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PCF Definability via Kripke Logical Relations (after O'Hearn and Riecke)

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The material presented here is an exposition of an application of logical relations to the problem of full abstraction for PCF. The point is to see Kripke logical relations as introduced by Jung and Tiuryn in [5] and used by O'Hearn and Riecke in [6] to give a logical characterization of sequentiality, within the mainstream of the classical notion of logical relations.

The main difference with respect to [6] is that we give a definition which is just a specialization of that of [5], and then we need the extensional collapse of "sequential" objects to get the fully abstract model. It is questionable whether the extensional collapse is a higher price to pay in comparison with the complex construction in [6].

As observed in [6], Jung and Tiuryn's Kripke logical relations are a special case of unary logical relations in a functor category. Since our exposition is very concrete, we avoid a strong commitment with categorical concepts, and use instead the definition of Kripke model and of Kripke logical relation given in [4] as our starting point.

The exposition is kept at an elementary level, even if familiarity with lambda-calculus and domain theory is assumed. As said before, we do not make any substantial use of category theory, but for some notation and for the use of diagrams that are of help to visualize some otherwise complex definitions and constructions.

The results are not original with the author, while this could be the case for errors.

1 PCF: Syntax and Semantics

PCF is a simply typed λ-calculus with constants and fixed points. The dialect of PCF we are using is that of [9, 6]. For the standard definition see [8]. Types are defined from the unique ground type ι, the type of natural numbers, closing under the arrow, i.e. $\sigma \to \tau$ is a type if σ and τ are types. $\boldsymbol{T}$ will be the set of types. Constants are: numerals $\underline{n}$, for all $n \in \mathbb{N}$, the successor succ, the predecessor pred, a test for zero ifz. The set of terms of type σ is defined according to the following rules:

$$\frac{}{\underline{n} : \iota} \qquad \frac{}{\mathsf{succ} : \iota \to \iota} \qquad \frac{}{\mathsf{pred} : \iota \to \iota} \qquad \frac{}{\mathsf{ifz} : \iota \to \iota \to \iota \to \iota}$$

$$\frac{}{x^\sigma : \sigma} \qquad \frac{M : \tau}{\lambda x^\sigma . M : \sigma \to \tau} \qquad \frac{M : \sigma \to \tau \quad N : \sigma}{MN : \tau} \qquad \frac{M : \sigma \to \sigma}{\mathsf{Y}\, M : \sigma}$$

Λ_σ is the set of terms of type σ; the set of closed terms of type σ is Λ^0_σ. We write Λ and Λ^0 for terms and closed terms of any type respectively. For any $M \in \Lambda$, the set of variables having at least one free occurrence in M is denoted by $FV(M)$.

To define the operational semantics we use a formal system defining the two place predicate $M \Downarrow C$ over closed terms, whose intended meaning is "M evaluates to the canonical form C", where canonical forms are either constants or abstractions:

$$\frac{}{C \Downarrow C} \qquad \frac{M \Downarrow \underline{n}}{\mathsf{succ}\, M \Downarrow \underline{n+1}} \qquad \frac{M \Downarrow \underline{n+1}}{\mathsf{pred}\, M \Downarrow \underline{n}}$$

$$\frac{F \Downarrow \lambda x.M \quad M[N/x] \Downarrow C}{FN \Downarrow C} \qquad \frac{M \Downarrow C \quad CN \Downarrow C'}{MN \Downarrow C'} \qquad \frac{M(\mathsf{Y}M) \Downarrow C}{\mathsf{Y}M \Downarrow C}$$

$$\frac{M \Downarrow \underline{0} \quad N \Downarrow C}{\mathsf{ifz} MNP \Downarrow C} \qquad \frac{M \Downarrow \underline{n+1} \quad P \Downarrow C}{\mathsf{ifz} MNP \Downarrow C}$$

We state some properties of $\Downarrow$ without proof.

Lemma 1.1 *Let $M \in \Lambda^0$ and C, C' any closed canonical forms.*

1. $M \Downarrow C \ \wedge \ M : \sigma \Rightarrow C : \sigma$
2. $M \Downarrow C \ \wedge \ M \Downarrow C' \Rightarrow C \equiv C'$
3. $M \Downarrow C \ \wedge \ M : \iota \Rightarrow \exists n \in \mathbb{N}.\ C \equiv \underline{n}$.

Remark 1 If one prefers, he can take the reduction based operational semantics of [8] and check that, for any closed term $M : \sigma$ and canonical form C

$$M \xrightarrow{*} C \ \Leftrightarrow \ M \Downarrow C$$

In particular, if $M : \iota$, then Plotkin's predicate

$$\mathit{Eval}(M) = \underline{n} \ \Leftrightarrow \ M \xrightarrow{*} \underline{n}$$

is equivalent to $M \Downarrow \underline{n}$.

A context $C[\ ]$ is a term with a hole (actually a typed hole). By $C[M]$ is meant the filling of the context $C[\ ]$ by the term M of the right type. A context $C[\ ]$ is said closing M if $C[M]$ is a closed term. A context is ground if it has type ι. The *operational preorder* is a binary relation over Λ which is defined by:

$$M \sqsubseteq_{op} N \ \Leftrightarrow \ \forall \text{ ground } C[\ ] \text{ closing } M, N \ \forall n \in \mathbb{N}.\ C[M] \Downarrow \underline{n} \Rightarrow C[N] \Downarrow \underline{n}.$$

Note that $\sqsubseteq_{op}$ is defined only among terms of the same type. Moreover it is a precongruence. The main syntactical lemma about the operational preorder is the following.

Lemma 1.2 (Context Lemma) *Let $\sigma = \sigma_1 \to \cdots \to \sigma_m \to \iota$, and $M, N \in \Lambda^0_\sigma$, then $M \sqsubseteq_{op} N$ if and only if*

$$\forall L_1 \in \Lambda^0_{\sigma_1}, \ldots, L_m \in \Lambda^0_{\sigma_m} \ \forall n \in \mathbb{N}.\ M L_1 \cdots L_m \Downarrow \underline{n} \Rightarrow N L_1 \cdots L_m \Downarrow \underline{n}.$$

Definition 1.3 *A* continuous applicative structure *is a structure*

$$\mathcal{A} = \langle \{A_\sigma\}_{\sigma \in \boldsymbol{T}}, \{app_{\sigma,\tau}\}_{\sigma,\tau \in \boldsymbol{T}} \rangle$$

where each A_σ is a cpo and $app_{\sigma,\tau} : A_{\sigma\to\tau} \times A_\sigma \to A_\tau$ is a Scott-continuous map for all $\sigma, \tau \in \boldsymbol{T}$, such that:

$$\forall a \in A_\sigma.\ app_{\sigma,\tau}(\bot, a) = \bot$$

$\mathcal{A}$ *is* order-extensional *iff for all $\sigma, \tau \in \boldsymbol{T}$*

$$\forall f, g \in A_{\sigma\to\tau}.(\forall x \in A_\sigma.app_{\sigma,\tau}(f, x) \sqsubseteq app_{\sigma,\tau}(g, x)) \Rightarrow f \sqsubseteq g$$

$\mathcal{A}$ *is said* extensional *if $\sqsubseteq$ is subtituted by equality in the formula above.*

