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On Polygonal Covers

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On polygonal covers[1]

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Abstract

A polygonal cover of a finite collection of pairwise disjoint convex compact sets in the plane is a finite collection of non-overlapping convex polygons such that each polygon covers exactly one convex set. We show that computing a polygonal cover with worst case minimal number of sides reduces to computing a pseudo-triangulation of the collection of convex sets. We obtain a similar reduction for two related problems concerning convex compact sets in the plane : computing a lighting set and computing a family of separating lines.

Keywords: Pseudotriangle, pseudo-triangulation, polygonal cover, translation query, Art gallery theorem, packing, covering.

1 Introduction

A *polygonal cover* of a finite collection of pairwise disjoint convex compact sets in the plane (convex disks for short) is a finite collection of non-overlapping convex polygons such that each polygon covers exactly one convex disk; see Figure 1. In 1950, L. Fejes Tóth [8] has

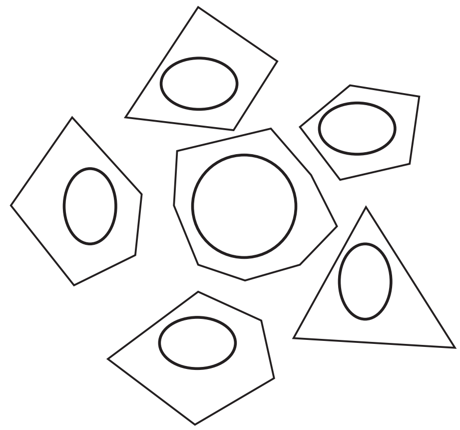

Figure 1: A polygonal cover of a collection of six disks.

shown the existence of polygonal covers with a total of no more than $6n-9$ sides for n disks, this bound being tight in the worst case [7]. He used this result in order to show that the upper density of a packing with congruent copies of a convex disk in the plane is bounded by the ratio of the area of the disk to the area of a hexagon of minimum area circumscribed round the disk (see [14] and the reference cited therein). More recently, polygonal covers have been used to show the existence of a disk separated by a line from at least $n/12$ other disks (see [12] and [1, 4, 5] for more separation problems) and to design efficient algorithms to answer translation queries for sets of polygons (see [6]).

The proof of L. Fejes Tóth proceeds by growing the convex disks unboundedly in all directions but the growth (in a given direction) is limited by the condition that the convex disks remain pairwise interior disjoint. In this way the convex disks expand into convex polygons that fill the plane except for a finite number of gaps that are also convex polygons. A simple application of the Euler relation for planar graph leads to the announced upper bound. The purpose of this note is to give a constructive proof of the result of L. Fejes Tóth. Our main result is the following.

Theorem 1 *A polygonal cover with worst case minimal number of sides for a collection of n convex disks is computable in $O(n \log n)$ time under the assumption that each disk has constant computational complexity.*

Our proof is based on properties of the so-called pseudo-triangulations introduced by the authors [15, 16] to design efficient algorithms to compute visibility graphs. In Section 2 we review the terminology on pseudo-triangulations. In Section 3 we give the proof of our main theorem, and in Section 4 we elaborate on two related problems concerning convex disks in the plane, namely computing a lighting set and computing a family of separating lines.

2 Background material

Consider a finite collection of pairwise disjoint convex disks in the plane. For the ease of the exposition we assume that the boundary of each disk is a smooth closed curve such that for each ϕ, $0 \leq \phi \leq 2\pi$, the curve has exactly one point where its tangent makes an angle of ϕ with the positive x-axis; furthermore we assume that no line is tangent to three disks. A *bitangent* is a closed line segment whose supporting line is tangent to two disks at its

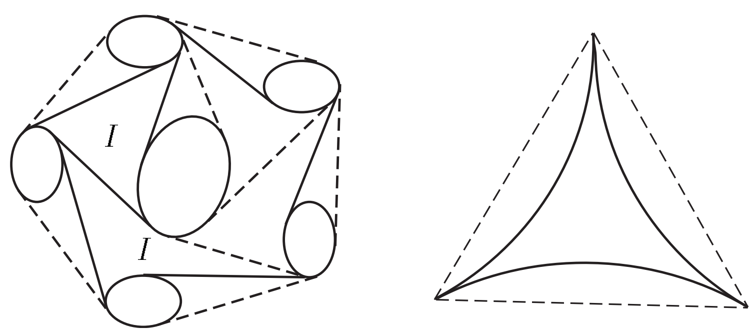

Figure 2: A pseudo-triangulation (exterior bitangents are shown dashed and type I pseudotriangles are labeled I) and a pseudotriangle.

