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Abstract. The edge-graphs of m-polygons and of 5 Platonic solids admit (unique) isometric embedding into a half cube (of dimension m and 6, resp.). The skeletons of α_n, β_n, γ_n are also ℓ_1-graphs but it was proved in [1] that 24-cell and 600-cell are not (since not 5- and not 7-gonal, resp.). We show here that the last remaining regular polytope, 120-cell, is also not ℓ_1-graph.

The 120-cell P_{120} is the largest one among the four-dimensional regular polytopes. It has 120 three-dimensional facets (cells) and 600 vertices. The skeleton G_{120} of this polytope is a regular graph of degree 4 on 600 vertices. It is not distance-regular.

The graph G_{120} is constructed from 480 pentagons such that each vertex belongs to 4 pentagons and each edge belongs to three pentagons. One can say that G_{120} is a 4-dimensional fullerene.

We use the following description of vertices P_{120} by four- dimensional vectors given in [2] (Ch.22.3, Exerc.5). There are two groups of vertices. Vertices of the first group are all permutations of coordinates of the vectors of the shapes 1.–4. below. Vertices of the second group are all even permutations of coordinates of the vectors of shapes 5.–7.

1. $(\pm 2, \pm 2, 0, 0)$
2. $(\pm(2\tau - 1), \pm 1, \pm 1, \pm 1)$
3. $(\pm\tau, \pm\tau, \pm\tau, \pm\tau^{-2})$
4. $(\pm\tau^2, \pm\tau^{-1}, \pm\tau^{-1}, \pm\tau^{-1})$
5. $(\pm\tau^2, \pm\tau^{-2}, \pm 1, 0)$
6. $(\pm(2\tau - 1), \pm\tau^{-1}, \pm\tau, 0)$
7. $(\pm 2, \pm 1, \pm\tau, \pm\tau^{-1})$

Here $\tau = \frac{\sqrt{5}+1}{2}$ such that $1.6 < \tau < 1.62$. Besides

$$\tau^{-2} = 2 - \tau,\ \ \tau^2 = \tau + 1,\ \ \tau^{-1} = \tau - 1,\ \ 2\tau - 1 = \sqrt{5}.$$

Two vertices given by vectors v and v' are adjacent in the skeleton G_{120}, i.e. are connected by an edge of P_{120}, if the inner product vv' is maximal, i.e. if $vv' = 6\tau - 2$. The norm (=squared length) of an edge of P_{120} is equal to $(v - v')^2 = 20 - 12\tau$, i.e. $0,56 < (v - v')^2 < 0,8$.

We shall identify vertices with the vectors describing them. We say that a vertex v *belongs* to the layer (x) if the first coordinates of v is x. All the 600 vertices of P_{120} are partitioned into 15 layers. The vertices of the k-th and the $(15 - k)$-th layers have the same, up to sign, first coordinate. The first layer (τ^2) and the last layer $(-\tau^2)$ are facets of P_{120}.

We give in Table below adjacencies of vertices of these layers.

Table. Adjacencies of the layers.

(x)	$\lvert(x)\rvert$	adjacencies	$G(x)$
(τ^2)	20	$3(\tau^2)+1(2\tau-1)$	*Dod*
$(2\tau-1)$	20	$1(\tau^2)+3(2)$	empty
(2)	30	$2(2\tau-1)+2(\tau)$	empty
(τ)	60	$1(2)+2(\tau)+1(1)$	$12C_5$
(1)	60	$1(\tau)+1(1)+2(\tau^{-1})$	$30K_2$
(τ^{-1})	60	$2(1)+1(\tau^{-2})+1(0)$	empty
(τ^{-2})	20	$3(\tau^{-1})+1(-\tau^{-2})$	empty
(0)	60	$1(\tau^{-1})+2(0)+1(-\tau^{-1})$	$12C_5$
$(-\tau^{-2})$	20	$3(-\tau^{-1})+1(\tau^{-2})$	empty
$(-\tau^{-1})$	60	$2(-1)+1(-\tau^{-2})+1(0)$	empty
(-1)	60	$1(-\tau)+1(-1)+2(-\tau^{-1})$	$30K_2$
$(-\tau)$	60	$1(-2)+2(-\tau)+1(-1)$	$12C_5$
(-2)	30	$2(-2\tau+1)+2(-\tau)$	empty
$(-2\tau-1)$	20	$1(-\tau^2)+3(-2)$	empty
$(-\tau^2)$	20	$3(-\tau^2)+1(-2\tau-1)$	*Dod*

We use in Table the following notation. Dod, $12C_5$ and $30K_2$ are the skeleton of a dodecahedron, 12 disjoint 5-cycles (*pentagons*) C_5 and 30 disjoint edges K_2, respectively. Dod is a strongly regular planar graph on 20 vertices of degree 3. It consists of 12 pentagons C_5.