An environment ρ *for $\mathcal{A}$ is a mapping from the set of term variables into $\bigcup_{\sigma\in\boldsymbol{T}} A_\sigma$ such that $\rho(x^\sigma) \in A_\sigma$ for all σ. A* continuous model *for PCF is a pair $\langle \mathcal{A}, [\![\cdot]\!]^{\mathcal{A}} \rangle$ where $\mathcal{A}$ is a continuous applicative structure and $[\![\cdot]\!]^{\mathcal{A}}$ is a map from terms and environments to $\bigcup_{\sigma\in\boldsymbol{T}} A_\sigma$ which satisfies:*

1. $[\![M]\!]^{\mathcal{A}}_\rho \in A_\sigma$ *whenever* $M : \sigma$
2. $[\![x^\sigma]\!]^{\mathcal{A}}_\rho = \rho(x^\sigma)$
3. *if* $\rho|_{FV(M)} = \rho'|_{FV(M)}$ *then* $[\![M]\!]^{\mathcal{A}}_\rho = [\![M]\!]^{\mathcal{A}}_{\rho'}$
4. $app_{\iota,\iota}([\![\mathsf{succ}]\!]^{\mathcal{A}}_\rho, \bot) = \bot$, $[\![\mathsf{succ}\ \underline{n}]\!]^{\mathcal{A}}_\rho = [\![\underline{n+1}]\!]^{\mathcal{A}}_\rho$
5. $app_{\iota,\iota}([\![\mathsf{pred}]\!]^{\mathcal{A}}_\rho, \bot) = \bot$, $[\![\mathsf{pred}\ \underline{0}]\!]^{\mathcal{A}}_\rho = \bot$, $[\![\mathsf{pred}\ \underline{n+1}]\!]^{\mathcal{A}}_\rho = [\![\underline{n}]\!]^{\mathcal{A}}_\rho$
6. $app_{\iota,\iota\to\iota\to\iota}([\![\mathsf{ifz}]\!]^{\mathcal{A}}_\rho, \bot) = \bot$, $[\![\mathsf{ifz}\ \underline{n}\ M N]\!]^{\mathcal{A}}_\rho = \begin{cases} [\![M]\!]^{\mathcal{A}}_\rho & \textit{if } n = 0 \\ [\![N]\!]^{\mathcal{A}}_\rho & \textit{otherwise} \end{cases}$
7. $[\![\mathsf{Y}\ M]\!]^{\mathcal{A}}_\rho = \bigsqcup_{n\in\mathbb{N}} app^n_{\sigma,\sigma}([\![M]\!]^{\mathcal{A}}_\rho, \bot)$ *where*

$$app^0_{\sigma,\sigma}(f, a) = a$$
$$app^{n+1}_{\sigma,\sigma}(f, a) = app_{\sigma,\sigma}(f, app^n_{\sigma,\sigma}(f, a))$$

8. $[\![MN]\!]^{\mathcal{A}}_\rho = app_{\sigma,\tau}([\![M]\!]^{\mathcal{A}}_\rho, [\![N]\!]^{\mathcal{A}}_\rho)$ *if* $M : \sigma \to \tau$ *and* $N : \sigma$

9. $\forall a \in A_\sigma.\ app_{\sigma,\tau}([\![\lambda x^\sigma.M]\!]^{\mathcal{A}}_\rho, a) = [\![M]\!]^{\mathcal{A}}_{\rho[a/x]}$

10. *if* $S \subseteq \Lambda_\sigma$, $N \in \Lambda_\sigma$ *and* ρ *is an environment such that the set* $B = \{[\![M]\!]^{\mathcal{A}}_\rho \mid M \in S\}$ *is directed and* $[\![N]\!]^{\mathcal{A}}_\rho = \bigsqcup B$, *then*

$$\forall C[\,].\ \bigsqcup_{M\in S} [\![C[M]]\!]^{\mathcal{A}}_\rho = [\![C[N]]\!]^{\mathcal{A}}_\rho$$

A continuous model is order-extensional *if the underlying applicative structure is such.*

Remark 2 The previous definition is not aimed to be very general, and, as it stays, it might appear too restrictive. It is however sufficiently comprehensive for the present purposes. For other definitions see e.g. [1]. Observe that condition 1.3 (10) is redundant when the model is order-extensional.

The *standard continuous model* of PCF $\langle \mathcal{D}, [\![\cdot]\!]^{\mathcal{D}}\rangle$ is defined as follows: $D_\iota = \mathbb{N}_\perp$, $D_{\sigma\to\tau} = [D_\sigma \to D_\tau]$, the set of all Scott-continuous maps from D_σ to D_τ; $app_{\sigma,\tau}(f,d) = f(d)$. $\langle \mathcal{D}, \{app_{\sigma,\tau}\}\rangle$ is also called the *full continuous type hierarchy* in [3]. Finally to specify the interpretation map it suffices to put $[\![\underline{n}]\!]^{\mathcal{D}}_\rho = n$ for all $n \in \mathbb{N}$. The standard model is order-extensional (indeed it is a frame: see [3]).

Any model $\langle \mathcal{A}, [\![\cdot]\!]^{\mathcal{A}}\rangle$ induces a preorder over Λ_σ for all σ:

$$M \sqsubseteq_{\mathcal{A}} N \ \Leftrightarrow\ \forall \rho.\ [\![M]\!]^{\mathcal{A}}_\rho \sqsubseteq [\![N]\!]^{\mathcal{A}}_\rho.$$

Definition 1.4 *Let* $\mathcal{A} = \langle \mathcal{A}, [\![\cdot]\!]^{\mathcal{A}}\rangle$ *be any model:*

1. $\mathcal{A}$ *is* computationally adequate *iff*

$$\forall \sigma \in \boldsymbol{T}\ \forall M, N \in \Lambda_\sigma.\ M \sqsubseteq_{\mathcal{A}} N \Rightarrow M \sqsubseteq_{op} N$$

2. $\mathcal{A}$ *is* fully abstract *iff*

$$\forall \sigma \in \boldsymbol{T}\ \forall M, N \in \Lambda_\sigma.\ M \sqsubseteq_{op} N \Rightarrow M \sqsubseteq_{\mathcal{A}} N$$

Theorem 1.5 (Plotkin 1977) *If* $M \in \Lambda^0_\iota$ *(i.e. it is a* program*) then* $[\![M]\!]^{\mathcal{D}} = n$ *of and only if* $M \Downarrow \underline{n}$. *Therefore the standard continuous model of PCF is computationally adequate. However it is not fully abstract.*

Proof. See [8]. The adequacy follows immediately from the previous assertion and the lemma 1.2.

□

By the adequacy, the failure of full abstraction of $\mathcal{D}$ has to be a consequence of the existence of a non definable object discriminating between two definable functions. Now in $D_{\iota\to\iota\to\iota}$ there is a function, called *parallel-or*, written *por*, satisfying:

$$\begin{array}{cccc} f & 0 & \bot & = & 0 \\ f & \bot & 0 & = & 0 \\ f & 1 & 1 & = & 1 \end{array} \tag{1}$$

On the other hand there exist two terms P_1, P_2 defined as follows:

$$P_i \equiv \lambda x^{\iota\to\iota\to\iota}.\ \mathsf{ifz}\ [(x\,\underline{0}\,\Omega) = \underline{0} \wedge (x\,\Omega\,\underline{0}) = \underline{0} \wedge (x\,\underline{1}\,\underline{1} = \underline{1})]\ \underline{i}\ \Omega$$

where partial equality and conjunction over D_ι (taking 0 as true and 1 as false) are clearly definable. Then

$$[\![P_i]\!]^{\mathcal{D}}\ f = \begin{cases} i & \text{if } f \text{ satisfies (1)} \\ \bot & \text{otherwise} \end{cases}$$

If one can prove that *por* is not definable, then for any closed M of type $\iota \to \iota \to \iota$ we have $[\![P_iM]\!]^{\mathcal{D}} = \bot = [\![P_iM]\!]^{\mathcal{D}}$ and therefore $P_1 \simeq_{op} P_2$ by Plotkin's theorem and the context lemma.

