ECOLE NORMALE SUPERIEURE

Face-regular bifaced polyhedra

Michel DEZA
Viacheslav GRISHUKHIN

LIENS - 97 - 10

Département de Mathématiques et Informatique

CNRS URA 1327

Face-regular bifaced polyhedra

Michel DEZA
Viacheslav GRISHUKHIN*

LIENS - 97 - 10

October 1997

Laboratoire d'Informatique de l'Ecole Normale Supérieure
45 rue d'Ulm 75230 PARIS Cedex 05

Tel : (33)(1) 44 32 30 00

Adresse électronique : deza@dmi.ens.fr

* CEMI, Russian Academy of Sciences, Moscow

Face-regular bifaced polyhedra

Michel Deza
Ecole Normale Supérieure, Paris, France

and Viacheslav Grishukhin
CEMI, Russian Academy of Sciences, Moscow

Abstract

Call *bifaced* any k-valent polyhedron, whose faces are p_a a-gons and p_b b-gons only, where $3 \le a < b$, $0 < p_a$, $0 \le p_b$. We consider the case $b \le \frac{2k}{k-2}$ covering applications; so either $k = 3 \le a < b \le 6$, or $(k; a, b, p_a) = (4; 3, 4, 8)$. For all these cases $p_a \le 12$. Call such a polyhedron aR_i (resp. bR_j) if each of its a-gonal (resp. b-gonal) faces is adjacent to exactly i a-gonal (resp. j b-gonal) faces. The preferable (i.e. with isolated pentagons) fullerenes are the case aR_0 for $(k; a, b) = (3; 5, 6)$. We classify bifaced polyhedra which are both aR_i and bR_j, and also all a- or b-face-regular bifaced polyhedra (except aR_0, aR_1 for (k, a)= (3,4), (3,5), (4,3), and, for fullerenes, $6R_4$ with $52 \le n \le 78$, or $6R_3$ with $52 \le n \le 58$).

1 Introduction

Denote by $(k; a, b; p_a, p_b)$ and call *bifaced* any k-valent polyhedron whose faces are only p_a a-gons and p_b b-gons with $3 \le a < b$ and $0 < p_a$, $0 \le p_b$. Any polyhedron $(k; a, b; p_a, p_b)$ with n vertices has $\frac{1}{2}kn = \frac{1}{2}(ap_a + bp_b)$ edges and satisfies the Euler relation $n - \frac{1}{2}kn + (p_a + p_b) = 2$, i.e.

$$p_a(2k - a(k-2)) + p_b(2k - b(k-2)) = 4k. \quad (1)$$

Note, that if $a \ge \frac{2k}{k-2}$, then $b > \frac{2k}{k-2}$, and the left hand side of above equality is less than zero. Hence $a < \frac{2k}{k-2} = 2 + \frac{4}{k-2}$ and (3,3), (3,4), (3,5), (4,3), (5,3) are only possible (k, a). We consider only the case when, moreover, $b \le \frac{2k}{k-2}$; it covers bifaced polyhedra mentioned in chemical applications. It is easy to see, that all possible such bifaced polyhedra belong to one of the following three classes.

A) If $k = 3 \le a < b \le 5$, then (**??**) takes the form $p_a(6 - a) + p_b(6 - b) = 12$ and, for $p_b > 0$, the class consists only the following 6 simple polyhedra (all but no.2 are duals of all 5 non-Platonic convex deltahedra; each no.i below, i=3,4,5, comes from no.$i+1$ by truncation of some edge).

1. $Prism_3$ for $(a, b) = (3, 4)$ with $p_3 = 2$, $p_4 = 3$, $n = 6$;

2. the Dürer octahedron (i.e. the cube truncated in 2 opposite vertices) for $(a, b) = (3, 5)$ with $p_3 = 2$, $p_5 = 6$, $n = 12$;

and 4 polyhedra for $(a,b) = (4,5)$:
3. dual of 2-capped $APrism_4$ with $p_4 = 2$, $p_5 = 8$, $n = 16$;
4. dual of 3-augmented $Prism_3$ with $p_4 = 3$, $p_5 = 6$, $n = 14$;
5. dual of bidisphenoid with $p_4 = 4$, $p_5 = 4$, $n = 12$;
6. $Prism_5$ with $p_4 = 5$, $p_5 = 2$, $n = 10$.

B) If $k = 3 \leq a < b = 6$, then (**??**) takes the form $p_a(6-a) = 12$ and there are 3 infinite families a_n: 3_n, 4_n, 5_n with $(a, p_a) = (3,4)$, (4,6), (5,12), respectively, and with the unbounded number p_6 of hexagons.

C) If $k = 4$, then $3 \leq a < b \leq \frac{2k}{k-2} = 4$ implies $a = 3$, $b = 4$ and (**??**) takes the form $p_a = 8$, i.e. there is only one infinite family $(3,4)_n$ with $(a, b, p_a) = (3,4,8)$ and unbounded p_4.

The minimal polyhedra of the families 3_n, 4_n, 5_n and $(3,4)_n$ are polyhedra 3_4, 4_8, 5_{20} and $(3,4)_6$ with $p_b = 0$, when $n = \frac{ap_a}{k}$. Clearly, they are Platonic polyhedra: tetrahedron, cube, dodecahedron and octahedron, respectively.

[?] gives that other 3_n, 4_n, 5_n and $(3,4)_n$ exist iff $12 \leq n \equiv 0 \pmod 4$, $12 \leq n \equiv 0 \pmod 2$, $24 \leq n \equiv 0 \pmod 2$ and $8 \leq n$, respectively. The equality $kn = ap_a + bp_b$ implies that the number n of vertices is equal to

$$n(3_n) = 4 + 2p_6, n(4_n) = 8 + 2p_6, n(5_n) = 20 + 2p_6 \text{ and } n((3,4)_n) = 6 + p_4. \qquad (2)$$

5_n are *fullerenes* well-known in Chemistry (see, for example, [?], [?]); 4_n are mentioned in [?]. In [?], [?], [?] we consider isometric embedding (up to scale 1 and 2) of skeletons of some bifaced polyhedra into the vertex-set of hypercubes. It turns out that all known fullerenes such that it or its dual is embbeddable are face-regular in the sense considered below. In fact, 5_{20}=20:1, 5_{26}=26:1, 5_{44}=44:73, 5_{80}=80:7 and duals of 20:1, 28:2, 36:15, $5_{60}(I_h)$ are embeddable into a half m-cube for m=10, 12, 16, 22 and 6, 7, 8, 10, respectively. (The notations $n : k$ of a 5_n are taken from [?]). Moreover, all known (see [?]) bifaced polyhedra such that it or its dual is embeddable turn out to be face-regular: ## 1, 2, 5, 6 and duals of ## 1, 2 in case **A**); 4 polyhedra 4_n (n=12, 24, 32, 32), 5 dual 3_n (n=12, 16, 16, 28, 36); all t-hex-elongated cubes and their duals, in addition to 5 others $(3,4)_n$ (n=8, 9, 10, 16, 24) and to the dual cuboctahedron embeddable into H_4.

The graphs of all polyhedra of classes **A**), **B**), **C**) and their duals (except undecided 5_n and dual $(3,4)_n$) have a Hamiltonian circuit; this follows from the results surveyed in Section 5.3 of [?].

Here we want to identify aR_i and bR_j bifaced polyhedra; aR_i (resp. bR_j) means that each a-face (resp. b-face) is edge-adjacent to exactly i a-faces (resp. j b-faces). Sometimes, it is convenient to distinguish aR_i and bR_j bifaced polyhedra by graphs G_a and G_b of the edge-adjacency of a- and b-faces, respectively. These graphs have p_a and p_b vertices, respectively. *Face-regular* bifaced polyhedra are those having regular graphs either G_a or G_b. (This combinatorial notion has nothing to do with the affine notion of *regular-faced* polyhedra.)