The first, the second and the fourth columns of Table give layers (x), their cardinality $|(x)|$ and the graph $G(x)$ induced in G_{120} by vertices of the layer (x).

The third column of Table shows layers in which the four edges incident to a vertex of the layer (x) go. The number $n(x,y)$ in the row (x) before the layer (y) denotes the number of edges going from a vertex of the layer (x) to the layer (y). Note that $\sum_y n(x,y)=4$ is the degree of G_{120}.

The number $n(x,x)$ is the degree of the graph $G(x)$. Since the graph $G_{120}=\cup_x G(x)$ is connected and its degree is 4, we have that $n(x,x)\leq 3$. Table shows that $n(x,x)=3$ only for two layers $(\pm\tau^2)$, i.e. for the facets of P_{120}.

If $n(x,x)=2$, then the graph $G(x)$ consists of k disjoint pentagons C_5, where $5k=|(x)|$. This is the case for $x=\pm\tau,0$, when $|(x)|=60$, and hence $k=12$.

If $n(x,x)=1$, then the graph $G(x)$ consists of m disjoint edges K_2, where $2m=|(x)|$. This is the case for $x=\pm 1$, when $|(x)|=60$, and hence $m=30$.

If $n(x,x)=0$, then the graph $G(x)$ has no edges, i.e. it is empty. This is so for $x=\pm(2\tau-1),\pm\tau^{-1},\pm\tau^{-2}$.

If $n(x,y)=1$ and $x\neq y$, then a unique vertex of the layer (y) corresponds to each vertex $v\in(x)$. Denote this vertex as $q_y(v)$

Since $n(\tau^2, 2\tau-1) = n(2\tau-1, \tau^2) = 1$, we have a one- to-one correspondence between vertices of the layers (τ^2) and $(2\tau-1)$.

Using Table we can describe the graph $G(2\tau-1,2)$ induced in G_{120} by the layers $(2\tau-1)$ and (2). The vertices of this graph from the layer $(2\tau-1)$ have degree 3, and the vertices from the layer (2) have degree 2. Hence the graph $G(2\tau-1,\tau)$ is a dodecahedron with one vertex (of degree 2) on each edge of this dodecahedron. Each pentagon C_5 of the dodecahedron is transformed into a 10-gon C_{10}.

Each vertex $w \in (2\tau-1)$ is adjacent in G_{120} to the unique vertex $q_{\tau^2}(w) \in (\tau^2)$, and each vertex $v \in (2)$ is adjacent in G_{120} to two vertices of the layer (τ). Since $n(\tau,2) = 1$, the vertex $q_2(v) \in (2)$ is uniquely determined for each vertex $v \in (\tau)$, and, since $n(2,\tau) = 2$, there are two vertices $v, v' \in (\tau)$ such that $q_2(v) = q_2(v')$.

The graph $G_1' := G(2\tau-1, 2, \tau)$ induced in G_{120} by the three layers $(2\tau-1)$, (2) and (τ) contains 110 vertices and it is as follows. Recall that $G(2\tau-1,2)$ contains 12 10-gons C_{10}. The graph G_1' contains a pentagon $C_5 \subset G(\tau)$ corresponding to a 10-gon C_{10} such that each vertex $v \in C_5 \subset G(\tau)$ is adjacent to the unique vertex $q_2(v) \in C_{10}$. Recall that the vertices of C_{10} distinct from $q_2(v)$ are adjacent to vertices of $Dod = G(\tau^2)$.

Denote by G_1 the graph induced in G_{120} be vertices of the 4 layers (τ^2), $(2\tau-1)$, (2) and (τ). The vertices of G_1 have degrees 3 and 4. The degree 3 is the degree of vertices of the layer (τ). The forth edges incident to these vertices in G_{120} go to the layer (1).

Table shows that there is a one-to-one correspondence between vertices of the layers (τ) and (1), since $n(\tau,1) = n(1,\tau) = 1$. Hence, for $v \in (\tau)$, the vertex $q_1(v)$ belongs to the layer (1), and $q_\tau(u) \in (\tau)$ for each $u \in (1)$.

Recall that $q_2(v) \in (2)$ is the unique vertex of the layer (2) corresponding to $v \in (\tau)$, and there is once more vertex $v' \in (\tau)$ such that $q_2(v') = q_2(v)$. The vertices $q_1(v)$ and $q_1(v')$ of the layer (1) are adjacent in $G(1)$. There are two edges going from the layer (1) into the layer (τ^{-1}). Note that the 5 vertices $v, v' \in (\tau)$, $q_2(v) = q_2(v') \in (2)$, $q_1(v), q_1(v') \in (1)$ induce an (isometric) pentagon.