We pospone the proof that *por* cannot be defined by any PCF term, and state Milner theorem, claiming that definability of finite elements, togather with extensionality, are necessary and sufficient conditions for a model to be fully abstract.

Theorem 1.6 (Milner 1977) *A continuous extensional model $\mathcal{A}$ of PCF is fully abstract iff it is order-extensional, ω-algebraic and such that all the finite elements of the A_σ are definable. Consequently, if $\mathcal{A}$ and $\mathcal{B}$ are two such models, then $A_\sigma \simeq B_\sigma$ for all $\sigma \in \boldsymbol{T}$.*

Proof. See [2].

□

2 Logical Relations for PCF

Logical relations originated from Kreisel's work concerning the HEO model and are ubiquitous in the theory of simply typed lambda calculus. They have been employed in [9] to prove the non definability of combinators like the parallel-or and to provide a characterization of sequentiality up to types of third order.

The reader is referred to [3] for an introduction and pointers to the literature. Here we build over [9] and give a domain theoretic version of logical relations which fits better with the definition of continuous models in the last section.

Definition 2.1

An n-ary logical relation $\mathcal{R}$ *over the continuous applicative structures* $\mathcal{A}_1, \ldots, \mathcal{A}_n$ *(that will be indicated simply by* $\mathcal{R} \subseteq \mathcal{A}_1 \times \cdots \times \mathcal{A}_n$*) is a family of n-ary relations*

$$R_\sigma \subseteq A_{1,\sigma} \times \cdots \times A_{n,\sigma}$$

such that:

1. R_σ *is a sub-cpo of* $A_{1,\sigma} \times \cdots \times A_{n,\sigma}$ *for all* σ*, that is*
   (a) $R_\sigma(\perp, \ldots, \perp)$
   (b) *if* $D \subseteq A_{1,\sigma} \times \cdots \times A_{n,\sigma}$ *is directed and such that* $R_\sigma(d_1, \ldots, d_n)$ *for all* $\langle d_1, \ldots, d_n \rangle \in D$ *and* $\bigsqcup D = \langle e_1, \ldots, e_n \rangle$ *then* $R_\sigma(e_1, \ldots, e_n)$*;*
2. $R_{\sigma \to \tau}(f_1, \ldots, f_n)$ *iff*
   $$\forall \langle d_1, \ldots, d_n \rangle.\ R_\sigma(d_1, \ldots, d_n) \Rightarrow R_\tau(app_{\sigma,\tau}(f_1, d_1), \ldots, app_{\sigma,\tau}(f_n, d_n))$$

The standard definition of logical relations is given over arbitrary applicative structures, which are not necessarily continuous. Therefore condition (2) is the only requirement. By the way we should speak of continuous logical relations: for simplicity, we abuse terminology. Observe that, because of clause (2), a logical relation is always determined by the relation R_ι. Moreover to meet condition (1) at all types it suffices that it is satisfied at type ι.

The following theorem is known as the main lemma of logical relations. Since the meaning of a closed term does not depend on the environment, when $M \in \Lambda^0$ we shall write simply $[\![M]\!]^{\mathcal{A}}$ instead of $[\![M]\!]^{\mathcal{A}}_\rho$.

Theorem 2.2 *Let $\mathcal{R}$ be an n-ary logical relation over $\mathcal{A}_1, \ldots, \mathcal{A}_n$ such that, for all PCF constants $c : \sigma$, it is the case that $R_\sigma([\![c]\!]^{\mathcal{A}_1}, \ldots, [\![c]\!]^{\mathcal{A}_n})$; then*

$$\forall \sigma \in \boldsymbol{T}\ \forall M \in \Lambda^0_\sigma.\ R_\sigma([\![M]\!]^{\mathcal{A}_1}, \ldots, [\![M]\!]^{\mathcal{A}_n}).$$

Proof. If $\rho_1, \ldots, \rho_n$ are environments wrt $\mathcal{A}_1, \ldots, \mathcal{A}_n$, we write $R(\rho_1, \ldots, \rho_n)$ if for any variable x^σ it holds that $R_\sigma(\rho_1(x^\sigma), \ldots, \rho_n(x^\sigma))$. Then it is easy to show, by induction on $M \in \Lambda$, that if $R(\rho_1, \ldots, \rho_n)$ then $R_\sigma([\![M]\!]^{\mathcal{A}_1}_{\rho_1}, \ldots, [\![M]\!]^{\mathcal{A}_n}_{\rho_n})$. The theorem immediately follows.

□

An immediate consequence of theorem 2.2 is the proof of non definability of parallel-or given in [9]. Recall that *por* is the least continuous function satifying

$$\begin{array}{ccccc} f & 0 & \perp & = & 0 \\ f & \perp & 0 & = & 0 \\ f & 1 & 1 & = & 1 \end{array}$$

Then take the 3-ary logical relation determined by

$$R_\iota(d_1, d_2, d_3) \ \Leftrightarrow\ d_1 = \perp \ \vee\ d_2 = \perp \ \vee\ d_1 = d_2 = d_3.$$

Then it is easy to check that $R_\sigma([\![c]\!]^{\mathcal{D}}, [\![c]\!]^{\mathcal{D}}, [\![c]\!]^{\mathcal{D}})$ for all constants c of type σ, so that, by the theorem, the same holds for any closed M. We conclude that $f \neq [\![M]\!]^{\mathcal{D}}$ for any closed M of the right type, since $R_\iota(0, \perp, 1)$ and $R_\iota(\perp, 0, 1)$, but not $R_\iota(0, 0, 1)$.

As shown by the example of the parallel-or, the interesting case when studying definability is that of logical relations of the form $\mathcal{R} \subseteq \mathcal{A}^n$. As the standard model $\mathcal{D}$ plays for PCF the same role as the full type hierarchy in the theory of simply typed lambda-calculus, we concentrate on n-ary relations over $\mathcal{D}$.

Definition 2.3 *For all $n \in \mathbb{N}$ and pair of sets $A \subseteq B \subseteq \{1, \ldots, n\}$ let $S^n_{A,B} \subseteq \underbrace{D_\iota \times \cdots \times D_\iota}_{n}$ be the relation:*

$$S^n_{A,B}(d_1, \ldots, d_n) \ \Leftrightarrow\ (\exists i \in A.\ d_i = \perp) \vee (\forall i, j \in B.\ d_i = d_j).$$

Then an n-ary logical relation $\mathcal{R}$ over $\mathcal{D}$ is called a sequentiality relation *iff R_ι is the intersection of relations of the form $S^n_{A,B}$.*

An element $d \in D_\sigma$ is logically sequential *iff for all sequentiality relations $\mathcal{R}$ it is the case that $R_\sigma(d, \ldots, d)$.*

The relation used above to prove the non definabilty of the parallel-or is logically sequential, since $R_\iota = S^3_{\{1,2\},\{1,2,3\}}$.

Theorem 2.4 (Sieber 1992) *An n-ary logical relation $\mathcal{R}$ over $\mathcal{D}$ is such that for all types σ and PCF constant $c : \sigma$*

$$R_\sigma([\![c]\!]^{\mathcal{D}}, \ldots, [\![c]\!]^{\mathcal{D}})$$

iff $\mathcal{R}$ is logically sequential.

