ECOLE NORMALE SUPERIEURE

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LIENS - 97 - 12

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CNRS URA 1327

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November 1997

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THE CONTINUUM: FOUNDATIONS AND APPLICATIONS

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November 26, 1997

This report includes two papers (a preprint and a reprint). The first paper is phisolophically oriented and concerns the foundations of the continuum. The second one is a technical survey of applications and methods based on the use of continuous structures.

Invited paper, in *Naturalizing Phenomenology: Issues in Contemporary Phenomenology and Cognitive Science* (J. Petitot et al., eds.), Stanford University Press, 1998.

Revised version of an Invited Lecture, in *Ninth Colloquium on Trees in Algebra and Programming*, (B. Courcelle, ed.), Cambridge University Press, 1984.

Chapter 1

THE MATHEMATICAL CONTINUUM: FROM INTUITION TO LOGIC

> *.... the conceptual world of mathematics is so foreign to what the intuitive continuum presents to us that the demand for coincidence between the two must be dismissed as absurd. Nevertheless, those abstract schemata which supply us with mathematics must also underlie the exact science of domains of objects in which continua play a role.*
> Hermann Weyl, Das Continuum, 1918.

In this paper[1] we will deal with some foundational problems concerning the mathematical continuum, in relation to Phenomenology. There are three main reasons which justify such an entreprise:
(1) First, as it is explained in the introductory essay to this volume, mathematics have always played an essential role in Husserlian Phenomenology. In fact Phenomenology roots itself in the Philosophy of Arithmetic.
(2) Secondly, the question of the *continuum* is central for Phenomenology. The flux of phenomenological data, what Husserl calls in Ding und Raum the pre-phenomenal space, and the internal time consciousness rely upon the originary intuition of the continuum. In what concerns the problematic of transcendental constitution, the link between the intuition and the formalization of the continuum yields the exemple "par excellence" of relation between what is constituting and what is constituted. Now it is well known that the formalization of the continuum always encountered, from Aristotle to Weyl, via Leibniz and Cantor, the greatest difficulties. One of the main thesis of this paper will be that these difficulties are, so to say, the formal "symptom" of the inherent difficulties in the "intuitions" of the continuum.
(3) Finally, the logical problems concerning formalization and axiomatization play a crucial role in Husserl's *formal ontology*. Husserl's opposition between formal apophantics and formal ontology is parallel to the model-theoretic one between syntax and semantics. But Husserlian semantics is not purely set-theoretic. It wants to be

[1]Invited paper, in **Naturalizing Phenomenology: Issues in Contemporary Phenomenology and Cognitive Science** (J. Petitot et al., eds.), Stanford University Press, 1998.

a true formal theory of constructing principles of objects. Now, we will see that categorical semantics provides a far more satisfactory formal ontology than Set Theory. Moreover we will analyse the correlation between proof principles (formal syntax) and construction principles (categorical semantics). This correlation is analog to Husserl's noetico-noematic one.

Dag Follesdal, when discussing Kurt Gödel's and Hermann Weyl's conceptions of the continuum, mentioned several important aspects of Husserl's influence on the thinking of mathematicians. I will approach the discussion of the continuum from the perspective of trying to obtain a *foundation for 'mathematical knowledge' as part of our way of interpreting and reconstructing the world, and not just as a 'purely logical', (meta)mathematical investigation of Mathematics.* Nonetheless, some references to technical work in Pure Mathematics and Mathematical Logic will be inevitable.

The starting point for this article are comments about the continuum made by Hermann Weyl in the book Das Kontinuum [Weyl, 1918]. Weyl, a mathematician of great stature, was strongly influenced by Husserl in his numerous foundational and philosophical reflections. In particular the 'phenomenology' of the continuum is at the heart of the most interesting, and modern, observations in [Weyl,1918]. Other important references for these notes will be the articles by René Thom, Jean Petitot and Jacques Bouveresse in the book Le Labyrinthe du Continu, [Salanskis, Sinaceur,1992], as well as the reflections of Wittgenstein (in different places, to be cited in the text) and in [Chatelet, 1993].

1.1 The Intuition

Our intuition about the continuum is built from common or stable elements, from invariants which emerge from a plurality of acts of experience: the perception of time, of movement, of a line extended, of a trace of a pencil...

Time. Weyl considers "time as a fundamental continuum", and the "phenomenal time" of Husserl and Bergson as "conscious experience" of the present which coexists with "memory of the instant gone". Its measure is based on the comparison of temporal segments [Weyl, 1918; p.109–111]. Weyl describes the intuition of time as a continuous flux, an "experience in transformation". For Weyl, phenomenological time is a duration without points, made out of parts that link together, that superimpose over each other, because "this is now, but meanwhile now is no more" [Weyl, 1918; p. 111][2] (footnotes on page 22).

Weyl's insistence on non-compositionality of the continuum develops a fundamental thesis of Husserl's. There is in Husserl an essential opposition between time and space as (pre-)phenomenal life-experiences, which are *constituent* existences, and time and space as *construed* entities. The pre-phenomenal ones, time in particular, are non-compositional, whilst the second ones, as a result of a mathematical construction, are made out of ultimate elements (the points). Weyl accepts the (mathematical) hypothesis which forces a one-to-one correspondence between the real line, as defined following Cantor and Dedekind, and pre-phenomenal space, but he considers unsatisfactory the extension of this correspondence to time.

Movement. We can 'see' the continuum in the movement of an object. In Physics, since Aristotle time presupposes movement: movement gives the measure and even the concept of time. The continuum which we derive reminds us again of flux, the passage from the power to the act: but the direct vision of movement has no need to appeal to memory. Weyl then proposes an interesting distinction: the continuous line which is there, the "tracks of the tramway" (an image also dear to René Thom [Thom, 1990]) and the curve, a potential path, "which a pedestrian walks on... the trajectory of a point in movement". When this point "finds itself in a determined position, it coincides with a determined point of the plane, without being itself this point of the plane". "In movement, the continuum of points on a trajectory recovers in a continuous monotone fashion the continuum of instants" [Weyl, 1918; ch.II par. 8]. But this is just superposition: for Weyl the temporal continuum does not have points, the instants are merely "transitions", the present is only possible due to the simultaneous perception of the past and of the future.

The String extended. A thread, a string extended (another of Thom's images), is another experience of the continuum. By its tension, it cannot have jumps nor holes.

The Pencil on a sheet. This is the most common experience of the continuum: no one entertains discourse or conscious reflection of the continuum before having drawn lines on pieces of paper thousands of times. The experience is neat: a set of black points transforms the curve into a line, in the sense of Weyl. The points are collected in the trace, which makes their individuality disappear. These points become evident again, as isolated points, when two lines cross each other.

Cauchy in his first demonstration of the Theorem of the Mean Value (see par.2 below) does not go further than the intuition of the continuum that comes from strings and curves traced by a pencil and their crossings.

Viewing the traces of pencil over paper suggests from where our intellectual experience of points — isolated and without dimensions — could have come: from the crossing of two lines. The points are not part of our intuition of the continuum, clearly at least not from the temporal continuum, as Weyl tells us, but also not part of the spatial continuum, as Wittgenstein explains. For Wittgenstein, a curve is a *law*, it is not made out of points; "the intersection point of two lines is not the common element of two classes of points, but the *intersection of two laws*" [Wittgenstein,1964; quoted by Bouveresse, 1992]. If the line or the curve of the movement has only one dimension, that given by the law that describes it, then we are forced to conceive of a point, as the crossing of two lines, as devoid of dimension. This is also suggested by two lines or by pencil traces that cross each other, on a paper sheet. *The point without dimension is a conceptual construction, a necessary consequence of a line as a one dimensional law.* It is a posterior construction, specific to Set Theory, which 'puts together' the points to reconstruct the line. From this construction comes the set-theoretical inversion of priorities – the continuum as a set of points — an inversion rejected, as we mentioned, by Weyl, Wittgenstein and Thom, for different reasons.

1.2 The Mathematics

One of the most important theorems about the mathematical continuum is intuitively obvious: if on a plane a continuous line has one of its extremities in one side of a right line and the other on the other side of the same right line, then the continuous line cuts through the right line.

Theorem of the Mean Value *If the function $f(x)$ is continuous with respect to the variable x between a and b, and if we call c an intermediary value between $f(a)$ and $f(b)$, then we can alway satisfy the equation $f(x) = c$, for at least one value of x between a and b.*

Proof (Cauchy, 1821) It is enough to see that the curve which has equation $y = f(x)$ will meet one or more times the line $y = c$, inside the interval between a and b; now, it is evident that this will be what will happen when the hypotheses are met. QED

This proof is not a proof. It is not that the reasoning is faulty, it is the definitions that are missing: Cauchy does not have (yet) a rigourous notion of continuity, nor of a curve (Weierstrass). He appeals to the evidence of threads and traces of pencil. Fortunately the theorem in Analysis is true, we can demonstrate it rigourously. Poinsot, in a course in the Ecole Polytechnique in 1815, believed he had demonstrated, in a similar fashion, that every continuous curve is differentiable everywhere, on the left or on the right. The counterexample is well-known[3].

Actually, at the beginning of the XIX century, the 'intuition' about the continuum in Mathematics needed to be made precise. The Ether of Physics was also in the scientific spirit of everyone, with the homogeneity of a perfect continuum. There was a choice to be made: in one side Leibniz infinitesimals, on the other the limits, the continuity in terms of 'for all ϵ, there exists a δ' (Cauchy, Weiertrass).

What is the invariant, the stable among the many experiences of the world that refer to the continum? Certainly a invariance of scale: all the little bits of time, of a line, even of a string ... keep the same properties that of a longer one (with the perception of continuity of an extended string, we don't see the atoms). In general, the magnifying glass does not change our intuition of the continuum. Or more formally all homotheties preserve the structure of the continuum. Then, the absence of jumps and of holes: no stop to jump further (the jumps), no abyss in which Zeno's arrows can be lost (the lacunas or absence of individual points).

There we have the formidable invention of Cantor and Dedekind. It will make people forget Leibniz's ideas until the invention of Nonstandard Analysis, a century later, because of its conceptual simplicity, its precision, its constructivity. Take the totally ordered set of the integers, N, the rationals, Q, as fractions of integers. The set Q is also totally ordered and has already some interesting properties for the continuum : it is in effect a dense order (between any two rationals, there is always a third rational), hence invariant by homothety and without jumps. But Q has an uncountable number of holes or lacunas. Add on all the limits, in the sense of Cauchy, or, what turns out to be the same, define a real number as the set of rationals that are smaller than itself (a Dedekind cut). This is the set-theoretical construction of Cantor-Dedekind which is the standard formalisation of the continuum, that of the real line R of Analysis. It satisfies the invariance of scale, it has no jumps or lacunas. A curve in space will be continuous, if it is described by a law, which does not introduce jumps nor lacunas and is parametrised by this line[4].

1.2.1 The Impredicative Definition of the Real Line

There is still a problem with the construction we have sketched: if a real number is the limit of all the rationals that precede it, we are using and we are preparing the ground for a ‘circular definition’.

Firstly, there is always an infinity of (positive) rationals smaller than whichever (positive) real: hence we need to use, when defining it, the collection N of all the integers, in its totality. And the classical definition of this totality has the following structure: N is the smallest set that contains zero and which if it contains n it contains $n+1$. Said in a different way, N is the intersection of all sets that contain zero and that are closed under the successor operation. But N has also this property: to define it using the phrase ‘all the sets that..’ we quantify over a collection that contains N itself. The *defeniens* uses the *defeniendum.*

Secondly, once the real line has been constructed, whenever we define, for example, least upper bounds or greatest lower bounds, we do it once again using the quantifications which can make reference at what is being defined (the collection of the upper bounds or of the lower bounds includes the ‘definiendum’, the smallest or greatest bound, which is also an upper or a lower bound).

Poincaré and Weyl, who were well aware of these problems in Analysis, gave a rigorous definition of ‘impredicative notion’ in mathematics [5]. Poincaré observes that these definitions are not always contradictory, but they always present the dangers of circularity. Lebesgue, in 1902, built the General Theory of Integration over an essentially impredicative definition (the Lebesgue measure). The question was hotly discussed at the beginning of the century, in particular under the impulse of Russell. We will hint to the consequences of this discussion in Weyl’s books.

Does this circularity separate the Cantor-Dedekind construction and hence Analysis, from the ‘intuition of the continuum’? Clearly not. Already in Aristotle we notice a circularity in the discourse on the continuum: the continuum is presented as one “totality already formed, which, on its own, gives meaning to its components ” [Panza,1992][6].

The same way the present time of Weyl is circular: none of its parts (past, present, future) has meaning without mutual reference to each other; time itself *is* the simultaneous perception of the past, the present and of the future. The present time that it is not there anymore, it is past, or that it is not there yet, it is future, and that we only understand when inserted in the whole of time or within a segment of time. The same is true about the continuity of the string or the line, which is not conceived of points, but globally, or at least through an ‘enlarged locality’. *The impredicativity of Analysis proposes a possible formalisation of this intuitive circularity, in particular of phenomenological time; it is one of its expressive richness, another point of contact between intuition and mathematics.* Thus, in our views, the intuitively non-compositional nature of time is reflected in the apparent paradoxes of the mathematical construction; these circularities need not to be avoided but analysed and developped, in particular by the tools of Logic and Type Theory (see below).

This way the division between time and Mathematical Analysis, which disturbs Weyl (the absence of points in the phenomenological time in comparison with the points which form the real line) is in part, but only in part, reduced: the real points can be, a posteriori, isolated, but their definition and their Analysis, à la Cantor-Dedekind, requires ‘a global look’ at the continuum, the same way the intuition of the present requires that of the past and of the future. In Das Kontinuum Weyl is worried, as most mathematicians at the beginning of the century, about the necessity of rigour in the mathematical definitions: too many paradoxes have disrupted the foundational work, the definitions tinged with circularity are suspected. For this reason, he tries a novel approach, which avoids impredicativity, as it is based on a predicative approach of Mathematical Analysis. This attempt will not affect his concrete work in Mathematics (see [Chand.,1987]) nor his futher foundational reflections (see the next footnote). Weyl is probably missing, in that temporary restriction of his mathematical tools, a common element between Analysis and the intuition of (temporal) continuum, of which he particularly cares. However, given his mathematical talent, the few pages he sketched on this point will be considered a paradigm by other logicians that, later on, will continue to prefer the stratified certitudes to the expressive circularities of impredicativity (see [Feferman,1988]). But the challenge of his book is primarily his insatisfaction with the mathematical analysis of the temporal continuum and in his critique of the artificial unity of the space-time, a very important (and very criticable) acquisition of the Mathematical Physics of his time[7]. Time, due to its irreversibility, to the nature of its continuum, is very different from space, as many thinkers, from St. Augustine to Weyl, have made the effort to tell us.

