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Clin d'Oeil on ℓ_1-embeddable Planar Graphs

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Clin d'oeil on L_1–embeddable planar graphs

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Abstract. In this note we present some properties of L_1–embeddable planar garphs. We show that every such graph G has a scale 2 embedding into a hypercube. Further, under some additional conditions we show that for a simple circuit C of G the subgraph H of G bounded by C is also L_1–embeddable. In many important cases, the length of C is the dimension of the smallest cube in which H has a scale embedding. Using these facts we establish the L_1–embeddability of a list of planar graphs.

Graphs with their shortest–path metrics are particular instances of discrete metric spaces, and may be investigated from the metric point of view. The L_1–embeddability question for metric spaces lead to a characterization of L_1–graphs [21, 7]. A particular class of L_1–graphs, possessing special features and applications [11, 19], is formed by planar L_1–embeddable graphs. It is the purpose of this note to present some properties of this class of graphs, which would be applied for testing whether a given planar graph is L_1–embeddable or not. For other results on L_1–embeddable planar graphs we refer to [1, 2, 8, 13, 14, 20].

An l_1–*metric* d on a finite set X is any positive linear combination of cut metrics

$$d = \sum_{C \in \mathcal{C}} \lambda_C \cdot \delta_C$$

[1] on leave from the Universitatea de stat din Moldova, Chişinău

where the "cut" metric δ_C associated with the cut $C = \{A, B\}$ of X is defined as follows:

$$\delta_C(x, y) = \begin{cases} 0 & \text{if } x, y \in A \text{ or } x, y \in B, \\ 1 & \text{otherwise.} \end{cases}$$

More generally, a metric space (X, d) is said to be L_1–*embeddable* if there is a measurable space $(\Omega, \mathcal{A})$, a nonnegative measure μ on it and an application λ of X into the set of measurable functions F (i.e. with $\|f\|_1 = \int_\Omega |f(w)|\mu(dw) < \infty$) such that

$$d(x, y) = \|\lambda(x) - \lambda(y)\|_1$$

for all $x, y \in X$ [11]. A well–known compactness result of [5] implies that L_1–embeddability of a metric space is equivalent to l_1–embeddability of its finite subspaces. The path–metric of the infinite rooted binary tree is perhaps the simplest L_1–embeddable metric which cannot be embedded into an l_1–space of finite dimension.

Let $G = (V, E)$ be a connected (not necessarily finite) graph endowed with the distance $d_G(u, v)$ equals the length of a shortest path joining the vertices u and v. Given two connected graphs G and H and a positive integer λ, we say that G is *scale* λ *embeddable* into H if there exists a mapping

$$\phi : V(G) \to V(H)$$

such that

$$d_H(\phi(u), \phi(v)) = \lambda d_G(u, v),$$

for all vertices $u, v \in V(G)$. In the particular case $\lambda = 1$ we obtain the usual notion of *isometric embedding* . In what follows we consider scale or isometric embeddings of graphs into hypercubes, half–cubes, cocktail–party graphs and their Cartesian products. The *half–cube* $\frac{1}{2}H_n$ is the graph whose vertex set is the collection of all vertices in one part of the bipartite representation of the n–cube H_n and two vertices are adjacent in $\frac{1}{2}H_n$ if and only if they are at distance 2 in H_n. Recall also that the *cocktail–party graph* $K_{m\times 2}$ is the complete multipartite graph with m parts, each of size 2. Both notions can be extended in an evident fashion to infinite graphs, too (it suffices to let n and m be cardinal numbers).

According to [2] the L_1–embeddable graphs are exactly those graphs which admit a scale embedding into a hypercube. Evidently, every scale 2 embeddable into a hypercube graph is an isometric subgraph of a half–cube. This analogy is much deeper: according to [21, 7] a graph G is an L_1–graph if and only if it is an isometric subgraph of the (weak) Cartesian product of half–cubes and cocktail–party graphs.

An L_1–embeddable graph G is called L_1–*rigid* [10] if it has an essentially unique L_1–representation. Every isometric subgraph of a hypercube is L_1–rigid [10]. On the other hand, as is shown in [21], every L_1–rigid graph is an isometric subgraph of a half–cube.

Some further terminology. Recall that a subset S of vertices of a graph G is *convex* if for any vertices $u, v \in S$ all vertices on shortest (u, v)–paths belong to S. If G is the Cartesian product of two graphs G_1 and G_2, then $d_G = d_{G_1} + d_{G_2}$, and any convex set S of G has the form $S_1 \times S_2$, where S_1 and S_2 are convex sets of G_1 and G_2, respectively [22]. For a vertex v of G_1 we will say that $\{v\} \times G_2$ is the *fibre* of v in $G = G_1 \times G_2$.

If G is an L_1–graph then for every cut (A, B) occuring in the L_1–decomposition of d_G both sets A and B are convex (we call such cuts *convex*). As was established in [3, 14] a graph G is scale λ embeddable into a hypercube if and only if there exists a collection $\mathcal{C}(G)$

of (not necessarily distinct) convex cuts of G, such that every edge of G is cutted by exactly λ cuts from $\mathcal{C}(G)$ (a cut (A,B) cuts an edge (u,v) if $u \in A$ and $v \in B$ or $u \in B$ and $v \in A$). For $\lambda = 1$ we obtain the well–known Djokovic characterization [16] of graphs isometrically embeddable into hypercubes. In fact, a similar characterization is valid for weighted graphs. Namely, let each edge (u,v) has a positive integer length $l(u,v)$. Define the distance between two vertices be the length of a shortest (weighted) path connecting the given pair of vertices. Assume in addition that the distance between any adjacent vertices u and v is $l(u,v)$. Then just repeating the proof from [3] we can show that the obtained metric space with integer–valued distances is scale λ embeddable into a hypercube if and only if there is a collection $\mathcal{C}$ of convex cuts, such that every edge (u,v) is cutted by exactly $\lambda l(u,v)$ cuts from $\mathcal{C}$.

For a given nonnegative integer k let T_k denotes the following metric space defined on the set $X = \{a_0, a_1, a_2, a_3, a_4, b_0, b_1, b_2, b_3, b_4\}$:

$$d(a_i, a_j) = d(b_i, b_j) = 1 \quad (i, j \in \{0,1,2,3,4\}),$$

$$d(a_0, b_i) = d(b_0, a_i) = k+1 \quad (i \in \{1,2,3,4\}), \qquad d(a_0, b_0) = k+2$$

while $d(a_i, b_i) = k$ and $d(a_i, b_j) = k+1$ if $i \neq j$. Note that T_0 is the graph $K_6 - e$ (i.e., a complete graph on 6 vertices minus an edge). Actually, $K_6 - e$ is the unique L_1–graph with at most 6 vertices having scale larger than 2.

Using the abovementioned results from [3, 14] and [7, 21] we can state the following characterization of scale 2 embeddable graphs (alias isometric subgraphs of half–cubes).