Proof. See [9].

□

In [9] there is a characterization of PCF-definable elements in the full continuous type hierarchy up to order 2, by means of logical relations.

Theorem 2.5 (Sieber 1992) *Define the order $o(\sigma)$ of a type as follows: $o(\iota) = 0$, $o(\sigma \to \tau) = \max\{o(\tau), o(\sigma) + 1\}$. Let $\tau = \sigma_1 \to \cdots \to \sigma_n \to \iota$ be such that $1 \leq o(\tau) \leq 2$ and let $f \in D_\tau$ be a logically sequential function. Then for any finite subset $\{(d_{i,1}, \ldots, d_{i,n}) \mid i = 1, \ldots, m\}$ of $D_{\sigma_1} \times \cdots \times D_{\sigma_n}$ there exists an $M \in \Lambda^0_\tau$ such that*

$$\forall i \leq m.\ f d_{i,1} \cdots d_{i,n} = [\![M]\!]^{\mathcal{D}} d_{i,1} \cdots d_{i,n}.$$

Using this result Sieber gives a caracterization of the fully abstract model of PCF up to types of order 3, which is the same as theorem 4.7 at the end of this paper, but for the logical relations, which are fixed arity relations as defined above in case of [9]. It is an open problem whether the characterization holds at higher types.

3 Kripke-style Models and Logical Relations for PCF

Kripke-style models for simply typed lambda calculus have been introduced in [4] to cope with the problem of completeness of equational theories in presence of constants and empty types. In the same paper the notion of logical relation is extended to the case of Kripke models in a natural way. Plotkin's I-relations in [7] turn out to be a particular case of Mitchell-Moggi Kripke logical relations.

An apparently different notion of Kripke logical relation has been introduced by Jung and Tiuryn in [5] to give a new characterization of lambda definability in the full type hierarchy. Although Jung and Tiuryn's definition generalizes Kripke logical relations of [4], the particular relations involved in the proof of their main theorem (theorem 5) can be seen as a particular case of Mitchell-Moggi relations.

Here we specialize the definition in [4] to the domain theoretical case. The idea is to have a set of "worlds" $\mathcal{W}$ and a partial order relation over it, the "accessibility" relation. Then for each $w \in \mathcal{W}$ we have a continuous applicative structure (possibly a model). Finally a suitable set of continuous maps will relate structures associated to accessible worlds in such a way that the application is preserved. The main difference with [4] is that, as the interpretation of a type at any world will be a cpo, we do not have empty types.

Definition 3.1 *A* Kripke continuous applicative structure *is a tuple*

$$\langle \mathcal{W}, \leq, \{\mathcal{A}^w\}_{w \in \mathcal{W}}, \mathcal{I} \rangle$$

where:

1. $\langle \mathcal{W}, \leq \rangle$ *is a poset (the poset of "worlds")*
2. *for all* $w \in \mathcal{W}$ *the tuple* $\mathcal{A}^w = \langle \{A^w_\sigma\}_{\sigma \in \boldsymbol{T}}, \{app^w_{\sigma,\tau}\}_{\sigma,\tau \in \boldsymbol{T}} \rangle$ *is a continuous applicative structure*
3. $\mathcal{I}$ *is a family of maps* $\{i^{w,v}_\sigma\}_{\sigma \in \boldsymbol{T}, w \leq v \in \mathcal{W}}$ *where each* $i^{w,v}_\sigma : A^w_\sigma \to A^v_\sigma$ *is a map; moreover:*
   (a) $i^{w,w}_\sigma = id_\sigma$

(b) $i_\sigma^{w',w''} \circ i_\sigma^{w,w'} = i_\sigma^{w,w''}$ *whenever* $w \le w' \le w''$

4. *for all* $\sigma, \tau \in \boldsymbol{T}$ *and* $w, v \in \mathcal{W}$, *if* $w \le v$ *then*

$$\forall f \in A^w_{\sigma\to\tau}\ \forall a \in A^w_\sigma.\ i_\tau^{w,v}(app^w_{\sigma,\tau}(f,a)) = app^v_{\sigma,\tau}(i^{w,v}_{\sigma\to\tau}(f), i^{w,v}_\sigma(a))$$

that is the following diagram commutes:

$$\begin{tikzcd}
A^v_{\sigma\to\tau} \times A^v_\sigma \arrow[r, "app^v_{\sigma,\tau}"] & A^v_\tau \\
A^w_{\sigma\to\tau} \times A^w_\sigma \arrow[r, "app^w_{\sigma,\tau}"'] \arrow[u, "i^{w,v}_{\sigma\to\tau} \times i^{w,v}_\sigma"] & A^w_\tau \arrow[u, "i^{w,v}_\tau"']
\end{tikzcd}$$

Given a Kripke continuous applicative structure an environment $\rho = \{\rho^w\}_{w\in\mathcal{W}}$ *is a* $\mathcal{W}$*-indexed family of environments, such that* ρ^w *is an environment for* $\mathcal{A}^w$ *and*

$$\rho^v = i^{w,v} \circ \rho^w \qquad \textit{for } w \le v$$

A Kripke continuous model is then a tuple

$$\langle \mathcal{W}, \le, \{\mathcal{A}^w\}_{w\in\mathcal{W}}, \mathcal{I}, [\![\cdot]\!]\rangle$$

where $[\![\cdot]\!] = \{[\![\cdot]\!]^{\mathcal{A}^w}\}_{w\in\mathcal{W}}$ *is a family of interpretation maps such that for all* w $\langle \mathcal{A}^w, [\![\cdot]\!]^{\mathcal{A}^w}\rangle$ *is a (classical) model and, for any environment* ρ

$$i^{w,v} \circ [\![\cdot]\!]^{\mathcal{A}^w}_{\rho^w} = [\![\cdot]\!]^{\mathcal{A}^v}_{(i^{w,v}\circ\rho^w)} = [\![\cdot]\!]^{\mathcal{A}^v}_{\rho^v} \qquad \textit{for } w \le v.$$

Remark 3 Observe that the condition $i^{w,v} \circ [\![\cdot]\!]^{\mathcal{A}^w}_{\rho^w} = [\![\cdot]\!]^{\mathcal{A}^v}_{\rho^v}$, when $v \ge w$, implies

$$\forall v \ge w\ \forall a \in A^v_\sigma.\ app^v_{\sigma,\tau}(i^{w,v}_{\sigma\to\tau}([\![\lambda x^\sigma.\, M]\!]^{\mathcal{A}^w}_{\rho^w}), a) = [\![M]\!]^{\mathcal{A}^v}_{\rho^v[a/x]}$$

which is in the definition of the abstraction clause in [3].

The notion of Kripke-logical relation now arises as a natural extension of the classical notion to the case of Kripke (continuous) applicative structures. The crucial point is that Kripke relations are predicates over Kripke models: therefore the universal quantification and the logical implication, involved in the definition of relations at higher types, are interpreted exactly as it is the case for the intuitionistic first order logic. Since the Kripke logical relations we are going to use in the next section are just unary relations, we give the definition only for that case. The reader will have no difficulty to extend the present definition to the general case.