1.3 Between Intuition and Mathematics

Cantor and Dedekind have proposed a precise mathematical formalisation of the intuitive continuum, with at least three points of contact with our intuitive demands: the invariance of scale, the absence of jumps and of holes. This formalisation is based on very clear 'construction principles' : the sequence of natural numbers, quotients, limits of convergent sequences. Because <u>iteration</u> gives us the integers (we will come back to this point) and <u>quotients</u> give us the rationals; a <u>convergence criterium</u> for a sequence given by a rule gives us a *method to construct* the reals. A convergence criterium for a sequence, even if the sequence is not known a priori, indicates, without ambiguity, by retracing the interval, what we define as 'the real limit' of this sequence[8].

The theoretical import of this construction is massive and its conceptual force rekindles our vision of the world. Because *not only Mathematics and its structures, it is our knowledge that is not stratified.* Once a language and a expressive geometry intervene with the description of the world, they enrich it with forms, which acquire an objective autonomy. This is the basis and the result of the intersubjectivity, it emerges from the world, it is full of history and because of this, it is not absolute nor arbitrary. But above all *this language, this geometry will influence our original intuition*, for a dynamic game is then played. This game goes from our intuitions to their formalisations and when it returns to the the intuition, it modifies it. A 'classical' mathematician does not see a trace of a pencil, without seeing the continuum of $\underline{\mathrm{R}}$, which parametrizes the trace as a curve. He will talk about the continuity of this trace, of space, of time, of movement, directly in terms of his analytical language. Also the trace over the sheet, the comtemplation of movement are instruments for his own reflection, 'eyes for the mind' for the construction that he is trying to master, Analysis. And, before any proofs, he starts to use his intuition over the mental spaces of Analysis and Geometry, trying to understand them as he understands the string, as if they were realities of the same level[9]. From this comes the usual platonic ontology of most mathematicians. It is a formidable help to formulating conjectures and even proofs: Cauchy has 'seen' the right Theorem of the Mean Value. René Thom also has 'lived' for a long time amongst the continuous and differentiable varieties. His deep immersion into this conceptual space, his mathematical genius, have allowed him to 'see', first, and classify the singularities (the catastrophes), an exceptional mathematical (and cognitive) performance. For him, as for many mathematicians of the continuum, "the Continuum precedes ontologically the discrete", for the latter is merely an "accident coming out of the continuum background", "a broken line" ... "the archetypical continuum is a space that has the property of a perfect qualitative homogeneity", hence it gives us a vision, more than a logico-mathematical construction [Thom, 1992]. Actually Thom goes further "any demonstration is a revelation of a novel structure, where the elements solidify the intuition and where the reasoning reconstructs the progressive genesis" [Thom,1990;p.560]. An intuition, non emergent from the world, but observation of the universe of Mathematics where the "form of existence is without doubt different from the concrete and material existence of the world, but nevertheless subtly and deeply linked to the objective existence." For this reason "the mathematician must have the courage of his inner convictions; he will affirm that the mathematical structures have an existence independent of the mind that has conceived them; ... the platonist hypothesis ... is ...the most natural and philosophically the most economical" [Thom,1990;p.560]. Dana Scott more prudent said to this author: "it does no harm".

The advantages of the platonic hypothesis in the 'linguistic synthesis' for the everyday communication amongst mathematicians are enormous, due to the efficacy of the objective signification that it can give to the language and to the crucial 'scribbles in the blackboard'. But the foundational and philosophical drawbacks that it entails are also very important, for all transcendent ontology disguises the historical and cognitive process, the project of intellectual construction, of which Mathematics is rich, and in particular the 'proof principles' and the 'construction principles' which are at the basis of its nature.

1.3.1 Other Constructions of the Continuum

Discussing the continuum we have tried to describe how the mathematical intuition is built in our relation with the world, by "these acts of experience ... within which we live as human beings" [Weyl,1918;p113]. *On the basis of these life experiences, we propose descriptions and deductions, we make wagers, not arbitrary, but full of history and of intersubjectivity, of invariance within a plurality of experiences.* Those wagers, organised in mathematical theories, are our linguistic (Algebra, Analysis) and spatial formalisations (Geometry). *The 'transcendental objectivity' (in the Husserlian sense) but not transcendent, which emerges by these intellectual constructions and which modifies itself and enriches itself in history, will give (mathematical) forms to the world: forms that are not 'already there' and which will also modify and enrich our original intuitions.*

These proposals, these constructions, which aim to an objectivity not absolute anymore, but strong, full of intellectual and cognitive paths, of theorems, of intersubjective communication, are not unique. In the case of the continuum, Leibniz had proposed another construction, in an way too incomplete to resist the very robust construction of Cauchy, Weierstrass, Cantor and Dedekind. It was necessary to wait for the Mathematical Logic of this century, so that an alternative proposal became a new Mathematical Analysis, Non-Standard Analysis[10]. The non-standard analyst describes the continuum differently: despite a number of conservative extension results for the new theories with respect to Standard Analysis (they prove the same theorems, within the standard fragment of the language), his real numbers like 'halos of integers' are a different thing altogether and it is possible to demonstrate new theorems. The ordered set of non-standard numbers, the new real line, loses, for example, the invariance of scale (Hartong), one of the strong invariants of our different views of the continuum, see [Barreau&Hartong,1989]. The non-standard analysist hence view the geometrical space, the physical world in effect, in a different way; this change of theory and of intuition of the mathematical continuum seems to offer new insights in Mathematical Physics (see [Cutland,1988] and the articles by Lobry, Lutz and Reeb in [Salanskis&Sinanceur,1992]; [Salanskis,1991] proposes an epistemological analysis of the non-standard continuum).

Thom himself does not believe that the standard analysis, yet at the heart of his work, gives a definitive representation of the continuum. Dissatisfied by the arithmetical (and logical) generativity of the (non-)standard continuum, exactly like Weyl was by the treatment of the "continuous flux" and of the phenomenal time as a set of real points, he will suggest new ideas (see [Thom,1992]). By his mathematical audacity, he sketches a new conceptual construction. This conception is built from his mathematical work experience, which is comparable, for him, for its force and its evidence, to the experience of the world.

But the intuition that is constructed in the praxis of Mathematics is different from that which emerges directly from our relation with the physical world, even if they do get mixed up in our 'working mathematicians' minds. The first one, in what concerns standard analysis for example, is based on the Cantor-Dedekind construction and the work derived from that in more than a century. If Cauchy in his 'proof' of the Theorem of the Mean Value had made reference to well-defined notions of curve and continuity, if he could have appealed to the rigorous mathematical intuition of the standard reals, built over the correct definitions given some decades later, then his proof would have been a proof. He would have used the 'informal rigor' of the practice of mathematics. In a somewhat different understanding of this notion from Kreisel's, *the informal rigor is based on observations 'from above and from a distance' of definitions and constructions that we know to be potentially rigorous and then by the development of an informal deduction*: the rigor stays more in the precision of the notions than of the deductions. This method is so typical of work in Mathematics, so much based on 'intuition', because *it is built on the history and the practice of Mathematics.* This mathematical intuition, and the informal rigor which is grounded on it, is not the one of the 'man in the street' (even less the one of the paleolithical man): all the training in Mathematics, from the student to the researcher, is to acquire this informal rigor, difficult balance between intuition and formal rigor, which permits a demonstration and its comprehensible expression.

The identification of these two kinds of intuition, the one of the trained mathematician and the other developed only in everyday life, into one single 'pure intuition', is the origin of the difficulties to developing a cognitive analysis, not purely psychological, not purely logical of Mathematics. For the analysis of mathematical intuition, which is not given, which is not an absolute, but it is built in the interplay of acts of experience, language, design and formalization, is actually part of the analysis of Mathematics as a form of knowledge. Moreover, the confusion between different levels or kinds of intuition, from the common sense one to the one in the experienced mathematician, beyond history, gives a comparable or identical level to the objectivity from the physical world and to the objectivity from Mathematics: in both cases the intuition of evidence will be the same, as well as the one of invariants and stabilities. One intuition 'pure and unique' forces us to believe in the unicity of the theory possible; it makes difficult a comparative analysis of different theories, or of wagers of representation, which are proposed to treat mathematically the world and our intuitions of it and which are full of history and of questioning, as the intuition of the continuum.

1.4 From Mathematics to Logic

Take the subsets, the parts, of the set N of the integers, $P(N)$. If two subsets A and B are strictly included into each other they differ by a finite or infinite subset, but, we would say, in 'a discrete way', by successive jumps: it is integers, well separated ones, that A is lacking to get to B. I hope the reader can 'see' this in his head, using his mathematical intuition. But, this is not really the case: $P(N)$ contains chains (totally ordered sets) with the same type of order as R (i.e. the kind of order of the continuum: dense, without jumps or holes). The proof is easy: Q is countable, choose a bijective enumeration of Q by N and associate to each real number the integers which enumerate its Dedekind cut. Then you have a bijection (an order isomorphism) between R and a chain inside $P(N)$. Our construction principles have given us very rich structures, R and $P(N)$, so rich that they escape the intuitive naïve observation. Actually these structures do not exist: the property that we just 'saw' is not there, it is not *explained* as we explain a property of the world, *we have demonstrated it*, as we have built these structures, as conceptual constructions. The well-trained analyst can short-circuit this proof and see immediately the continuous chain, for the Dedekind cuts are as concrete for him as this table (to paraphrase Gödel). In any case, to construct the chain in $P(N)$, we have made some 'choices'. We have presented Q as a set of pairs (fractions) of the integers N. Each rational corresponds actually to an infinity of equivalent fractions; hence to give a bijective enumeration of Q we must enumerate $N \times N$ (easy) and *choose* a representant for each set of equivalent fractions. This choice is effective, for these equivalence classes are decidable – and the Theory of Recursively Enumerable Sets (and Recursive Functions) realises the Axiom of Choice[11]. This axiom, this principle, is a construction, or allows a construction, that of the "set of choices", composed of one element for each set in the collection considered (see the note). Hence it is a construction principle, since, by using a specific mathematical structure, it allows the construction of new structures. But it is also a principle of proof: once presented 'in abstracto' (that is, at formal level, with no intended domain of interpretation, as if it held for all collection of sets, without any hypothesis on decidability nor on order that allowed the choice of the 'first element' of each set) it becomes a purely logical stake: further than the finite (or decidable or ordered) it completely cuts itself off from the practices of life and it acquires a level of abstraction that makes it independent of the formalisations 'without structure' of mathematics (the *formal set theories*, see paragraph 5). Nonetheless the trained mathematician uses it everyday, without fear of error, knowing without knowing that he's using a powerful proof principle, which only the specific structure of certain constructions makes applicable. And he confuses his cognitive performance, the *vision* of the conceptual structures of his daily intellectual practices, with a mystical ontology.

Let us try again: cardinality is in first approximation the number of elements of a set. Cantor has shown, by a simple diagonal construction that R has more elements than N, the integers. The reader clearly sees the real line and the 'integer points' well isolated and regularly spaced. The rationals Q are dense and hence give an approximation for each real number. But they are as numerous as the integers. Is it true that if a subset of R is larger than N or Q then it has necessarily the cardinality of R? What does say the observation about this object universe? What says the pure intuition? Nothing. But still the reals are there, God at least must know them all, with their subsets.

To answer these questions it is necessary to make precise the 'frame of the set theoretical construction', to make precise our 'basic principles': if we consider the reals inside the universe of construtibles of Gödel we say yes, if we consider the reals inside the set theoretical universe of Cohen we say no. We do not know which framework God prefers. The question, from Cantor to Frege, Gödel, Cohen and D.Scott has been a key issue in Mathematical logic: it is the challenge of the Continuum Hypothesis (HC). We will refer again to it in section 5.2; but before that we must discuss 'iterations' and 'horizons'.

1.5 Construction Principles and Proof Principles

One of the 'theoretical situations' that gives 'certainty' or 'structural solidity' in the work of mathematicians and logicians is the joining of different methods, which converge to the same construction. When very different ideas with technical and cultural origins very much apart can be translated into each other, possibly up to isomorphisms, we are sure that we have in our hands a significant construction. For these connections, with different degrees of proximity, sometimes just embeddings without isomorphisms, are to be found in all interesting domains of Mathematics. That is the unity of Mathematics: these bridges, these translations, this to-and-from, these intellectual percourses through rough tracks, sometimes in parallel, which may arrive by shortcuts to well-known valleys. The audacious explorers (constructors!) will be rejoined by others, which proposed totally different paths, with (sometimes) independent goals.

The relationship between Intuitionistic Logic and Theory of Categories (by means of the Theory of Types) gives one of the more interesting and elegant examples of this kind of correspondence. A few remarks on this subject will allow us to clarify the notions of 'proof principle and construction principle', to mention a categorical semantics of impredicative definitions and of the notion of 'variation', which are the heart of the analysis of the continuum.

After that, we will briefly go back to the Axiom of Choice and the Continuum Hypothesis as logical axioms and as mathematical constructions.

In our views, the relationship we sketch below may be clarified by (and help to understand) some concepts of Husserl's. First, as we already said, the categorical semantics of formal languages offers a 'formal ontology' far more adequate to Husserl's prespective than the set-theoretic one. Category Theory is grounded on principles of constructions, within a structural understanding of mathematics, which allow to go far beyond the 'point-wise', unstructured and basically compositional, set-theoretic approach. Moreover, *the interplay between proof principles (syntax) and principles of constructions (closer to semantics) may be understood by means of a parallelism with the Husserlian correlation noesis/noema.* One has to interpret mathematical constructions as 'act of a subjet', in intuitionistic terms. However, *the 'objectivity' of these constructions, in our approach, is due to their relation to the regularities of the world and to their 'intersubjective' content, which is built by and in the dialogue with other subjective experiences.* (Of course, this significantly departs from Brouwer's solipsistic philosophy). Thus Husserl's 'noeses' may be understood, in mathematics, as systems of rules, possibly implemented in mental acts, and 'noemas' are the correlated objectivities, the intersebjective constituent of the mental constructions.