Proposition 1. *An L_1 graph G is an isometric subgraph of a half–cube if and only if it does not contain any T_k $(k \geq 0)$ as an isometric subspace. In particular, every planar L_1–graph is scale 2 embeddable into a hypercube.*

Proof. We start by showing that T_k $(k \geq 0)$ is not scale 2 embeddable into a hypercube. This can be verified in a straightforward way for $T_0 = K_6 - e$. Suppose by way of contradiction that we can select the smallest T_k which has a scale 2 embedding into a hypercube. Equivalently, there is a collection $\mathcal{C}(T_k)$ of convex cuts of T_k such that every edge (u,v) is cutted by $2l(u,v)$ cuts from this collection. Consider an arbitrary edge (a_i, b_i) $(i \in \{1,2,3,4\})$. Then

$$d(b_i, a_j) = d(b_i, a_i) + d(a_i, a_j),$$

$$d(a_i, b_j) = d(a_i, b_i) + d(b_i, b_j)$$

for any $j \in \{0,1,2,3,4\}$. The unique convex cut (A,B) with the property that $a_i \in A$ and $b_i \in B$ has the form $A = \{a_0, a_1, a_2, a_3, a_4\}$ and $B = \{b_0, b_1, b_2, b_3, b_4\}$. Since $d(a_i, b_i) = k$ the cut (A,B) is included in $\mathcal{C}(T_k)$ $2k$ times. Removing two occurences of (A,B) in $\mathcal{C}(T_k)$ we obtain a family of convex cuts which define a scale 2 embedding of T_{k-1} into a hypercube, contrary to the choice of T_k.

Conversely, assume that G is an L_1–graph which does not contain any T_k $(k \geq 0)$ as an isometric subspace. Since the Cartesian product of half–cubes is isometrically embeddable into a larger half–cube, by the result of [21, 7] we can assume that G is isometrically embeddable into a graph $\Gamma = K_{m \times 2} \times H$, where $m \geq 5$ and H is a Cartesian product of some half–cubes and cocktail–party graphs. (Recall that the cocktail–party graph $K_{4\times 2}$ coincides with $\frac{1}{2}H_4$). Moreover, in $K_{m\times 2}$ we can find a subgraph $K_{m+1} - e$, such that the fibre $\{v\} \times H$ of any vertex $v \in K_{m+1} - e$ contains at least one vertex of G. Indeed, otherwise we are in a position to reduce $K_{m\times 2}$ to a smaller cocktail–party graph. Let $K \subseteq K_{m+1}-e$ be a complete subgraph with m vertices (*m–clique* for brevity). We assert that $V(G)\cap(\cup_{v\in K}(\{v\}\times H))$ contains an m–clique sharing a common vertex with each fibre $\{v\}\times H$, where $v\in K$. Suppose the contrary, and consider a maximal clique C of $V(G)\cap(\cup_{v\in K}(\{v\}\times H))$. Assume that C does not intersect the fibre $\{w\}\times H$ of some vertex $w\in K$. In this fibre pick a vertex x of G as close as possible to C. Since the fibres of vertices of K as well as their unions are convex sets of Γ, we deduce that x is equidistant to all vertices of the clique C. By definition, x and any vertex $y'\in C\cap(\{y\}\times H)$ $(y\in K)$ can be connected in Γ by a shortest path consisting of vertices of G only. Employing the same convexity argument as above, we conclude that this path is completely contained in $(\{w\}\times H)\cup(\{y\}\times H)$. The first edge (x,y'') of this path cannot belong to $\{w\}\times H$, because of the choice of the vertex x. Therefore, y' and y'' belong to the same fibre. Again, using the convexity property of fibres we obtain that x together with the vertices y'' $(y'\in C)$ constitute a larger clique of $V(G)\cap(\cup_{v\in K}(\{v\}\times H))$, contrary to the choice of C. Hence, the clique C intersects every fibre $\{v\}\times H$ $(v\in K)$ in exactly one vertex v'.

One can extend C (in a unique way) to the cocktail–party subgraph of Γ isomorphic to $K_{m\times 2}$. Denote this extension by C^*, preserving the notation v' for the unique vertex of C^* from the fibre of $v\in K_{m\times 2}$. Let z and t be the nonadjacent vertices of $K_{m+1}-e$, and suppose that $z\notin K$. Hence $t'\in C$. In the fibre $\{z\}\times H$ select a vertex x of the graph G as close as possible to the clique C. As in the first part of the proof, one can show that x is at the same distance $d>0$ to all vertices v' of the set $C^*-\{t'\}$ (indeed, z is adjacent to any vertex v of $K_{m\times 2}$ except t, so we can use convexity of fibres). In addition, x and t' are at distance $d+1$. This is so, because any shortest path connecting them in Γ necessarily passes through a fibre of some vertex $v\in K_{m\times 2}-\{z,t\}$. Since G is an isometric subgraph of Γ, the vertex x and any $y'\in C$ can be connected inside G by a shortest path. Again, from the choice of x we conclude that the neighbour y'' of x in this path belongs to the same fibre as y'. Since the unions of fibres of the vertices from K are convex, we obtain that all vertices y'' $(y'\in C)$ are pairwise adjacent. But then the subspace generated by x, t', arbitrary 5 vertices y' from C and their corresponding vertices y'' is isomorphic to T_{d-1}. This leads us to a contradiction, because all selected vertices belong to G. □

The second assertion of Proposition 1 for finite graphs has another proof via Delaunay polytopes (for notions and results in this direction the reader can consult [9]). If a graph G is an l_1–graph, then it generates a Delaunay polytope $P(G)$, and contain an affine basis of $P(G)$. If the scale of G is larger than 1, then $P(G)$ is a Cartesian product of polytopes of half–cubes and cocktail–party graphs (the latter polytopes are the well–known cross–polytopes). An affine basis of a Cartesian product is a union of affine bases of the components with one point in common. Any affine basis of a cross–polytope of dimension n contains an $(n-1)$–dimensional simplex. The skeleton of this simplex is the complete graph K_n. Hence, if G is planar, the corresponding direct product $P(G)$ can contain cross–polytopes of dimension smaller than 5. The skeletons of such polytopes are isometric subgraphs of half–cubes.

Now, let G be a planar locally–finite (all vertices have finite degree) L_1–graph, embedded in the Euclidean plane. A *face* of G is meant an induced cycle of G which bounds a simply connected region. This immediately implies that any edge of G belongs to at most two faces. By Proposition 1 G is scale 2 embeddable into a hypercube. Let $\mathcal{C}(G)$ be a family of convex cuts of G defining such an embedding. For a cut (A,B) of $\mathcal{C}(G)$ let $E(A,B)$ be the set of edges cutted by (A,B). Evidently, removing $E(A,B)$ from G we obtain a graph with at least two connected components, i.e. $E(A,B)$ is a cutset of edges.