A motivation for this work comes from fullerenes studies; see, for example, [?] (pentagonal and hexagonal indices and their connection to the steric strain). In fact, these indices give the number of vertices with degree 0,...,5 and 0,...,6 for graphs G_a and G_b of fullerenes; they where introduced in [?] as a try to mesure the steric strain of isomers of 5_{84}.

See [?] for terms used here for polyhedra. We identify a polyhedron with the graph of its skeleton. According to the famous Steinitz' Theorem, a graph is the skeleton of a (3-dimensional) polyhedron if and only if it is planar and 3-connected. For a simple polyhedron P we denote by cham$P$ and call *chamfered* P the polyhedron obtained by putting prisms on all faces of P and deleting original edges (see [?], [?] for more details).

For $t \geq 1$, denote by 2-$Prism_4^t$ the *t-elongated* octahedron, i.e. the column of t cubes, capped in 2 most opposite faces. It is $(3,4)_{4t+6}$, and besides it is $3R_2$ in our terms.

Similarly, for $t \geq 1$, denote by $(APrism_3^{t+1})^*$ the *t-hex-elongated cube*, i.e. the cube with t triples of hexagons inserted as belts between 2 triples of squares incident to 2 opposite vertices (in other words, the dual of the column of $t+1$ octahedra $\beta_3 = APrism_3$). It is tetrahedral 4_{6t+8}, which is $4R_2$.

Finally, for $t \geq 1$, denote by $(2\text{-}APrism_6^{t+1})^*$ the *t-hex-elongated* 5_{24}, i.e. the dual of the column of $t+1$ $APrism_6$'s capped on 2 opposite 6-faces. It is 5_{12t+24} and is $5R_2$.

2 Face-regular polyhedra 3_n, 4_n and 6 sporadic ones

At first, one can verify by a direct check the following

Fact 1. *All 6 bifaced polyhedra of the class* **A**) *above are both aR_i and bR_j for (i,j) =(0,2), (0,4), (0,4), (0,3), (1,2) and (2,0), respectively. Their graphs G_b are, respectively: C_3, the skeletons of the octahedron, of $APrism_4$, $K_2 \times K_3$, C_4, C_5.*

From now on we consider only classes **B**) and **C**). So $(k,b) = (3,6)$ or (4,4), i.e. we consider only 3_n, 4_n, 5_n and $(3,4)_n$. Studing polyhedra with bR_j we, naturally, exclude the four Platonic polyhedra 3_4, 4_8, 5_{20} and $(3,4)_6$ having no b-faces.

First observations about bR_j give

Lemma 1 *For a bifaced polyhedron bR_j the following holds:*

(i) $p_b \leq a\frac{p_a}{b-j}$ with an equality iff it is aR_0; if $k=3$, then $p_b \geq \frac{p_a}{b-j}$;

(ii) $p_b = (a-i)\frac{p_a}{b-j}$ if the polyhedron is also aR_i;

(iii) if $(3,4)_n$ is $4R_j$ for $j=1,2$, then it is not $3R_0$;

(iv) if $k=3$, then bR_j for $j<\frac{b}{2}$ excludes aR_0 and bR_{b-1} implies aR_0; similarly, aR_i for $i<\frac{a}{2}$ excludes bR_0 and aR_{a-1} implies bR_0.

Proof. At first, (i) comes from counting the number of a-a edges as $\frac{1}{2}p_a a - \frac{1}{2}p_b(b-j) \geq 0$. The lower bound comes from the upper bound $\frac{1}{2}p_a(a-1)$ on the number of a-a edges. In fact, the only 3_n, 4_n and 5_n which contain an a-face surrounded by a-faces only, are Platonic polyhedra. Now, (ii) comes from double counting of the number of a-b edges. (iii) and (iv) can be easily checked case by case. □

The case (i) of Lemma **??** imples the finiteness of the number of bifaced polyhedra bR_j. The same upper bound holds even without our restriction $b \leq \frac{2k}{k-2}$, but only this restriction limits p_a.

The doubble counting of the number of a-b edges also implies the equality $2e(G_a) - 2e(G_b) = ap_a - bp_b$ for the numbers $e(G_a)$, $e(G_b)$ of edges of G_a, G_b. So, they are equal iff $kn = 2ap_a$.

2.1 Face-regular polyhedra 3_n

For 3_n, Lemma **??** gives a full answer on our problem of classifying of aR_i and bR_j.

Proposition 1 *Any 3_n (except the tetrahedron 3_4) is $3R_0$. There are exactly 4 polyhedra 3_n which are $6R_j$:*

the truncated tetrahedron 3_{12} for $j = 3$,

the chamfered tetrahedron 3_{16} and its twist 3_{16} (coming by a truncation of the cube on 4 vertices pairwise at distance 2 for the first 3_{16} and on 4 vertices being endvertices of two opposite edges for the second 3_{16}) for $j = 4$,

a 3_{28} (coming by a truncation of a dodecahedron on 4 vertices pairwisely at distance 3) for $j = 5$.

The graphs G_b of these polyhedra are the skeletons of the tetrahedron, the octahedron and the icosahedron for $j = 3, 4, 5$, respectively.

Proof. Recall that for 3_n, $p_3 = 4$ by (**??**), $n = 4 + 2p_6$, and by Lemma **??**(i) $p_6 \leq \frac{12}{6-j}$. Hence we have

$$n \leq 4 + \frac{24}{6-j} \leq 28.$$

If $j \leq 2$, this bound implies $n \leq 10$, i.e. $n = 4$, and we have a tetrahedron having no hexagons. Hence $j \geq 3$, and $n \geq 12$. We just checked all 3_n for $n \leq 28$ (there are 1, 2, 1, 2, 2 polyhedra 3_n for $n = 12, 16, 20, 24, 28$, respectively). □

2.2 Face-regular polyhedra 4_n

The cube is unique 4_n which is $4R_4$. There is no 4_n which is $4R_3$.

Proposition 2 *The only polyhedra 4_n which are $4R_2$ are either $4_{12} = Prism_6$ or the family $(APrism_3^{t+1})^* = 4_{8+6t}$, $t \geq 1$, of t-hex-elongated cubes.*

Proof. Let q_0 be a quadrangle of 4_n with $4R_2$. Then q_0 is adjacent to two quadrangles q_1 and q_2. These quadrangles are adjacent other qudrangles. There are two cases: either q_1 and q_2 are adjacent or not. In the first case, we obtain a configuration of 3 quadrangles surrounded by 3 hexagons. This configuration generates the family $(APrism_3^t)^*$. In the second case, we obtain a ring of six quadrangles that uniquely gives $Prism_6 = 4_{12}$. □

There is also an infinity of $4R_0$ and $4R_1$. If a polyhedron 4_n is $4R_1$, then the 6 quadrangles are partitioned into 3 pairs of adjacent quadrangles. The pairs are separated by hexagons. Each pair of quadrangles is surrounded at least by one ring of hexagons. There are 3 polyhedra 4_n for n=18, 20, 26, where each pair of quadrangles is surrounded exactly by one ring of hexagons. Remark that a deletion of 3 edges, separating quadrangles in the 3 pairs, (and 6 their endvertices) produces bifaced polyhedra from above 4_{18} and 4_{26}, namely, $Prism_6$ and dual of 3-augmented $Prism_3$, respectively.

Let there be at least two rings of hexagons around one pair of quadrangles. Then one can find a chain of hexagons connecting two pairs of quadrangles such that the two end hexagons of the chain are adjacent exactly to one quadrangle of each pair. Let P be such chain containing k hexagons. We can consider P as follows. Let P_k e a chain of $k+4$ quadrangles, where 2+2 end quadrangles are the original ones. Each inner quadrangle has an upper and a lower edges, and two side edges by that it is adjacent to two neighbouring quadrangles. We have to set two new vertices on the upper and lower edges of each inner quadrangle of P_k for to otain the chain of hexagons. There are 3^k possibilities of setting new vertices. Some of these 3^k chains of hexagons are isomorphic. Not each obtained chain is *feasible*, i.e. it can be enlarged upon a polyhedron 4_n. But if a chain of hexagons with two end pairs of quadrangles is feasible, then it and the third pair of quadrangles defines uniquely a polyhedron 4_n. Nonisomorphic chains can define isomorphic polyhedra 4_n.