1.5.1 Conjunction, Quantification and Products

In Intuitionistic Logic[12] we say to have a 'proof' of a conjunction $A \wedge B$ (in a unique, canonical way), if we have a proof of A, a proof of B *and* the possibility of reconstructing from a proof of $A \wedge B$ a proof of A and one of B (we have projections). Let A and B be two sets, two spaces, any two mathematical structures: what we have just defined is simply the cartesian product $A \times B$ of A and B with its projections, which associate to each element, or proof of $A \times B$, one element or proof of A and one of B. More precisely it is the Category Theory version of the cartesian product, the product invented by the geometers, which thanks to its categorical generalisation gives us also the product of two topological spaces, two partial orders, of any two mathematical structures ... in their categories (of structures). We have already gone to constructions, having started with proofs: $A \times B$ is the *categorical* (in fact geometric) *semantics* (interpretation) of the intuitionist conjunction $A \wedge B$.

In Mathematics, in Algebra, in Geometry when we have a construction, we usually have another, its dual, for free. Category Theory says that it is enough to reverse all the arrows, that is the direction of all morphisms or functions between objects, to obtain the dual of a given construction. In the case of the product, we reverse the direction of the projections. This way we obtain the categorical coproduct, which can be constructed in several categories. This corresponds to the notion of intuitionistic disjunction: the famous intuitionistic disjunction $A \vee B$, of which we have a proof if and only if we have a proof of A or a proof of B and we know of which one the proof is. In particular, for this notion of disjunction, $A \vee \neg A$ (A or not A) is not demonstrable: to prove it, it is necessary to have a proof of A or a proof of $\neg A$, hence the 'excluded third option' is not valid. More formally, write $\underline{S} \vdash C$ to mean '$\underline{S}$ demonstrates C'; then, in full generality,

$$S \vdash A \vee B \text{ if and only if } S \vdash A \text{ or } A \vdash B$$

and hence the theoretic 'or' ($\vee$) corresponds to the metatheoretic 'or'. In a classical system this beautiful intuitionist symmetry theory/metatheory is lost, for the implication from left to right is false. Hence this intuitionistic 'or' is not so odd: it is simply the dual of a very familiar geometric construction, the cartesian product, and it transfers the metatheoretical disjunction into the theory.

We can also show that the intuitionist implication can be interpreted as the exponential objects in the categories closed under this construction. In intuitionistic systems, a proof of $A \to B$ is a 'computation' which takes any proof of A into a proof of B or, in Type Theory, a term of type $A \to B$; informally, it is then a 'computable' function from A to B. In Category Theory, the exponential object, which may also be written as $A \to B$, represents exactly the set of morphisms or functions between two objects of a category.

But Mathematics needs variables. The syntactic entity represented by $x, y, \ldots$, which is an <u>individual variable</u> in Mathematical Logic, is a projection in Category Theory. When it appears within a formula, this generalises to the notion of <u>fibration</u>, a categorical way of talking about *variation*. Thus the universal quantification $\forall x \in B.A(x)$ (for all x in B we have $A(x)$) corresponds to a <u>fibred product</u> (or pullback), a notion well-established in Geometry, a kind of 'generalised cartesian product': actually universal quantification generalises conjunction, for $A(x)$ must be true at the same time for any x in B. This is an infinite conjunction or a limit: very informally it corresponds to $A(b) \wedge A(b') \wedge \ldots$ for all elements $b, b', \ldots$ in B.

How do we understand the existentencial quantification $\exists x \in B.A(x)$ (there exists a x in B such that $A(x)$ holds), always in the first-order case (that is when the variables are individual ones) ? The seminal observation of Lawvere is that this is nothing but the dual of the product above, with respect ot the operation of substitution (formally $\forall x$ and $\exists x$ correspond, respectively, to right and left adjoints to the substitution functor, see [Lambek&Scott,1986]). Thus, once more, syntactic principles from Logic, indeed Frege's first-order universal quantification, nicely corresponds to actual constructions in geometry.

Matters get more complicated when we consider variables over propositions or sets (we will write them with capital letters $X, Y, \ldots$). Why this extra work? When discussing the continuum from the logical point of view this is inevitable: the real numbers of Analysis are sets of integers, the numerical codes of (equivalence classes of) Cauchy sequences of rationals. For this reason the Arithmetic of second order, with variables ranging over propositions is considered the logical counterpart of Cantor-Dedekind's Analysis. Here comes the difficulty: the variables do not vary over a set or predicate, as in $\forall x \in B.A(x)$, but instead they vary inside the collection $Prop$ of all the sets or propositions, including $\forall X \in Prop.A(X)$, the proposition that we are trying to define, for $\forall X \in Prop.A(X)$ is in $Prop$. Danger, danger: impredicativity got us. No problem: we will sort things out in two different ways. Through a normalisation theorem (we will see this in paragraph 5.3) and through a construction, that does not depend on the logic, and which has its origin in Geometry (the Grothendieck topos and the ideas of Lawvere). Inside these geometrical categories we can give a structural meaning, as a closure property of certain categories, to this stake that worries many logicians (but very few mathematicians and computer scientists). Briefly the variation will happen now over a category and not simply over an object of a category, as in the first order case, for we need to give meaning to the variables over propositions and each proposition is an object; thus it is necessary that this category be closed under products indexed over itself. All this gives a new structure for the variation and a strong closure property. The circularity of the impredicative definitions becomes then a theorem, the closure of certain categories under generalised products, whose origin is geometrical (see [Asperti&Longo, 1991]).

The only difficulty is that the construction cannot be done inside a classical Set Theory ([Reynolds, 1984]), instead one needs an intuitionistic environment ([Pitts, 1987], [Hyland, 1988], [Longo, Moggi, 1991]). Once again, but this is complicated, the geometrical symmetry between $\forall X$ and $\exists X$ can be represented as left and right adjunctions, with respect to a functor that also generalises the cartesian product, the diagonal functor.

Here we have a game of principles of proof and principles of construction that have very different origins and motivations. We understand ones through the others and this way we obtain one of these conceptual chains that are the kernel of the mathematical construction.

We have used implicitly in this sketch of a mathematical semantics of proofs, some constructions that take us back to infinity and the continuum. We have touched the continuum in two main ways: the semantics of the notion of variation or change, which is one of the elements of the phenomenon of the continuum, and the impredicative definitions. But there are also passages to the limit, which are implicit in the categorical constructions of the product, as the universal quantification $\forall x \in B.A(x)$ is an infinite conjunction, a categorical limit.

We then go back to infinity, to limits and to the continuum in Mathematics and Logic.

1.5.2 Limits and Closings of the Horizon

Despite the supporting references to systems of Intuitionstic Logic, the reader should not suppose that the author is a 'devoted intuitionist' as we can still find them (and of great scientifique value) in Northern Europe. The notion of conceptual construction discussed here is the one which emerges from the practice of Mathematics and it is more general than the one of Brouwer or as formalised by Heyting. The interest for Intuitionism is first *mathematical*: these systems have a correspondence in Geometry (Topos Theory) which is hard to find for other logical systems. But the interest in Intuitionism is also *methodological*, because of the emphasis it puts on the notion of construction[13]. But we should not make a limiting religion of our extraordinary creative possibilities, when it comes to mathematical constructions. Infinity for example has been part of our practices of language and of our perception of space for a long time, too long for us to try and expel it from our mathematical practice or from the logical theoretizations.

Consider the sequence $1, 2, 3, \ldots$ that we can iterate without any reason to stop. Its closure, on the horizon, which we call ω, is it not as clear and certain as the finite iteration? Nowadays with computers that do iteration so well, we can observe what happens after iteration more easily than in the past: the finitist engagement in Logic this century is in the origin of (the development of) these formidable digital machines that have changed our daily life[14]. This finitist effort should remain with the machines! We can continue, as mathematicians have done forever, using this construction, this going to the limit, without fear of losing our "unshakeable certainties". And we can state with no problem:

$$\omega + 1, \omega + 2, \ldots, \omega + \omega = \omega \times 2$$

But now the playing is easy, the construction evident:

$$\omega \times 2, \omega \times 3, \ldots, \omega\omega = \omega^2$$

Why not carry on? The rule is there:

$$\omega^2, \omega^3, \ldots, \omega^\omega$$

So long as we have a persisting iteration, we, human beings, we get bored. This is one of the differences between us and the computers: boredom. Computers don't get bored: iteration is their strongest point. We, once we have understood, once we have detected a regularity, we look further afield, we see the horizon, ω or even ω^ω, as we see the image of poplars in [Chatelet, 1993;ch.2.2]: we enclose into one single look the range that repeats itself in the direction of the horizon and we project it over an actual infinity. This is a human experience which is gradually made explicit in concepts through the centuries; maybe it has its origins in the Oriental religions, as Weyl would have it[15]; in any case, this experience has developed because of and within the mathematical practice, where religious commitment and platonist ontology can mix up and justify a conceptual construction, as with Galileo, Newton or Cantor. But what happens if we continue the iteration of the exponentials? We have ω to the power ω to the power ω ... on the limit, in the horizon this will be simply ω to the power ω, ω times. This ordinal we call ϵ_0, it gives the smallest solution to the equation $x = \omega^x$. Do we need a transfinite ontology to describe and use this construction? No, a simple principle of going to the limit, to the horizon, suffices, if we have an explicit iteration (as in this case) or a criterium of convergence (as in the case of Cauchy sequences). The ordinal ω is not in the world, it is not a convention, nor merely a symbol: *it synthetises a principle of construction*, a "disciplined gesture" to paraphrase Chatelet, rich of history. Its rigorous use in Mathematics has given it a meaning, has inserted it inside operative contexts, has shown us its different points of view, briefly has found it a place within the conceptual network we call Mathematics. This gesture reiterated gives us ...$\omega \times 2$,... ω^2,...ω^ω... ϵ_0. And whatever follows[16].

The utilisation of ϵ_0 in proofs has huge consequences. To begin with, Gentzen showed, in 1936, the consistency of Arithmetic, by induction up to ϵ_0, hence using methods beyond the finite ones, which are below ω[17]. Next, this "skeletons of infinity" can be found in the minimal construction of a model of Set Theory: Gödel's constructibles, which takes us back to the continuum. Gödel's idea in 1938 was briefly as follows: starting from the empty set, repeat by induction up to ϵ_0 the constructions formalised by Set Theory in their language and noting else[18]. The real numbers built inside this mathematical structure do satisfy the Axiom of Choice (and the Continuum Hypothesis), for reasons of minimality that we can guess: the sets of real numbers have minimal cardinality (see [Jech,1973], [Devlin,1973]). Cohen in 1966 proposed another construction for Set Theory: he adds generic or arbitrary elements, whose properties are "forced" bit by bit, during the construction of the model, in a way that *does not realise* the Continuum Hypothesis (or the Axiom of Choice).

We normally say that these two major results show the independence of the Continuum Hypothesis (and of the Axiom of Choice) from the formal Set Theories (Zermelo-Frankel, etc..). But this is not the most interesting aspect: *the meaning of these theorems is in their proofs.* They consist of mathematical (set-theoretical) constructions inside which certain properties are realised and through this they give us precise information about the nature of these properties (in particular about the structure of the continuum and the cardinality of subsets of the Cantorian reals: they depend on the construction made). The fact that these properties are independent from the Formal Set Theories concerned (the independence) says nothing about the continuum, but simply underlines the poverty of these formalisations, which are independent of any structure and which were born exactly to answer the questions about the continuum and about choice. Frequently formalism forgets the constructive and structural nature of Mathematics: Gödel and Cohen's constructions remind us of this.

1.5.3 The Infinite in the Trees

Yet another relevant construction, in Mathematics, is that of 'tree'. Mathematical trees have their root on top: a unique node, which branches downwards. A tree is finitely branching if each node has a finite number of nodes below it; a branch is a sequence of consecutive nodes, a path with no interruptions that starts at the root and develops to the bottom. Consider now the following principle, known as König's Lemma (KL): "in a finitely branching, infinite tree, there is an infinite branch".

The reader certainly understands, 'sees' this geometrical property of trees: if the infinite tree cannot grow infinitely horizontally (since it is finitely branching) it must grow infinitely vertically. This is an easy observation about the construction of trees, by an 'insight' onto the plane or the structure of planar trees. However, we cannot, in general, effectively produce (construct by a calculable process) the infinite branch, even if the nodes are labelled and the tree is effectively produced (recursively enumerable). More precisely: one cannot give an algorithmic rule, write a program that generates the infinite branch, for the computer will have to go down paths for exploration and returns, erasing and reconstructing its memory in a non-effective way. Hence this principle, even if evident, goes beyond usual effectiveness; it is not intuitionistically acceptable[19].

Yet this principle has several applications. One is implicit in the categorical analysis of the impredicative definitions, mentioned in 5.1: a somewhat similar principle, the Uniformity Principle (UP, see the last note above), is used in the construction of the categories closed under products indexed over themselves ([Rosolini, 1986], [Hyland, 1988], [Longo&Moggi, 1991]; see [Longo, 1987] for a partly informal exposition). The principle hence contributes to giving structural semantics to the syntax of impredicativity: as we said in section 5.1, we can 'understand' the impredicative definitions as closure properties of certain categories. Moreover, Tait-Girard proof of the normalisation theorem for impredicative Type Theory (see [Girard&al, 1989]) uses König's Lemma and one "comprehension" axiom over Sets of the following form

$$\exists X \in Sets. \forall x.(x \in X \iff A(x)).$$

The naive platonists, which accept this axiom, and the limitative constructivists of different schools, that reject it, all attribute to it an ontological content, on the basis of a "prejudice (in fact a medieval one) according to which the *same* logic holds for Mathematics and the real world – this implies, as a consequence, that an existential quantification must refer to singular individual entities really existing as separated, independent and transcendent entities" [Petitot, 1992]. This mistake that Petitot describes very well, is based on forgetting the role of proofs in Mathematics; it is sufficient to observe closely the argument for "strong normalisation" in Type Theory, in [Girard&al, 1989; par. 14] for example, to see that this axiom is simply a *principle of proof*: it 'just" permits to *replace* one variable over propositions (or types) for a given collection of terms, defined during the proof. Where is the ontological miracle?

A major consequence of the Strong Normalisation Theorem for Girard's system, and also for other systems starting with Gödel's 1958 system, is a demonstration of the consistency of Arithmetic of first and second order, and hence of Formal Analysis (see [Girard&al, 1989])[20].

In summary, non-effective insights or conceptual constructions are part of the mathematical practice and the metamathematical theoretization, with no need to refer to 'ontological' principles. The consequences of an 'existentially quantified' assertion (a comprehension axiom, say) are *logical consequences* of a possible (or assumed) constructions. The insight into trees may be as certain as an effective procedure[21].