endpoints. It is called *free* if it lies in free space, i.e., the complement of the union of the interior of the disks. A bitangent is said to be *interior* if its supporting line separates the two convex disks to which it is tangent, otherwise the bitangent is said to be *exterior*. A *pseudo-triangulation* is a maximal subset of non-crossing free bitangents—according to our assumptions two bitangents are disjoint or they cross at an interior point.

The boundaries of the convex disks and the bitangents in a pseudo-triangulation induce a regular cell decomposition of the plane, still called a pseudo-triangulation; see the left part of Figure 2. According to [15, 16], a bounded free face of a pseudo-triangulation is a *pseudotriangle*, that is a simply connected subset of the plane whose boundary consists of three smooth convex curves that are tangent at their endpoints, and which is contained in the triangle formed by the three endpoints of these convex curves; the convex curves are called the *sides* of the pseudotriangle, and their endpoints are called the *cusp points* of the pseudotriangle; see the right part of Figure 2. The number of pseudotriangles is $2n-2$ for n disks, and the number of bitangents is $3n-3$. A pseudo-triangulation of a collection of n disks is computable in $O(n \log n)$ time under the assumption that each disk has constant computational complexity in the sense that the tangent to a disk with a given direction, the common tangents to two disks or the common tangents to a disk and a point are computable in constant time. See [15, 16] for more details. (If we assume that the convex disks are composed of a total of m arcs, each of constant computational complexity, then a pseudo-triangulation is computable in time $O(m + n \log m)$ time.)

A pseudotriangle of a pseudo-triangulation is said to be of type I (II) if none (one) of its sides is a bitangent. The boundary of a pseudotriangle of type I contains exactly three interior bitangents, namely the bitangents at its cusp points. The bitangent side of a pseudotriangle of type II is an exterior bitangent and the boundary of a pseudotriangle of type II contains exactly one interior bitangent, namely the bitangent at the opposite cusp point of the bitangent side. Note that an exterior bitangent is the side of a unique type II pseudotriangle. The pseudo-triangulation depicted in Figure 2 has two pseudotriangles of type I. The following lemma gives a lower bound on the number of exterior bitangents in a pseudo-triangulation.

Lemma 1 *Consider a pseudo-triangulation of a collection of at least three disks. Then its number of exterior bitangents is at least 3. Furthermore if the number of exterior bitangents lying on the convex hull of the convex disks is 2 then this number is at least 4.*

Proof. Consider the graph whose nodes are the convex disks and whose edges are the pairs of convex disks connected by an interior bitangent of some pseudo-triangulation. Clearly this graph is a plane graph without multiple edges; thus its number of edges is at most $3n-6$ where n is the number of disks. Since the total number of bitangents in a pseudo-triangulation is $3n-3$ the result follows. A similar argument applies if the number of bitangents on the boundary of the convex hull is 2. □

Collections of n interior disjoint circles whose contact graphs are triangular planar graphs on n points are examples of configurations of n disks with exactly three free exterior bitangents (according to the 'Koebe-Andreev-Thurston Circle Packing Theorem' [18, page 117] or [14, pages 95–96] any triangular planar graphs on n points is realizable as the contact graph of a collection of n interior disjoint circles).

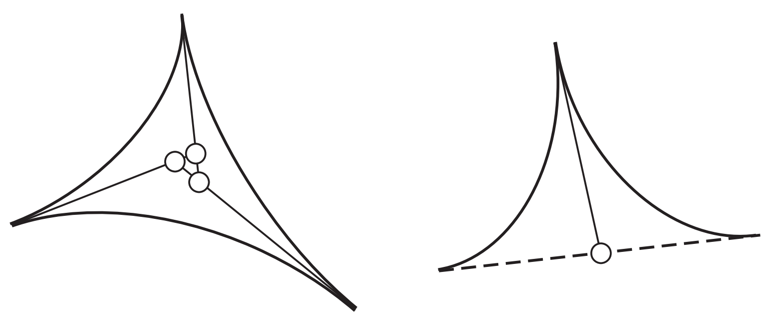

Figure 3: The edges of the screen graph inside a type I (left) or a type II (right) pseudotriangle.