Definition 3.2 *A* Kripke logical relation $\mathcal{R}$ *over the Kripke continuous applicative structure* $\langle \mathcal{W}, \mathcal{A}, \mathcal{I} \rangle$ *is a family of relations* $\{R^w_\sigma\}_{\sigma \in \in \boldsymbol{T}, w \in \mathcal{W}}$ *indexed over types and worlds such that:*

1. R^w_ι *is a sub-cpo of* A^w_ι
2. $\forall v \geq w \; \forall a \in A^w_\iota .\; R^w_\iota(a) \Rightarrow R^v_\iota(i^{w,v}_\iota(a))$
3. $R^w_{\sigma\to\tau}(f) \;\Leftrightarrow\; \forall v \geq w \; \forall a \in A^v_\sigma .\; R^v_\sigma(a) \Rightarrow R^v_\tau(app^v_{\sigma,\tau}(i^{w,v}_{\sigma\to\tau}(f), a))$

Remark 4 If $\mathcal{R}$ is a Kripke logical relation over $\langle \mathcal{W}, \mathcal{A}, \mathcal{I} \rangle$, then

1. R^w_σ is a sub-cpo of A^w_σ for all σ and w
2. $\forall v \geq w \; \forall a \in A^w_\sigma .\; R^w_\sigma(a) \Rightarrow R^v_\sigma(i^{w,v}_\sigma(a))$ for all σ and w
3. the family $\{R^w_\sigma\}_{\sigma \in \boldsymbol{T}}$ is a classical logical relation over $\mathcal{A}^w$, for all w.

The strictness of $app^w_{\sigma,\tau}$ in its first argument is needed to prove 1. A family of relations $\{R^w_\sigma\}$ such that 1-3 of the present remark are true, is not in general a Kripke logical relation.

Theorem 3.3 *If* $\mathcal{R}$ *is a Kripke logical relation over the model* $\langle \mathcal{W}, \mathcal{A}, \mathcal{I}, [\![\cdot]\!] \rangle$ *such that for all constants* $c : \sigma$, $R^w_\sigma([\![c]\!]^{\mathcal{A}^w})$, *then* $R^w_\tau([\![M]\!]^{\mathcal{A}^w})$ *for all* $M \in \Lambda^0_\tau$.

Proof. An easy exercise for the reader.

□

4 Full Abstraction via Logical Relations

In this section we consider varying arity logical relations as introduced in [5] and used for PCF in [6]. We introduce them as a particular case of unary Kripke logical relations.

Consider the full continuous type hierarchy $\mathcal{D}$. It is known that the domains D_σ are SFP objects. Shortly this is summarized as follows. For all $n \in \mathbb{N}$ define the family of continuous maps $\{p^n_\sigma\}_{\sigma \in \boldsymbol{T}}$ as follows:

$$\begin{array}{lcl} p^n_\iota(d) & = & \begin{cases} d & \text{if } d \in \{0, \ldots, n-1\} \\ \bot & \text{otherwise} \end{cases} \\ p^n_{\sigma\to\tau}(f) & = & p^n_\tau \circ f \circ p^n_\sigma \end{array}$$

Write D^n_σ for $p^n_\sigma(D_\sigma)$. The mappings p^n_σ are called *finitary projections* since

- $p^n_\sigma \circ p^n_\sigma = p^n_\sigma$;
- $p^n_\sigma \sqsubseteq id_\sigma$, where $\sqsubseteq$ is the pointwise ordering;
- D^n_σ is always finite.

The fact that D_σ is an SFP object is now expressed by

$$id_\sigma = \bigsqcup_n p^n_\sigma.$$

In what follows it is essential that projections are definable. This is shown by considering the following combinators P^n_σ:

$$\begin{array}{lcl} L_0 & = & \lambda x^\iota.\, \Omega \\ L_{n+1} & = & \lambda x^\iota.\, \mathsf{ifz}\ x\ x\ (L_n(\mathsf{pred}\ x)) \\ & & \\ P^n_\iota & = & \lambda x^\iota.\, \mathsf{ifz}\ (L_n x)\ x\ \Omega \\ P^n_{\sigma\to\tau} & = & \lambda x^{\sigma\to\tau}.\, P^n_\tau \circ x \circ P^n_\sigma \end{array}$$

where $\Omega \equiv \mathsf{Y}(\lambda x^\iota . x)$.

For each $n \in \mathbb{N}$, we define a Kripke continuous applicative structure $\langle \mathcal{W}^{(n)}, \mathcal{D}, \mathcal{I} \rangle$ as follows:

- the elements $w \in \mathcal{W}^{(n)}$ are of the form $w = D^n_{\sigma_1} \times \cdots \times D^n_{\sigma_m}$ considered as sets. In particular $\mathbf{1} \in \mathcal{W}^{(n)}$ (the empty product) is a singleton set.

- the accessibility relation is defined by

$$w \le v \;\Leftrightarrow\; \exists v' \in \mathcal{W}^{(n)}.\; v = w \times v'.$$

- $D^w_\sigma = [w \to D_\sigma]$, where w is taken with the discrete ordering.
- for $g \in D^w_{\sigma\to\tau}$ and $h \in D^w_\sigma$

  $$app^w_{\sigma,\tau}(g,h) = \varepsilon \circ (g \times h) \circ \Delta_w$$

  which is illustrated in the following diagram:

$$\begin{tikzcd} w \arrow[r, dashed, "app^w_{\sigma,\tau}(g,h)"] \arrow[d, "\Delta_w"'] & D_\tau \\ w \times w \arrow[r, "g \times h"'] & D_{\sigma\to\tau} \times D_\sigma \arrow[u, "\varepsilon"'] \end{tikzcd}$$

- let $w \le v$ and $\pi_{v,w} : v \to w$ be the cartesian projection, namely $\pi_{v,w}(d_1, \ldots, d_m, d_{m+1}, \ldots, d_{m+k}) = \langle d_1, \ldots, d_m \rangle$ when $w = D^n_{\sigma_1} \times \cdots \times D^n_{\sigma_m}$ and $v = w \times D^n_{\sigma_{m+1}} \times \cdots \times D^n_{\sigma_{m+k}}$; then, for any $g \in D^w_\sigma$ put

  $$i^{w,v}_\sigma(g) = g \circ \pi_{v,w}$$

To show that this is a Kripke continuous applicative structure we have first to check the equation

$$\begin{aligned} i^{w,v}_\tau(app^w_{\sigma,\tau}(g,h)) &= app^w_{\sigma,\tau}(g,h) \circ \pi_{v,w} = \\ &app^v_{\sigma,\tau}(g \circ \pi_{v,w}, h \circ \pi_{v,w}) = app^v_{\sigma,\tau}(i^{w,v}_{\sigma\to\tau}(g), i^{w,v}_\sigma(h)) \end{aligned}$$

which is immediately seen from the commutativity of the following diagram:

$$
\begin{tikzcd}
v \arrow[r, "\Delta_v"] \arrow[dd, "\pi_{v,w}"'] \arrow[drrr, bend left, "app^v_{\sigma,\tau}(g \circ \pi_{v,w}, h \circ \pi_{v,w})"] & v \times v \arrow[dd, "\pi_{v,w} \times \pi_{v,w}"'] \arrow[dr, "(g \circ \pi_{v,w}) \times (h \circ \pi_{v,w})"] & & \\
 & & D^n_{\sigma\to\tau} \times D^n_\sigma \arrow[r, "\varepsilon"] & D^n_\tau \\
w \arrow[r, "\Delta_w"'] \arrow[urrr, bend right, "app^w_{\sigma,\tau}(g, h)"'] & w \times w \arrow[ur, "g \times h"'] & &
\end{tikzcd}
$$

Finally, let $g = \lambda x \in w. \perp_{\sigma\to\tau}$, that is the bottom of $D^w_{\sigma\to\tau}$; then for any $h \in D^w_\sigma$ and $\vec{d} \in w$ we have:

$$app_{\sigma,\tau}(g, h)(\vec{d}) = g(\vec{d})(h(\vec{d})) = \perp_{\sigma\to\tau}(h(\vec{d})) = \perp_\tau,$$

hence $app_{\sigma,\tau}(g, h) = \lambda x \in w. \perp_\tau$.