1.6 The Logical Independence

The first great result of incompleteness, or of undecidability with respect to an interesting formal system, is Gödel's result, in 1931. In particular, Gödel's "first incompleteness theorem" shows that formal Arithmetic, which can code all effective processes, contains an undecidable propositions, call it G, if it is consistent[22]. The second incompleteness theorem says that Arithmetic does not show its own consistency. More precisely, the second theorem shows that Gödel's undecidable proposition G is provably equivalent, in Arithmetic, to consistency. Hhence the second theorem shows that G is true, in the standard model, if we suppose the consistency of Arithmetic.

Later, to show the consistency from Gentzen to Girard, it was necessary to come out of the effective finitism and make use of stronger principles of proof, as hinted above.

We have also mentioned two other major results of independence, as consequence of Gödel and Cohen's constructions: the Continuum Hypothesis and the Axiom of Choice are not demonstrable nor refutable within Formal Set Theories.

By this, are there mathematical truths that we cannot reach through 'demonstration'? How would this be implied by the results of incompleteness or independence, if we just mentioned the existence of constructions which *demonstrate* the consistency of Arithmetic, of the Continuum Hypothesis and of the Axiom of Choice? *There are no propositions that are 'true and not demonstrable' in Mathematics.* True and demonstrable with respect to what, with respect to which construction and which proof principles ? One must make this precise.

There is in the usage of the this phrase, 'true but not provable', a 'slipping of meaning', very relevant and typical of naive Set Theory. We only have a precise notion of 'truth of a proposition' with respect to given mathematical structures (there are in fact several notions: Tarski's, Kripke's, Brouwer-Heyting-Kolmogorov's...). But we believe naively that there exists a *set* of true propositions. And hence the mystical reasoning: we move from the *notion of truth* to the *collection*, which exists in God's mind and which contains, one by one, the true propositions, in a well-ordered fashion. Which one, then, between the Continuum Hypothesis (CH) and its negation $\neg CH$ belongs to this collection?

In Mathematics when we talk about the truth of a proposition, it is necessary to say what we mean by this (that is, with respect to which notion of truth and with respect to which structures) and moreover it is necessary to show the truth with respect to this structure, to this notion. That was what Gödel did with the proposition G ("this proposition is not provable"), which he showed to be codifiable and undecidable in formal Arithmetic. He also proved that this proposition is true in the standard model *under the hypothesis of consistency*, an obvious consequence of the second theorem of incompleteness (the equivalence, in Arithmetic, of consistency and G). But Gödel did not say that the consistency of Arithmetic, undemonstrable in Arithmetic, is "true": for that it would be necessary to use Gentzen's proof, based on stronger principles. Gödel (and Cohen) will give us structures where the Continuum Hypothesis and the Axiom of Choice are true (or false) and they *proved it.*

Hence, what is this phenomenon of incompleteness, so important for the treatment of the continuum in Logic?

When discussing the intuitionist conjunctions and disjunctions, we saw a perfect correspondence between proof principles and categorical constructions. But this is not always this perfect. *The incompleteness of a formal theory, with respect to a precise structure, appears when we have a rift, a gap, between proof principles and construction principles.* Formal axioms, abstract principles, syntax for the manipulation of symbols and proofs in one side, constructions, in general geometrical or structural ones, in the other. The mathematical and logical difficulty lies in 'putting the finger on' the gap by providing theorems, making precise the proof principles and the construction principles utilised.

Actually, even once the principles of proof and construction are well-described, there is not always a clear demarcation between them; think for example of the Principle of Uniformity or of König's Lemma, or the Axiom of Choice, which are always between proof and construction. Amongst the ones we have seen, perhaps only the axioms of comprehension do not look like principles of construction and are pure 'proof techniques'. Also the rules and the formal axioms of the Arithmetic of Peano-Dedekind or the logical systems of first order of Frege and Hilbert, the Set Theories, are very clearly principles of proof, derived from mathematical constructions (Number Theory, Analysis, ..). *It is in the difficult to detect, but possible gaps, between formal proofs and mathematical constructions, that incompleteness theorems can be found*[23].

The incompleteness theorems of Gödel (the first: under the hypothesis of consistency, there is an undecidable proposition; the second: the consistency is undecidable) and the consistency proofs from Gentzen to Girard show that in the construction of the integers and their properties we use, or we can use, if we we accept them, strong principles, beyond formal arithmetic: we show hence that the consistency is a *true* property over the integers (and hence we show the *truth* of the undecidable proposition given by the first theorem of Gödel, which is non-demonstrable in Arithmetic and equivalent to consistency).

The constructions of Gödel and Cohen prove the same thing about CH and AC: they show that they are true (or false in Cohen's case) on certain structures, contructed using certain principles, but that they are non-demonstrable using simply the axioms and rules, the proof principles, of Formal Theories of Sets[24]. In other words, all these results (and many others: Paris-Harrington, Kruskal-Friedman ...) by proving the truth or validity of certain propositions over *possible* mathematical structures (universes of sets or of numbers) or by proving their unprovability within given formal systems (described by *possible* proof-principles), 'simply' display the gap between mathematical constructions and formal theories.

Thus, *one should never say in Philosophy of Mathematics the phrase 'there are true but non-demonstrable propositions', for this phrase makes no sense in Mathematics.* A working mathematician (not on Sundays, for then he does the usual naive platonic philosophy) asks immediately ' *Non-demonstrable* with respect to which system (to which proof principles)? *True* in which structure (using which construction principles and notion of truth)?'

This century, the formalism, in Logic and Proof Theory, which going further, has found in finitism and formalism its origins, have without doubt helped to answer these questions. But why logical formalism, a philosophical indirect springing of the mathematical practice, should be the ultimate source of our certainties, of our analyses of proof and of construction in Mathematics?

The conceptual networks, inside which the mathematical constructions are embedded, do not give us the ultimate certainties, but insert each construction within other forms of knowledge. These give it a meaning, several meanings, whose connections and compatibilities, form the net, relatively solid, of our relation with the world. It is the practical unity of Mathematics and its emergence from the world which contitutes its foundations: this frame and the balances of theories, which translate each other, interpret each other, give root to each of its nodes in our forms of knowledge.

The analysis of proofs, Proof Theory, is one of its instruments. The different structural semantics will provide others. But it is necessary to insert Mathematics in the triangular relation history–individual–world, by reconstructing the cognitive and historic percourses which are at the origin of the mathematical invention.

Our effort towards the comprehension of the world is like a walk over quicksand: when we throw the net of our knowledge, of which mathematics is but a small part, this net will permit us to advance a few steps, just by its extension. The challenge of naturalization, as cognitive analysis, and as analysis of the historical and collective construction of concepts (mathematical ones in particular) consists in finding a few supporting points for this net.

1.7 Three Levels and the Richness of the Continuuum

In this article we have underlined, initially, the non-unicity of the intuition of the continuum. Then we have developed an analysis which emphasized three levels: the intuition one, the construction principles one, and the proof principles one.

On the first level, the richness of the world and of points of view from which to observe it, compatible points of view, non isolated, but built from a dialogue with evolution and history, suggest a plurality of intuitive approaches and ground mathematics in our relation to the world. In part we find these points of view in the different mathematical constructions of the continuum, which constitute the second level. These constructions enrich and modify the original intuitions, which are not that simple when the mathematical praxis adds to them its depth. But thanks to Logic there is a third level, where the analysis of the proof (as well as the Formal Set Theories, their axioms, their rules of inference) plays an essential role. Clearly, the incompleteness results lay in between the second and the third level, as a precise form of indetermination of mathematical constructions by formal theories.

Speaking transcendentally, the mathematical objectivity does not find its origin in the unicity of the intuitive giving, nor in the categoricity (or unicity) of the psycho-physical genesis of that one, but instead in the common, historical and cognitive (hence intersubjective) process of the conceptual construction. That is, *in the mathematical construction the value and the objective realities are not to be found in the mathematical entities (the integers, the real numbers, ω or ϵ_0 for example) but in the process of constitution of these so-called entities, as conceptual constructions:* the iterations, the passages to the limit, the closures of horizons, the constitutions of invariants[25]. In the case of the continuum, the mathematical objectivity is also in the richness of interaction of three levels we mentioned: intuition, Mathematics, Logic. This interaction is not a vicious circle, but a virtuous one, extraordinary example of the dynamicity of our forms of knowledge: Logic, for example, which only extracts formal rules from the constructive practices of Mathematics, offers, thanks to the incompleteness theorems and Non-standard Analysis, new mathematical structures, which suggest a new intuition about the continuum. A further starting point, through games of dynamic reflections, for other constructions and formalisations.

1.8 Conclusion

The logicist and formalist philosophies of Mathematics, in this century, provided the conceptual philosophical background of major scientific achievements. The birth of computability and, hence, of Computer Science is one example. Another is the remarkable proposal for a precise notion of mathematical rigour: formal rigour, as the 'potential deduction of a theorem by a mechanical (in principle) system of axioms and rules', tells us what it may exactly mean that a proof can be carried on with 'absolute' certainty. We also know what is a good definition and the paradoxes are distant. Now that formal rigour has been objectivized in formalisms (and computers) *we can reconstruct the meaning and the practice of demonstrations and widen our notion of rigour, by encompassing also diagrams, metaphors, images; by trying to understand the role of the 'geometric insight'.* This is not about opposing a new Proof Theory to the old one, but about enriching Logic and Proof Theory, making it come out of the formalist cage which generated the so-called "logico-computational hypothesis" for the human intelligence: "Intelligence ... is effectively defined as that which can be manifested by the communication of discrete symbols" ([Hodges, 1992]). Hence the direct, geometric proof of Pitagora's theorem do not contain "explicit" intelligence, even less foundational interest for Mathematics: intelligence develops only after its traduction in finite algebraic languages, if necessary pixel by pixel, over discrete cartesian coordinates.

We found traces of a different analysis of foundation (and of mathematical intelligence) in the work of logicians who have insisted on the role of Geometry. For example, in the denotational (or categorical, see section 5.1) semantics of Lawvere-Scott for intuitionistic systems (or for programming), geometry and/or continuity give signification to lists of symbols 'without meaning', for "Geometry is more compelling", as Dana Scott suggested once. Or also in the geometry of proof nets by J.Y. Girard, where the symmetries and the direct manipulation of images (networks over the plan) come into the play of logical derivations, in an essential way. Moreover geometry is central in the recent mathematical development of husserlian analysis of knowledge, as in Petitot's work[26]. It is perhaps 'vision' that is more compelling, as some neurophysiologists claim.

But this widening of Proof Theory should not be just a new game of mathematical rules, as this would only give us a new mathematical discipline. Wittgenstein had forseen this happening with the hilbertian metamathematics [Shanker, 1988] and it has in fact happened. Metamathematics became a new and beautiful kind of mathematics, where the principal results have been indirect: a precise notion of formal rigour and ... Computer Science, but not the explicit foundation of the mathematical practice, as was Hilbert's dream. We can not 'found' mathematics (its "rules of the game" as Wittgenstein says) over a mathematical discipline, a logical-mathematical system also made up of mathematical "rules of the game". *There cannot be an internal foundation, purely formal and mathematical, of Mathematics: the incompleteness theorems are not accidents, they underline the gap between the metamathematical principles of proof (once transformed into a mathematics of formal rules) and the rigourous practice of mathematical constructions.* It is then necessary to increase the variety of tools for the foundation of mathematics, first by the direct constructions of Geometry (which is being done), then with other forms of knowledge; that is, retaking the metaphor of knowledge as a network (end of Sect. 6) *it is necessary to insert the partial network of Mathematics in the wider one of the other forms of knowledge.* The project to aim at should take mathematics out of its 'auto-foundational' game (metamathematics as a form of mathematics) and look for its cognitive origins in our relation to the regularities of the world, in the connections to different conceptual constructions, in the mental invariants that we build while living and historical beings.

Acknowledgments

I would like to thank warmely Jean Petitot for his many detailed comments on an earlier version of this paper, in particular on Husserl's philosophy, as well as for his constant and critical encouragement. Thierry Coquand, Françoise Longy and Valeria DePaiva made further helpful comments.

Notes

1. A preliminary version of this paper, in french, is available from the author.

2. St. Augustine in his Confessions, which Weyl unfortunately does not cite, has the same point of view. Time is a primary notion, independent of movement. "There are three forms of time:... the present time of things gone, which is our memory; the present time of present things, which is our vision; the present time of future things, which is our wait" [St. Augustine,401;Lib XI, ch.XX]. Evidently St. Augustine is not talking about intuition: time, Mathematics and the knowledge of God himself reside in *memory*. More precisely, the measure of time is given in *memory*, because on remembering we compare the temporal segment of a short syllable, "which is not there anymore", to a long one [St. Augustine, 401; Lib. XI]. Contrary to Aristotle's opinion in Physics, for St. Augustine movement

is not a primary notion: it is time that permits us to describe it as velocity (in modern terms, as a continuous function of space and time).

3. An early counterexample is due to Weierstrass. One variant of his example is the function $f(x) = \Sigma \frac{sin(2^n x)}{2^n}$. One was surprised until Poincaré by the fact that this function is nowhere differentiable (and with good reasons...).

4. It is interesting to note how usually we talk, in Mathematics and in Logic, about the 'reduction' (à la Cantor-Dedekind) of the real numbers to the integers, as if the reals were already 'there', as if the 'informal practice' of the mathematical continuum (see Cauchy's demonstration of 1821) made reference to an external objectivity, that we must understand by reduction (the same way we reduce some chemical realities to Physics). This is comprehensible in the naif platonic practice of Mathematics, but it it is less so for the formalist/definitionist vision of mathematics still prevailing in Logic.

5. In Set Theory, in writing $\forall y$ for 'for all y', a set b is defined impredicatively if, typically, it is given in the form

$$b = \{x | \forall y \in A \ P(x, y)\}$$

   where b can be an element of A (the same set or collection A of sets which appears in the definition of b). Briefly in an impredicative theory there is no stratification of the mathematical universe and it is acceptable to define one element b using a predicate or set A which can contain b. Informally we can not comprehend the parts, the elements, without comprehending at the same time the whole, or a big part of the whole.

6. Also for Leibniz and Kant the continuum cannot be decomposed into its elements, it is not formed from simpler unities: it presents itself simultaneously as a totality and its parts (see [Panza,1989]).