3 Proof of our theorem

The idea is to generate a convex tessellation of the plane that covers and separates the convex disks. To this end we start from the geometric graph consisting of the collection of *interior* bitangents of a pseudo-triangulation and we extend its edges along their supporting lines until their endpoints hit the current extended geometric graph. The resulting geometric graph realizes a convex tessellation of the plane. However in order to generate a tessellation that covers and separates the convex disks the bitangents have to be extended in a specific order. This is done in two steps. First we extend each interior bitangent inside the (two) pseudotriangles to which it belongs, as illustrated in Figure 3. A pseudotriangle of type I is split into three curved triangles and one (straight) triangle, possibly reduced to a point. A pseudotriangle of type II is split into two curved triangles. The resulting geometric graph is unique and clearly computable in $O(n)$ time (assuming that each pseudotriangle is represented by its three cusp points). We call this graph the "screen graph", for a reason that will be clarified in a moment. Note that the vertices of the screen graph are of degree one or three. The number of vertices of degree three is three times the number of type I pseudotriangles while the number of vertices of degree one is the number of type II pseudotriangles. The second step is completely free : we just have to extend the half-edges incident to vertices of degree one until they hit the current extended geometric graph. There are several ways to realize this second step, leading to different geometric graphs. Such a graph is called an "extended screen graph"; see Figure 4 for an illustration. An extended screen graph can be computed in $O(n \log n)$ time using a straightforward adaptation of the Bentley-Ottmann sweep algorithm to compute the intersections of a collection of line segments.

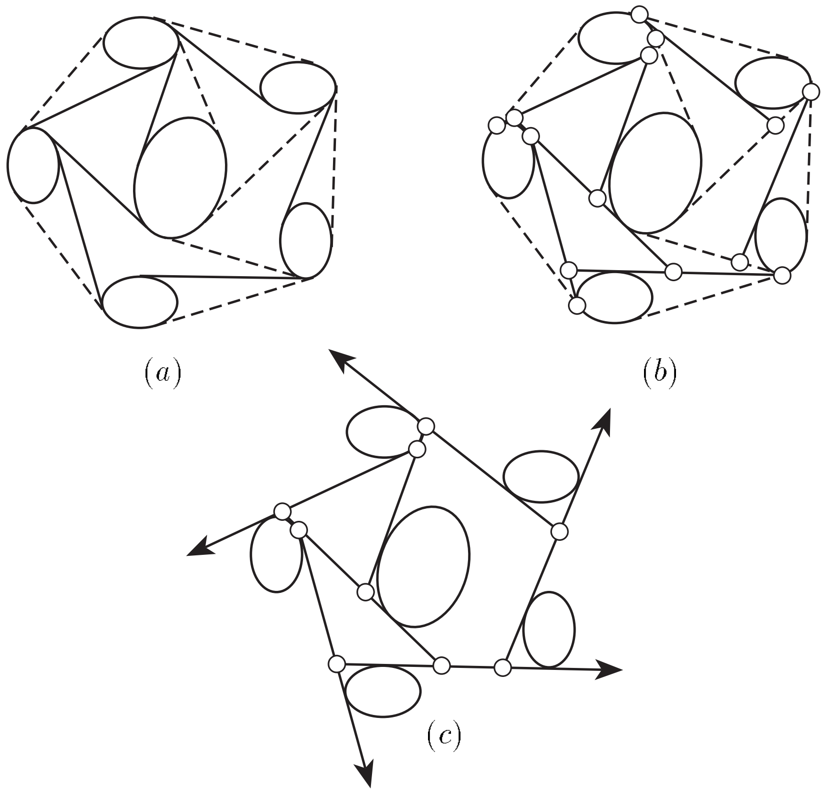

Figure 4: (a) A pseudo-triangulation, (b) its screen graph, and (c) an extended screen graph.