LEMMA 1. *For any face F of G and any cut (A,B) of $\mathcal{C}(G)$ $|E(A,B)\cap E(F)|=0$ or 2.*

PROOF. Suppose by way of contradiction that (A,B) cuts at least three edges $(a_1,b_1),(a_2,b_2)$ and (a_3,b_3) of F, where $a_1,a_2,a_3\in A$ and $b_1,b_2,b_3\in B$ (then $|E(A,B)\cap E(F)|\geq 4$, because it is an even integer). Pick arbitrary shortest paths P_1,P_2, and P_3 between the vertices a_1,a_2,a_3 and arbitrary shortest paths Q_1,Q_2, and Q_3 between the vertices b_1,b_2,b_3. Because any of these two triplets cannot cover the vertices of F, necessarily two paths from different triplets intersect. Since $P_1\cup P_2\cup P_3\subseteq A$ and $Q_1\cup Q_2\cup Q_3\subseteq B$, we obtain a contradiction with $A\cap B=\emptyset$. □

Further we assume that G is a planar graph, embedded in the Euclidean plane with the property that

(a) any face is an isometric cycle of G.

(Although natural, one can construct planar graphs which do not admit a planar embedding obeying the condition (a): for this take a book, i.e., a collection of 4–cycles sharing a common edge.)

Two edges $e'=(u',v')$ and $e''=(u'',v'')$ on a common face F are called *opposite* if $d_G(u',u'')=d_G(v',v'')$ and equal the diameter of the cycle F. If F is an even face, then any of its edges has a unique opposite, otherwise, if F has an odd length, then every edge $e\in F$ has two opposite edges e^+ and e^-. In the latter case, if F is clockwise oriented, for e we distinguish the *left opposite edge* e^+ and the *right opposite edge* e^-. If a convex cut (A,B) of G intersects the face F, then convexity of A and B yields that (A,B) cuts F in two opposite edges.

For a given cut (A,B) denote by $Z(A,B)$ the family of faces of G cutted by (A,B). With same abuse of language, one can say that $Z(A,B)$ is the *zone* of the cut (A,B). If every face of $Z(A,B)$ is intersected by (A,B) in two opposite edges, then we say that (A,B) is an *opposite cut* of G. One can easily show that in a planar graph with isometric faces all convex cuts are opposite.

If G is a planar graph with isometric faces of even length only (i.e., G is bipartite), then G is an L_1–graph if and only if every opposite cut is convex. This already present a useful way to verify if G is L_1–embeddable or not. For example, using this we obtain that the first graph presented in Figure 1 is not L_1–embeddable (this is the skeleton of the smallest convex polyhedron with an odd number of faces, all of which are quadrangles; see [17]).

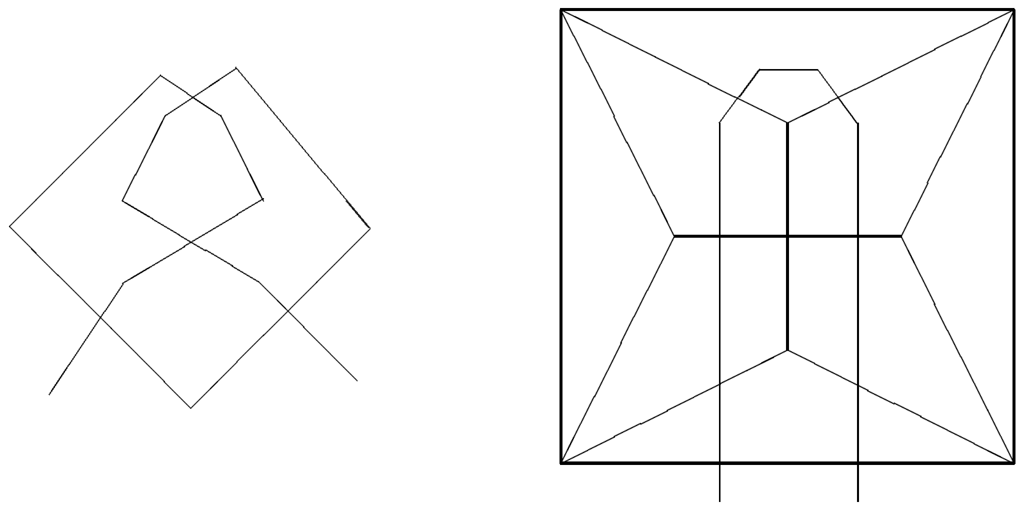


FIGURE 1.

The property that any edge of G belongs to at most two faces (due to the requirement that all faces are isometric cycles) ensures that for each convex cut (A, B) the dual graph of $Z(A, B)$ is either an induced path or a cycle. In addition to the condition (a) we will suppose that G fulfils the following condition:

(b) the faces of G meet only along common edges or vertices.

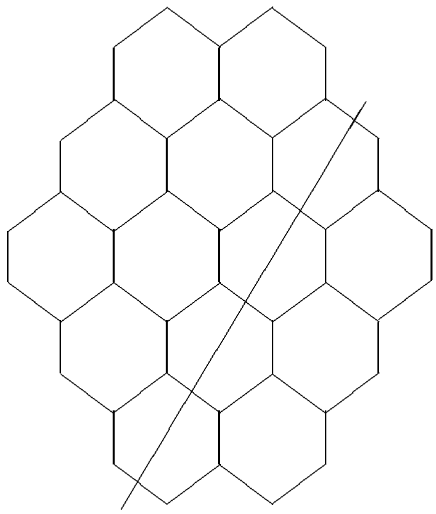
In this case the planar graph G regarded as a cell complex is a 2–dimensional pseudomanifold. The second condition as well as the definition of faces are enjoyed by skeletons of convex polyhedra.

Further we investigate the conditions under which a planar graph G satisfying the conditions (a) and (b) is an L_1–graph. We start by defining a special type of opposite cuts of G. For a given opposite cut (A, B) the edges and the faces intersected by (A, B) can be consecutively numbered in accordance with their occurance in the dual graph of $Z(A, B)$. We label the faces of the zone $Z(A, B)$ with "+" ,"0", and "-" in accordance with the following rules:

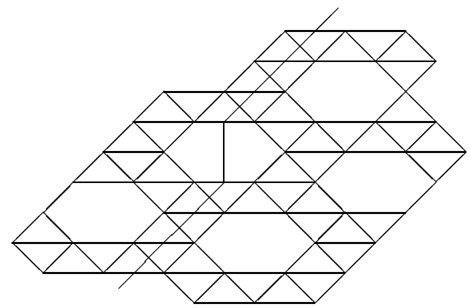
(1) if $F \in Z(A, B)$ is an even face, then we label it with "0";

(2) if $F \in Z(A, B)$ is an odd face and e', e'' are the opposite edges of F cutted by (A, B), then we label F with "+" if e'' is the left opposite edge of e', and with "-" if e'' is the right opposite edge of e' in F (we assume that e' preceeds e'' in the ordering of the edges cutted by (A, B)).