Remark 5 (Jung-Tiuryn, O'Hearn-Riecke Logical Relations). Once we have the structure $\langle \mathcal{W}^{(n)}, \mathcal{D}, \mathcal{I} \rangle$, definition 3.2 carries over, giving a notion of Kripke logical relation which is equivalent to that one introduced in [5, 6]. To see this last point let us recall their definitions.

Let $\mathcal{W}$ be a small category of sets (the category of "worlds"), and D be a cpo. Then a $\mathcal{W}$-indexed Kripke relation over D is a family of relations $R = \{R^w\}_{w\in\mathcal{W}}$ such that:

1. R^w is a sub-cpo of $[w \to D]$, for all $w \in \mathcal{W}$;

2. for all $f \in \mathcal{W}[v, w]$ and $g \in R^w$ it is the case that $g \circ f \in R^v$.

In case of $\mathcal{W}^{(n)}$ the category is a posetal category, hence falls under definition 3.2. Indeed this case is exactly the relevant one.

The family $R = \{R_\sigma\}_{\sigma \in \boldsymbol{T}}$ is a $\mathcal{W}$-indexed Kripke logical relation over the continuous applicative structure $\mathcal{D}$ iff:

3. R_ι is a $\mathcal{W}$-indexed Kripke relation over D_ι;

4. $R_{\sigma\to\tau}$ is the family of $\mathcal{W}$ indexed relations over $D_{\sigma\to\tau}$ such that

$$g \in R^w_{\sigma\to\tau} \;\Leftrightarrow\; \forall f \in \mathcal{W}[v,w], h \in R^v_\sigma.\ \lambda x \in v.g(f(x))(h(x)) \in R^v_\tau.$$

Now condition (4) is illustrated by the following diagram:

$$\begin{tikzcd}
v \arrow[r, "\langle f, id_v\rangle"] \arrow[rrr, bend right, "\lambda x \in v.g(f(x))(h(x))"'] & w \times v \arrow[r, "g \times h"] & D_{\sigma\to\tau} \times D_\sigma \arrow[r, "\varepsilon"] & D_\tau
\end{tikzcd}$$

Mitchell-Moggi's definition gives us a similar condition, namely

$$R^w_{\sigma\to\tau}(g) \;\Leftrightarrow\; \forall v \geq w\ \forall h \in D^v_\sigma.\ R^v_\sigma(h) \Rightarrow R^v_\tau(app^v_{\sigma,\tau}(g \circ \pi_{v,w}, h))$$

which is equivalent to the previous one as shown by the following diagram:

$$\begin{tikzcd}
 & v \times v \arrow[dd, "\pi_{v,w} \times id"] \arrow[dr, "(g \circ \pi_{v,w}) \times h"] & & \\
v \arrow[ur, "\Delta_v"] \arrow[dr, "\langle \pi_{v,w}, id\rangle"'] & & D_{\sigma\to\tau} \times D_\sigma \arrow[r, "\varepsilon"] & D_\tau \\
 & w \times v \arrow[ur, "g \times h"'] & &
\end{tikzcd}$$

Sieber notion of sequentiality relation can be adapted to the new scenario as follows (see [6])

Definition 4.1 *Let w be a finite set. For any subsets $A \subseteq B \subseteq w$, define $S^w_{A,B} \subseteq [w \to \mathbb{N}_\perp]$ as follows:*

$$S^w_{A,B}(g) \;\Leftrightarrow\; (\exists i \in A.\; g(i) = \perp) \vee (\forall i, j \in B \; g(i) = g(j)).$$

Then $R \subseteq [w \to \mathbb{N}_\perp]$ is a sequentiality relation *iff it is the intersection of a collection of relations of the form $S^w_{A,B}$.*

Definition 4.2 *Let $\mathcal{R}$ be a Kripke logical relation over $\langle \mathcal{W}^{(n)}, \mathcal{D}, \mathcal{I} \rangle$. Then we say that it is* logically sequential *iff for all $w \in \mathcal{W}^{(n)}$, R^w_ι is a sequentiality relation.*

*An element $d \in D_\sigma$ is called n-*sequential *iff for all $w \in \mathcal{W}^{(n)}$ and for all Kripke logical relation $\mathcal{R}$ which is logically sequential, the mapping*

$$w \xrightarrow{\;!_w\;} \mathbf{1} \xrightarrow{\;d\;} D_\sigma$$

is in the relation R^w_σ.

An element $d \in D_\sigma$ is called sequential *iff it is n-sequential for all n.*

In the new setting, Sieber's theorem 2.4 reads as follows:

Proposition 4.3 *Let $\mathcal{R}$ be a Kripke logical relation over $\langle \mathcal{W}^{(n)}, \mathcal{D}, \mathcal{I} \rangle$. Then $\mathcal{R}$ is logically sequential iff for all PCF constants $c : \sigma$, $R^w_\sigma([\![c]\!]^{\mathcal{D}} \circ \,!_w)$, where $[\![\cdot]\!]^{\mathcal{D}}$ is just the interpretation map of the standard model $\mathcal{D}$.*

Let $w = D^n_{\sigma_1} \times \cdots \times D^n_{\sigma_m}$ and $g : w \to D_\sigma$ be any map. Then we say that g is PCF-definable iff there exists a closed term $M : \sigma_1 \to \cdots \to \sigma_m \to \sigma$ such that

$$\begin{tikzcd} w \arrow[r, hook] \arrow[rr, bend right, "g"'] & D_{\sigma_1} \times \cdots \times D_{\sigma_m} \arrow[r, "\Lambda_m^{-1}([\![M]\!]^{\mathcal{D}})"] & D_\sigma \end{tikzcd}$$

where Λ_m is the isomorphism

$$[D_{\sigma_1} \times \cdots \times D_{\sigma_m} \to D_\sigma] \simeq [D_{\sigma_1} \to \cdots \to D_{\sigma_m} \to D_\sigma].$$

When dealing with PCF-definable tuples we shall write simply $[\![M]\!]^{\mathcal{D}}$ for $\Lambda_m^{-1}([\![M]\!]^{\mathcal{D}})$. For $w \in \mathcal{W}^{(n)}$ define $R_\iota^w \subseteq D_\iota^w$ as follows:

$$R_\iota^w(g) \;\Leftrightarrow\; g \text{ is PCF-definable.}$$

In what follows the notation $\mathcal{R}$ is referred to the relation just defined.

The following results are clearly parametric with respect to the choice of n in $\mathcal{W}^{(n)}$ etc.

Lemma 4.4 *The family $\{R_\iota^w\}_{w \in \mathcal{W}^{(n)}}$ determines a Kripke logical relation over $\langle \mathcal{W}^{(n)}, \mathcal{D}, \mathcal{I} \rangle$. Moreover it is a logically sequential relation.*

Proof. Let $w = D_{\sigma_1}^n \times \cdots \times D_{\sigma_m}^n$. To see that R_ι^w is a sub-cpo of D_ι^w first observe that the bottom element, that is $\lambda x \in w.\ \bot_\iota$, is definable by $\lambda x_1^{\sigma_1} \ldots x_m^{\sigma_m}.\ \Omega$, where Ω is such that $[\![\Omega]\!]^{\mathcal{D}} = \bot_\iota$.