For $k=1$, there is only one feasible chain that defines the polyhedron 4_{26}.

For $k=2$, there are 3 nonisomorphic chains. All they are feasible and define polyhedra 4_n for n=32, 36, 44.

For $k=3$, there are 6 feasible nonisomorphic chains defining 5 nonisomorphic polyhedra 4_n for $n=36$, 48, 50, 56, 66, where 4_{36} is isomorphic to 4_{36}, obtained for $k=2$.

We can explicitly define an infinite sequence of feasible chains P^0_k, $k\geq 1$, such that each inner quadrangle of P^0_k has by one new vertex on the upper and the lower edges. These chains define an infinite sequence of polyhedra 4_n for $n=2(k^2+6k+6)$, $k\geq 1$.

There are another sequences of $4R_1$-polyhedra 4_n for $n=2(k^2+4k+4)$, $n=2(k^2+5k+4)$, $k\geq 1$, and $n=2(k^2+3k+6)$, $k\geq 3$. For $k=1$, the first three sequences give 3 smallest $4R_1$-polyhedra 4_{26}, 4_{18}, 4_{20}.

All six $4R_1$ polyhedra 4_n with $n<46$ (see Table 1 below) come as first two members of those 3 sequences. Perhaps, for any large even n, there are $4R_1$ and $4R_0$ polyhedra. Similar assertion is even more probable for more rich families 5_n and $(3,4)_n$.

Proposition 3 *(i) All $(i,j;n)$ such that there exists 4_n both $4R_i$ and $6R_j$ are:*
(2,0;12), (2,2;14), (1,3;20), (2,4;20), (0,3;24), (1,4;26), (0,4;32), (0,5;56).

(ii) Each of above 8 cases is realized by unique 4_n, except the case (0,4;32) realized by the chamfered cube and its twist.

(iii) $Prism_6=4_{12}$ (the first case in (i) above) is only $6R_0$. The dual tetrakis snub cube (the last case in (i)) is unique $6R_5$. There are no $6R_1$. The unique $4_{14}=(APrism^2_3)^*$ (which is also $4R_2$) and unique 4_{16} are only $6R_2$. Only $6R_3$ are cases (1,3;20) with G_b being $Prism_3$, and the truncated octahedron (0,3;24) above. All $6R_4$ are the cases (2,4;20)=$(APrism^3_3)^*$, (1,4;26) with G_b being the unique $(3,4)_9$ and (0,4;32) (the chamfered cube and its twist having as G_b the cuboctahedron and its twist). □

All examples of 4_n for $n\leq 44$ are given in Table 1. The last 3 columns of Table 1 give numbers of 4_n, $n\leq 44$, with the graph G_a of the edge-adjacency of 4-gons being $2K_2+2K_1$, K_2+4K_1, P_3+P_3, respectively. The polyhedra 4_n, $n\leq 44$ are taken from [?].

Table 1

n	$\#4_n$	$\#4R_0$	$\#4R_1$	$\#4R_2$	$2K_2+2K_1$	K_2+4K_1	P_3+P_3
12	1	–	–	1	–	–	–
14	1	–	–	1	–	–	–
16	1	–	–	–	–	–	1
18	1	–	1	–	–	–	–
20	3	–	1	1	–	–	1
22	1	–	–	–	1	–	–
24	3	1	–	–	1	–	1
26	3	–	1	1	1	–	–
28	3	–	–	–	1	1	1
30	2	1	–	–	1	–	–
32	8	2	1	1	3	–	1
34	3	1	–	–	1	1	–
36	7	3	1	–	1	1	1
38	7	1	–	1	3	2	–
40	7	2	–	–	2	2	1
42	5	2	–	–	2	1	–
44	14	3	1	1	8	1	–

So, all face-regular 4_n (except two infinite sets for $4R_0$, $4R_1$) are 9 polyhedra from Proposition **??**(i), the unique 4_{16} and all t-hex-elongated cubes for $t > 2$.

3 Face-regular fullerenes 5_n

The dodecahedron 5_{20} is the unique fullerene which is $5R_5$.

The hexagonal barrel $Barrel_6 = 5_{24}$ is the unique 5_n which is $5R_4$. It is also $6R_0$.

The only fullerenes with $5R_3$ are $5_{28}(T_d)$ and $5_{32}(D_{3h})$, which are $6R_0$ and $6R_2$, respectively. All fullerenes $5R_i$ for $i = 3, 4$ are the first 3 cases in (i) of Proposition **??** below.

The fullerenes which are $5R_2$ are distinguished by graphs G_a which are cycles formed by pentagons. There are the following 5 cases:

(1) $G_a = 4C_3$,
(2) $G_a = 2C_3 + C_6$,
(3) $G_a = C_3 + C_9$,
(4) $G_a = 2C_6$,
(5) $G_a = C_{12}$.

There are infinitely many fullerenes in the case (1): $5_{48}(D_2)$ and at least one tetrahedral 5_n for $n = 4(a^2 + ab + b^2) - 8$ starting with $5_{40}(T_d)$, $5_{44}(T)$, $5_{56}(T_d)$, $5_{68}(T)$, $5_{76}(T)$; see [?] for the case (1). The fullerenes $5_{44}(T)$ and $5_{56}(T_d)$ are also $6R_3$ and $6R_4$, respectively.

Clearly, any fullerene 5_n in the case (1) comes by collapsing into a point of all 4 triangles of a $3_{(n+8)}$; so, n is divided by 4. Actually, all face-regular 5_n (besides 26:1, 30:1, 38:16 and (0,-), (1,-)) have n divisible by 4.

Proposition 4 *There is no fullerene in the case (2).*

Proof. The 6-cycles of pentagons can be considered as a ring with 6 *tails*, i.e. edges connecting the vertices of the ring with other vertices. Similarly, each 3-cycle of pentagons is a circuit of 9 vertices, six of which are endpoints of 6 tails. We have to connect the 6 tails of the 6-cycles with 12 tails of two 3-cycles for to obtain a net of hexagons.

The 6-cycles C_6 has two domains: outer and inner. There are two cases: either the two 3-cycles lie in distinct domains or both lie in the same, say, outer, domain. In the first case, by symmetry, we can consider only outer domain. The Euler relation shows that the boundary circuit of the ring of pentagons should have 3 tails. It is easy to verify that it is not possible to form a net of hexagons using 3 tails of the 6-cycle and 6 tails of the 3-cycle.

In the second case by the Euler relation we have the 6-cycle with 6 tails and two 3-cycles C_3^A and C_3^B each with 6 tails. Suppose there is a fullerene containing this configuration. Then, in this fullerene, there are chains of hexagons connecting a pentagon of the 6-cycle and a pentagon of a 3-cycle. Consider such a chain of minimal, say, q, length. In this case, the 6-cycle is surrounded by q rings each containing 6 hexagons. If we dissect the q-th ring of hexagons into two 6-cycles each with 6 tails, we obtain the 6-cycle surrounded by $q-1$ rings. The boundary of the $(q-1)$-th ring contains 6 tails.

Let the chain of q hexagons connect the 6-cycle with C_3^A. At least two tails of the $(q-1)$-th ring correspond (are connected) to tails of C_3^A. Since the boundary of the $(q-1)$-th ring with 6 tails is similar to the boundary of the 6-cycle with 6 tails, our problem is reduced to the case when two tails of the 6-cycle are connected to two tails of C_3^A. There are two cases: either endpoints of the two tails of C_3^A are separated on the boundary of C_3^A by a vertex or not. We obtain two configurations each consisting of a circuit with vertices having or not having tails. Both these configurations have unique enlarging by hexagons which cannot be glued with the cycle C_3^B having 6 tails. $\square$

We proved in [?] that there is the unique fullerene $5_{38}(C_{3v})$ (which is $6R_2$) in the case (3), and exactly 4 fullerenes $5_{36}(D_{2d})$, $5_{44}(D_{3d})$, $5_{44}(D_2)$ and $5_{48}(D_{6d})$ (which are not $6R_j$ for all j) in the case (5).