7. In the sequence of his fundamental reflections Weyl first joined the ranks of Intuitionism, then he embraced a more open view of mathematical knowledge. But the mathematics of Brouwer and the logical systems of Heyting are compatible with the impredicative notions: in fact even the definition of an intuitionist proof is impredicative (see [Longo, 1987] for more: yet the interplay is between theory and metatheory, so it is acceptable for many). After that, in his logical-philosophical writings (see the French version of [Weyl, 1918] for many references), Weyl will never go back to his Predicative Analysis. On the contrary, he will develop a very rich vision of the connections between Mathematics and Physics which will culminate in [Weyl, 1953] his last book, a Husserlian masterpiece, clearly anti-formalist: Mathematics emerge from the effort to know the world (physically, chemically, artistically ...) as a 'transcendental objectivity'. See also [Weyl, 1985] for a very balanced and 'secular' view of the instruments of demonstration in mathematics.

8. In Intuitionistic Mathematics we distinguish between sequences given by a 'law', or 'lawlike', and 'lawless' ones. Here the 'law' is an algorithmic (or effective) rule. For example, π is the limit of a lawlike sequence (the algorithm for constructing

it), whereas a real whose decimals are given by successively playing a die is the limit of a lawless sequence. But even a convergent lawless sequence obeys a rule and follows a convergence criterium: in the lawless sequence above, one plays the dice and add its results as decimal numbers. The limit is unique and well-defined: the criterium of convergence is given by the fact that 'we add as decimals' the results of the throw. It is the existence (of the limit) that is weak, non-effective.

9. "Among the usual spaces that better embody the ideal of the continuum, there are two that appear almost immediately: the euclidian line and the euclidian plan; the line for its mechanical and physical realisations (the extended thread, the luminous ray) ..." [Thom, 1992; p.142].

10. Leibniz infinitesimals became the new real numbers, smaller than any other standard real number. Then $x \approx y$ if $x - y$ is infinitesimal and hence a function is continuous if $f(x) \approx f(x+h)$ for all infinitesimal h.

11. Axiom of choice (AC): "For all non-empty collection of non-empty sets, we can contruct a set which constains exactly one element from each set of the family". The Axiom of Choice is essential in many demonstrations, including some that concern the continuum: without AC the definitions of limit based on neighbourhoods and the one based on sequences are not equivalent (it is necessary to *construct* a sequence, by *choosing* a point for each succesive environment).

12. The few technical notions in this section will not be used in the sequel: they are just examples of elementary connections between principles of proof and principles of construction. For more details on intuitionistic systems of types and Category Theory, see [Lambek&Scott,1986], [Asperti&Longo,1991].

13. We could say the same about Girard's Linear Logic as its nature makes even Classical Linear Logic ... "constructive" [Girard, 1991].

14. The research on the "unshakeable certainties" of Hilbert and Brouwer (see [Brouwer, 1927]) has given us this century a very solid notion of mathematical rigor: the finitist deduction, formal and effective. Over this basis, the amazing Thirties of Logic have seen the birth of one precise notion of calculation and of machine, the foundations of the modern programming languages (Turing: the imperative languages; Church: the functional ones; Herbrand: the logic ones).

15. See the note "D'Anaxagore à Dedekind", 1926, in the French version of [Weyl, 1918].

16. We can continue with ϵ_1, ϵ_2,..., ϵ_ω and having understood the mechanism, which after ϵ_0 is not that simple, we can continue with ϵ_{ϵ_0} ... .

17. But this method of proof was not considered very convincing by many, for the heart of the problem of consistency for Arithmetic is the *consistency of induction*, the key principle of Peano's Arithmetic: one shouldn't use an even more powerful induction to show it. There are other ways of proving it, tough, using formally equivalent, yet more conving, methods.

18. This means reiterate the constructions formalised by the axiom of power of a set, of replacement– image of a set by a function, etc ... so long as they are definable in the language of Set Theory.

19. Consider the contrapositive of König's Lemma (KL), which is called the FAN Theorem since Brouwer: "if in a finitely branching tree, each branch is finite, then the tree is finite". FAN says that if we 'get stopped' along each of the descendent branches, then a finitely branching tree is uniformly limited: it is thus a compactness property. Most intuitionists (e.g. [Troelstra, 1973]) accept FAN, which does not imply KL, for the equivalence between FAN and KL, its classical contrapositive, is not intuituionistically valid. (In general, in Intuitionistic Logic, $\neg\neg A$ is not the same as A and we cannot go from $\neg B \rightarrow \neg A$ to $A \rightarrow B$ (read KL as "A [infinite tree] implies B [there exists an infinite branch]").) In [Troelstra, 1973] a relevant variant of FAN is proposed, the Uniformity Principle (UP): under certain circumstances, "$\forall x \exists y$..." may be replaced by "$\exists y \forall x$..." (or, y is *uniform* for all x; compare with the alternation of quantifiers in FAN: for every branch there is an end *implies* there exists an end for any branch).

20. The theorems of cut-elimination and normalisation for the systems of higher order give extremely solid bases to the impredicative definitions. The consequence is that every proof in the system can be simplified to a 'minimal form' (a normal form or without cuts), or that there are no 'incontrollable propositions' that can introduce themselves into proofs. We must note that the second principle of proof mentioned here is sufficient to prove the theorem of normalisation, but the proof of Girard, which uses both principles displays very clearly, for its elegance, the issues of the construction. See also [Fruchart&Longo, 1995] for an application of a recent theorem to the justification of impredicative definitions.

21. As a matter of fact, even much stronger properties of trees than the previously described compactness property, (Köng's Lemma), may be acceptable. This is too complex a matter to be described in short, but it may be worth hinting that also the so called "determinacy for Δ_0 trees" bases its reliability on an insight into the planar structure of trees. Consider a well-founded tree (roughly, a tree with no infinite branch) and let two players play the following game. Player one moves downwards from the root by choosing one node; player two moves further by choosing a node below. The first player that cannot move anymore (he is on a leaf) has lost. Fact: there is always a winning strategy for one of the players. The proof goes by an 'easy', but powerful induction: it is trivial for 'one node trees'; given a tree, assume the thesis for all the trees obtained by erasing the root, then prove it for the whole tree (obvious). Surprisingly enough, one may derive from this fact the consistency of Arithmetic (and even more): the expressiveness (and difficulties) depend on (the careful - impredicative - definition of) the rich structure of well-founded trees and the use of induction on them (see [Moschovakis,1980], for example). The latter turns out to be convincing, even certain, by the insight into the planar structure of trees.

22. In particular the formalised proposition, which says "this proposition is not provable".

23. Computer Science has given new motivations to the work of logicians, since, without reference to the mathematical structures, they try to analyse the practice of programming and the conception of the architectures of computers (and to propose new designs). However, in the good practice of computing the unity of Mathematics imposes itself again, in the research for a mathematical and structural meaning for these theories (semantics of programming languages: denotational, algebraic,..). In some cases, the incompleteness of theories, w.r.t. semantic models, has suggested relevant extensions of the formal or programming theory or language.

24. We could mention that beyond products and coproducts, which correspond so well to the intuitionist conjunction and disjunction, the Effective Topos, which is the basis of the sketched constructions of second order, is constructed using principles that go beyond the other 'pure' intuitionist rules (the principles of Uniformity and of Markov, amongst others): hence the Effective Topos shows the truth of non-demonstrable propositions of the systems of intuitionistic logic, of which it is a model. The Genericity Theorem, see [Longo, 1995; Fruchart&Longo, 1997], gives another mismatch between the Topos and Intuitionistic Logic.

25. See also [Petitot,1992; par. II.1].

26. For a philosophical introduction to the "geometry of perception", and also for its numerous applications to which it referes, see [Petitot, 1995]. About the role of the continuum in linguistics, see [Fuchs&Victorri, 1994].

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Chapter 2

C0NTINUOUS STRUCTURES AND ANALYTIC METHODS IN COMPUTER SCIENCE[1]

Abstract

Most of the structures one deals with in Computer Science have a discrete and effective character: (finite) graphs, nets, trees in programming and formal languages, algorithms over finite strings or natural numbers, circuits etc... By continuous structures we mean "smooth" spaces, usually of cardinality not less than continuum, where interesting topological or order properties give some information on, say, classes of functions over them. By discussing results from various areas of Mathematical Computer Science, we stress the role of continuous structures as tools for proving results about discrete or even finite structures. In particular we overview results concerning functionals in computability theory, trees in lambda calculus, boolean circuits in complexity theory and relate the finitary/combinatorial nature of the problems with their continuous solutions. We mostly focus on the methodology, and just hint to the technical aspects of the results presented.

2.1 Introduction

In order to outline the purpose of this lecture, it is probably worth mentioning first what we are not going to discuss. We are not going to look at the most widely known application of continuous structures in Computer Science. By this we mean the solution of the basic domain equation $X = X \to X$. Scott's solution of this equation, in the early 70's (to within isomorphism, of course; Scott (1972)), and his preliminary work on it, e.g. Scott (1970), set the mathematical basis for the Scott-Strachey approach to programming languages (see Stoy (1978), say). The work done within (or motivated by) denotational semantics by several authors includes, in a rather broad sense, purely

[1]Revised version of an Invited Lecture, in **Ninth Colloquium on Trees in Algebra and Programming** (CAAP 84), pages 1–22 (Courcelle, editor). Cambridge University Press, 1984.

mathematical developments (see Gierz et al. (1980)) as well as methods which influenced the actual design of programming languages (e.g. Gordon et al. (1979)). Specific areas are concerned with original applications of Category Theory to the semantics of programming languages (see Plotkin & Smyth (1982) for more results and references) and computability in abstract structures (see Barendregt & Longo (1982) or Longo & Martini (1984) (for simple approaches and references), just to mention two broadly construed topics.

What is the relation between the equation $X = X \to X$ and continuity properties? The point was to find a model of type free λ-calculus. This language already was (and more become later) a paradigmatic or stylistic language for functional programming.

Approximation properties suggested the way to find a non trivial space isomorphic to its own "function space": by allowing only the continuous functions w.r.t. a suitable topology, that isomorphism was made possible without contradicting Cantor's theorem, though obtaining a sufficiently rich function space for the interpretation of all λ-definable functions.

The required approximation properties may be given by the notion of domain as topological space. In general, a partially ordered set Y (poset) is **directed** if $\forall x, y \in Y$ there exists $z \in Y$ such that $x, y \leq z$.

Definition 2.1.1 *(i) A poset $(X, \leq)$ is* **complete** *if any directed set $Y \subseteq X$ has a least upper bound (l.u.b.), $\sqcup Y$. We then say that $(X, \leq)$ is a complete partial order (c.p.o.).*

(ii) Let $(X, \leq)$ be a c.p.o.. $A \subseteq X$ is open in the **Scott topology** *if*

1. $x \in A$ **and** $x \leq y \Rightarrow y \in A$
2. $Y \subseteq X$ *directed* **and** $\sqcup Y \in A \;\Rightarrow\; A \cap Y \neq \emptyset$.

(iii) A c.p.o. $(X, \leq)$ is **algebraic***, with base $X_0 \subseteq X$, if $\emptyset$ and $\{y \in X \mid x_0 \leq y\}$, for x_0 ranging over X_0 , form a basis for the Scott topology on $(X, \leq)$. X_0 is the set of* **finite** *(or algebraic) elements. $(X, \leq)$ is* **consistently complete** *if every bounded subset has a l.u.b..*

(iv) A **domain** *is a consistently complete algebraic c.p.o., with the Scott topology.*

Remark 2.1.2 *Given domains $\underline{X} = (X, X_0, \leq)$ and $\underline{Y} = (Y, Y_0, \leq)$, $f : X \to Y$ is continuous iff $\forall x \in X \quad f(x) = \sqcup\{f(x_0) | x_0 \leq x\}$. The Scott topology on non trivial domains is T_0 , but not T_1. Recent work in denotational semantics has been dealing with stronger separation properties (e.g. metric spaces) or weaker closure properties not necessarily topological.*

Examples. The set of partial functions from ω (the natural numbers) to ω, and $P\omega$, the powerset of ω, are domains. ($P\omega$ is also a lattice, of course, and $P\omega_0 = \{A : A \subseteq \omega$ **and** A is finite$\}$).

The category of domains, with continuous maps, is Cartesian closed. That is, if $\underline{X}$ and $\underline{Y}$ are domains, then also $\underline{X \times Y}$ and $\underline{Cont(X, Y)}$, the continuous functions from X to Y, are domains, with suitable bases, and they are nicely related (see Mac Lane (1971), Scott (1981), for details[2]).

[2] Added in reprint: see also Asperti & Longo (1991)

Most of the research issues mentioned above involve the Category of domains or related structures essentially as foundational tools for Computer Science or, say, analyze the relation between computational and denotational properties of programming languages. More discussion on this matter may be found in several places, see Longo (1984). In the present lecture we would like to motivate the use of continuous structures by their direct applications to results on finite or discrete spaces. Namely, we are going to survey a few results which may be stated in elementary/combinatory terms over discrete or finite structures and which have been proved by using continuous spaces. The mathematical notions used for this investigation are borrowed from (or related to) Mathematical Logic.

Probably a rough analogy with a basic topic in Mathematics may help to understand our methodological viewpoint.

The assertions of Number Theory deal with integers and are mostly elementary and of a combinatory character. The fundamental theorem of arithmetic, say, proves that any number is the product of a unique finite sequence of primes. In this case the proof is both elementary and simple; the relevance, though, of this old discovery is clear to everybody. Thus more knowledge about properties of prime numbers is surely important; in particular, say, the way they distribute over ω. Gauss, in 1849, first conjectured that, if $p(n)$ denotes the number of primes not exceeding n, then

$$(G) \qquad\qquad \lim_{n\to\infty} \frac{p(n)\log n}{n} = 1$$

that is $p(n)$ is asymptotic to $n/logn$.

(G) may be clearly stated in elementary formal number theory, Peano Arithmetic, as

$$\forall x\ \exists y\ \forall z\ (x > 0 \wedge z > y\ \rightarrow\ |1 - \frac{p(z)\log z}{z}|\ <\ \frac{1}{x})$$

(G) is known as the Prime Number Theorem. It was first proved by Hadamard and de la Vallée Poussin in 1896 by using Complex Function Theory, i.e. by the strong topological algebraic properties of the field of complex numbers and the functions over it.

Only much later, in 1949, Selberg and Erdos gave an elementary proof of (G), that is a proof not using properties of functions of complex variables.

This pattern occurred several times in number theory; first analytic proof, then an elementary argument, with at most a little use of real valued functions.

