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Metrics on Permutations, a Survey

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Abstract: This is a survey on distances on the symmetric groups S_n together with their applications in many contexts; for example: statistics, coding theory, computing, bell-ringing and so on, which were originally seen unrelated. This paper initializes a step of research toward this direction in the hope that it will stimulate more researchs and eventually lead to a systematic study on this subject.

§0. Introduction

Distances on S_n were used in many papers in different contexts; for example, in statistics (see [Cr] and its references), coding theory (see [BCD] and its references), in computing (see, for example [Kn]), bell-ringing and so on. Here we attempt to give a brief bird's view of distances on S_n according to types of problems considered:

§1. *Bi-invariant semi-metrics*: consider, especially, extreme rays of the cone formed by them; some of such extreme rays coming from graph metrics are given in §4.

§2. *Right-invariant metrics*: lists many examples of such metrics, their connection with statistics and some properties and inequalities of them.

§3. *Ball and cliques*: collect some known information of volumes of balls for some right-invariant metrics, and also on maximal sizes of subsets of S_n having given pairwise distances.

§4. *Graphic and hamiltonian distances*: survey possibilities of derving metric spaces (S_n, d) from graphs or of sorting it out as a kind of hamiltonian circuits.

§5. *Metric basis, permutation approximation and symmetries:* relates, via underlying concept of metric basis of (S_n, H), several papers concerning approximation and the symmetric groups of (S_n, H), (S_n, ℓ_1).

§6. *Commutation distance*: treats separately this distance on S_n, usually considered for other groups, actually $d(a,b)=1$ if and only if $d_{com}(ab,ba)=0$.

§1. Bi-invariant semi-metrics on S_n

Call semi-metrice d on S_n *bi-invariant* if $d(a,b)=d(ca,cb)=d(ac,bc)$ for any $a,b,c\in S_n$. So $d(a,b)=d(ab^{-1},e)$ and weight values $d(a)=d(a,e)$, where e is the identity, determine completely a bi-invariant semi-metric d on S_n. Now, d is bi-invariant if and only if $d(a)=d(b^{-1}ab)$ for all $a,b\in S_n$, *i.e.* if and only if the weight $d(a)$ is constant on conjugacy classes. Let $C_1,\cdots,C_{p_n-1}$ be all nontrrivial conjugacy classes of S_n, where p_n is the number of partitions on n. So any bi-invariant semi-metric can be seen as a vector $(d(C_1),\cdots,d(C_{p_n-1}))$ of length p_n-1. It was noted in $[CD1]$ that all bi-invariant semi-metrics on S_n (in the above form of weight functions on conjugacy classes) form a polyhedral convex cone B_n of dimension $p=p_n-1$ with vertex O.

We are interested in finding extreme rays of the cone B_n, which is exactly the set of vectors $(x_1,\cdots,x_p)$ such that $x_i\geq 0$ for $1\leq i\leq p$, and $x_i\leq x_j+x_k$ if $1\leq i,j,k\leq p$, and $C_i\subseteq C_jC_k$. We take extreme semi-metric, *i.e.* the point d on extremal ray such that $\min\{d(a)\mid d(a)>0\}=1$ as a representative of an extreme ray. So an extremal semi-metric takes only rational values with degrees 2 as denominators.

Some examples of bi-invariant semi-metrics are given in the following as weight functions, for each $a\in S_n$:

1) the Hamming weight $H(a):=|\{1\leq\alpha\leq n, a(\alpha)\neq\alpha\}|$;

2) the Cayley weight $T(a):=$ the minimum numbers of transpositions such that their product is a;

3) the semi-metric $Q(a):=0$ if $a\in A_n$, and $:=1$ otherwise.

We have $H(a)=T(a)+N(a)$ where $N(a)$ is the number of cycles of $a\in S_n$. Moreover,

i) T is an extremal metric, but H does not belong to an extremal ray [CD1].

ii) Q is an extremal semi-metric, and for $n \neq 4$, any bi-invariant semi-metric which is not a metric is a multiple of Q [BC].