In the case (4) we have an analogue of Proposition **??**.

Proposition 5 *The only fullerenes 5_n which are $5R_2$ and have $G_a = 2C_6$ are t-elongated 5_{24}, i.e. (2-$APrism_6^{t+1})^* = 5_{12t+24}$, $t \geq 1$.*

Proof. Let C_6 and C_6' be the inner and the outer circuits of a ring of 6 adjacent quadrangles. We have to set 6 new vertices on edges of C_6 and C_6' such that the 6 quadrangles of the ring are transformed into 6 pentagons. It is not difficult to verify that all new vertices should be set on one of the circuits C_6 and C_6'. Let C_6 have no new vertices. Then it is the boundary of a hexagon. A similar assertion is true for the other 6-cycles of pentagons. This configuration of two 6-cycles of pentagons and the condition $5R_2$ uniquely give the family (2-$APrism_6^{t+1})^*$ consisting of a 6-cycle of pentagons surrounded by t rings of hexagons (each containing 6 hexagons) and by the other 6-cycle of pentagons. $\square$

It looks too hard to describe all fullerenes $5R_0$, and even simpler fullerenes $5R_1$. (All 130 of such $5R_1$ fullerenes for $n \leq 72$ are listed in [?]; the two smallest are a $5_{50}(D_3)$ and the $5_{52}(T)$.) Clearly, any fullerene which is $5R_1$ comes by collapsing into an edge each of all 6 quadrangles of a $4_{(n+12)}$.

Now we consider fullerenes which are $6R_j$.

It is not difficult to show that the barrel 5_{24}, the 5_{26} and a $5_{28}(T_d)$ are the only fullerenes which are $6R_0$. A not great enumeration shows that only fullerenes which are $6R_1$ are the $5_{28}(D_2)$ and the $5_{32}(D_3)$.

$6R_2$-configurations G_b of hexagons are sums of cycles C_m.

Proposition 6 *The only fullerenes which are $6R_2$ are the following fullerenes:*
$5_{32}(D_{3h})$ *with* $G_b = 2C_3$,
$5_{38}(C_{3v})$ *with* $G_b = C_3 + C_6$,
$5_{40}(D_{5d})$ *with* $G_b = 2C_5$,
$5_{30}(D_{5h})$ *with* $G_b = C_5$,
$5_{32}(D_{3d})$ *and* $5_{32}(D_2)$ *both with* $G_b = C_6$,
$5_{36}(D_{2d})$ *with* $G_b = C_8$,
$5_{40}(D_2)$ *with* $G_b = C_{10}$.

Proof. It can be done using $n \leq 50$ and the inequality for $6R_2$-fullerenes from Lemma **??**(i) and scanning the list of small fullerenes in [?]. But we cannot assert that the list contains all $6R_2$-fullerenes. Hence we give below a geometrical proof.

Let a fullerene contains a triple C_3 of mutually adjacent hexagons. Then the triple is surrounded by a ring of 9 pentagons. There are two cases: either the ring of pentagons is surrounded by a ring of hexagons or not. In the first case we otain uniquely the fullerene $5_{38}(C_{3v})$ with $G_b = C_3 + C_6$, which is the case (3) of fullerenes that are $5R_2$. In the second case, we obtain uniquely the fullerene $5_{32}(D_{3h})$ with $G_b = 2C_3$ such that the ring of 9 pentagons is surrounded by 3 pentagons and 3 mutually adjacent hexagons.

Now consider fullerenes containing only rings C_m of hexagons for $m > 3$. Of course, there is a ring R containing inside only pentagons. We distinguish cases by the number p of pentagons contained inside the ring R. If $p = 1$, then the ring R consists of 5 hexagons. Since the fullerene is $6R_2$, R is surrounded by a ring of 10 pentagons. We obtain a configuration which uniquely defines the fullerene $5_{40}(D_{5h})$ with $G_b = 2C_5$, where the ring of 10 pentagons is surrounded by the second ring of 5 hexagons, and the outer boundary of the ring is a pentagon.

If $p = 2$, we obtain, as above, uniquely the fullerene $5_{40}(C_2)$, which is not $6R_2$.

For $p = 3$, there are 2 configurations of 3 pentagons: three mutually adjacent pentagons and a chain of pentagons. We obtain again the fullerene $5_{38}(C_{3v})$ in the first case. The other case do not give fullerenes which are $6R_2$.

For $p = 4$ and $p = 5$, there are 4 and 7 configurations of pentagons, respectively. None of them gives a fullerene which is $6R_2$.

There are 18 connected configurations of 6 pentagons. Only 5 of them give fullerenes which are $6R_2$: $5_{30}(D_{5h})$ with $G_b = C_5$, $5_{32}(D_{3d})$ and $5_{32}(D_{2h})$ both with $G_b = C_6$, $5_{36}(D_{2d})$ with $G_b = C_8$, $5_{40}(D_2)$ with $G_b = C_{10}$.

For $7 \leq p \leq 12$, we do not obtain fullerenes which are $6R_2$. $\square$

We know the following 5 fullerenes which are $6R_3$: $5_{36}(D_2)$, $5_{44}(T)$ which is also $5R_2$, $5_{48}(D_3)$, $5_{52}(T)$ which is $5R_1$, and the buckminsterfullerene $5_{60}(I_h)$ which is $5R_0$. Remaining unchecked $6R_3$-fullerenes 5_n have $52 \leq n \leq 58$.

We know also the following 5 fullerenes which are $6R_4$: $5_{40}(D_{5d})$, $5_{56}(T_d)$ which is also $5R_2$, $5_{68}(T_d)$ which is also $5R_1$, $5_{80}(I_h)$ which is also $5R_0$, a $5_{80}(D_{5h})$ which is also $5R_0$. Remaining unchecked $6R_4$-fullerenes 5_n have $52 \leq n \leq 78$.

All face-regular fullerenes 5_n, $50 < n < 80$, with the symmetries T, T_h, T_d, I_h (besides 4 items in Proposition **??**(i) below) are the unique with their symmetries $5_{68}(T)$, $5_{76}(T)$, $5_{76}(T_d)$ which are (2,-), (2,-), (0,-) with $G_5 = 4K_3$, $4K_3$, $12K_1$, respectively.

The fullerenes 5_n which are both $5R_i$ and $6R_j$ have $n = 20 + 24\frac{5-i}{6-j}$. Taking such fullerenes from the above list of fullerenes which are $5R_i$ and/or $6R_j$, we obtain

Proposition 7 *(i) All $(i, j; n)$ such that there exists 5_n both $5R_i$ and $6R_j$ are:*

(4,0;24), (3,0;28), (3,2;32), (2,2;38), (2,3;44), (1,3;52), (2,4;56), (0,3;60), (1,4;68), (0,4;80), (0,5;140).

Their G_b are, respectively, $2K_1$, $4K_1$, $2C_3$, C_{10}, truncated tetrahedron, chamfered tetrahedron, a 4-valent polyhedron with p= (p_3=14, p_6=6), the dodecahedron, a 4-valent polyhedron with p= (p_3=20, p_6=6), the icosidodecahedron, its twist, snub dodecahedron.

(ii) Each of 11 above cases is realized by the following (unique with their symmetry) fullerenes:

$5_{24}(D_{6d})$, $5_{28}(T_d)$, $5_{32}(D_{3h})$, $5_{38}C_{3v})$, $5_{44}(T)$, $5_{52}(T)$, $5_{56}(T_d)$, $5_{60}(I_h)$, $5_{68}(T_d)$, $5_{80}(I_h)$, $5_{80}(D_{5h})$, $5_{140}(I)$.