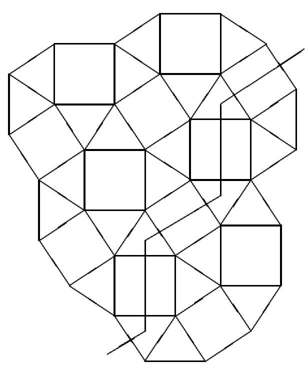
As a result the zone of each opposite cut (A, B) is represented by a sequence $\alpha(A, B)$ in the three–letter alphabet $\{+, 0, -\}$. One can say that (A, B) is an *alternating cut* of G, if after removing from $\alpha(A, B)$ of all 0's we will get an alternating sequence of "+" and "-". Evidently, every zone consisting of even faces only is an alternating cut. Consequently, if G is bipartite then the alternating cuts are exactly the opposite cuts of G. For an illustration of this concept we present a list of planar graphs, in particular of tilings (for other examples of L_1–graphs with isometric faces see [8, 13]).



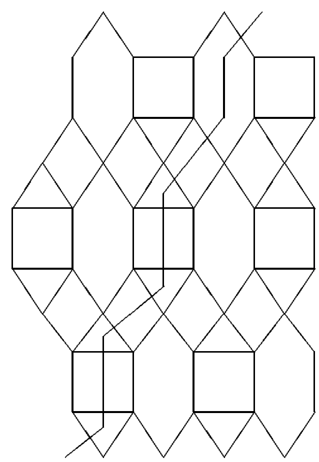
$\alpha(A, B) = (\ldots 0, 0, 0, 0 \ldots)$



$\alpha(A, B) = (\ldots - + - + - +0 - + - + - + \ldots)$



$\alpha(A, B) = (\ldots + 0 - 0 + 0 - 0 \ldots)$



$\alpha(A, B) = (\ldots + 0 - 0 + 0 - 00 \ldots)$

FIGURE 2.

Now, we present an algorithmic procedure to find the alternating cuts (if they exist) which intersect a given edge $e = (u, v)$. Namely, two subsets of edges $E'(e)$ and $E''(e)$ (not necessarily distinct) are constructed with the property that any alternating cut intersecting e cuts the graph G along the edges from $E'(e)$ or from $E''(e)$. We return also two pairs of paths (not necessarily simle) $(P'(e), Q'(e))$ and $(P''(e), Q''(e))$. To do this we proceed as follows. Initially set $E'(e) := \{e\}$ and $E''(e) := \{e\}$ and let e be the unique active edge. We go away from e in two directions (or in only one direction if e belongs to the exterior face of G) until we arrive to odd faces. Each time when we pass through an even face we continue the movement via the unique opposite edge e' to an active edge (each time we have one or two active edges). Then e' is included in both sets $E'(e)$ and $E''(e)$. Also e' becomes an active edge, while e losts the special status. Now, suppose that F' and F'' are the first odd faces which occur when moving in opposite directions. Assume that $e' \in F'$ and $e'' \in F''$ are the active edges. Then

(1) in $E'(e)$ we turn to left in F' and to right in F''.
(2) in $E''(e)$ we turn to left in F'' and to right in F'.

(Here, by a turning, say to left in F', is meant that the second edge of F' included in the corresponding set $E'(e)$ or $E''(e)$, say in $E'(e)$, will be the left opposite edge e^+ to the active edge e; in addition, e^+ becomes the active edge of $E'(e)$). After that we have only to alternate the directions when passing through odd faces of G. Namely, if say our last change of direction was to left, then comming to the next odd face we have to move to right and conversely. Each time we have to include new edges in $E'(e)$ and $E''(e)$ and to update the lists of active edges.

To derive $(P'(e), Q'(e))$ (the pair of paths $(P''(e), Q''(e))$ can be defined similarly) we have to follow the construction of $E'(e)$. Namely, let $e' = (u', v')$ and $e'' = (u'', v'')$ be consecutive edges of $E'(e)$. Then they are opposite edges of a face F. Assume without loss of generality that already $u' \in P'(e)$ and $v' \in Q'(e)$. Also suppose that the shortest paths P and Q of F between u', u'' and v', v'', respectively, are disjoint. Then add P to $P'(e)$ and Q to $Q'(e)$.

We assert that for any alternating cut (A, B) which cuts the edge e either $E(A, B) = E'(e)$ or $E(A, B) = E''(e)$ holds. Indeed, (A, B) cuts the edges from the common part of $E'(e)$ and $E''(e)$ until the faces F' and F''. In this moment, we have only two possibilities to continue the movement along $E(A, B)$, namely, (A, B) cuts the faces F' and F'' in the same fashion as $E'(e)$ or $E''(e)$, say as $E'(e)$. In this case necessarily $E(A, B)$ and $E'(e)$ coincide everywhere. Concluding, we obtain the following result.

Lemma 2. *Every edge e of a planar graph G satisfying the condition (a) and (b) is cutted by at most two alternating cuts, each of them defined by $E'(e)$ or $E''(e)$.*

Denote by $\mathcal{A}(G)$ the collection of all alternating cuts of G, where every cut (A, B) with $\alpha(A, B) = 0$ is counted twice. In general, we can construct planar graphs without alternating cuts. This is due to the fact that $E'(e)$ and $E''(e)$ do not necessarily define cutsets of G. In Figure 1 we present two examples of alternating "pseudo–cuts" constructed by our procedure, which are not cuts. As a consequence, they are not L_1–graphs. The second graph is taken from [4] and is a skeleton of a space–filler. The skeletons of many others space–fillers listed in this paper represent L_1–graphs. However, if all $E'(e)$ and $E''(e)$ $(e \in E(G))$ are cutsets, then Lemma 2 infers that the family of alternating cuts $\mathcal{A}(G)$ is rather complete: every edge of G is cutted by exactly two cuts from $\mathcal{A}(G)$. Unfortunately, only this property together with (a) and (b) do not imply L_1–embeddability of a planar graph G, because alternating cuts can be non–convex. To ensure the L_1–embeddability of G we have to impose some metric conditions on the pairs of paths $(P'(e), Q'(e))$ and $(P''(e), Q''(e))$ constructed by our procedure (fortunately, these natural requirements are easily verified in many important particular cases). By a *geodesic* is meant a (possibly infinite in one or two directions) simple path P with the property that $d_P(x, y) = d_G(x, y)$ for any $x, y \in P$.