Actually, for $n = 3$ there are two nontrivial conjugacy classes: all transpositions, C_1, and all cycles of length 3, C_2. B_3 has only two extremal semi-metrics, *i.e.* T and Q. For $n = 4$, there are four nontrivial conjugacy classes: $C_1 = (\cdot\cdot)$, $C_2 = (\cdot\cdot)(\cdot\cdot)$, $C_3 = (\cdot\cdot\cdot)$, $C_4 = (\cdot\cdot\cdot\cdot)$, and all extremal semi-metrics for $n = 4$ are listed below [BC]:

	C_1	C_2	C_3	C_4
Q	1	0	0	1
R	1	0	2	1
T	1	2	2	3
N	1	2	1	1
	3	2	2	1
	1	2	1	2
	2	2	1	1
	1	2	2	1

Remark that Q and R are the only two among the above eight extremal semi-metrics which are not metrics and that $H = T + Y$. It found in [Fac], by computer check, that there are 50 extremal semi-metrics for $n = 5$ and 805 extremal semi-metrics for $n = 6$. It was shown in [BC] that B_n has at least $2^{\frac{\alpha exp(\pi\sqrt{2n})}{n}}$ extremal semi-metrics as n approaches ∞. Some constructions of extremal metrics coming from graphs will be given below in §4.

§2. Right-invariant metrics on S_n

A semi-metric d on S_n is called *right-invariant* if $d(a,b) = d(ac,bc)$ for any $a,b,c \in S_n$. So $d(a,b) = d(ab^{-1},e)$ as in §1, and weight values $d(a) = d(a,e)$, $a \in S_n$, determine d completely. Some examples of right-invariant metrics are:

1) $\ell_1(a,b) = \sum\limits_{i=1}^{n} |a(i) - b(i)|$, called also *Manhattan, city-block* or *taxi-cab distance (n = 2)* and, in statistics, *Spearman footrule.*

2) $\ell_\infty(a,b) = \underset{i \leq i \leq n}{max} |a(i) - b(i)|$, the dual to ℓ_1 (spaces ℓ_p, ℓ_q are called dual if $\frac{1}{p} + \frac{1}{q} = 1$).

3) $\ell_2(a,b) = \sqrt{\sum_{i=1}^{n}(a(i)-b(i))^2}$, the usual euclidean distance; also called *Spearman's rank correlation* in statistics. Note that $\ell_1, \ell_2, \ell_\infty$ are *Minkowski-Hölder* distances (*i.e.*, $d(a,b) = \|a-b\|$) of normed spaces with $\|a\| = (\sum_{i=1}^{n} |a_i|^p)^{1/p}$ for cases $p = 1,\ 2,$ and ∞ respectively, restricted on vectors $a = (a_1, \cdots, a_n)$ where $a_1, \cdots, a_n$ are permutations of $\{1, 2, \cdots, n\}$.

4) $L(a,b) = \sum_{i=1}^{n} min(|a(i)-b(i)|, n - |a(i)-b(i)|)$, the *Lee distance* used in modulation.

5) $H(a,b) = |\{i \mid i \in \{1,2,\cdots,n\}, a(i) \neq b(i)\}|$, it is *Hamming distance* used in transmission. Note that $H(a,b) = n - |Fix(a^{-1}b)|$ and in case of binary vectors of length n, the distances ℓ_1, L, H coincide with the usual Hamming distance on binary sequences, *i.e.* the cardinality of the symmetric difference.

6) $T(a,b)$:= the minimum number of transpositions needed to obtain b from a, which is equal to n minus the number of cycles in ba^{-1}, *i.e.* the Cayley distance.

7) $I(a,b)$:= the minimum number of pairwise adjacent transpositions needed to obtain b from a, *i.e.*

$$I(a,b) = |\{(i,j) \mid 1 \leq i,j \leq n, a(i) < b(j), b(i) > b(j)\}$$

which correspond to *Kendall's* τ in statistics [Ke].

8) $UL(a,b)$:= n minus the length of the longest increasing subsequence in $(ba^{-1}(1), \cdots, ba^{-1}(n))$. It is the metric introduced by Ulam *et.al.*, [BSU] for DNA research in biology, called *evolutionary distance*, and by Levenstein [Le] for codes correcting errors, deletions and insertions of symbols. It is also used in linguistics as *editing distance*.

In the above list of eight metrics, only the last three are *graphic* (in the same defnied in §4 below.) Metrics d (especially $d = \ell_2, \ell_1, I$) are usually used in statistics in the form

$$1 - \frac{2d}{\max\limits_{a',b' \in S_n} d(a',b')}$$

in order to interpret them as a correlation coefficient. Moreover, metrics on S_n were used in statistics (see, for example, [DG], [Cr] and references there) to compare two permutatins considered as two ranking of the same n items by two judges. The right-invariance of the metric is cructial here since it means that the distance between rankings does not depend on the labellings of our n items. Metric $\ell_1, \ell_\infty, L, H, T$ are extended as right-invariant metrics on partial transformations in [CD2], and metric $\ell_1, \ell_2, H, T, I, UL$ are extended for partially ranked data in [Cr].