(iii) For $(i, j; n)$=(0,4;80) there is exactly one other fullerene: $5_{80}(D_{5h})$= twisted chamfered dodecahedron. The fullerenes of other 10 cases (except unchecked n =56 and n=68) are unique.

(iv) $5_{140}(I)$ is unique $6R_5$; 5_{24}, 5_{26}, $5_{28}(T_d)$ are only $6R_0$ (their duals and the icosahedron are called Frank-Kasper polyhedra in chemical physics; they appear also as disclinations (rotational defects) of local icosahedral order).

All 17 fullerenes 5_n with $n \leq 50$ and which are $6R_j$ are among 27 which are listed below in Table 2. Six more $6R_j$-fullerenes are the last 6 items of Proposition **??** (i). The only remaining possibilities should be either $6R_3$ with n=52, 54, 56, 58, or $6R_4$ with n between 52 and 78; in both these cases fullerenes are not $5R_i$, except possibly the triples (i, j, n)=(2,4,56) and (1,4,68).

For $n < 82$, there are 15, 4, 4 fullerenes from 3 respective infinite series: $(0, -)$, $(2, -)$ with $G_5 = 2C_6$, $(2, -)$ with $G_5 = 4C_3$; there are 75 fullerenes $(1, -)$ for $n < 74$.

Table 2. All face-regular fullerenes with at most 50 vertices

# in [?]	symmetry * if unique		(i,j) in $5R_i, 6R_j$	G_a	G_b
$20:1$	I_h	$*$	$(5,-)$	Icosahedron	$-$
$24:1$	D_{6d}	$*$	$(4,0)$	$APrism_6$	$2K_1$
$26:1$	D_{3h}	$*$	$(-,0)$	G_7	$3K_1$
$28:1$	D_2	$*$	$(-,1)$	G_8	$2K_2$
$28:2$	T_d	$*$	$(3,0)$	truncated K_4	$4K_1$
$30:1$	D_{5h}	$*$	$(-,2)$	$2\hat{C}_5$	C_5
$32:2$	D_2	$*$	$(-,2)$	$2G_1$	C_6
$32:3$	D_{3d}	$*$	$(-,2)$	$2G_2$	C_6
$32:5$	D_{3h}	$*$	$(3,2)$	G_4	$2C_3$
$32:6$	D_3	$*$	$(-,1)$	G_9	$3K_2$
$36:2$	D_2		$(-,3)$	$2G_1$	G_5
$36:6$	D_{2d}		$(-,2)$	$2G_3$	C_8
$36:14$	D_{2d}		$(2,-)$	C_{12}	$2P_4$
$36:15$	D_{6h}	$*$	$(2,-)$	$2C_6$	C_6+2K_1
$38:16$	C_{3v}	$*$	$(2,2)$	C_3+C_9	C_3+C_6
$40:1$	D_{5d}		$(-,4)$	$2\hat{C}_5$	$APrism_5$
$40:38$	D_2		$(-,2)$	$2P_6$	C_{10}
$40:39$	D_{5d}		$(-,2)$	$C_{10}+2K_1$	$2C_5$
$40:40$	T_d	$*$	$(2,-)$	$4C_3$	G_6
$44:73$	T	$*$	$(2,3)$	$4C_3$	truncated K_4
$44:85$	D_2		$(2,-)$	C_{12}	$2G_1$
$44:86$	D_{3d}		$(2,-)$	C_{12}	$2G_2$
$48:144$	D_2		$(2,-)$	$4C_3$	G_{11}
$48:186$	D_{6d}		$(2,-)$	C_{12}	$2\hat{C}_6$
$48:188$	D_3	$*$	$(-,3)$	$3P_4$	G_{10}
$48:189$	D_{6d}		$(2,-)$	$2C_6$	$APrism_6+2K_1$
$50:270$	D_3		$(1,-)$	$6K_2$	G_{12}
All fullerenes 5_n both $5R_i$ and $6R_j$ with $n>50$					
52	T	$*$	$(1,3)$	$6K_2$	chamfered K_4
56	T_d	$*$	$(2,4)$	$4C_3$	4-valent,$p=(p_3=14, p_6=6)$
60	I_h	$*$	$(0,3)$	$12K_1$	Dodecahedron
68	T_d	$*$	$(1,4)$	$6K_2$	4-valent,$p=(p_3=20, p_6=6)$
80	I_h	$*$	$(0,4)$	$12K_1$	Icosidodecahedron
80	D_{5h}	$*$	$(0,4)$	$12K_1$	its twist
140	I	$*$	$(0,5)$	$12K_1$	snub dodecahedron

In Table 2 we give also the graphs G_a and G_b (of adjacencies of 5- and 6-faces, respectively). We use the following notation:

C_n, P_n, K_n are the cycle, the path and the complete graph, all on n vertices. Truncated K_4 means the skeleton of the truncated terahedron. $\hat{C}_n$ is a wheel, i.e. C_n plus an universal vertex. We suppose below that the set of n vertices of a graph is the set $\{1, 2, ..., n\}$.

$G_1 = C_6$ with additional edges (1,5), (5,2), (2,4);

$G_2 = C_6$ with additional edges (1,3), (3,5), (5,1);

$G_3 = P_5$ with additional edges (2,6), (3,6), (4,6), where 6 is a new vertex;

$G_4 = C_9$ with additional edges (1,10), (2,10), (3,10), (4,11), (5,11), (6,11), (7,12), (8,12), (9,12), where 10, 11 and 12 are new vertices;

$G_5 = C_6$ with additional edges (1,7), (2,7), (3,7), (4,8), (5,8), (6,8), where 7 and 8 are new vertices;

$G_6 = C_8$ with additional edges (1,9), (9,5), (3,10), (10,7), where 9 and 10 are new vertices;

$G_7 = C_6$ and C'_6 with additional edges (1,3), (3,5), (5,1), $(1',3')$, $(3',5')$, $(5',1')$, $(2,2')$, $(4,4')$, $(6,6')$;

G_8= two isomorphic graphs G_1 and G'_1 connected by edges (x,x') and (y,y'), where x,y and x',y' are pairs of vertices of valency 2 in G_1 and G'_1, respectively;

$G_9 = C_8$ with additional edges (1,9), (2,9), (9,10), (10,11), (5,11), (6,11), where 9, 10, 11 are new vertices;

G_{10}= 3 isomorphic graphs G_3, G'_3 and G''_3 with identified vertices $x = x' = x''$ and $y = y' = y''$, where x,y, x',y' and x'',y'' are pairs of vertices of valency 1 in each of the 3 graphs;

G_{11}= two isomorphic graphs G_1 and G'_1 with additional edges $(3,3')$, $(6,6')$, (2,7), $(2',7)$, (5,8), $(5',8)$; without the new vertices 7 and 8 it is G_8;

G_{12}= Dürer octahedron plus 3 new vertices: the midpoints of 3 edges which are disjoint pairwise and with each triangle.

The asterisk * in the symmetry column of Table 2 means that the fullerene is unique with this symmetry and corresponding number of vertices.

Call a fullerene *quasi*-$6R_j$ if the number of 5-6 edges divided by the number of hexagons is $6-j$. If it is, moreover, not $6R_j$, then call it *proper quasi*-$6R_j$. There are at all 65 proper quasi-$6R_j$ with at most 50 vertices: 3, 2, 4, 17, 1, 38 for (n,j)= (36,2), (38,2), (40,3), (44,3), (48,4), (48,3), respectively. The unique case of $j=4$ above is the fullerene 48:1. Among above 65 fullerenes only 38:10 and 38:14 are also *quasi*-$5R_i$, i.e. the number of 5-6 edges divided by the number of 12 of pentagons. In fact, they are both proper quasi-$5R_2$ and quasi-$6R_2$. The second one looks as a double spiral: its 5-graph G_a is the path P_{12} with the additional edge (1,3), and its 6-graph G_b is the path P_9 with the edge (1,3) also. The fullerenes 44:85,86 in Table 2 are quasi-$6R_3$ while 32:2,3 and 36:2 are quasi $5R_3$.