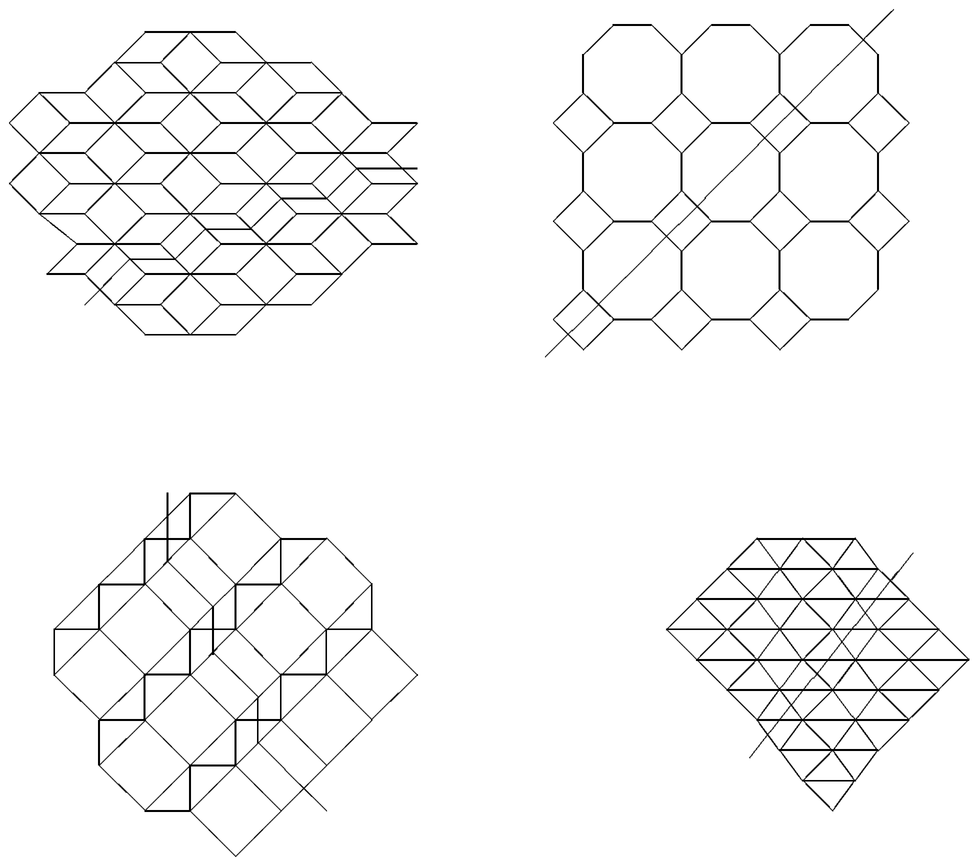


FIGURE 3.

PROPOSITION 2. *Let G be a planar graph satisfying the condition (b). If for each edge e the sets $P'(e), Q'(e), P''(e)$, and $Q''(e)$ constitute isometric cycles or geodesics, then G is an L_1–graph.*

PROOF. Since $P'(e), Q'(e), P''(e)$, and $Q''(e)$ are isometric subgraphs of G, we immediately conclude that $E'(e)$ and $E''(e)$ are cutsets of G. Moreover, from the same condition we obtain that all faces of G are isometric cycles. As we already showed the corresponding cuts (A', B') and (A'', B'') are alternating. By Lemma 2 the edge e is cutted only by these alternating cuts. To complete the proof we have to show that the alternating cuts are convex. Assume the contrary, and say the set A' is not convex: then we can find two vertices $x, y \in A'$ and a shortest path R between x and y such that $R \cap B' \neq \emptyset$. We can suppose without loss of generality that among the vertices of A' violating the convexity condition the vertices x and y are choosen as close as possible. Let x' and y' be the neighbours in R of x and y, respectively. Then x', y' and all vertices of R between them belong to the set B'. In particular, the edges (x, x') and (y, y') are cutted by (A', B'). This implies that $x, y \in P'$ and $x', y' \in Q'$. Since P' and Q' are isometric subgraphs of G, $d_{P'}(x, y) = d_G(x, y)$ and $d_{Q'}(x', y') = d_G(x', y')$. Since (A', B') is an alternating cut of G, one can easily conclude that

$$|d_{P'}(x, y) - d_{Q'}(x', y')| \leq 1.$$

This contradicts our supposition that x' and y' lie on the common shortest path R connecting x and y. □

To apply this result we have to construct the alternating cuts of a graph G, and to verify if all $P'(e), P''(e), Q'(e)$, and $Q''(e)$ are isometric cycles or geodesics. For example, if we consider $K_{2,3}$ with the vertices x_1, x_2, y_1, y_2, y_3, then for the alternating cut $\{x_1, y_2\}, \{y_1, x_2, y_3\}$ we have $P' = P'' = (y_1, x_2, y_3)$ and $Q' = Q'' = (x_1, y_2, x_1)$. The second path is not a geodesic (it is not even simple), so we cannot apply Proposition 2.

From Proposition 2 one can easily deduce the L_1–embeddability of many nice planar graphs, in particular tilings (some of them were already presented before in Figures 2–3); in all these cases the sets $P'(e), Q'(e), P''(e)$, and $Q''(e)$ represent geodesics. We continue by establishing the L_1–embeddability of still another class of planar graphs. Recall that a (finite) planar graph G is *outerplanar* if there is an embedding of G in the Euclidean plane such that all vertices of G belong to the exterior face.

PROPOSITION 3. *Any outerplanar graph G is L_1–embeddable.*

PROOF. Indeed, G enjoys the conditions (a) and (b). In fact, every interior face of G is convex. In addition, for each edge e the sets $P'(e), Q'(e), P''(e)$, and $Q''(e)$ cannot be cycles. If one of them, say $P'(e)$ is not a geodesic, then we can find a shortest path L between two vertices u and v of $P'(e)$, such that $L \cap P'(e) = \{u, v\}$. First suppose that L is disjoint from $P''(e)$. But then at least one of the vertices of $P'(e)$ or $P''(e)$ belongs to the interior of the region bounded by L and the second such path, contradicting that G is outerplanar. Otherwise, if L shares a vertex with $P''(e)$, then one can deduce that L consists of two edges $(u, x), (v, y)$ with $x, y \in P''(e)$ and the portion of $P''(e)$ between x and y. By the algorithmic construction of alternating cuts we conclude that the length of L must be larger tnan that of $P'(e)$, contrary to our assumption. Therefore, we are in a position to apply Proposition 2. □

Perhaps the most known class of planar graphs verifying the conditions (a) and (b) is that of *triangulations* (i.e., planar graphs in which all interior faces have length three) and their duals (i.e., *cubic planar graphs*) . It has been established in [3] that any finite (planar) triangulation with the property that all vertices which do not belong to the exterior face have degree larger than 5 is L_1–embeddable. Moreover, all such graphs are L_1–rigid. In fact, the latter holds true for all triangulations without K_4 as an induced subgraph.

PROPOSITION 4. *Any L_1–embeddable triangulation without K_4 as an induced subgraph is L_1–rigid.*

The proof is a consequence of the following result and Lemma 2.