Usually, as in the case of (G), the elementary proof is more intricate and not simple at all, but it gives a better insight into constructive aspects of the result (y, say, depends uniformly effectively on x , in the formal statement of (G)). The analytic approach, though, may be more transparent, just because the difficulties are scattered over whole nicely organized theory, such as the theory of complex functions.

Moreover, it may relate the results studied to other problems because of the unifying power of a large and deep theory. This is exactly what happened with (G), in view of the work done, in order to prove (G), on the Riemann zeta function and the Euler identity by using Fourier series, see Hardy & Wright (1960).

It should be clear, then, that elementary or combinatory does not mean simple (or simpler): it may even mean the opposite, because it is harder to prove something by means of only a few notions, a few tools.

The results below have also been proved by looking at continuous structures. In the first two cases one may even say that "analytic" methods have been used, for elements of domains are described by finite approximations, similarly to approximations by finite expansions of Fourier series in complex function theory. Of course, we do not yet deal with the width and the difficulties of number theory: the Theory of Numbers is the oldest and most basic topic in Mathematics. In Pisa, say, some beautiful results were already obtained by Fibonacci (1202). Mathematical Computer Science is surely not as old as that. Its growth, though, because of the relevance of Computer Science and its applications in every-day life, is much faster.

Section 2.2 presents an elementary notion of higher type computation. The basic properties of the type structure obtained by this, are proved by an essential use of continuous domains for denotational semantics and generalized computability. In section 2.3 below, a (preliminary) fact for the solution of recursive equations between programs is discussed. This is done for a paradigmatic functional language, λ-calculus. The proof uses a non-axiomatizable ("infinitary") consistent extension of λ-calculus and a continuous model.

Section 2.4 deals with the answer to a question concerning boolean circuits and their complexity. As a matter of fact, the finitary argument for relating size and depth is based on an analogy to hierarchies of infinite classes of open sets in suitable topological spaces.

References to the eight or more authors, including this author, who proved the results summarized in this introductory lecture, are given in the corresponding sections.

2.2 The Hereditary Partial Effective Functionals

Classical higher type recursion theory deals with total object. As a matter of fact, Gödel and Kreisel's motivation for this topic refers to the analysis of the meaning in constructive mathematics of Peano or Heyting arithmetic (see Kreisel's paper in Heyting (1959)). For Kreisel's "no counterexample" interpretation, say,
from $\exists x \forall y \exists z \forall w\ A(x,y,z,w)$
write $\exists F \exists G \forall f \forall g\ A(F(f,g), f(F(f,g)), G(f,g), g(F(f,g).G(f,g)))$,
where F, G, f, g are function variables in the due type (type 1, the type of number theoretic functions, as for f and g; (pure) type 2, as for the functionals F and G, since they are functions of functions). Then x and y are effectively recovered from f and g, if F and G are (higher type) total *computable* functionals. Thus the need to define this sort of computable maps.

In Computer Science, a new branch of mathematical research on these matters has been motivated by the semantics of typed and type free languages, for it is sound to investigate the effectiveness, over semantic domains, of the interpretation of (higher type) programs. In particular, since the work done by Scott in the semantics of programming languages and, in higher type recursion theory, by Ershov, who refers to early research by Scott, there has been a growing interest in partial objects. As a matter of fact, since the 30's, the notion of "partiality" turned out to be essential and helpful in the foundation of recursive function theory. A key point, say, is that the class of partial recursive functions (PR) can be effectively numbered; more precisely, the universal function is also in PR.

By using this idea, in Longo (1982) a direct elementary approach was proposed to higher type partial functionals (the HPEF). In Longo & Moggi(1984) the basic facts are proved and the type structure is compared to extant approaches.

The difference with Scott-Ershov approach (or Hyland's, for total functionals) is that the later first introduces the notion of continuity, possibly in abstract categorical settings, then defines the continuous and computable objects in any type by using PR; while the HPEF are presented by an inductive elementary construction, only based on the notion of partial recursive function. Interestingly enough, the key structural property of the HPEF, which may be expressed in purely elementary terms, could only be proved by using continuous structures. This may be considered a further motivation to the invention of the category of domains.

2.2.1 The problem

For sets A, B, write $f : A \to B$ if f is a total map from A to B. By setting $\omega^{\perp} = \omega \cup \{\perp\}$, we may write $f : \omega \to \omega^{\perp}$ for any partial number-theoretic function. As for higher types, integer (or, respectively, pure) types are defined by $n{+}1 : n \to n$ (or, respectively, $n + 1 : n \to 0$), for $n \in \omega$.

Let $<,>$ be any effective coding of pairs in ω; that is $<,>$ codes a pair of numbers by a single number.

Set $C^{(0)} = \omega$ and $C^{(1)} = PR$. We now define $C^{(n)}$, for all $n > 1$, as the set of **hereditary partial effective functionals** (HPEF) of integer type n, by induction on n, using a set $C^{(n.5)}$ of maps from $C^{(n-1)}$ to $C^{(n)}$ (functions of "intermediate" type). The maps in $C^{(n)}$ go from $C^{(n-1)}$ to $C^{(n-1)}$. The coding $<,>$ is extended to higher types in one of the several possible ways. In other words, the extented $<,>$ codes $C^{(n)} \times C^{(n)}$ by $C^{(n)}$, for any n, as it will be shown below.
Notation:

$$\lambda xy.g(x,y) \text{ is the map } < x, y > \vdash g(x,y).$$

Definition 2.2.1 *(i)* $C^{(n.5)} = \{\phi : C^{(n-1)} \to C^{(n)} \mid \lambda xy.\phi(x)(y) \in C^{(n)}\}$;
ii) $C^{(n+1)} = \{\tau : C^{(n)} \to C^{(n)} \mid \forall \phi \in C^{(n.5)} \tau \circ \phi \in C^{(n.5)}\}$.

In order to see that this definition makes sense, let's check the types.

Assume that $C^{(n-1)}$ and $C^{(n)}$ are given. Consider first $\phi : C^{(n-1)} \to C^{(n)}$ and set $\psi(< x, y >) = \phi(x)(y)$, for $x, y \in C^{(n-1)}$ (check the typing for exercise: e.g., observe that $\phi(x) : C^{(n-1)} \to C^{(n-1)}$).

Clearly $\psi : C^{(n-1)} \to C^{(n-1)}$. Then, by (i) in the definition above, $\phi \in C^{(n.5)}$ iff $\psi \in C^{(n)}$.

The following diagram should help in understanding how $C^{(n+1)}$, the "next" higher type, is defined in (ii), by using $C^{(n.5)}$, which has been defined in (i) from the given $C^{(n-1)}$ and $C^{(n)}$:

$$\begin{tikzcd}
C^{(n-1)} \arrow[d, "\phi"'] \arrow[dr, "\tau \circ \phi"] & \\
C^{(n)} \arrow[r, "\tau"'] & C^{(n)}
\end{tikzcd}$$

That is, $\tau \in C^{(n+1)}$ iff $\forall \phi \in C^{(n.5)} \;\; \tau \circ \phi \in C^{(n.5)}$, as given in (ii) in the definition.

Note that the pure type functionals $\{PC^{(n)}\}_{n\in\omega}$, i.e. for $n+1 : n \to 0$), may be also defined, by the technique above: just substitute $PC^{(n)}$ in the second instance of $C^{(n.5)}$ in the definition of $PC^{(n+1)}$ corresponding to definition 2.2.1 (ii) (see Longo & Moggi (1984; 3.12)).

In both cases, one only needs to know $C^{(0)} = \omega$ and $C^{(1)} = \mathrm{PR}$, to start with. Apparently, though, it may seem that the full strength of Set Theory is needed in order to define the sets $C^{(n.5)}$ and $C^{(n+1)}$ above. As a matter of fact, a consequence of the results mentioned below is that, for all n, $C^{(n)}$ and $C^{(n.5)}$ are countable and can be effectively numbered.

By this we do not need to look, say, at the set of *all* functions from $C^{(n-1)}$ to $C^{(n)}$ in order to define $C^{(n.5)}$, but a countable subset would do. A posteriori, this is all expressible in Peano Arithmetic. (Exercise: set $C^{(1)}$ = "the constant functions from ω to ω". What is the cardinality of $C^{(2)}$? And if one takes, as $C^{(1)}$, the primitive recursive functions?).

We want to know, now, what the $C^{(n)}$'s are and how they behave, for $n > 1$. It is easy to see that all acceptable Gödel-numberings of $C^{(1)} = \mathrm{PR}$ live in $C^{(1.5)}$. It would be nice, say, if one could extend the notion of effective numbering to maps in $C^{(n.5)}$, for $n > 1$. In view of the s-m-n (iteration) theorem, this corresponds to proving the property (P) below, which was conjectured in Longo (1982).

$$(P) \qquad \forall n \geq 0 \; \exists \phi_n \in C^{(n.5)} \; \forall \phi \in C^{(n.5)} \exists f_n \in C^{(n)} \phi = \phi_n \circ f_n$$

Recall now that for all $f_{n+1} \in C^{(n+1)}$ one has $f_{n+1} \circ \phi_n \in C^{(n.5)}$. Then by (P), for some $f_n \in C^{(n)}$, $f_{n+1} \circ \phi_n = \phi_n \circ f_n$.

By this, HPEF may be visualized in the integer types by the following diagram:

$$\begin{tikzcd}
\omega \arrow[r, "f"] \arrow[d, "\phi_1"'] & \omega \arrow[d, "\phi_1"] \\
C \arrow[r, "f_2"'] \arrow[d, dotted] & C \arrow[d, dotted] \\
C^{(n-1)} \arrow[r, "f_n"'] \arrow[d, "\phi_n"'] & C^{(n-1)} \arrow[d, "\phi_n"] \\
C^{(n)} \arrow[r, "f_{n+1}"'] & C^{(n)} \\
\vdots & \vdots
\end{tikzcd}$$

In order (P) to hold, the ϕ_n's have to be surjective as well and, by this, preserve the cardinality of ω at higher types. Again, (P) is elementary and has a combinatory flavour.

2.2.2 The method

In this section we give the basic hints for the main result concerning the HPEF, with the only aim to display the use of continuous structures in the proofs.

In Longo & Moggi (1984), (P) has been proved by an heavy induction loading, proposed by the second author, to be precise, who first pointed out that, by looking only at definition 2.2.1, the HPEF do not need to be ... well defined. As a matter of fact, $<,>: \omega^2 \leftrightarrow \omega$ is not trivially inherited at higher types. A first sight, one would just set $< f, g > (n) = < f(n), g(n) >$ as a definition for $<,>: C \times C \leftrightarrow C$ and so on. This is clearly wrong, for divergence of f or g cause essential problems. Set then $< f, g > (n) =$ **if** n is even **then** $f(\frac{n}{2})$ **else** $g(\frac{n-1}{2})$.

This is fine for $C^{(1)}$, but it doesn't automatically prove that
$< \tau, \sigma > (f) = < \tau(f), \sigma(f) >$
defines an (effective) coding of $C^{(n)} \times C^{(n)}$ into $C^{(n)}$, for $n \geq 2$. The proof of this goes inductively together with the proof of (P) and of two more facts, which need a few definitions.

Recall that a numbered set is a pair (D, e) where $e : \omega \to D$ is onto. $f : D \to D'$ is a **morphism** (of the numbered sets $(D, e), (D', e')$) if $\exists f' \in R\ f \circ e = e' \circ f'$ where R are the recursive functions.

Definition 2.2.2 *(i) Let (D, e) and (D', e') be numbered sets. Define then*

(i) $d : \omega \to D$ is an **acceptable numbering** *of D if $\exists f, g \in R\ e = d \circ g$ and $d = e \circ f$.*

(ii) $\phi : D \to D'$ is a **relative numbering** *(of (D', e') w.r.t. (D, e)) if $\phi \circ e$ is an acceptable numbering of (D', e').*

(iii) $\phi : D \to D'$ is a **principal numbering** *(of (D', e') w.r.t. (D, e)) if ϕ is a morphism and $\forall \psi \in Mor(D, D')\ \exists \theta \in Mor(D, D)\ \psi = \phi \circ \theta$*

Clearly acceptable, relative and principal numberings are onto maps. In presence of a relative numbering, any principal numbering is also relative. The converse doesn't hold. It is easy to see that, if (D, e) yields (a generalized version of) the recursion theorem and $f : D \to D'$ is a relative numbering, then also D' has the same property (see Longo & Moggi (1984; Intermezzo)).

The other assertion in the inductive proof need notions and facts from the Category of domains (see 1.1). Let $\underline{X} = (X, X_0, \leq)$ be a countably based domain, with a given numbering $\{x_n\}_{n \in \omega}$ of X_0. Under a few simple decidability conditions for $\{x_n\}_{n \in \omega}$, define $x \in X$ **computable** if $\{n \mid x_n \in X_0$ **and** $x_n \leq x\}$ is r.e. (see Scott (1981) for details). Let X_c be the collection of computable elements. Under these assumptions, call any such domain **effective**.

Since in Cartesian Closed Categories properties of objects are inherited at higher types, given $\underline{X}$ as above, each object in the type structure generated by $\underline{X}$ is an effective domain, with a corresponding notion of computable element. By this, Ershov nicely related the continuous (and computable) partial functionals in the type structure over $\omega^{\perp}$, the flat domain with $\perp$ as least element, to the classical Kleene-Kreisel countable functionals (and the Hereditary Effective Operators; see Ershov(1976) for an account). Of course, continuity properties are the key issue in the Category of domains.

Consider now $E = Cont(\omega, \omega^{\perp})$, where ω is given the discrete topology. Clearly, E is the effective domain of the partial functions $E_c =$ PR.

Set $E^{(n+1)} = Cont(E^{(n)}, E^{(n)})$. We can now state the main theorem in Longo & Moggi (1984).

Theorem 2.2.3 *For all $n \geq 1$ one has:*
(1) $C^{(n)} = E_c^{(n)}$
(2) $C^{(n)} \times C^{(n)} \cong C^{(n)}$ via $<,>$
(3) $C^{(n.5)} = Cont(E^{(n-1)}, E^{(n)})_c$
(4) $\exists\phi \in C(n.5)$ principal relative numbering.

Proof. Longo & Moggi (1984). □

Properties (1) - (4) are proved by combined induction. (1) and (3) give the continuity of the HPEF. More precisely, (2) at type n+1 is obtained from (1) + (2) at type n. by first showing that, in an effective domain, any effective coding of pairs of algebraic elements by algebraic elements is inherited to a corresponding effective coding at the higher type (the function space). (1) + (3) + (4) give (1) at the higher type. (1) + (2), again, prove (3) and so on, with various combinations, to prove the rest.