Some similarities between the graphs G_a, G_b and between symmetries in Table 2 indicate examples of operations on fullerenes:

(i) 30:1 and 40:1 have the same graph $G_a = 2\hat{C}_5$ and they belong (for $t=1,2$) to the sequence of *t-hex-elongated dodecahedron* 5_{20}, i.e. the dual of column of $t+1$ $APrism_5$ capped on 2 opposite 5-faces;

(ii) by deleting 4 points of $K_{1,3}$ in any triple of adjacent pentagons, we obtain 28:2, 40:40 from 44:73 and the $5_{56}(T_d)$, respectively;

(iii) by deleting all six 5-5 edges from the unique $5_{56}(D_{3d})$ which is $5R_1$, we obtain 44:86;

(iv) all 5 (respectively 4) fullerenes having a cycle as G_b (respectively G_a) are in Table 2. A cutting of each hexagon in the cycle into two pentagons produces 44:85, 44:86, 40:39 from 32:2, 32:3, 30:1, respectively. The same cutting of hexagons of the 6-cycle (of hexagons) produces 48:186 from 36:15. The same cutting of alternating hexagons of the 8-cycle (of hexagons) produces 5_{44} from 36:6. The 5-graph G_a of 40:38 is $2P_6$ while the 6-graph G_b of 36:14 is $2P_4$.

4 Face-regular polyhedra $(3,4)_n$

Proposition 8 *All polyhedra $(3,4)_n$ which are $4R_j$ are as follows:*

(i) 3 polyhedra $4R_0$: $APrism_4 = (3,4)_8$ which is $3R_2$ with $G_b = 2K_1$, the $(3,4)_9$ with $G_b = 3K_1$ and the cuboctahedron $(3,4)_{12}$ which is $3R_0$ with $G_b = 6K_1$;

(ii) 2 polyhedra $4R_1$: a $(3,4)_{10}$ with $G_b = 2K_2$, $G_a = 2P_4$ and a $(3,4)_{12}$ which is the twisted cuboctahedron with $G_b = 3K_2$;

(iii) 4 polyhedra $4R_2$: 2-$Prism_4 = (3,4)_{10}$ which is $3R_2$ with $G_b = C_4$, $(3,4)_{12}$ with $G_b = C_6$ and two nonisomorphic $(3,4)_{14}$, both are $3R_1$, with $G_b = C_8$ and $G_b = 2C_4$;

(iv) 3 polyhedra $4R_3$: 2-$Prism_4^2 = (3,4)_{14}$ which is $3R_2$, a $(3,4)_{22}$ which is $3R_1$ and the cross-capped truncated cube $(3,4)_{30}$ which is $3R_0$. G_b graphs of these polyhedra are planar cubic graphs consisting of quadrangles and hexagons. They are skeletons of the cube 4_6, the unique polyhedron 4_{16}, and the truncated octahedron 4_{24}, respectively.

Proof. (i) The condition $4R_0$ implies that each quadrangle is adjacent only to triangles. Consider a quadrangle q_0. Each vertex of q_0 is additionally a vertex of two triangles adjacent to q_0 and a triangle or a quadrangle lying between two triangles adjacent to q_0. We classify $(3,4)_n$ with $4R_0$ using the number s of quadrangles having a common vertex with q_0. If $s=0$, we uniquely obtain $APrism_4 = (3,4)_8$ which is $3R_2$. If $s=1$, then the vertex of the new quadrangle opposite to the vertex common with q_0 has valency 4 with two edges that cannot be connected to anything else. So, this case is impossible. If $s=2$, we uniquely obtain $(3,4)_9$. If $s=3$, we uniquely obtain $(3,4)_{11}$ which has 5 quadrangles such that the 5th quadrangle having no common vertices with q_0 is adjacent to other two quadrangles. Hence this polyhedron is not $4R_0$. If $s=4$, we uniquely obtain the cuboctahedron $(3,4)_{12}$ which is $3R_0$.

(ii) The condition $4R_1$ implies that there is a pair of two adjacent quadrangles such that these quadrangles are adjacent to six triangles. Call this configuration by T. If there is a triangle adjacent to T, we uniquely obtain $(3,4)_{10}$. Otherwise, we uniquely obtain the twisted cuboctahedron $(3,4)_{12}$.

(iii) Let P be a polyhedron $(3,4)_n$ with $4R_2$. Let q_0 be a quadrangle of P. The condition $4R_2$ implies that q_0 is adjacent to two triangles t_0, t_0' and two quadrangles q_1, q_2. There are two possibilities: either q_1 and q_2 have a common vertex or not.

Let S be the configuration consisting of q_0, q_1, q_2, t_0 and t_0', where q_1 and q_2 have no common vertex. Each of the quadrangles q_1 and q_2 is adjacent to two triangles. Without loss of generality we can suppose that one of triangles adjacent to q_1, say t_1, is adjacent to the triangle t_0, too. We have two possibilities: either one of triangles, adjacent to q_2, is adjacent to t_0, or not. Consider the first case. Let t_2 be the triangle adjacent to q_2 and

t_0. We obtain a configuration, say, S_1. It is easy to verify that the condition $4R_2$ imples that S_1 defines uniquely polyhedron 2-$Prism_4 = (3,4)_{10}$ such that the other triangles t'_1 and t'_2 adjacent to q_1 and q_2 are adjacent to the triangle t'_0, too. This polyhedron is also $3R_2$. It has $G_b = C_4$.

Now suppose that P has a configuration S, but no configuration S_1. Then both the triangles adjacent to q_2 are not adjacent to t_0, and one of these triangles, say t_2, is adjacent to t'_0. Since P has no S_1, the second of the triangles adjacent to q_1, say t'_1, is not adjacent to t'_0. Hence there are quadrangles q_3, q_4 adjacent to t_0, q_1 and t'_0, q_2, respectively. So, we obtain a configuration consisting of 5 quadrangles anf 6 triangles that defines uniquely a polyhedron $(3,4)_{14}$. This polyhedron is also $3R_1$. Its graph G_b is C_8.

Now suppose that P has no configuration S. Then the quadrangles q_1 and q_2 have a common vertex v_0 (which is a vertex of q_0, too). There are two possibilities: either v_0 is also a vertex of the fourth quadrangle or of a triangle. Both these cases and the condition that P has no configuration S imply uniquely either a polyhedron $(3,4)_{14}$ having 4 mutually adjacent quadrangles (which is also $3R_1$) or a polyhedron $(3,4)_{12}$ with a 6-belt of 4-gons. The graphs G_b of these polyhedra are $2C_4$ and C_6, respectively.

(iv) Let P be a polyhedron $(3,4)_n$ which is $4R_3$.

(a) Suppose P has a triangle adjacent to 3 quadrangles. Then we have a configuration of a triangle surrounded by six quadrangles. This configuration defines uniquely the cross-capped truncated cube $(3,4)_{30}$ (i.e. each of the six 8-faces of the truncated cube is capped alternating such that each vertex get valency 4), which is also $3R_0$. Its G_b is the skeleton of some 4_{24}.

(b) Let P has no triangle adjacent only to quadrangles. Suppose that P has a triangle adjacent to two quadrangles and a triangle. . The condition $4R_3$ implies that the second triangle is adjacent also to two quadrangles and there are two other quadrangles having common vertices with above triangles. We obtain a configuration consisting of two adjacent triangles surrounded by six quadrangles. This configuration defines uniquely a polyhedron $(3,4)_{22}$ which is also $3R_1$. Its graph G_b is the skeleton of 4_{16}.