To see direct completeness we first note that the cardinality of w is finite, say k, so that $w = \{\langle d_{i,1} \ldots, d_{i,m} \rangle \mid i \leq k\}$. Let $G \subseteq R_\iota^w$ be directed; then for all $i \leq k$, $(\bigsqcup G)(d_{i,1}, \ldots, d_{i,m}) = \bigsqcup_{g \in G} g(d_{i,1}, \ldots, d_{i,m})$. But D_ι is flat, hence

$$\forall i \leq k\ \exists g_i \in G.\ (\bigsqcup G)(d_{i,1}, \ldots, d_{i,m}) = g_i(d_{i,1}, \ldots, d_{i,m}),$$

so that $\bigsqcup G = \bigsqcup_{i \leq k} g_i$. Now $\{g_1, \ldots, g_k\}$ is a finite subset of the directed set G, therefore there exists a $g \in G$ which is greater than all g_i. Clearly $g = \bigsqcup G$.

Finally, to check condition (2) of definition 3.2, suppose that $v = D_{\sigma_1}^n \times \cdots \times D_{\sigma_m}^n \times D_{\sigma_{m+1}}^n \times \cdots \times D_{\sigma_{m+k}}^n$. Then the following diagram commutes for all closed M of the right type:

$$\begin{tikzcd}
v \arrow[d, "\pi_{v,w}"] & \hookrightarrow & D_{\sigma_1} \times \cdots \times D_{\sigma_{m+k}} \arrow[rr, "{[\![\lambda x_1^{\sigma_1} \ldots x_{m+k}^{\sigma_{m+k}}.\ M x_1 \ldots x_m]\!]^{\mathcal{D}}}"] & & D_\iota \\
w & \hookrightarrow & D_{\sigma_1} \times \cdots \times D_{\sigma_m} \arrow[urr, "{[\![M]\!]^{\mathcal{D}}}"'] & &
\end{tikzcd}$$

It remains to show that $\mathcal{R}$ is a sequentiality relation. By proposition 4.3, it suffices to prove that $[\![c]\!]^{\mathcal{D}} \circ !_w$ is in the relation R^w_σ, namely $R^w_\sigma(\lambda x \in w.\ [\![c]\!]^{\mathcal{D}})$ for all w and constant $c : \sigma$. As all cases are similar (and PCF constants are all first order) we just check this for $\mathsf{succ} : \iota \to \iota$. Let w be arbitrary and suppose that $v \geq w$. Let $h \in D^v_\iota$ be such that $R^v_\iota(h)$. Then h is PCF-definable, say by H. Therefore for all $\vec{d} \in v$

$$app_{\iota,\iota}([\![\mathsf{succ}]\!]^{\mathcal{D}} \circ !_w \circ \pi_{w,v}, h)(\vec{d}) = [\![\mathsf{succ}]\!]^{\mathcal{D}}([\![H]\!]^{\mathcal{D}}\vec{d})) = [\![\lambda\vec{x}.\ \mathsf{succ}(H\vec{x})]\!]^{\mathcal{D}}\vec{d}$$

that is $app_{\iota,\iota}([\![\mathsf{succ}]\!]^{\mathcal{D}} \circ !_w \circ \pi_{w,v}, h)$ is PCF-definable, and hence in the relation R^v_ι.

□

Remark 6 Suppose that $g \in [w \to D_{\sigma\to\tau}]$ is such that $g = p^n_{\sigma\to\tau} \circ g$, that is $g(w) \subseteq D^n_{\sigma\to\tau}$. Then, for all $\vec{d} \in w$,

$$g(\vec{d}) = (p^n_{\sigma\to\tau} \circ g)(\vec{d}) = p^n_\tau \circ g(\vec{d}) \circ p^n_\sigma.$$

Therefore, by the idempotency of both p^n_τ and p^n_σ, we have

$$p^n_\tau \circ g(\vec{d}) = g(\vec{d}) = g(\vec{d}) \circ p^n_\sigma.$$

Lemma 4.5 *For all $\sigma \in \boldsymbol{T}$ and $w \in \mathcal{W}^{(n)}$, if $g \in [w \to D_\sigma]$ is such that $g = p^n_\sigma \circ g$, then*

$$R^w_\sigma(g) \quad \Leftrightarrow \quad g \text{ is PCF-definable.}$$

Proof. By induction on σ. Case ι is immediate by definition. Case $\sigma \to \tau$:

($\Rightarrow$) Let $w = D^n_{\sigma_1} \times \cdots \times D^n_{\sigma_m}$ and $v = w \times D^n_\sigma$. Consider

$$\begin{tikzcd} v \arrow[r, hook] \arrow[rr, bend right, "h"'] & D_{\sigma_1} \times \cdots \times D_{\sigma_m} \times D_\sigma \arrow[r, "{[\![\lambda x_1^{\sigma_1} \ldots x_m^{\sigma_m} x^\sigma .\ x]\!]^{\mathcal{D}}}"] & D_\sigma \end{tikzcd}$$

Then, for all $d_1, \ldots, d_m, d \in v$:

$$\begin{aligned} h(d_1, \ldots, d_m, d) &= d \\ &= p^n_\sigma(d) \\ &= (p^n_\sigma \circ h)(d_1, \ldots, d_m, d) \end{aligned}$$

since $d \in D^n_\sigma$. Therefore $h = p^n_\sigma \circ h$ so that, by induction hypothesis, $h \in R^v_\sigma$. If $g \in R^w_{\sigma \to \tau}$ and h as above, then

$$\begin{tikzcd} v \arrow[r, "\langle \pi_{v,w}, id \rangle"] \arrow[rrr, bend right, "g'"'] & w \times v \arrow[r, "g \times h"] & D_{\sigma \to \tau} \times D_\sigma \arrow[r, "\varepsilon"] & D_\tau \end{tikzcd}$$

so that $R^v_\tau(g')$.

Moreover, if g is such that $g = p^n_{\sigma \to \tau} \circ g$, then, for all $d_1, \ldots, d_m, d \in v$:

$$\begin{aligned} (p^n_\tau \circ g')(d_1, \ldots, d_m, d) &= p^n_\tau(g'(d_1, \ldots, d_m, d)) \\ &= p^n_\tau(g(d_1, \ldots, d_m)(d)) \\ &= (p^n_\tau \circ g(d_1, \ldots, d_m))(d) \\ &= g(d_1, \ldots, d_m)(d) \\ &= g'(d_1, \ldots, d_m, d) \end{aligned}$$

by remark 6. It follows that $g' = p^n_\tau \circ g'$, so that induction hypothesis applies, that is for some closed M of the right type we have:

$$\begin{tikzcd} v \arrow[r, hook] \arrow[rr, bend right, "g'"'] & D_{\sigma_1} \times \cdots \times D_{\sigma_m} \times D_\sigma \arrow[r, "[\![M]\!]^{\mathcal{D}}"] & D_\tau \end{tikzcd}$$

Now, for all $d_1, \ldots, d_m \in w$ and $d \in D_\sigma$:

$$\begin{aligned}
g(d_1, \ldots, d_m)(d) &= g(d_1, \ldots, d_m)(p^n_\sigma(d)) \\
&= g'(d_1, \ldots, d_m, p^n_\sigma(d)) \\
&= [\![M]\!]^{\mathcal{D}} d_1 \cdots d_m ([\![P^n_\sigma]\!]^{\mathcal{D}} d) \\
&= [\![\lambda x_1^{\sigma_1} \ldots x_m^{\sigma_m} x^\sigma .\ M x_1 \ldots x_m (P^n_\sigma x)]\!]^{\mathcal{D}} d_1 \cdots d_m d
\end{aligned}$$

again by remark 6. Therefore g is definable.