Now, using Table, we describe the graph G_2 induced in G_{120} by the 4 layers (1), (τ^{-1}), (τ^{-2}) and (0). The subgraph $G(1,\tau^{-1})$ of G_2 induced by the layers (1) and (τ^{-1}) consists of 12 vertex disjoint 10-gons C_{10}. The vertices of (1) and (τ^{-1}) lie in each C_{10} in turn. The 5 vertices of C_{10} from the layer (1) have degree 3, and other 5 vertices of C_{10} from the layer (τ^{-1}) have degree 2. Since $n(1,1) = 1$, there is a pairing q_1 of vertices from the layer (1) in the graph $G(1,\tau^{-1})$. This pairing connects vertices of distinct C_{10}. Call two 10-gons *neighbouring* if they have a pair of adjacent vertices of $G(1)$.

Since $n(\tau^{-1}, \tau^{-2}) = 1$, there corresponds a unique vertex $q_{\tau^{-2}}(u)$ to each vertex $u \in (\tau^{-1})$. Table shows that the vertex $q_{\tau^{-2}}(u) \in (\tau^{-2})$ has degree

3 in the graph G_2. Let u_1, u_2, u_3 be vertices of (τ^{-1}) such that $q_{\tau^{-2}}(u_1) = q_{\tau^{-2}}(u_2) = q_{\tau^{-2}}(u_3) = w \in (\tau^{-2})$. The 3 vertices u_i, $1 \le i \le 3$, belong to 3 distinct pairwise neighbouring 10-gons C^i_{10}. Let v_{i1} and v_{i2} be two vertices of the layer (1) adjacent to the vertex $u_i \in (\tau^{-1})$, $1 \le i \le 3$. All the six vertices v_{ij} are distinct. Moreover, we can denote the vertices v_{ij} such that v_{ij} and $v_{i+1,j+1}$ are adjacent in G_2, where $i+1$ and $j+1$ are considered by modulo 3 and 2, respectively. Hence the 5 vertices w, u_i, u_{i+1}, $v_{i,j}$ and $v_{i+1,j+1}$ for the corresponding j induce a pentagon.

By Table, the subgraph $G(0) \subseteq G_2$ consists of 12 disjoint pentagons C_5. Since $n(0,\tau^{-1}) = 1$, there corresponds a unique vertex $q_{\tau^{-1}}(v)$ to each $v \in (0)$. Since $n(\tau^{-1}, 0) = 1$, there is a one-to-one correspondence between vertices of the layers (τ^{-1}) and (0) such that the 5 vertices of a pentagon of the graph $G(0)$ correspond to 5 vertices of (τ^{-1}) of the same 10- gon.

The graph G_2 is planar. It can be drawn as follows. There is a pentagon inside each of the 12 vertex disjoint 10-gons. The vertices of the layer (τ^{-2}) lie inside 9-gons formed by 3 pairs of adjacent edges of 3 pairwise neigbouring 10-gons and by 3 disjoint edges of $G(1)$.

Let G_{-1} and G_{-2} be graphs similar to G_1 nad G_2 but induced by the corresponding layers with negative signs.

The graph G_{120} is the following union of the graphs G_1, G_2, G_{-2} and G_{-1}. The vertices of G_1 (G_{-1}, respectively) from the layer (τ) ($(-\tau)$, respectively) are adjacent the uniquely determined vertices of G_2 (G_{-2}) from the layer (1) (from the layer (-1), respectively), since $n(\tau, 1) = n(1, \tau) = 1$ (and $n(-\tau, -1) = n(-1, -\tau) = 1$). The graphs G_2 and G_{-2} are glued by pentagons of the common subgraph $G(0)$. Besides, since $n(\tau^{-2}, -\tau^{-2}) = n(-\tau^{-2}, \tau^{-2}) = 1$, there is a one-to-one pairing the vertices of the layers (τ^{-2}) and $(-\tau^{-2})$ by edges.

Let $G_{1,2}$ ($G_{-1,-2}$) be a subgraph of G_{120} induced by vertices of G_1 and G_2 (G_{-1} and G_{-2}, respectively). Obviously $G_{1,2}$ and $G_{-1,-2}$ are isomorphic and $G_{1,2} \cap G_{-1,-2} = G(0)$. Since there is a one-to-one correspondence between paths in $G_{1,2}$ and $G_{-1,-2}$, we have the following obvious fact.