[DG] gives mean, max, variance and normality for distance ℓ_1 as $n \to \infty$; they also indicate also the asymptotic normality for T, I and (private communication from Diakonis) for H. [DG] also shows $I + T \leq \ell_1 \leq 2I$, where $d \geq d'$ means $d(a) \geq d'(a)$ for any $a \in S_n$, and that simultaneously equality of both bounds hold exponentially often, since $|\{a \mid a \in S_n, I(a) = T(a)\}| = F_{2n-2}$, where $F_0 = F_1 = 1$ and

$$F_n = F_{n-1} + F_{n_2}$$

are the Fibonacci numbers. It is easy to see ([CD2]) that $\ell_\infty \leq I \geq T$ also and $H/2 \leq T \leq H \leq L \leq \ell_1$.

§3. Balls and cliques for right-invariant distances

The right-invariance of the metric d means that any *sphere*

$$S_{d,n}(r, a_0) = \{a \mid a \in S_n, d(a, a_0) = r\}$$

with center a_0 and radius r has the same size $|S_{d,n}(r)|$ for any choice of the center $a_0 \in S_n$. Equivalently, all *balls* $B_{d,n}(r, a_0) = \bigcup_{i \leq r} S_{d,n}(i, a_0)$ have the same size $|B_{d,n}(r)|$ for any choice of the center $a_0 \in S_n$. It is easy to see that

$$|S_{H,n}(r)| = \binom{n}{r} r! \sum_{i=0}^{r} \frac{(-1)^i}{i} \approx \ell^{-1} \binom{n}{r} r!$$

and

$$|S_{T,n}(r)| = \sum_{\substack{(t_1, \cdots, t_n) \in \{1,2,\cdots,n\}^n \\ \sum_{i \leq i \leq n} t_i \,=\, n-r}}^{n} \frac{n!}{1^{t_1} t_1! \cdots n^{t_n} t_n!}.$$

The size of Hamming sphere $S_{H,n}(r)$ in S_n is just the number of derangements in S_r. We have $|B_{T,n}(1)| = |B_{H,n}(2)| = 1 + \binom{n}{2}$. It will be interesting to find *perfect packings* of S_n, *i.e.*, partitions of S_n into union of disjoint balls $B_{d,n}(r)$ in a given right-invariant metric d. But this is a difficult problem even for unit balls in metrics T and in $H/2$. Of course, we need divisibility of $n!$ by $1 + \binom{n}{2}$ for it, which is possible for example $n = 11$. But [RT] proved that such perfect packing is not possible if $1+n$ is divisible by a prime exceding $\sqrt{n}+2$, and hence $n = 11$ is ruled out. Now,

$$|S_{I,n}(r)| = \sum_{i=0}^{n-1} |S_{I,n-1}(r-i)|,$$

see [Ke], and an explicit formula for it can be fond in [Kn, p. 16].

In addition to H, T, I, the size of ball was studied only for L_∞. It is clear that

$$|B_{L_\infty,n}(1)| = |B_{L_\infty,n-1}(1)| + |B_{L_\infty,n-2}| = F_n,$$

the Fibonacci numbers, refer to §2. [La] gives

$$|B_{L_\infty,n}(2)| = 2|B_{L_\infty,n-1}(2)| + 2|B_{L_\infty,n-3}(2)| - |B_{L_\infty,n-5}(2)|.$$

In the remainder of this section, we consider bounds on maximal size of a *D-clique* $A(D)$ in the metric space (S_n, d), *i.e.* $max\,|A(D)|$ where $A(D) \subset S_n$ with the property that all $d(a,b)$ belong to D whenever $a, b \in A(D)$. Let $|A_S(D)|$ be the size of the D-clique $A_S(D)$ contained in $S \subseteq S_n$. Then, from the *density bound*, $\frac{|A_{S_n}(D)|}{|S_n|} \leq \frac{|A_S(D)|}{|S|}$, it follows [CD1] that

$$|A(D)| \leq \max_{S \subseteq S_n} |A_S(D)| n!/|S|$$

if either d is bi-invariant or $A(D)$ is symmetric (*i.e.* it contains a^{-1} whenever it contains $a \in S_n$). Let $q : S_n^2 \to \mathbf{R}$ be a right-invariant function such that the matrix $[q(a,b)]$ of order $n!$ has only nonnegative eigenvalues and that $g(a,b) \leq 0$ whenever $d(a,b) \in D$. Then the *averaging bound* from [GS] gives

$$|A(D)| \leq (n!)^2 \frac{\max_{a \in S_n} q(a,a)}{\sum_{a,b \in S_n} q(a,b)}.$$