Note, now, that once proved that each $C^{(n)}$ is a countable set, with the natural numbering of constructive domains, (P) above amounts to say that each $C^{(n.5)}$ contains a principal numbering. In fact, by generalized Myhill-Shepherdson theorem (see Giannini & Giuseppe Longo (1984)), the continuous and computable functions over the computable elements of effective domains coincide with their morphisms, as numbered sets.

There may be direct arguments for proving (P), i.e. with no use of the continuous structure. In Longo & Moggi (1984), though, it seemed essential to the role of finite approximants (algebraic elements) and their effectiveness properties to deal with the functions in the $C^{(n)}$'s, which are infinite objects.

Properties of effective computations over (possibly) infinite inputs are naturally handled by looking at finite pieces and, then, taking (topological) limits. Apparently, there is no way to escape from this, also when direct (non topological nor limit) definitions are given, as in our case.

Remark. Topological or limit structures are implicit in the definition of the Kleene-Kreisel countable functionals, when given by the "associates" (see Normann (1980)). This is not so for the HPEF, where in the definition, there is no flavour of continuity nor approximation. Though, as soon as we just want to check if the definition makes sense in any type, finite approximants, that is the algebraic elements of suitable domains, come in. Of course, Myhill-Shepherdson and Kreisel-Lacombe-Shoenfield theorems, and their generalizations, are the key facts to understand this.

Question. Fix C as your preferred set of partial maps from ω to ω. What are, then, the $C^{(n)}$'s defined as in definition 2.2.1? (Take, say, C equal to the primitive recursive functions or the polynomial time computable functions. The properties of C are naturally inherited at higher types via the intermediate types $C^{(n.5)}$).

2.3 Invertible terms in λ-calculus

Terms of λ-calculus are defined by variables, formal application and the binding operator λ. The computation (reduction) rules are axiomatized as follows:

$(\beta)\quad (\lambda x.M)N \geq [N/x]M \quad$, for N free for x in M, as usual;

$(\eta)\quad \lambda x.Mx \geq M \quad$, for x not free in M.

The reduction predicate "$\geq$" is reflexive, transitive and substitutive. "$=$" is the least equivalence relation such that $M \geq M' \Rightarrow M = N$; "$=$" also is substitutive.

A term is in **(β-)normal form** ($\beta\eta$-normal form) if no subterm is the LHS of axiom $(\beta)((\beta)$ or $(\eta))$.

A term M **possesses a ($\beta\eta$-) normal form** if, for some M' in ($\beta\eta$)normal form, one has $\lambda\beta(\eta) \vdash M = M'$. Finally, M is **solvable** (or has head normal form), if, for some $P_1, ..., P_q, \vec{x}$ and y, $\lambda\beta \vdash M = \lambda x_1 ... x_n.yP_1...P_q$, M is **unsolvable**, otherwise.

The paradigmatic role of λ-calculus and Combinatory logic, as functional programming languages, is well known, see Backus (1978).

2.3.1 The problem

Combinatory Logic and λ-calculus, as type free applicative languages, have a set of terms which forms a monoid with $I \equiv \lambda x.x$ as identity and composition "$\circ$" as application, where $M \circ N = BMN$ for $B = \lambda xyz.x(yz)$. Church proved that the closed terms of $\lambda\beta\eta$-calculus form a recursively presented monoid, with an unsolvable word problem. Note, now, that equations among terms give a standard methods for defining terms, as well as for programs. Equations, though, as well known from algebra, can be more easily solved when working in a group.

Consider, for example, the following definition of the (unknown) term $X : M \circ X = X \circ P$. If M, say, possesses an inverse, M^{-1} , an easy application of Curry's fixed point combinator gives X from $X = M^{-1} \circ X \circ P$, since we obtain an ordinary recursive definition of X.

It is then an important question, raised by Curry & Feys (1958), under what conditions a term in the monoid has an inverse.

Definition 2.3.1 *Let T be (an extension of) λ-calculus. M is T-**invertible** if $\exists N\ T \vdash M \circ N = I$ and $T \vdash N \circ M = I$. (In this case we that M, N are T-**inverse**).*

Dezani (1976) gives a characterization of invertible terms in $\lambda\beta\eta$-calculus, among those which *possess a normal form* (for $\lambda\beta$-calculus, the group coincides with I). This was done by an (implicit) use of the notion below of tree for terms. More recently, Böhm & Dezani (1984), by similar techniques, characterized invertibility w.r.t. a large class of associative combinators, besides composition "$\circ$". By this, more general tools for solving equations are provided[3].

[3] Added in revision: a rather unexpected application of invertible terms is given in Bruce et al.(1992), as these terms are used to caracterize the isomorphisms which hold in all Cartesian Closed Categories.

Definition 2.3.2 *(Informal) The* **Böhm tree** *of M is given by:*

$BT(M) = Q$ *if M has no head normal form*

$$BT(M) = \begin{array}{c} \lambda x_1 ... x_n . y \\ \diagup \quad \diagdown \\ BT(M_1) \quad \cdots \quad BT(M_p) \end{array} \qquad if\ M =_\beta \lambda x_1 ... x_n . y M_1 ... M_p$$

The definition is informal, since Böhm-trees may be infinite (see Barendregt (1984) for a precise definition). In particular, $BT(M)$ finite and Ω-free iff M has a normal form.

Thus, there are plenty of interesting terms with infinite Böhm-trees. Take, say the fixed point combinator $Y = \lambda y.(\lambda x.y(xx))(\lambda x.y(xx))$ or the solution of $X = \lambda z.zX$, and a lot more. Since Wadsworth and Hyland's work, Böhm-trees have been a basic tool in the comparison of operational and denotational semantics of λ-calculus (see Barendregt (1984), Longo (1983)).

Roughly, this is because all terms, as all programs, are finite Böhm-trees "display" their computational (operational) behaviour, which may be infinite (see Longo (1984) for a discussion).

Böhm-trees may be partially ordered by setting Ω, the undefined tree, as the least one and, then, by setting $BT(M) \subseteq BT(N)$ if $BT(N)$ is obtained from $BT(M)$ by replacing Ω in some leaves of $BT(M)$ by some Böhm-tree.

Dezani's result (see theorem 2.3.4, below) uses a difficult combinatory technique, where the finiteness of the Böhm-trees considered has an essential role. The question remained of characterizing all invertible terms, not necessarily with normal form; equations like the example above are perfectly sound also if $BT(M)$ is infinite, for M could still possess a head normal-form.

Bergstra & Klop (1980) fully answered the general questions, by using results in Dezani (1976). Our present interest in the methods used in Bergstra & Klop (1980) relies on the essential application made of infinitary, continuous structures to solve this typically combinatotial problem. What's nice is that the characterization in Bergstra & Klop (1980) confirms the result in Dezani (1976), since invertible terms turn out to live only among normal forms. To see this, though, one needs to look also at terms with an infinite computational behaviour.

2.3.2 The method

Let $\sigma_1, ..., \sigma_n$ be a permutation of $1, ..., n$ and $\vec{z}$ be a finite string of variables.

Definition 2.3.3 *(i) The* **hereditary permutators** *(HP) is the set of λ-terms defined by:*

$$M \in HP \quad if \quad BT(M) = \begin{array}{c} \lambda z x_1 ... x_n . z \\ \diagup \quad \diagdown \\ \lambda \vec{y_1} . x_{\sigma 1} \quad \cdots \quad \lambda \vec{y_n} . x_{\sigma n} \\ \diagup\diagdown \qquad\qquad \diagup\diagdown \end{array}$$

and each subtree is the tree of a term in HP.

(ii) The **finite hereditary permutators** *(FHP) are the terms in HP with a finite, Ω-free Böhm-tree.*

Theorem 2.3.4 *Assume that M has a normal-form. Then M is $\beta\eta$-invertible iff $M \in FHP$.*

Proof. Dezani (1976) (see also Bergstra & Klop (1980)). □

Theorem 2.3.5 *M is $\beta\eta$-invertible iff $M \in FHP$.*

Proof. Bergstra & Klop (1980) (see also Barendregt (1984)). □

It is clear that in order to prove theorem 2.3.5 from theorem 2.3.4 one only has to show

(O) M is $\beta\eta$-invertible $\Rightarrow$ M has a normal form, since any M in FHP automatically has a normal form.

Bergstra and Klop give two arguments for (O), by embedding λ-terms into different continuous models, D_∞ and $P\omega$. The discussion below borrows from the presentation in Barendregt (1984) and the authors second proof. It should be clear that we only try to give some transparency, if possible, to what is not immediately transparent from the very technical presentations by the authors, namely the mathematical significance of the methods used.

$P\omega$ may be turned into a model of type-free λ-calculus by a classical recursion theoretic notion of set-theoretic application and by defining abstraction accordingly (see Scott (1976) and Longo (1983), for recent work).

Definition 2.3.6 *Let $<,>: \omega^2 \leftrightarrow \omega$ be a coding of pairs (the "little diagonal", say) and $\{e_n\}_{n\in\omega}$ the canonical numbering of the finite subsets of ω. Set then, for $A, B \in P\omega$ and $f \in Cont(P\omega, P\omega)$,*

$$A \cdot B = \{m | \exists e_n \subseteq B < n, m >\in A\} \quad \text{and} \quad \underline{\lambda} x.f(x) = \{< n, m > | m \in f(e_n)\}$$

Clearly, $\underline{\lambda} x.f(x) \cdot B = f(B)$, by remark 2.1.2, that is, $F(A) = (\lambda)x.A \cdot x$ and $G(f) = \underline{\lambda} x.f(x)$ give a retraction $Cont(P\omega, P\omega) \triangleleft P\omega$, for $F \circ G(f) = f$. For the given F,G is actually the unique such (see Longo (1983)). Of course, the cardinality of $P\omega$ is $2^{\aleph_0}$ and its topological structure is the key point for defining the given retraction.

The true equalities between λ-terms in $P\omega$ are characterized by the following theorem.

Theorem 2.3.7 *(Hyland) $P\omega \models M = N$ iff $BT(M) = BT(N)$.*

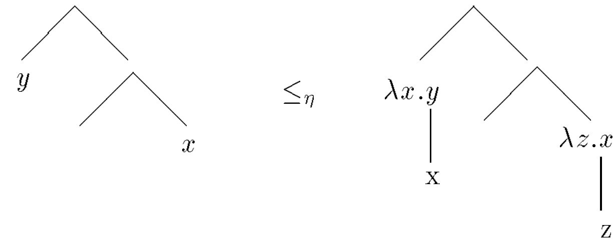
Proof. (see Barendregt (1984)). □

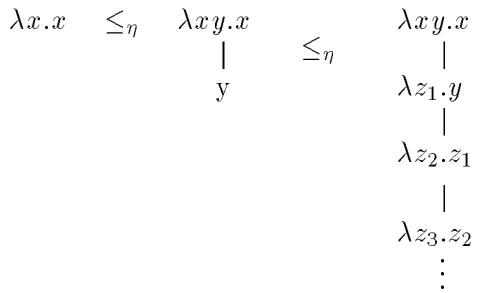
We also need to introduce an interesting extension of $\lambda\beta\eta$-calculus, which may be defined in terms of Böhm-trees. The reader who wants to avoid the few technical notions below, at first reading, may skip them, and go to comment 2.3.11.

Note first that, once the (possibly infinite) operational behaviour of a λ-term has been displayed by its Böhm-tree, one may also consider infinitary reduction rules (or expansion roles, their inverse operations).

In particular, one may η-expand (possibly infinite) nodes of a tree. Call "$\leq_\eta$" such a (possibly infinitary) rule between trees.

Examples





(infinitary)

It is easy to see that the last tree is $BT(J)$, for $J = \lambda xy.x(Jy)$.

Definition 2.3.8 *Let A, B be Böhm-tree. Define then*

(i) $A \;^\eta\!\subseteq B$ if $\exists A'\ A \leq_\eta A' \subseteq B$, where $\subseteq$ is defined after definition 2.3.2,

(ii) $A \;^\eta\!\subseteq^\eta B$ if $\exists A', B'\ \ A \leq_\eta A' \subseteq B' \geq_\eta B$

(iii) $A =_\eta B$ if $A \;^\eta\!\subseteq^\eta B \;^\eta\!\subseteq^\eta A$

(iv) $H^\star = \{M = N \mid M,\ N$ are λ-terms and $BT(M) =_\eta BT(N)\ \}$

$H^\star$ is a consistent extension of $\lambda\beta\eta$-calculus. It is actually the largest consistent extension of the theory H, where all unsolvable terms are equated (Barendregt (1984)). Thus $H^\star$ is the unique Hilbert-Post completion of H.

Lemma 2.3.9 (Main Lemma) *Let M, N be $H^\star$-invertible (see def. 2.3.1). Then for some $n, m \geq 0$), say $n \geq m$, there are permutations π, σ and M_i $(1 \leq i \leq n), N_j$ $(1 \leq j \leq m)$ such that*

(i) $\lambda\beta \vdash M = \lambda z x_1 ... x_n.z(M_1 x_{\pi 1}) \cdots (M_n x_{\pi n})$,

$\quad\lambda\beta \vdash N = \lambda z y_1 ... y_m.z(N_1 y_{\sigma 1}) \cdots (N_m y_{\sigma m})$,

$\qquad\pi \circ \sigma = \sigma \circ \pi = id$, *and*

(ii) $N_i, M_{\sigma i}$, for $1 \leq i \leq m$, and M_i, I, for $m < i \leq n$, are $H^\star$-inverse, modulo some substitution instances.

Notation: $M \in_k HP$ if M is in HP up to level k of $BT(M)$, where the root is level 0 of a tree.

Theorem 2.3.10 *If M is $\beta\eta$-invertible, then $M \in HP$.*

Proof. (Sketch) Assume that M, N are $\beta\eta$-inverse. Then M, N are also $H^\star$-inverse, for $\lambda\beta\eta \subseteq H^\star$. If we prove

$$1) \qquad \forall k\ (M\ H^\star\text{-invertible} \Rightarrow M \in_k HP),$$

we are done, by the (informal) inductive definition of HP.

1) above holds for $k = 1$, by (a lot of) combinatory work on $BT(M)$, up to the first two levels (see Dezani (1976), Bergstra & Klop 1980) or Barendregt (1984)).