($\Leftarrow$) Let $g \in [w \to D_{\sigma\to\tau}]$ be definable by the closed term M. Suppose again that $w = D^n_{\sigma_1} \times \cdots \times D^n_{\sigma_m}$. To prove that $R^w_{\sigma\to\tau}(g)$, we have to show that for any $v = w \times D^n_{\sigma_{m+1}} \times \cdots \times D^n_{\sigma_{m+k}}$, if $h \in D^v_\sigma$ is such that $R^v_\sigma(h)$, then the mapping $app^v_{\sigma,\tau}(g \circ \pi_{v,w}, h) \in D^v_\tau$ is in the relation R^v_τ.

First note that, by the hypothesis that $g = p^n_{\sigma\to\tau} \circ g$ and by remark 6, for all $d_1, \ldots, d_{m+k} \in v$,

$$g(d_1, \ldots, d_m)(h(d_1, \ldots, d_{m+k})) = g(d_1, \ldots, d_m)((p^n_\sigma \circ h)(d_1, \ldots, d_{m+k}))$$

therefore there is no loss of generality in supposing that $h = p^n_\sigma \circ h$. This implies, by induction hypothesis, that h is definable, say by the closed term H. It follows that $g' = app^v_{\sigma,\tau}(g \circ \pi_{v,w}, h)$ makes the following diagram to commute:

$$\begin{tikzcd}
w \times v \arrow[r, hook] & D_{\sigma_1} \times \cdots \times D_{\sigma_{m+k}} \arrow[r, "{[\![M]\!]^{\mathcal{D}} \times [\![H]\!]^{\mathcal{D}}}"] & D_{\sigma\to\tau} \times D_\sigma \arrow[d, "\varepsilon"] \\
v \arrow[u, "{\langle \pi_{v,w}, id \rangle}"] \arrow[rr, "g'"'] & & D_\tau
\end{tikzcd}$$

This means that g' is definable by $\lambda x_1^{\sigma_1} \ldots x_{m+k}^{\sigma_{m+k}} .\ M x_1 \ldots x_m (H x_1 \ldots x_{m+k})$. On the other hand, for all $\vec{d} \in v$, we have, by remark 6:

$$(p^n_\tau \circ g')(\vec{d}) = (p^n_\tau \circ g(\pi_{v,w}(\vec{d}))(h(\vec{d})) = g'(\vec{d})$$

We conclude that $g' = p^n_\tau \circ g'$ so that, by induction hypothesis, $R^v_\tau(g')$.

$\square$

Corollary 4.6 *For all type σ and $f \in D^n_\sigma$,*

$$R^{\mathbf{1}}_\sigma(f) \;\Leftrightarrow\; f \text{ is PCF-definable}$$

where f is identified with $\mathbf{1} \xrightarrow{f} D_\sigma$.

Theorem 4.7 *Let $f \in D_\sigma$ be logically sequential; then it is the lub of a directed set of definable elements.*

Proof. f is sequential by hypothesis and for all n, $\mathcal{R}$ is a sequentiality relation by lemma 4.4; hence for all n and $w \in \mathcal{W}^{(n)}$, $R^w_\sigma(\lambda x \in w.\ f)$, which in particular holds for $w = \mathbf{1}$. Note that $p^n_{\sigma\to\sigma}(p^n_\sigma) = p^n_\sigma \circ p^n_\sigma \circ p^n_\sigma = p^n_\sigma$, therefore $p^n_\sigma \in D^n_{\sigma\to\sigma}$. Since each p^n_σ is definable, by corollary 4.6 $R^{\mathbf{1}}_{\sigma\to\sigma}(\lambda x \in \mathbf{1}.\ p^n_\sigma)$. Therefore

$$\begin{tikzcd}
\mathbf{1} \arrow[r, "\langle p^n_\sigma, f\rangle"] \arrow[rr, bend right, "app^{\mathbf{1}}_{\sigma,\sigma}(p^n_\sigma, f)"'] & D_{\sigma\to\tau} \times D_\sigma \arrow[r, "\varepsilon"] & D_\sigma
\end{tikzcd}$$

so that $app^{\mathbf{1}}_{\sigma,\sigma}(p^n_\sigma, f) = p^n_\sigma(f)$ is in the relation $R^{\mathbf{1}}_\sigma$. Now $p^n_\sigma(f) \in D^n_\sigma$, so that, by corollary 4.6, it is definable. Hence the thesis follows since, being D_σ an SFP object, $f = \bigsqcup_n p^n_\sigma f$.

$\square$

Corollary 4.8 *Let $L_\sigma \subseteq D_\sigma$ be the set of logically sequential elements of type σ, for all type σ. If $\sigma = \sigma_1 \to \cdots \to \sigma_m \to \iota$ and M, N are closed terms of type σ, then*

$$M \simeq_{op} N \;\Leftrightarrow\; \forall(d_1, \ldots, d_m) \in L_{\sigma_1} \times \cdots \times L_{\sigma_m}.\ [\![M]\!]\, d_1 \cdots d_m = [\![N]\!]\, d_1 \cdots d_m.$$

Therefore the (continuous) applicative structure $\mathcal{L}$ (where application is just functional application) is fully abstract.

Proof. The ($\Rightarrow$) part is proved from 4.7 by contrapposition, using algebraicity of $\mathcal{D}$, continuity and the adequacy theorem by Plotkin. The ($\Leftarrow$) part is immediate by the context lemma.

□

This is half of the requirements in Milner theorem, the second half being (order) extensionality. Now $\mathcal{L}$ is not extensional, as e.g. the denotations of the two tests for *por* are different as functions in $\mathcal{D}$, even if they coincide when restricted to the "sequential" objects in $\mathcal{L}$. To get the fully abstract model, we have to identify sequential functions, whose behavior is the same when restricted to sequential arguments: this is the *extensional collapse* (also called the *Gandy hull*) of $\mathcal{L}$.

Let $\mathcal{E} \subseteq \mathcal{L} \times \mathcal{L}$ denote the classical logical relation induced by:

$$E_\iota(d, d') \;\Leftrightarrow\; d = d'.$$

Then it easily shown that this is an equivalence relation at all types. So let $\mathcal{L}/\mathcal{E}$ be the structure whose carriers are quotients L_σ/E_σ, let $[f]$ indicate the equivalence class of the element $f \in L_\sigma$ under E_σ and define

$$app_{\sigma,\tau}/\mathcal{E}([f],[d]) = [app_{\sigma,\tau}(f,d)] = [f(d)].$$

As $\mathcal{E}$ is a logical relation, this is well defined. Now $L_\iota/E_\iota \simeq D_\iota$, which induces an ordering over L_ι/E_ι; suppose inductively that the ordering of L_τ/E_τ has been defined, then, for $[f],[g] \in L_{\sigma\to\tau}/E_{\sigma\to\tau}$ put

$$[f] \sqsubseteq [g] \;\Leftrightarrow\; \forall [d] \in L_\sigma/E_\sigma.\ [f(d)] \sqsubseteq [g(d)].$$

The proof of the last theorem is now immediate.

Theorem 4.9 *The quotient structure $\mathcal{L}/\mathcal{E}$ is a continuous applicative structure which is an ω-algebraic, order-extensional model of PCF, in which all finite elements are definable. Therefore it is the fully abstract model of PCF.*

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