 *The clique graph $\mathcal{C}$ of Γ_5 is the Petersen graph.*

Moreover, for other such 4×4 matrices in the list, let

$D = [A_{ii}]_{1 \le i \le 10}$; i.e., the diagonal-like matrix with 10 copies of identity matrix of order 4 along its main diagonal, and zero elsewhere;

$M = A - D$;

$N =$ the matrix obtained from M by interchanging 0, 1 in those nonzero A_{ij};

$R = J - (I + D + M + N)$.

Note that all these four matrices are symmetric, the following lemma can be checked easily by some computer work.

Lemma 3.2 *The ordinary products among $\{J, D, M, N, R\}$ are given in the following table:*

$\bullet$	J	D	M	N	R
J	$40J$	$3J$	$12J$	$12J$	$12J$
D	$3J$	$3I + D$	$M + 2N$	$2M + N$	$3R$
M	$12J$	$M + 2N$	$12I + 2M + 4(J - I - M)$	$8D + 4M + 2N + 4R$	$4M + 4N + 4R$
N	$12J$	$2M + N$	$8D + 4M + 2N + 4R$	$12I + 2M + 4(J - I - M)$	$4M + 4N + 4R$
R	$12J$	3R	$4M + 4N + 4R$	$4M + 4N + 4R$	$12I + 12D + 4M + 4N$

This lemma shows that the vector space spanned by $\{I, D, M, N, R\}$ over $\mathbf{C}$ is also closed under the ordinary product of matrices; hence it also carries some algebraic structure. It is straightforward to check that

$$\begin{aligned} A &= D + M; \\ A^2 &= 4J + 11I + 2D + 4N; \\ A^3 &= 76J - 10I + 31D + 21M \text{ and} \\ A^4 &= 1224J + 261I + 52D + 104N. \end{aligned}$$

The following Theorem follows immediately.

Theorem 3.3 *The adjacency algebra of Γ_5 is the symmetric Bose-Mesner algebra of dimension 5 generated by I, D, M, N and R.*

As a consequence, a symmetric association scheme of 4 classes is obtained with $\{I, D, M, N, R\}$ as its adjacency matrices. Note also that

$$N^2 = M^2 = 12I + 2M + 4(J - I - M)$$

in the table, it can be interpreted in terms of strongly regular graph. Let $G(M)$ (respectively $G(N)$) be the graph with M (respectively N) as its adjacency matrix.

Corollary 3.4 *$G(M)$ is a strongly regular graph $SRG(40, 12, 2, 4)$ and hence its adjacency algebra is a symmetric Bose-Mesner algebra of dimension 3 generated by $\{I, M, J - I - M\}$.*

This equation also shows that $G(N)$ is 12-regular such that each pair of vertices has either 3 or 4 common neighbors depending on whether they are adjacent in $G(M)$, rather than in $G(N)$ itself.

4. Quasi strongly regular graphs (QSRG)

It is well known that distance-regular graphs can be seen as a generalization of strongly regular graphs by allowing diameters larger than 2. Following the prototype example Γ_5 given in section 3, we will introduce another interesting generalization of strongly regular graphs.

Let us give the definition first, and we then consider this generalization in a broader sense. A connected graph Γ with adjacency matrix M is called *quasi strongly regular* if

$$M^2 = kI + \lambda A + M(J - I - A)$$

for some symmetric $(0,1)$-matrix A where $\langle I, J, A\rangle$ form a Bose-Mesner algebra of dimension 3.

Let us now consider this generalization in terms of incidence matrices of some incidence structures. Let M be a $(0,1)$-matrix of order $v \times b$, indexed by $X \times \mathcal{B}$, such that $MJ = rJ$, $JM = kJ$, and

$$M^t M = kI + xA + y(J - I - A).$$

This matrix M can be interpreted as a k-uniform, r-regular incidence structure $\Pi = (X, \mathcal{B})$ such that any pair of distinct blocks have either x or y points in common. It is interesting to note that some classical combinatorial structures can be derived by posing some additional conditions over MM^t. For example, if in addition $MM^t = rI + \lambda(J-I)$, then M is an incidence matrix of a *quasi symmetric* $2-(v, k\lambda)$ design with sizes x and y of intersection between any two blocks. A well known theorem of Seidel and Goethals showed that both MM^t and M^tM lie in the symmetric Bose-Mesner algebra of dimension 3 generated by I, J and A.