Let $\overline{D}$ denote the set of all nonzero values of d on S_n which are not in the set D, then $|A(D)||A(\overline{D})| \leq |S_n| = n!$ from the duality bound [DF], it follows that $|A(D)| \leq n!/max|A(\overline{D})|$ if either d is bi-invariant or $A(D)$ is symmetric. For example, let $D = \{r+1, \cdots, n\}$, A_1 the ball $B_{d,n}(\lfloor \frac{r}{2} \rfloor)$, A_2 the stablizer of the smallest subset M of $\{1, 2, \cdots, n\}$ such that its stablizer is $A(D)$. Both A_1, A_2 are symmetric cliques $A(\overline{D})$. Specifiying further $d = H$, we have $A(D) \leq n!/|B_{H,n}\lfloor r/2 \rfloor|$ with equality corresponding to perfect packing of S_n for even r and $A(D) \leq n!/|A_2| = n!/(n-r)!$ with equality if and only if $A(D)$ is a sharply $(n-r)-$transitive subset of S_n.

§4 Graphic and Hamiltonian distances

§4.1 Graphic distance

A distance d on S_n is called *graphic* if $d(a,b)$ is the length of a shortest path joining a and b in the simple graph with vertex set S_n, and edge set $\{(c,d) \mid d(c,d)=1\}$. For example, the commutation distance (defined in §6) on $S_n - Z(S_n)$ is not graphic, since $d_{com}(a,b)$ is the length of the shortest path avoiding the center $Z(S_n)$ in the above graph. It is known [KC] that an integer-valued metric on any set X is graphic if and only if $d(a,b)>1$ implies $d(a,c)+d(c,b)=d(a,b)$ for some c. For any finite graphic metric d, the set $\{a \mid a \in S_n, d(a,e)=1\}$ generates S_n.

On the other hand, for any symmetric generating subset E of S_n (*i.e.* $a \in E$ implies $a^{-1} \in E$), define d_E to be the graphic distance on S_n such that the edge-set is exactly $\{(c,d) \mid ac=d \text{ for some } a \in E\}$. Then d_E is a right-invariant distance. Any finite $d_E(a,b)$ is the smallest k such that $a^{-1}b$ is the product of at most k elements of E. d_E is finite if and only if E generates S_n. d_E is bi-invariant if and only if E is a union of conjugacy classes; so, bi-invariant d_E is, moreover, finite if and only if $E \not\subseteq A_n$, the alternating group. For example, d_E with E being the set of all transpositions is exactly (extremal bi-invariant) Cayley metric $T(a)$ considered above in §2. Another example of d_E with E being the set of all *adjacent* transpositions $(i,i+1)$ in the right-invariant metric $I(a)$ from §2 corresponding to Kendall's τ in statistics [Ke, Cr]; it is the shortest path metric of the Cayley graph of S_n generated by E (*i.e.* of the skeleton of the permutahedron - the Voronai polytope of the lattice A^*_{n-1}).

A refreshing example of other graphic metric on S_5 is the shortest path metric of the skeleton of truncated icosadodecahedron - 120-vertices simple zonotope, the largest Archimedean solid. This graph is the Cayley graph of S_5 genrated by $(12)(34)$, $(23)(45)$ and (34). In campanology, it corresponds to the Plain Bob method for 5 cells. Do not confuse it with the permutahedron on S_5 - another (4-dimensional) 120-vertrices simple zonotope. An example of right-invariant graphic metric in S_n which is not of form d_E is the Ulam-Levenstein metric $UL(a)$ in §2 considered in genetic [BSU] and coding [Le].

Some examples of bi-invariant d_E which are extremal (in the cone of all bi-invariant semi-metirc on S_n) are given below:

[CD1]: If C is a conjugacy class of S_n, $C \not\subset A_n$, then d_c is extremal

[BC]: If C is a conjugacy class of S_n, $C \not\subset A_n$ and $C^2 = A_n$, then $d_{c'}$ is extremal where C' is a union of conjugacy classes containing C but not containing more than two classes from A_n.

The above bound is good since d_{A_5} is not extremal for S_5, but A_5 consists of exactly three conjugacy classes. We remark also that 5 is the smallest n such that there is nongraphic extremal bi-invariant semi-metric on S_n.