LEMMA 3. *If a triangulation G does not contain K_4 as an induced subgraph, then any convex cut of G is alternating.*

PROOF. Pick a convex cut (A, B) of G and consider two faces F_1 and F_2 of $Z(A, B)$ sharing a common edge (u, v). Suppose that $u \in A$ and $v \in B$. Let x and y be the vertices of F_1 and F_2, respectively, distinct from u and v. Evidently, $d(x, y) = 2$, otherwise we get a forbidden K_4. Since A and B are convex and both u and v belong to shortest paths connecting the vertices x and y, we deduce that x and y must be separated by (A,B). This shows that F_1 and F_2 will be labeled in $\alpha(A,B)$ by different signs "+" and "-". Therefore, (A,B) is an alternating cut. $\Box$

Replace every edge $e=(u,v)$ of a cubic planar graph G by two arcs $e'=(u,v)$ and $e''=(v,u)$. Denote the resulting oriented graph by Γ. A simle circuit C of Γ is said *alternating* if every face of G is either disjoint or shares with C exactly two consecutive arcs. The graph H dual to G is a planar triangulation. Every alternating circuit of Γ corresponds to an alternating cut of H, and, conversely, any convex alternating cut of H defines an alternating circuit of Γ. Threfore, we obtain the following property of G :

If the dual of a finite cubic planar graph G is L_1–embeddable, then there is a family of alternating circuits of Γ such that any arc of Γ is covered by exactly one circuit.

A *benzenoid system* (alias *hexagonal system*) is a planar graph in which every (interior) face is bounded by a regular hexagon of side length 1. Equivalently, a benzenoid system is a subgraph of the hexagonal grid which is bounded by a simle circuit of this grid. That the benzenoid systems are L_1–graphs (namely, they are isometrically embeddable into hypercubes) was established in [19]. Moreover, it has been shown how to apply this embedding to compute the Wiener index of a benzenoid system G. Recall that the *Wiener index* $W(G)$ (often used in mathematical chemistry) of G is the sum of distances $d_G(u,v)$ taken over all pairs of vertices u,v of G. Our final purpose is to extend these results to much larger classes of planar graphs. For a finite L_1–graph G let size(G) denotes min $\frac{n}{\lambda}$ taken over all scale embeddings of G into a hypercube (here λ is the scale, while n is the dimension of the host hypercube).

Proposition 5. *Let H be a planar graph satisfying the condition (b), and such that the sets $P'(e), Q'(e), P''(e)$, and $Q''(e)$ are geodesics for all edges $e \in E(H)$. Let G be a subgraph of H bounded by a simple (nondegenerated) cycle C of length p of H. Then*

(1) G endowed with its own metric d_G is an L_1–graph;

(2) size$(G)=p/2$;

(3) if $\mathcal{A}(G)$ is the collection of alternating cuts of G then

$$W(G)=\frac{1}{2}\sum_{(A,B)\in\mathcal{A}(G)}|A|\cdot|B|.$$

Proof. We show how to derive $\mathcal{A}(G)$ from $\mathcal{A}(H)$. Pick an alternating cut (A,B). From Lemma 2 and Proposition 2 we know that (A,B) is defined by a cutset $E'(e), e\in E(G)$. Let $Z_1,\ldots,Z_p$ be the connected components of the zone $Z(A,B)$. Each of them is the zone of a cut from G. Denote the resulting cuts by $(A_1,B_1),\ldots,(A_p,B_p)$, so that $Z_1=Z(A_1,B_1),\ldots,Z_p=Z(A_p,B_p)$. By the definition, each of these cuts is an alternating cut of G. Suppose without loss of generality that (A_i,B_i) is defined by the cutset $E'(e_i)$ of G, where e_i is an arbitrary edge cutted by (A_i,B_i). Since $P'(e)$ and $Q'(e)$ are geodesics of H and P_i' and Q_i' are subpaths of P' and Q', respectively, we conclude that both P'_i and Q'_i are geodesics of G (note that $d_G(u,v) \geq d_H(u,v)$ for any vertices u, v of G.) Thus, we are in a position to apply Proposition 2. This shows that G is an L_1–graph.

To prove (2) first note that every alternating cut of G starts and ends with edges which lie on C. If G is bipartite, then the alternating and opposite cuts coincide and G is isometrically embeddable into a hypercube of dimension $p/2$. Therefore, in this case size$(G) = p/2$. Otherwise, if G has an odd face, then G is scale 2 embeddable into a hypercube. Then every edge of C takes part in two alternating cuts (not necessarily distinct). This means that G has a scale 2 embedding into a hypercube of dimension p. This implies that again size$(G) = p/2$.

To establish (3) we have to rewrite the expression for $W(G)$, taking into account that $\mathcal{A}(G)$ defines a scale 2 embedding of G into a hypercube:

$$W(G) = \frac{1}{2} \sum_{u \in V} \sum_{v \in V} d_G(u,v) =$$

$$= \frac{1}{2} \sum_{u \in V} \sum_{v \in B} |(A,B) \in \mathcal{A}(G)| = \frac{1}{2} \sum_{(A,B) \in \mathcal{A}(G)} |A||B|.$$

Actually, this proof of (3) inspired by [19] and [6] can be extended to all L_1–graphs. $\square$

In Figure 4 we present few examples of graphs verifying the conditions of Proposition 5.

The assertion (1) of Proposition 5 does not hold for all planar L_1–graphs H. For example, let H be the prism $C_6 \times K_2$ embedded in the Euclidean plane, and suppose that G is obtained from H by deleting a boundary vertex. Then G is not an L_1–graph, however it is obtained from H using the operation from Proposition 5. It would be interesting to investigate the planar L_1–graphs which verify the hereditary property described in Proposition 5(1).

An operation in some sense inverse to the previous one is that of gluing planar L_1–graphs along common (isometric) faces. Again, it does not preserve L_1–embeddability, so, the question is to find under which conditions the resulting planar graph is L_1–embeddable, too.