Assume now (1) for $k > 1$. By lemma 2.3.9, M, N $H^\star$-inverse $\Rightarrow N_i, M_{\sigma i}$ are $H^\star$-inverse, as in lemma 2.3.9(ii).

Then, by induction hypothesis, $N_i, M_{\sigma i} \in_k HP$ and, clearly, $M, N \in_{k+1} HP$. □

Comment 2.3.11 *Interestingly enough, most of the proof of the Main Lemma may be directly borrowed from Dezani's argument, which never mentions the theory $H^\star$. What is then the use of it? The point is in the induction loading. If one assumes that M and N are $\beta\eta$-inverse, one doesn't get, in general, that $M_i, N_{\sigma i}$, defined as in lemma 2.3.9 (ii), are $\beta\eta$-inverse, but just $H^\star$-inverse (modulo some substitution instances). Thus the argument cannot be iterated inductively. In Dezani (1976) this can be done under the strong assumption that M and N possess a normal form.*

Moreover, by the infinitary character of the provable equalities, $H^\star$ allows to compare terms with infinite Böhm-trees. Roughly, it is like looking at full decimal expansions of rational numbers, without restricting the attention to numbers with finite expansions (normal forms). The proof, though, works, at finite levels, exactly as for the finitary result and it is then analytically extended to the infinite expansions.

Church original work on λ-conversion used a more restrictive notion of λ-term. Namely, λI-terms are defined similarly to λ-terms, provided that $\lambda x.M$ is a λI-term iff x occurs free in M.

It is easy to see that a λI-term M has a normal form iff all of its subterms have a normal form. By looking at the definition of HP, one may work out the following fact.

Theorem 2.3.12 *If $M \in HP$, then $BT(M)$ is Ω-free and there exists a λI-term M' such that $BT(M') = BT(M)$.*

Let's now follow stepwise Bergstra and Klop's argument for

(O) If M is $\beta\eta$-invertible, then M has a normal form. Assume that M, N are $\beta\eta$-inverse and that M has no normal form. Then

(1) There exist λI-terms M', N' s.t. $BT(M') = BT(M)$ and $BT(N') = BT(N)$, by theorem 2.3.12.

(2) $BT(M')$ is infinite, by assumption and theorem 2.3.10, and, hence M' has no normal form.

(3) $M' \circ N'$ is a λI-term, for $M' \circ N' \equiv BMN$ where $B \equiv \lambda xyz.x(yz)$, and, hence, it has no normal form, by (2).

(4) $P\omega \models M' = M$ and $P\omega \vdash N' = N$. by (1) and theorem 2.3.7.

(5) $P\omega \models M' \circ N' = M \circ N$. by (4).

(6) $BT(M' \circ N') = BT(M \circ N)$, by (5) and theorem 2.3.7, again.

(7) $BT(M \circ N)$ is infinite, by (3) and (6), and, hence , $M \circ N$ has no normal form.

The conclusion in (7) is clearly impossible, for $\lambda\beta\eta \models M \circ N = I$.

Comment 2.3.13 *The role of $P\omega$, as continuous model should be clear. It essentially allows to go from (1) to (6), by using theorem 2.3.7. Since BT-equal terms are the same object in $P\omega$ by theorem 2.3.7, one has (4) and, hence, (5) and (6) trivially follow.*

It is like looking at a property of natural numbers by studying them as elements of a more structured continuum they live in, the real or complex field, say. Where does continuity come in, in this case? The proof of theorem 2.3.7 heavily relies on continuity properties of $P\omega$. To see this more closely, one may consult Barendregt (1984) or look at results which distillate the notion of "approximable application" in Longo (1983). This notion relates, in a general setting, the interpretation of formal application to the limit structure of models.

Can one directly go from (1) to (6) by working over Böhm-trees with some smart combinatory technique? This is done in Barendregt (1984; §.18.4), but the continuity argument is important as well. As a matter of fact one has to take the (completion of the) set of Böhm-trees, partially ordered as after definition 2.3.2, and heavily work with approximation and Scott continuity over this domain.

A final remark on cardinality. It is possible to take a countable model instead of $P\omega$, in the argument above, for $RE \subseteq P\omega$. the set of recursively enumerable sets, is an equationally equivalent sub-model of $P\omega$. That is, by the same notions of application and abstraction, RE yields a model which satisfies the same equalities as $P\omega$. More generally, for any infinite cardinal α, there is an applicative structure of cardinal α which yields lots of different λ-models, including one equationally equivalent to $P\omega$ (see Longo (1983)). The use of RE, though, instead of $P\omega$ would be like looking at rational numbers (or complex numbers with rational coordinates), for the analysis of the behaviour of continuous functions or predicates over the real (or complex) numbers. Those countable subsets are dense and they perfectly determine continuous functions or predicates on the latter sets. Similarly, RE is dense in $P\omega$ w.r.t. the Scott topology (and contains all algebraic elements).

2.4 Circuit complexity

By circuits one may compute functions. Actually, boolean circuits form the core of computers, when printed as chips. It is then worth studying the complexity of circuits, e.g. by looking at their size and depth.

2.4.1 The problem

A boolean circuit is usually described as a finite directed acyclic graph, with input nodes and gates, labelled by $\wedge$ or $\vee$, as other nodes. One node is the output node. The size is given by the number of gates and the depth is the maximum length of a directed path from an input to the output.

One may also define boolean circuits in a more set-theoretic style.

Definition 2.4.1 *For $n > 0$, let $\{x_1, \overline{x}_1, ..., x_n, \overline{x}_n\}$ be the* **inputs** *or* **litterals**. *The litterals are $\wedge_0(\vee_0)$-***circuits**. *For $i > 0$, an $\wedge_i, (\vee_i)$-***circuit** *is a finite nonempty collection of $\wedge_{i-1}(\vee_{i-l})$-circuits (write* **i-circuit** *for a $\wedge_i$-or $\vee_i$-circuit).*

*A Σ_i-***circuit** *is a $\vee_{i+l}$-circuit. π_i-circuits are defined dually. A $\Sigma_i(\Pi_i)$-***family** *is a collection of $\Sigma_i(\Pi_i)$-circuits all of whose 1-circuits are uniformly bounded in size (in some n). The* **depth** *of an i-circuit is i and the* **size** *is the cardinality of its sub-circuits.*

Let $I = \{0, 1\}$. Clearly, each boolean circuit with n inputs computes a function from I^n to I. A reasonable bound, in the size growth for circuits, is a polynomial growth rate in n. $\Sigma_i(\Pi_i)$-families may be considered with this polynomial growth limitation. The question to ask then is how depth and size relate in computing functions. In particular, are there functions computed by polynomial size, depth $k + 1$ circuits which cannot be computed by polynomial size, depth k circuits?

This question does not have an obvious answer since one may expect that a sufficiently large polynomial for the size growth may compensate the little loss in depth.

2.4.2 The method

In this final example the use of infinitary-continuous structures is indirect. More precisely, the proof in the finitary case is built up by an analogy to the infinitary result. The step towards infinity is made by looking at I^ω instead of I^n , for $n < w$. I^ω, of course, is the set of infinite sequences of 0 and 1's.

Infinite circuits are defined similarly as in definition 2.4.1, by dropping the finitary restrictions. Namely, inputs are from an infinite set $\{x_1, \overline{x}_1, x_2, \overline{x}_2...\}$ and, at each $\wedge_i(\vee_i)$ level, one may take infinite collections from the level below. We only require that an (infinite) Σ_i-circuit has all of its 1-subcircuits which are finite.

Clearly, each circuit in the present sense defines a function $f : I^\omega \to I$, similarly as finite circuits define functions from I^n to I, for suitable n. $A \subseteq I^\omega$ is **defined by** a circuit C, if $A = f^{-1}(1)$ for f defined by C.

Definable subsets of I^ω are in a simple relation to a well known σ-algebra of subsets of I^ω as topological space.

Definition 2.4.2 *Let X be a topological space. Define then*
$\Sigma^0_1 = \{A \mid A \subseteq X$ **and** A *is open*$\}$
$\Pi^0_1 = \{A \mid A \subseteq X$ **and** A *is closed*$\}$,
for $i > 1, \Sigma^0_i = \{\cup^\omega_1 A_n \mid \forall n < \omega\ \exists j < i\ A_n \in \Pi^0_j\}$
$\Pi^0_i = \{A \mid \overline{A}$ *(the complement of A) is in* $\Sigma^0_i\}$.

If X is a countably based complete metric space, then $A \subseteq X$ is **Borel** *if, for some $i < \omega_1, A \in \Sigma^0_i$.*

Borel sets are extensively used in several areas of Mathematics and in Descriptive Set Theory. Consider now ω and I with the discrete topology.

Note then that I^ω , with the product topology, is a countably based compact metric space (the Cantor space). The distance between $\alpha, \beta \in I^\omega$ is given by $d(\alpha, \beta) = \frac{1}{2^n}$, where n is the least such $\alpha(n) \neq \beta(n)$. Thus the basic open sets are the collections of extensions of finite sequences of 0 and 1's.

Just looking at the definitions, one may easily compare circuits and Borel sets in I^ω, by the following lemma.

Lemma 2.4.3 *$A \subseteq I^\omega$ is in Σ_i^0 iff it is definable by a Σ_i-circuit.*

A classical result for the Borel hierarchy of sets states the non trivial fact that for $i < j$, Σ_i^0 is strictly smaller than Σ_j^0.

The proof is usually given by diagonalization (Jech (1978)). Namely, one first shows that for each i, there exist a universal set U in Σ_i^0. That is, for every Σ_i^0 set A, for some a in the given space, $A = \{x \mid (x, a) \in U\}$. Then, by the usual Cantor-Russell diagonal argument, take $K = \{x \mid (x, x) \in U\} \in \Sigma_i^0$. Clearly, $\overline{K} \in \Sigma_{i+l}^0$ but $\overline{K} \notin \Sigma_i^0$ otherwise, for some k, $\overline{K} = \{x \mid (x, k) \in U\}$ and $k \in K$ iff $k \in \overline{K}$.

In Sipser (1983) a more combinatorial proof is given of the strict inclusion between Σ_i-circuits and Σ_{i+1}-circuits.

Theorem 2.4.4 *For all $i > 0$, there exist functions definable by Σ_i-circuits but not by Σ_{i-1}-circuit.*

The proof actually gives, for each $p > 0$, a function $f_p : I^\omega \to I$ with the required property, at level p, as follows. Let $<,>: \omega^2 \leftrightarrow \omega$ be a pairing function, extended to ω^n in the usual way. Define, for $\alpha \in I^\omega$,

$$f_p(\alpha) = 1 \ \text{ iff } \ \exists i_1 \forall i_2 \ldots Q i_p \ \alpha(< i_p, \ldots, i_p >) = 1.$$

For $p = 1$, f_p is clearly Σ_1 definable, but not Σ_0. The basic work is in the inductive step, where, assuming the result for $p-1$, a Σ_{p-2}-circuit is given for f_{p-1} from a Σ_{p-1}-circuit for f_p. This is done by completing functions constructively defined at finitary levels (Sipser (1983)).

The finitary case deals with circuits, Σ_i-circuits in particular, defined as in definition 2.4.1.

The input strings, then, and all higher levels of $\wedge, \vee$ collections are finite. Since we want to look at polynomial growth rate, the attention is restricted to circuit families whose size is bounded by a polynomial in the number of input variables. Thus a Σ_i-family, say, is a collection of $\vee_{i+1}$-circuits as in definition 2.4.1, with this polynomial size limitation.

Theorem 2.4.5 *For all $i > 0$, there exist functions definable by a Σ_i-circuit family but not by a Σ_{i-1}-circuit family.*

The proof goes similarly as for theorem 2.4.4. The finitary limitations come in by the finite nature of the counterexamples and by the uniform bounds in their definition. Namely, for $p, m > 0$ and $n = m^p$, define $f_p^n : I^n \to I$ as follows:

$$f_p^n(\alpha) = 1 \ \text{ iff } \ \exists i_1 < m \ \forall i_2 < m \ldots Q i_p < m \ \alpha(< i_1, \ldots, i_p >) = 1$$

(if n is not a power of m, then f_p^n is everywhere 0).

A finitary analogue to the work done for the infinitary statement, in particular in the inductive step, proves that the f_p^n have the required property (Sipser (1983)).

Comment 2.4.6 *Infinite binary sequences, i.e. the elements of I^ω , are limits of finite sequences in the obvious sense. This is made more precise by taking I^ω with the Cantor topology and by the topological notion of limit: just recall the definition of basic neighbourhood in I^ω.*

Thus also in this case one looks at the completion ("take all limits") of a given set of finite objects. Similarly, in section 2.3 the infinite Böhm-trees considered were limits, w.r.t. the Scott topology, of the finite Böhm-trees used in the original finitary result. In both cases, the finite objects to be extended were clearly provided by the extant result or the problem itself. As for the result sketched in section 2.2, no explicit hint was given by the problem to the continuous structure which could be used. Myhill-Shepherdson theorem and Scott's work, though, suggested the method and the (heavy) induction loading.

In the result just presented, the rich topological structure of I^ω appears, since the Borel hierarchy theorem is given for (spaces homeomorphic to) countably based complete metric spaces. This is only used for an analogy, though, more than for a rigorous infinitary extension of finitary facts, as in the previous cases.

Proofs as well as conjectures by analogy are fruitfully used in Mathematics. As for the present finite/infinite analogy, applied to circuit complexity, more work may be found 1n Furst et. al. (l981).

The leading idea in this approach may be extended to the fundamental $P = ?NP$ question in complexity theory, by comparing exponential growth to uncountability (Sipser (1954)). In short, typical NP-complete problems, such as the various satisfiability problems are decided by deterministic algorithms enumerating and checking all subsets of a given set. The exponential growth arises just because the subsets of a set of cardinality n are 2^n.

Analogously, uncountability arises from the powerset operation over infinite sets.

Thus polynomial growth may be related to exponential growth similarly as countability to uncountability. By this analogy Furst et al. (1981) answer a question on the complexity of parity functions. We look forward to seeing further applications of this method to the $P = ?NP$ question.

It should be clear that this informal survey is far away from being exhaustive as far as the use of "analytic" methods in Computer Science is concerned. We hope, though, that the examples mentioned could give some motivations for looking at combinatorial facts also in infinitary frameworks or by transporting combinatorial methods on infinite objects, where they are often simpler, to corresponding methods on finite objects.

Acknowledgements

The first part of this lecture was directly inspired by the joint work and a thousand and one discussions with Henk Barendregt.

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