§4.2 Hamiltonian graphs on S_n

A distance d on S_n is called *hamiltonian* if S_n can be cyclically ordered in such a way that any two consecutive permutations have distance 1. So, graphic d is hamiltonian if and only if the corresponding graph has a hamiltonian circuit. Let $H(a)/i$ denote the graphic metric on S_n with $b, c \in S_n$ adjacent if and only if their Hamming distance is i.

[EW]: $H(a)/i$ is hamiltonian for $n \geq 2$, and any integer $i \in [2, n] - \{3\}$. $H(a)/3$ is not hamiltonian since all 3-cycles in S_n generate A_n but not S_n.

[Sl]: d_E, with E being a set of transpositions, is hamiltonian if the graph G_E with veretx set $\{1, 2, \cdots, n\}$ and edge-set $\{(i,j) \mid \alpha(i) = j, \alpha(j) = i$ for some $\alpha \in E\}$ is connected. Cayley distance $T(a)$ corresponds to $G_E = K_n$, the distance $I(a)$ corresponds to G_E being a path of length n, so both $T(a)$ and $I(a)$ are hamiltonian. $L_\infty(a)$ is also Hamiltonian following from $L_\infty \leq I(a)$.

[CD1]: if $d_E, d_{E'}$ are hamiltonian, then $d_{EE'}$ is hamiltonian on A_n.

Some special hamiltonian circuits in (S_n, L_∞) correspond to good ringring of n bells in [Ja]; see also, for example, [CSW] and references [32, 49, 53, 57-60] there.

§5 Metric basic, permutation approximation and symmetries

Call a subset $B \subseteq S_n$ a *d-metric basis* if the validity of $d(a, c) = d(b, c)$ for any $c \in S_n$ implies $a = b$, *i.e.* an element of S_n is uniquely determined by its distance from elements of B.

The utility of this concept can be seen by considering works on permutation approximation [GSM, Mi], and on the symmetries of (S_n, H) [Far]. They proved independently for different purposes and in different terms (see, for example, lemma 3.1 [Far] and Theorem 1[GSM]) that e, all transpositions and all cycles of length 3 form a metric basis for Hamming metric. We now describe those works briefly in the following:

A) Approximatin of almost commuting permutations using Hamming distance: The following problem was considered in [GSM, Mi] - let $a, b \in S_n$, if $H(ab, ba)$ is small, *i.e.* if $a, b \in S_n$ almost commute, is

$$H_a(b) = \min_{C \in C(a)} H(b, c)$$

necessarily small? *i.e.*, can b be approximated by an element of $C(a)$? where $C(a)$ is the centralizer of a. Gorenstein et.al. [GSM] gave negative answer if $|C(a)|$ is small, and positive answer if a is a product of m disjoiont cycles of length $t = n/m$ for large m. More precisely, let $H_a = \max_{b \in C(a)} H_a(b)/H(ab, ba)$ in the later case for any $a \in S_n - \{e\}$. Then for $m > 1$, we have

a) $H_a = t/4$ if $t \;=\; n/m$ is even [GSM],

b) $(t-1)/4 \leq H_a \leq t/4$ if $t > 1$ is odd [GSM],

c) $H_a = (t-1)^2/(4t-6)$ if $t > 1$ is odd and $m \geq t - 2$ [Mi].

The main idea of [Mi] is that the determination of $H_a(b)$ is equivalent to the optional assignment problem in linear programming.

B) the symmetries of the metric spaces $(S_n, H), (S_n, \ell_1)$:

Farahat [Far] proved that the symmetry group $I_S(S_n, H)$ has, for $n \geq 3$, order $2(n!)^2$. For distance ℓ_1, [Dj] gave $|I_S(S_n, \ell_1)| = 2n!$ for $n \geq 3$ and also that all values of ℓ_1 on S_n are all even integers from 0 to $2\lfloor n^2/4 \rfloor$.