Proposition 1 *The graphs $G_{1,2}$ and $G_{-1,-2}$ are isometric subgraphs of G_{120}.*

Consider the vertices of the graphs G_1 and G_2 from the layers (τ) and (1), respectively. Recal that since $n(\tau, 1) = n(1, \tau) = 1$, there is a one-to-one correspondence q_1 between these sets of vertices. Moreover, this correspondence generates a one-to-one correspondence between the 12 disjoint pentagons of the graph $G(\tau) \subset G_1$ and the 12 disjoint 10-gons of the graph $G(1, \tau^{-1}) \subseteq G_2$.

By an inspection of the graphs G_1 and G_2 on can prove the following

Lemma 1 *For $v_1, v_2 \in (\tau)$ let $q_1(v_1)$, $q_1(v_2)$ be the corresponding vertices of the layer (1). Let $l_1(v_1, v_2)$, $l_2(q_1(v_1), q_1(v_2))$ be the lengths of the shortest* *paths connected v_1 and v_2 in G_1 and $q_1(v_1)$ and $q_1(v_2)$ in G_2. Then*

$$l_2(q_1(v_1), q_1(v_2)) \geq l_1(v_1, v_2) \text{ if } l_1(v_1, v_2) > 2.$$

Note that $l_2(q_1(v_1), q_1(v_2)) = 1$ if $l_1(v_1, v_2) = 2$.

Let $p(v, v'))$ be a path connecting in G_{120} vertices $v, v' \in G_1$. If $p(v, v')$ contains vertices of the graph G_2, then there exist at least two vertices $u, u' \in (\tau) \subset G_1$ such that the subpath $p(u, u') \subset p(v, v')$ contains besides u, u' no other vertex of G_1. The vertices $q_1(u)$ and $q_1(u')$ are adjacent to u and u' in the path $p(u, u')$. Now the above Lemma 1 implies

Proposition 2 *The graph G_1 is an isometric subgraph of G_{120}.*

Consider a subgraph G' of G_1 induced by a pentagon of $G(\tau)$ and the corresponding 10-gon C_{10} of $G(2\tau - 1, 2)$. It is easy to verify that C_{10} is an isometric subgraph of G'. This implies

Proposition 3 *The graph G_0 induced by the 3 layers (τ^2), $(2\tau - 1)$ and (2) is an isometric subgraph of G_1.*

Theorem 1 *The graph G_0 is not 5-gonal.*

Proof: We give 5 vertices a, B, a, b, c of G_0 that violate the following pentagonal inequality

$$\sum_{u,v \in S} d(u, v) + \sum_{u',v' \in S'} d(u', v') \leq \sum_{v \in S, v' \in S'} d(v, v'),$$

where $S = \{A, B\}$, $S' = \{a, b, c\}$.

Two 10-gons of G_0 either disjoint or have 3 common vertices (and two common adjacent edges). Call two 10-gons *adjacent* if they are not disjoint. Consider four 10-gons C^i_{10}, $1 \leq i \leq 4$, such that C^1_{10} is adjacent to C^i_{10} for $2 \leq i \leq 4$, C^3_{10} is adjacent to C^2_{10} and C^4_{10}, but C^2_{10} and C^4_{10} are disjoint.

The intersections $(C^1_{10} \cap C^3_{10}) \cap C^2_{10}$ and $(C^1_{10} \cap C^3_{10}) \cap C^4_{10}$ are nonempty, and each contains one vertex, say v^2 and v^4, respectively. The intersection $C^1_{10} \cap C^3_{10}$ consists of 3 vertices v^2, v^4 and a third vertex that we call c.

Let $a \in C^2_{10}$ and $b \in C^4_{10}$ be vertices at distance 5 in the corresponding 10-gons from the vertices $v^2 \in C^2_{10}$ and $v^4 \in C^4_{10}$, respectively. Let $A \in C^2_{10}$ be a vertex adjacent to a. We have two vertices in C^4_{10} adjacent to the vertex $b \in C^4_{10}$. Let $B \in C^4_{10}$ be those of these vertices that is farther in G_0 from the vertex A.

It is easy to veryfy that the distances in G_0 between the 5 vertices A, B, a, b, c are as follows:

$$d(a, c) = d(b, c) = 6;\; d(a, b) = 8;\; d(A, B) = 7;$$

$$d(a,B)=d(b,A)=7;\ d(c,A)=d(c,B)=5;\ d(a,A)=d(b,B)=1.$$

Hence $\sum_{u,v\in S} d(u,v)+\sum_{u'v'\in S'} d(u',v') = 2\times 6+8+7 = 27 > 26 = 2\times(7+5+1) = \sum_{v\in S,v'\in S'} d(v,v')$. $\square$

Since G_0 is an isometric subgraph of G_{120}, we have

Corollary 1 *The skeleton G_{120} of the 120 cell is not 5-gonal and so not ell$_1$-graph (see [3] for the context of it).*

References

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