(c) Now suppose that each triangle of P is adjacent at most to one quadrangle. Then we otain uniquely the polyhedron 2-$Prism_4^2 = (3,4)_{14}$ which is also $3R_2$. Its G_b is the skeleton of a cube. □

Taking from polyhedra of Proposition **??** ones that are $3R_i$, we obtain

Corollary 1 *(i) All $(i,j;n)$ such that there exists $(3,4)_n$ both $3R_i$ and $4R_j$ are as follows: (2,0;8), (2,2;10), (0,0;12), (1,2;14), (2,3;14), (1,3;22), (0,3;30).*

(ii) Each, except (1,2;14), of above 7 cases is realized by unique $(3,4)_n$:

$APrism_4 = (3,4)_8$, 2-$Prism_4 = (3,4)_{10}$, the cuboctahedron=$(3,4)_{12}$, 2-$Prism_4^2 = (3,4)_{14}$, a $(3,4)_{22}$ and the cross-capped *truncated cube=$(3,4)_{30}$.* □

It looks too hard to describe all $(3,4)_n$ which are $3R_0$; for example, the *ambo* (i.e. the convex hull of the mid-points of all edges) of $(3,4)_n$ is $(3,4)_{2n}$ which is $3R_0$.

There are other operations to obtain one $(3,4)_n$ from another. Let P be a polyhedron $(3,4)_n$. Consider an alternating cut which is not self-intersecting. It is a set C' of edges of P such that if an edge $e \in C'$ belongs to a quadrangle, then the edge opposite to e in this quadrangle belongs to C', but the other two edges of this quadrangle do not belong to C'. Similarly, if an edge of C' belongs to a triangle, then exactly two edges of this triangle belong to C'. Now we transform the cut C' into a circuit C as follows. We set a vertex in the middle of each edge of C' and connect the new vertices consecutively. Then each quadrangle having (two) edges of C' is partitioned into two quadrangles adjacent by the new edge. Each triangle intersecting with C' is partitioned into a triangle and a quadrangle adjacent by the new edge. We obtain a new polyhedron $(3,4)_{n'}$, where $n' = n + |C|$ and $|C|$ is the number of vertices (and edges) of the circuit C.

The circuit C is such that it is not self-intersecting and each its edge is an adjacency-edge of a quadrangle either with a triangle or with a quadrangle. Call a circuit of a polyhedron $(3,4)_n$ with this property *admissible*.

Call a circuit C of a polyhedron P $(3,4)_n$ *alternating* if any two edges of C belong to distinct faces of P. Let v be a vertex of C. The valency of v is 4 in P and is 2 in C. The 4 edges incident to v belonging and not belonging to C alternate. Any admissible circuit of P is alternating. If P has an admissible circuit C, we can delete C from P such that two quadrangle adjacent by an edge of C are glued into a new quadrangle. Similarly, a triangle and a qudrangle adjacent by an edge of C are glued into a new triangle. We obtain a new polyhedron $(3,4)_{n'}$ with $n' = n - |C|$.

Call a polyhedron $(3,4)_n$ *reducible* if it has an admissible circuit. Otherwise the polyhedron is called *irreducible*. An irreducible polyhedron cannot be obtained from other polyhedron by transforming an alternating cut into a circuit. Any reducible polyhedron can be obtained from another polyhedron with smaller number of vertices.

We know two infinite families of irreducible polyhedra $(3,4)_{3k}$, $k \geq 2$, and $(3,4)_{4k}$, $k \geq 2$, $k \neq 3$, and 3 irreducible polyhedra not belonging to these families.

The first polyhedron $(3,4)_6$ of the family $(3,4)_{3k}$ is the octahedron β_3. The polyhedron $(3,4)_{3(k+1)}$ is obtained from $(3,4)_{3k}$ by inscribing a triangle into one of two triangles of $(3,4)_{3k}$ adjacent to 3 other triangles.

The first polyhedron $(3,4)_8$ of the family $(3,4)_{4k}$ is the antiprism $APrism_4$. The polyhedron $(3,4)_{4(k+1)}$ is obtained from $(3,4)_{4k}$ by inscribing a quadrangle into one of two quadrangles of $(3,4)_{4k}$ adjacent to 4 triangles.

The transformation of an alternating cut into a circuit shows that there are infinitely many polyhedra with $3R_1$. In fact, let P be $(3,4)_n$ which is $3R_1$. Let p be a path consisting of adjacency edges of quadrangles with quadrangles. Let p connects two vertices of two pairs of adjacent triangles such that these vertices are endvertices of adjacency edges of these triangles. Then the cut generated by the set of vertices of the path p is alternating and transforms P into $(3,4)_{n'}$ $(n' = n + 2|p| + 1)$ which is $3R_1$ again. But we have the following

Proposition 9 *The only polyhedra $(3,4)_n$ which are $3R_2$ are either $(3,4)_8 = APrism_4$ or the family 2-$Prism_4^t$, $t \geq 1$, of t-elongated octahedra.*

Proof. Let t_0 be a triangle of $(3,4)_n$ with $3R_2$. Then t_0 is adjacent to two triangles t_1 and t_2. These triangles are adjacent other triangles. There are two cases: either t_1 and t_2 have or have not a common second adjacent triangle. In the first case, we obtain a configuration of 4 triangles surrounded by 4 quadrangles. This configuration generates the family 2-$Prism_4^t$. In the second case, we obtain uniquely $APrism_4 = (3,4)_8$. $\square$

Consider alternating (as it is defined after Corollary 1) cuts of the octahedron $\beta_3 = (3,4)_6$. Each cut partitions the set of 6 vertices of β_3 into two parts. We distinguish a cut C by the cardinality $n(C)$ of the smallest part. We have 4 cases, when $n(C) = 1,2,3$. If $n(C) = 2$, then the two vertices are adjacent, otherwise we have nonalternating cut. If $n(C) = 3$, then there are two cases, when β_3 is partitioned either into two triangles or into two 3-paths.

The alternating cut with $n(C) = 1$ provides $(3,4)_{10}$ which is $3R_2$ and $4R_2$. The alternating cut with $n(C) = 2$ provides $(3,4)_{12}$ which is $4R_1$ but not $3R_i$. The cut with $n(C) = 3$ vertices of which form a triangle provides $(3,4)_{12}$= the cuboctahedron which is $3R_0$ and $4R_0$. This polyhedron is the second (reducible) polyhedron of the family $(3,4)_{4k}$ for $k = 3$. The cut with $n(C) = 3$ vertices of which form a 3-path provides $(3,4)_{14}$ which is $3R_1$ and $4R_2$ with $G_b = 2C_4$. (cf. Proposition **??**).

Denote by T_i the polyhedron obtained from β_3 by a cut C with $n(C) = i$, $i = 1,2$. Similarly, let T_3 and T_3' be polyhedra corresponding to cuts with $n(C) = 3$ vertices of which form a triangle and a 3-path, respectively.

The reducibility of the 12 polyhedra of Proposition **??** are as follows:

(i) $(3,4)_8 = (3,4)_{4k}$, $k = 2$, and $(3,4)_9 = (3,4)_{3k}$, $k = 3$, are irreducible, and $(3,4)_{12} = T_3$;

(ii) $(3,4)_{10}$ is irreducible, and $(3,4)_{12} = T_2$;

(iii) $(3,4)_{10} = T_1$, $(3,4)_{14} = T_3'$ with $G_b = 2C_4$, but $(3,4)_{12} = (3,4)_{3k}$, $k = 4$, and $(3,4)_{14}$ with $G_b = C_8$ are irreducible;

(iv) $(3,4)_{14}$, $(3,4)_{22}$ and $(3,4)_{30}$ are reduced to T_1, T_4 and T_2, respectively, i.e. they are reduced to β_3.