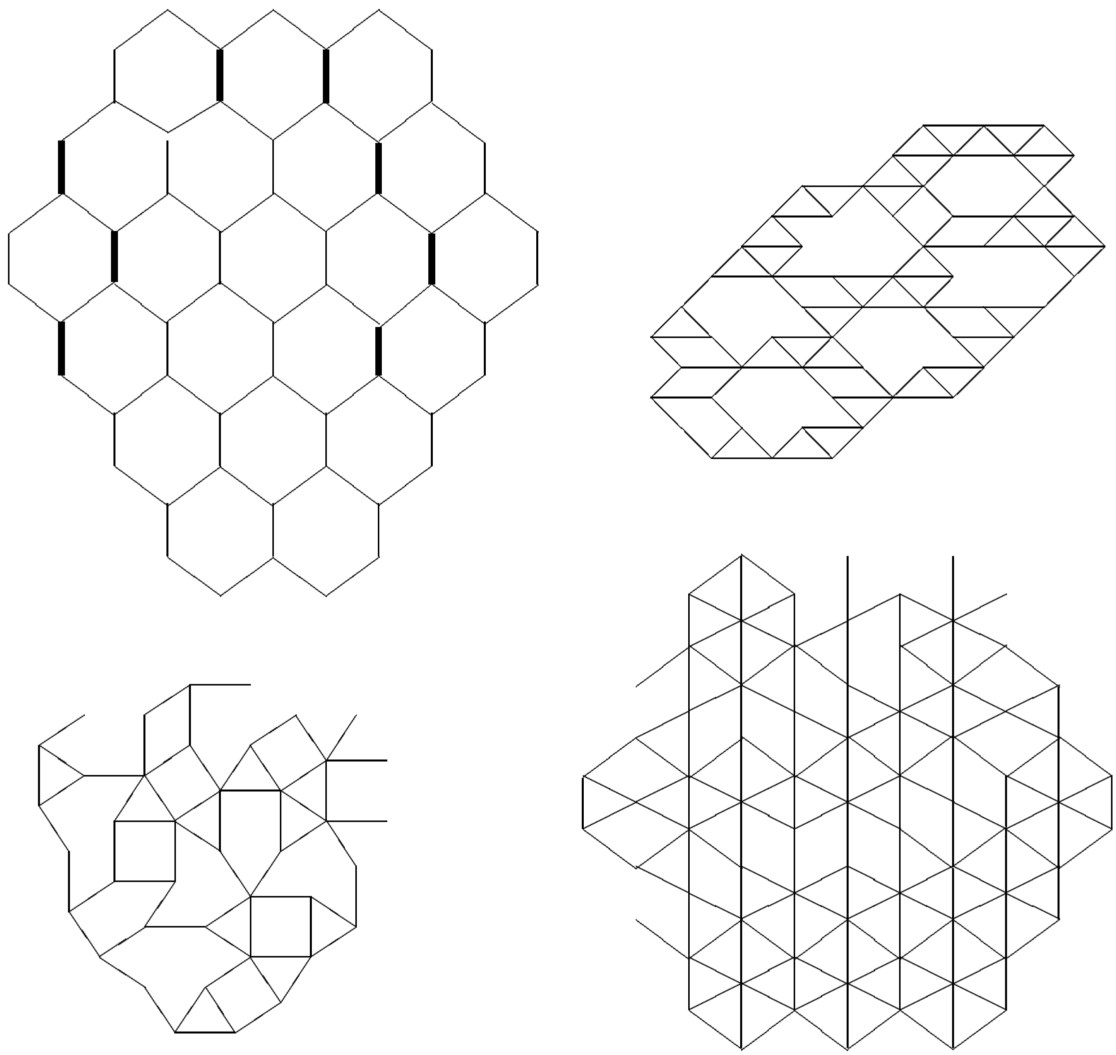


FIGURE 4.

A particular instance of this gluing operation is that of *capping* of a planar graph G (it corresponds to gluing a planar graph and a wheel): add a new vertex inside a given face and connect this vertex to all vertices of this face. An *omnicapping* of G is capping of all faces of G. When capping preserves L_1–embeddability? We know only that all partial cappings of skeletons of regular polyhedra are L_1–graphs, except the cube. Capping one, two, or three pairwise non–opposite faces of H_3 results into L_1–graphs; all other cappings give non–L_1–graphs.

Nowadays the chemical graph theory present the richest source of planar graphs. Using our approach one can establish L_1–embeddability of many chemical graphs. Call a *corona* $Cor(p,q)$ (p and q are positive integers, $p \geq 4$) the graph defined in the following way: $Cor(p,1)$ is the cycle of length p. Then $Cor(p,q)$ is obtained by surrounding $Cor(p,q-1)$ with a ring of p-cycles. From Proposition 2 we obtain that all coronas are L_1–graphs. We wish to conclude our note with two examples of chemical L_1–graphs of size 5 and 10 taken from [15, 18].

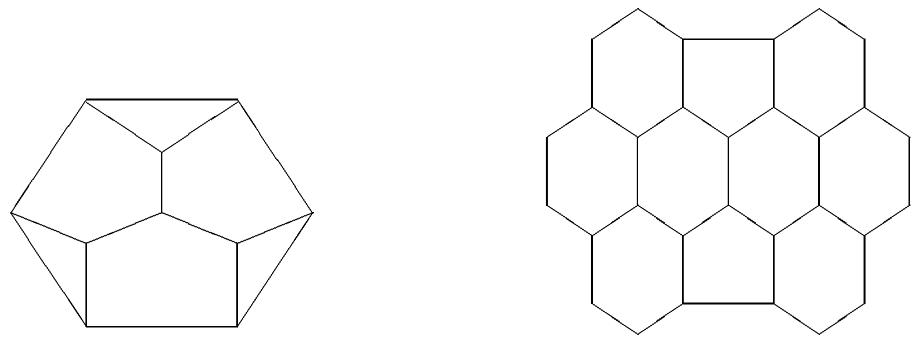


FIGURE 5.

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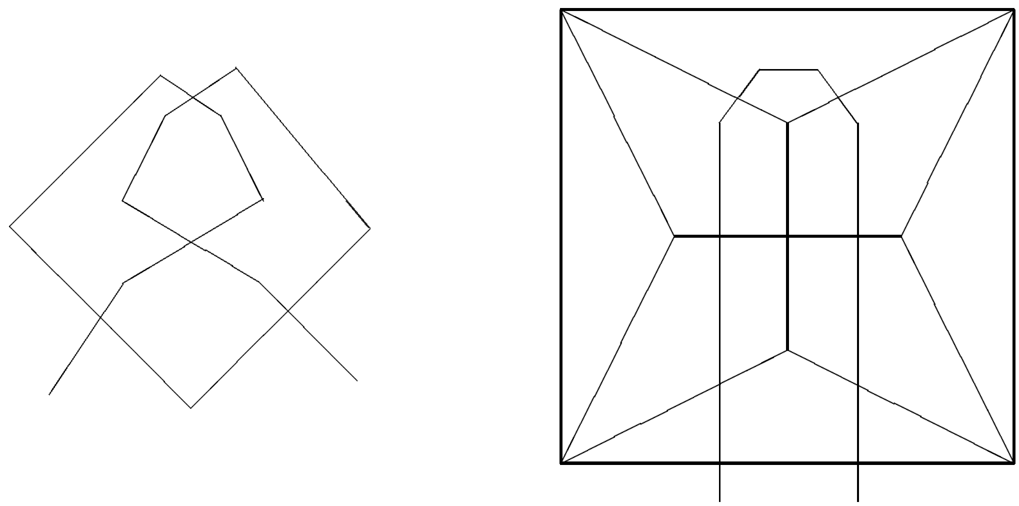

Figure 1.