§6 Commutation distance on S_n

The following distance on any finite group G was considered [BF, Na, ES, Ne, Ti, Bi] and by others in various context and terms. Consider the *commutation graph* of G, with vertex set G, and distinct elements $a, b \in G$ are connected by an edge whenever they commute, *i.e.* $ab = ba$. Any two distinct elements $a, b \in G$ which are not commute, are connected by the path (a, c, b) where c is any element of the center $Z(G)$ of G. Call *N-path* any path $(a, c_1, \cdots, c_t, b)$ where all $c_1, \cdots, c_t$ do not belong to $Z(G)$; call $a, b \in G \backslash Z(G)$ *N-connected* if they are connected by some N-path and define their *commutation distance* $d_{com}(a,b)$ as the minimum length of N-path connecting a and b. Define

$$d_{com}(a,b) = \begin{cases} 0 & \text{if } a=b, \\ 1 & \text{if } a \neq b, ab=ba, \end{cases}$$

and, $d_{com}(a,b) = \infty$ if $a,b \in G\backslash Z(G)$ are not connected by any N-path. A representation $G = \bigcup_{i=1}^{k} M_i$ is called an $N-partition$ of the group G if $M_i \cap M_j = Z(G)$ whenever $i \neq j$, and $G\backslash Z(G)$ splits into maximal N-connected disjoint subset $M_i\backslash Z$, $1 \leq i \leq k$. $M_i, 1 \leq i \leq k$, are called a $N-components$. The case $Z = <e>$ and all M_i being subgroups corresponds to the partitions of G considered by R. Baer, M. Suzuki and others.

Problem A: to find diameter $d(G)$ of a group G (i.e. max $d_{com}(a,b)$ for all N-connected pairs $a,b \in G$) and to find all N-components M_i, $1 \leq i \leq k$, of G.

$N-partition$ of G were studied for S_n, A_n and Weyl groups $W(B_n), W(D_n)$ in [Na] and, independently, for S_n, A_n, $GL(2,q), PGL(2,q)$, $PSL(2,q)$ and infinite groups $PGL(3,K)$ in [Bi]. Among other things, Bianchi [Bi] proved also that $d(S_n) \leq 8$, $d(A_n) \leq 8$ for any $n \geq 2$; both Sym(M) and Alt(M) are N-connected with $d(G) \leq 2$ for infinite M. Furthermore, those N-components for $S_n, n \geq 5$, are:

a) S_n itself is N-connected if and only if $n, n-1$ are composite numbers;

b) in the case of prime n:
$(n-2)!$ N-components are subgroups of order n and one N-component consists of all permutations which are not cycles of length n;

c) in the case of prime $n-1$:
$n(n-3)!$ N-components are subgroups of order $n-1$ and one N-component consists of all permutations which are not cycles of length $n-1$.

Now, S_2 is abelian, S_3 has one N-component, which is subgroup of order 3, three N-components which are subgroups of order 2 and $d(S_3) = 1$. S_4 has four N-components which are subgroups of order 3, one N-components (not a group) consisting of all permutations which are not cycles of length 3 and $d(S_n) = 3$.

N-partitions of A_n are also known [Na, Bi], but more messy to describe. In particular, $A_n (n \geq 3)$ is N-connected if and only if either $n = 3$ or $n, n-1, n-2$ are composite numbers. In fact, A_3 is abelian, *i.e.* it is N-connected and $d(A_3) = 1$,

A_4 has five N-components which are all Sylow subgroups and $d(A_4) = 1$. A_5 has 21 N-components which are all subgroups and $d(A_5) = 1$. A_6 has 42 N-componetns (not all are groups) and $d(A_6) \leq 4$.

Other way to study commutation graph of a group G was started by Erdös [ES, Er]. If $Z(G) \neq < e >$, then a coset decompoition $G = \cup < x, Z >$ is a covering of G by abelian subgroups.

Problem B_1 : to estimate the minimal cardinality $\beta(G)$ of coverings of G by abelian subgroups;

Problem B_2 : to estimate the maximum cardinality $\alpha(G)$ of a set of pairwise non-commuting elements of G, i.e. the independence numbers of the communication graph.

The bounds on $\alpha(G), \beta(G)$ were given in [Es, Ma, Ne, Be, Ry]. Brown [Br] concentrated on the case $G = S_n$ in which we are interested here; the following asymptotic bounds for $\alpha(S_n) = \alpha_n$, $\beta(S_n) = \beta_n$ were given in [Br] too:

1) $(n-2)! \log\log n \gg \beta_n \gg \alpha_n \gg (n-2)!$;

2) for infinitely many n, one has $(n-2)! \gg \beta_n \geq \alpha_n$;

3) for infinitely many n, one has $\beta_n \geq \alpha_n \gg (n-2)! \log\log n$.

He also showed that $\alpha_n = \beta_n$ for all $n \geq 1$ if the (bounded, as he proved) sequence $\{\beta_n/\alpha_n\}$ has a limit. The exact values of $\alpha_n = \beta_n$ for all $n \leq 9$ and the equality $\alpha_{11} = \beta_{11} = 4212330$ were also given.

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