So, the polyhedra $(3,4)_{10}$ of (ii) and $(3,4)_{14}$ with $G_b = C_8$ of (iii) are two irreducible polyhedra not belonging to the families $(3,4)_{3k}$ and $(3,4)_{4k}$. There exists the third irreducible polyhedron $(3,4)_{16}$ not belonging to these families. This polyhedron has $G_a = G_3 = 4K_1 + 2K_2$ and $G_b = G_4$ is C_8 with two pendant edges incident to two adjacent vertices of C_8.

So, all face-regular $(3,4)_n$ (except two infinite sets for $3R_0$, $3R_1$) are 8 polyhedra from Corollary 1, the unique $(3,4)_9$, the twisted cuboctahedron $(3,4)_{12}$, a $(3,4)_{10}$, a $(3,4)_{12}$ and all t-elongated octahedra for $t > 2$. The rhombicuboctahedron and its twist are $(3,4)_{24}$ which are $3R_0$. Their graphs $G_b = G_4$ are the octahedron plus a new vertex on each edge and, respectively, $C_8 + K_{9;1,3,5,7} + K_{10;2,4,6,8}$. The snub cube is a 5-valent polyhedron with p=(p_3=32, p_4=6) obtained from the rhombicuboctahedron by cutting 12 of its squares into two triangles; its graph G_4 is the cube plus two new vertices on each edge.

5 Some remarks

Among face-regular bifaced polyhedra, which are both aR_i and bR_j, the most interesting are those that satisfy the following conditions:

a) $(i,j)=(0,b-1)$: dual 2-capped $APrism_4$, Dürer's octahedron, the dual 4-triakis snub tetrahedron 3_{28}, the dual tetrakis snub cube 4_{56}, the dual pentakis snub dodecahedron $5_{140}(I)$ and the cross-capped truncated cube $(3,4)_{30}$.

b) $p_a = p_b$: dual bisdisphenoid, the truncated tetrahedron 3_{12}, two 4_{20}, $5_{44}(T)$ and three $(3,4)_{14}$ with $(i,j;p_a)$= (1,2;4), (0,3;4), (2,4;6), (1,3;6), (2,3;12), (1,2;8), (1,2;8), (2,3;8), respectively.

c) $i=j$: $4_{14}=(APrism_3^2)^*$, $5_{38}(C_{3v})$ and $(3,4)_{10}$=2-$Prism_4$, each has $i=2$ and consists of concentric belts of a- or b-gons (belts sizes are 3,3,3 for above 4_{14}, 3,6,9,3 for $5_{38}(C_{3v})$ and 4,4,4 for 2-$Prism_4$).

All bifaced polyhedra with all a-faces forming a ring are: $Prism_5$, $Prism_6$, $APrism_4$ and 4 fullerenes with 36, 44, 44, 48 vertices. All bifaced polyhedra with all b-faces forming a ring are: $Prism_3$, dual of bidishpenoid, the 4_{14}, the 4_{16}, 2-$Prism_4$, a $(3,4)_{12}$, a $(3,4)_{14}$ and 5 fullerenes with 30, 32, 32, 36, 40 vertices.

It turns out that all bR_j bifaced polyhedra with $b>2$ have as G_b the skeleton of a $bR_{j'}$ bifaced (or regular) polyhedra. In particular, all 3 bR_4 polyhedra 4_n (n=26, 32, 32) have as G_b polyhedra $(3,4)_n$, and all 3 bR_3 polyhedra $(3,4)_n$ (n=14, 22, 30) have as G_b polyhedra 4_n. The face-regularity aR_i or bR_j can be compared with other, relevant for applications topological indices of a bifaced polyhedron. For example, it can be compared with description of vertices by the vertex type or with the pair (q_a,q_b), where q_a (resp. q_b) is the number of maximal connected sets of a-gons (resp. b-gons). Here we call a set of faces *connected* if, seen as a graph with adjacency being the edge-adjacency of faces, it is connected. For example, $q_a+q_b\le p_a+p_b$ with equality for the cuboctahedron; $(q_a,q_b)=(p_a,1)$ or $(\frac{1}{2}p_a,1)$, if a bifaced polyhedron is simple and aR_0 or aR_1, respectively.

It will be interesting to identify face-regular polyhedra among the following simple polyhedra, generalizing those considered in this paper (see [?] for the existence):

1) with $p=(p_3=2,p_4=3,p_6)$; it exists unless $p_6=1,3,7$;

2) with $p=(p_4,p_5=12-2p_4,p_6)$; it exists unless (p_4,p_6) =(1,0), $(i,1)$ for $i=0,1,5,6$;

3) with $p=(p_3,p_5=12-3p_3,p_6)$; it exists unless (p_3,p_6) =(0,1), $(1,i)$ for $i=0,1,2,4$ or (2,1), $(3,i)$ for $i=0,2,4$ or (4,2), (4, any odd).

One can look for face-regular bifaced infinite polyhedra, i.e. partitions of Euclidean plane. For example, all 6 bifaced Archimedian plane partitions are face-regular. In fact, (3.6.3.6), (4.8.8), (3.12.12), (3.3.3.3.6), (3.3.3.4.4), (3.3.4.3.4) are (0,0), (0,4), (0,6), $(-,0)$, (2,2), (1,0), respectively.

Finally, there are face-regular simple polyhedra with $p=(p_5,p_bi), b>6$. Icosahedral polyhedra with $p=(p_5=72,p_{6+i}=\frac{60}{i})$ for i=1,2,3 (so,$n=140+\frac{120}{i}$) with G_b being snub dodecahedron, Icosidodecahedron, Dodecahedron, respectively, are b-face-regular $(6+i)R_{(6-i)}$. On the other hand, following 3 simple bifaced polyhedra are (3,0) face-regular.They have $p=(p_5=12,p_6=4)$, $(p_5=24,p_8=6)$, $(p_5=60,p_{10}=12)$ and n=28, 56, 140, respectively. Their G_5 is truncated Tetrahedron, truncated Cube, truncated Dodecahedron and they have tetrahedral, octahedral, icosahedral symmetry, respectively. They come as a triakon decoration of all 4,8,20 hexagons of truncated Tetrahedron, truncated Octahedron, truncated Icosahedron, respectively.

References

[BaLe93] M.M.Bayer and C.M.Lee, *Combinatorial aspects of convex polytopes*, in: P.M.Gruber and J.M.Wills, eds., Handbook of Conves Geometry, vol.A, pp. 485–534, North Holland, Amsterdam, 1993.

[DDGr] A.Deza, M.Deza and V.P.Grishukhin, *Embedding of fullerenes and coordination polyhedra into half-cubes*, Discrete Mathematics 1997, to appear.

[DeGr97a] M.Deza and V.P.Grishukhin, *A zoo of l_1-embeddable polyhedral graphs*, *Bull. Inst. Math. Acad. Sinica* **25** (1997).

[DeGr97b] M.Deza and V.P.Grishukhin, *A zoo of l_1-embeddable polyhedra II*, Report LIENS-97-9, Ecole Normale Supérieure, Paris, 1997.

[Dill96] M.B.Dillencourt, electronic mail (1996).

[Fowl93] P.W.Fowler, *Systematics of fullerenes and related clusters*, Phil. Trans. R. Soc. London A (1993) 39–51.

[FoMa95] P.W.Fowler and D.E.Manolopoulos, *An Atlas of fullerenes*, Clarendon Press, Oxford, 1995.

[FoCr97] P.W.Fowler and J.E.Cremona, *Fullerenes containing fused triples of pentagonal rings*, J. Chem. Soc., Faraday Trans. **93** (1997) 2255–2262.

[GaHe93] Ying-Duo Gao and W.C.Herdon, *Fullerenes with four-membered rings*, J. Amer. Chem. Soc. **115** (1993) 8459–8460.

[Grün67] B.Grünbaum, *Convex polytopes*, Interscience, New York, 1967.

[Gold35] M.Goldberg, *An isoperimetric problem for polyhedra*, Tohoku Math. Journal **40** (1935) 226–236.

[Ragh92] K.Raghvachari, Chemical Physics Letters **190** (1992) 397–400.