Theorem 2 *An extended screen graph induces a convex tessellation of the plane that covers and separates the convex disks.*

Proof. This follows from the observation that the screen graph is a *screen* with respect to the collection of convex disks, that is *it intersects any line segment joining two distinct convex disks*. Indeed, follow a ray in free space starting from a point on the boundary of some disk: if we enter a type I pseudotriangle the ray is going to hit the screen graph (and consequently the extended screen graph) thanks to the first step of the extension process; now what happens if the ray enters a pseudotriangle of type II? If we enter the pseudotriangle along one of its two non-bitangent sides the ray is going to hit the screen graph (thanks to the first step of the extension process) or is going to *leave* the pseudotriangle along its bitangent side; in the latter case we *enter* a type I pseudotriangle or a type II pseudotriangle *along one of its two non-bitangent sides* since, as observed in the previous section, an exterior bitangent determines a unique pseudotriangle of type II. It follows by induction that we can't enter a type II pseudotriangle along its bitangent side, and therefore we will eventually hit the screen graph. This proves our theorem. □

Proof of Theorem 1. In view of Theorem 2 it suffices to show that the polygonal cover induced by an extended screen graph has a worst case minimal number of sides, i.e., at most $6n-9$ sides for n disks. To count the number of sides we count the number of vertices (counting multiplicities) of the polygonal cover. Each vertex of degree three of the screen graph counts for one and each vertex of degree one of the screen graph generates a vertex of the extended screen graph that counts for 2. Therefore the number of sides is $2a+3(2n-2-a)=6n-6-a$ where a is the number of type II pseudotriangles or the number of exterior bitangents. Since a is at least three the result follows. □

4 Two related problems

4.1 Separating lines

A family of *separating lines* for a collection of disks is a collection of lines such that each pair of convex disks is separated by at least one of the lines of the family; see Figure 5. It is well known (and easy to prove) that the family of supporting lines of the sides of a polygonal cover is a family of separating lines. From the previous discussion we deduce the following theorem.

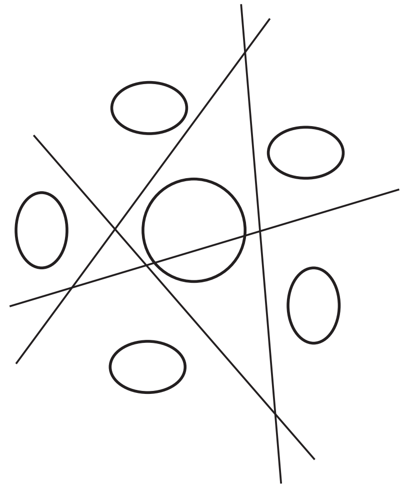

Figure 5: A family of separating lines.

Theorem 3 *The collection of supporting lines of the interior bitangents of a pseudo-triangulation of a collection of pairwise disjoint convex disks is a family of separating lines for the collection of disks.* □

4.2 L. Fejes Tóth illumination problem revisited

Partitioning theorems and algorithms are key tools to solve Art gallery problems [2, 11, 13, 17]. For example Fisk's proof [10] of 'Chvátal's Art Gallery Theorem' [3], which asserts

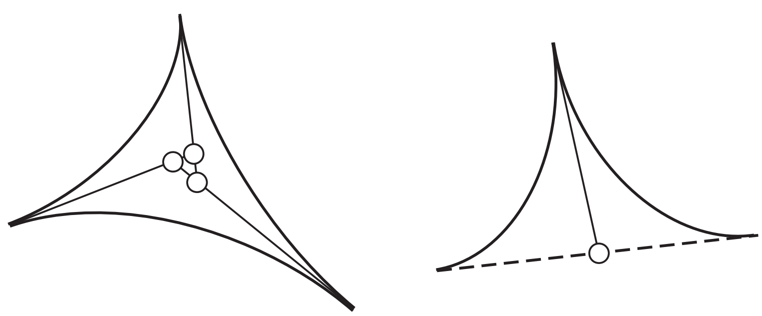

Figure 6: Two (one) lighting points are sufficient to illuminate the boundary of a type I (II) pseudotriangle.

that $\lfloor n/3 \rfloor$ lighting points are occasionally necessary and sometimes sufficient to illuminate a polygon of n vertices, can be summarized as follow: computing a lighting set of at most $\lfloor n/3 \rfloor$ lighting points for a given polygon of n vertices reduces in $O(n)$ time to computing a triangulation of the polygon. We give now a similar reduction for L. Fejes Tóth's art gallery theorem [9], which asserts that the boundary of a set of $n \geq 3$ pairwise disjoint convex disks can be illuminated by $4n-7$ points. As the reader might have suspected the suitable partition for this theorem is a pseudo-triangulation. Our result is the following.