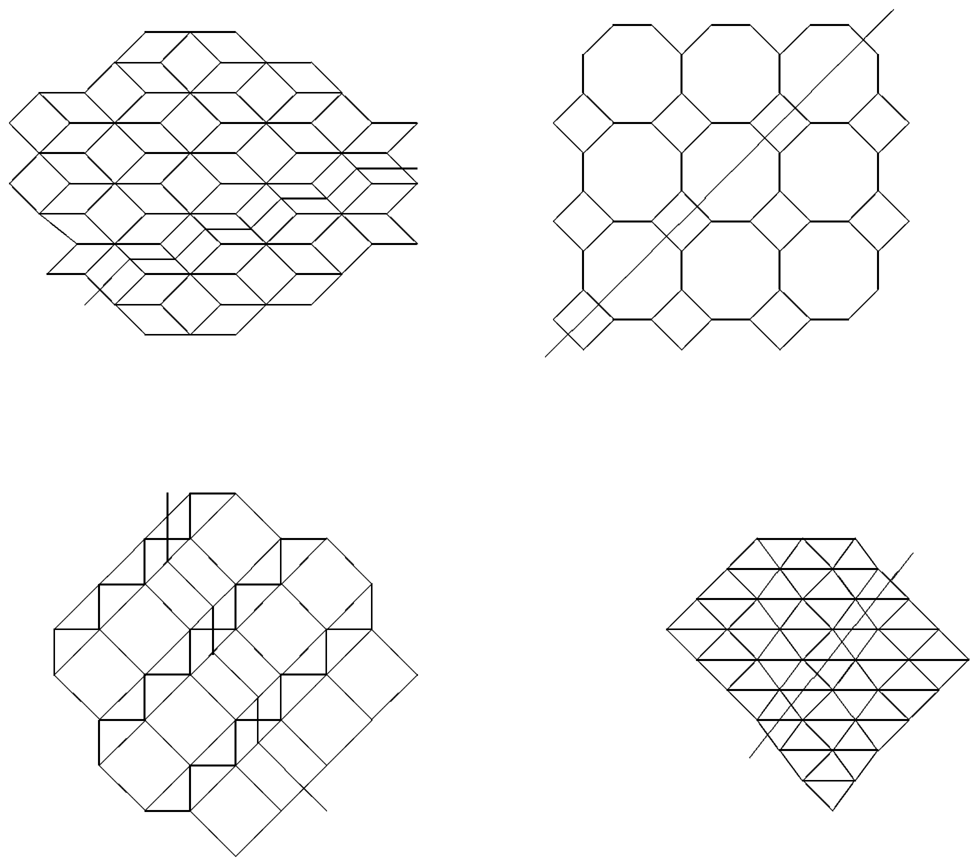

Figure 3.

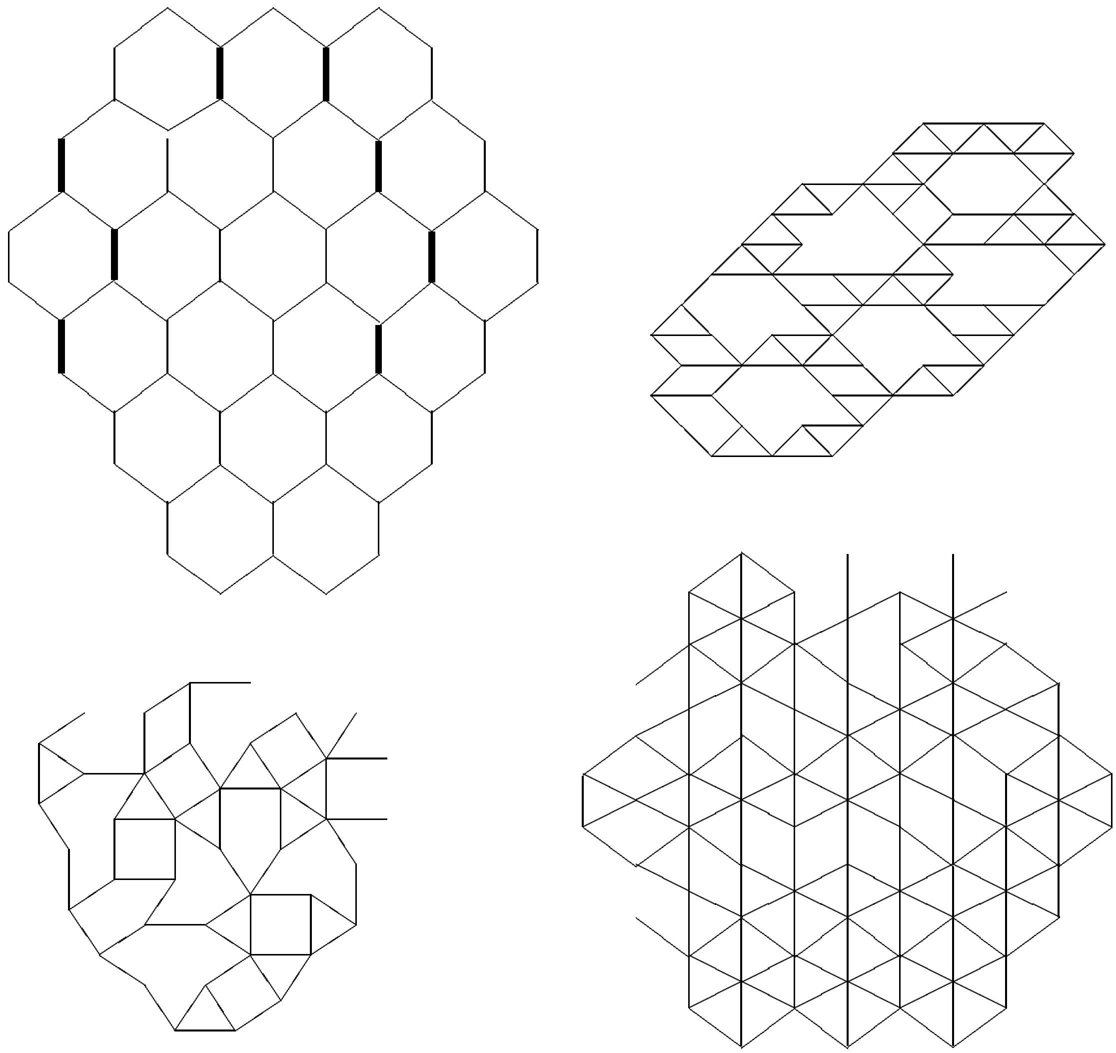

FIGURE 4.

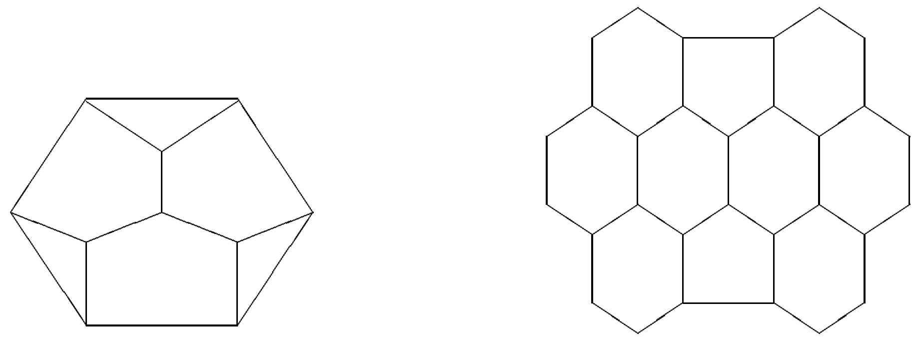

Figure 5.