The above observation for (0,1)-matrices and designs motivate us to study those matrices $M \in \mathcal{M}_{n\times m}(\mathbf{C})$ with constant column and row sums such that either M^tM or MM^t lie in some Bose-Mesner algebras and their potential combinatorial structures. Let M be a (0,1)-matrix indexed by $X \times \mathcal{B}$ such that $MJ = rJ$, $JM = kJ$ and

$$M^tM = (k-\beta)I + (\alpha-\beta)A + \beta J$$

$$MM^t = (r-\mu)I + (\lambda-\mu)B + \mu J$$

for some square matrices A (indexed by $\mathcal{B}\times\mathcal{B}$) and B (indexed by $X \times X$). Note that

$$A = \frac{1}{(\alpha-\beta)}(M^tM - (k-\beta)I - \beta J),$$

and hence

$$A^2 = \frac{1}{(\alpha-\beta)^2}(M^tMM^tM - (k-\beta)^2 I + (\beta b^2 - 2kbr)J - 2(k-\beta)(\alpha-\beta)A) \quad (*)$$

where $\beta = |\mathcal{B}|$. Substituting

$$M^tM(M^tM) = (\alpha-\beta)M^tMA + (k-\beta)^2 I + (\beta(k-\beta) + \beta rk)J + (\alpha-\beta)(k-\beta)A,$$

$$M^t(MM^t)M = (\lambda-\mu)M^tBM + (r-\mu)(k-\beta)I + (\beta(r-\mu) + \mu k^2)J + (\alpha-\beta)(r-\mu)A,$$

into (*), we have

$$\begin{aligned} A^2 &= \frac{1}{(\alpha-\beta)^2}((\alpha-\beta)M^tMA + (b(k-b) + \beta b^2)J - (\alpha-\beta)(k-b)A) \\ &= \frac{1}{(\alpha-\beta)^2}((\lambda-\mu)M^tBM + ((r-\mu)(k-\beta) - (k-\beta)^2)I + (\beta(r-\mu) + dk^2 \\ &\quad -(\beta b^2 - 2kbr))J + ((\alpha-\beta)(r-\mu) - (k-\beta)(\alpha-\beta))A \end{aligned}$$

It follows that $A^2 \in \langle I, J, A\rangle$ if and only if M^tMA (and hence M^tBM) $\in \langle I, J, A\rangle$. Note also that $M^tMA(B_1, B_2)$ is the number of flags (x, C) such that $x \in B_1 \cap C$ and $C \cap B_2 \neq \emptyset$ (to be specified); and $M^tBM(B_1, B_2)$ is the number of adjacent pairs (x, y) with $x \in B_1$ and $y \in B_2$.

Two graphs can be associated with M naturally. The one with B as an incidence matrix is called the *point graph* of Π. The other with A as an incidence matrix is called the *block graph* of Π. Note that those with $J-I-B$ or $J-I-A$ as their adjacency matrices are simply the complements of those under consideration.

Lemma 4.1 *The block graph of $\Pi = (X, \mathcal{B})$ is strongly regular if and only if M^tMA (and hence M^tBM) belong to the algebra $\langle I, J, A\rangle$ generated by I, J and A.*

Theorem 4.2 *If $\Pi = (X, \mathcal{B})$ is an incidence structure with MB in $\langle M, J\rangle$ (or equivalently AM in $\langle M, J\rangle$), then $A^2 \in \langle I, J, A\rangle$, $B^2 \in \langle I, J, B\rangle$ and hence both the point graph of and the block graph Π are strongly regular.*

From now on, we assume that M is symmetric and hence $M^2 = kI + \lambda A + \mu(J - I - A)$. In other words, M is the adjacency matrix of a connected k-regular graph such that any two distinct vertices has λ or μ common neighbors, called *Deza graphs* in [5].

Theorem 4.3 *If Γ is a connected simple graph with an adjacency matrix M such that $M^2 = kI + \lambda A + \mu(J - I - A)$ and MA or AM lying in $\langle M, J\rangle$, then Γ is quasi strongly regular.*

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