Theorem 4 *Computing a lighting set with at most $4n-7$ lighting points for a collection of n pairwise disjoint convex disks reduces in linear time to computing a pseudo-triangulation of the collection of convex disks.*

Proof. Since the boundary points of the convex disks are the boundary points of the pseudotriangles (forget for a moment the points lying on the convex hull of the convex disks) it is sufficient to find a lighting set for the pseudotriangles of a pseudo-triangulation. How many lighting points are necessary for a pseudotriangle? Two in general (computable in $O(1)$ time) but only one if a side of the pseudotriangle reduces to a line segment, as illustrated in Figure 6. Let a be the number of pseudotriangles with a line segment side or, equivalently, the number of exterior bitangents in the pseudo-triangulation. From the above discussion a lighting set with $4n-4-a$ lighting points exists and is computable in $O(n)$ time from the pseudo-triangulation. According to Lemma 1 the number a is at least three from which the result follows (We take care to send the lighting points on the convex hull far enough to infinity in suitable directions to guarantee that they illuminate also the boundary of the convex hull). $\square$

References

[1] N. Alon, M. Katchalski, and W. R. Pulleyblank. Cutting disjoint disks by straight lines. *Discrete Comput. Geom.*, 4:239–243, 1989.

[2] I. Bjorling-Sachs and D. L. Souvaine. An efficient algorithm for guard placement in polygons with holes. *Discrete Comput. Geom.*, 13:77–109, 1995.

[3] V. Chvátal. A combinatorial theorem in plane geometry. *J. Combin. Theory Ser. B*, 18:39–41, 1975.

[4] J. Czyzowicz, H. Everett, and J-M. Robert. Separating translates in the plane: Combinatorial bounds and an algorithm. Report DIM-94-2, Université du Québec à Chicoutimi, Chicoutimi, 1994.

[5] J. Czyzowicz, E. Rivera-Campo, J. Urrutia, and J. Zaks. Separating convex sets in the plane. *Discrete Comput. Geom.*, 7:189–195, 1992.

[6] M. de Berg, H. Everett, and H. Wagener. Translation queries for sets of polygons. *Internat. J. Comput. Geom. Appl.*, 5:221–242, 1995.

[7] H. Edelsbrunner, A. D. Robison, and X. Shen. Covering convex sets with non-overlapping polygons. *Discrete Math.*, 81:153–164, 1990.

[8] L. Fejes Tóth. Some packing and covering theorems. *Acta Scientiarum Mathematicarum (Szeged)*, 12/A:62–67, 1950.

[9] L. Fejes Tóth. Illumination of convex discs. *Acta Math. Acad. Sci. Hungar.*, 29(3–4):355–360, 1977.

[10] S. Fisk. A short proof of Chvátal's watchman theorem. *J. Combin. Theory Ser. B*, 24:374, 1978.

[11] F. Hoffmann, M. Kaufmann, and K. Kriegel. The art gallery theorem for polygons with holes. In *Proc. 32nd Annu. IEEE Sympos. Found. Comput. Sci.*, pages 39–48, 1991.

[12] R. Hope and M. Katchalski. Separating plane convex sets. *Math. Scand.*, 66:44–46, 1990.

[13] J. O'Rourke. *Art Gallery Theorems and Algorithms.* Oxford University Press, New York, NY, 1987.

[14] J. Pach and P. K. Agarwal. *Combinatorial Geometry.* John Wiley & Sons, New York, NY, 1995.

[15] M. Pocchiola and G. Vegter. Topologically sweeping visibility complexes via pseudo-triangulations. *Discrete Comput. Geom.*, December 1996. To appear in the special issue devoted to ACM-SoCG'95.

[16] M. Pocchiola and G. Vegter. The visibility complex. *Internat. J. Comput. Geom. Appl.*, 6(3):279–308, September 1996.

[17] T. C. Shermer. Recent results in art galleries. *Proc. IEEE*, 80(9):1384–1399, September 1992.

[18] G. M. Ziegler. *Lectures on Polytopes*, volume 152 of *Graduate Texts in Mathematics*. Springer-Verlag, 1994.