ECOLE NORMALE SUPERIEURE

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Neil GHANI

LIENS - 96 - 10

Département de Mathématiques et Informatique

CNRS URA 1327

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June 1996

Laboratoire d'Informatique de l'Ecole Normale Supérieure
45 rue d'Ulm 75230 PARIS Cedex 05

Tel : (33)(1) 44 32 00 00
Adresse électronique : ... @dmi.ens.fr

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Neil Ghani
LIENS-DMI, Ecole Normale Superieure
45, Rue D'Ulm, 75230 Paris Cedex 05
France
e-mail:ghani@dmi.ens.fr

May 17, 1996

Abstract

The use of expansionary η-rewrite rules in various typed λ-calculi has become increasingly common in recent years as their advantages over contractive η-rewrite rules have become apparent. Not only does one obtain simultaneously the decidability of $\beta\eta$-equality and also a procedure for the calculation of the long $\beta\eta$-normal form of a term, but rewrite relations using expansions generalise more easily to other type constructors, retain key properties when combined with other rewrite rules, and are supported by a categorical theory of reduction.

In this paper we prove strong normalisation and confluence for a rewrite relation on the terms of System F consisting of traditional β-reductions and η-expansions satisfying certain restrictions. In addition, we obtain a natural characterisation of the second order long $\beta\eta$-normal forms as precisely the normal forms of our restricted rewrite relation.

These results are an important step towards a new theory of reduction in type theories which mix algebraic rewrite rules with $\beta\eta$-equality.

1 Introduction

Extensional equality for terms of the simply typed λ-calculus requires η-conversion, whose interpretation as a rewrite rule has traditionally been as a contraction:

$$\lambda x : T.fx \ \Rightarrow f \qquad\qquad (x \not\in \mathtt{FV}(f)) \tag{1}$$

When combined with the usual β-reduction, the resulting rewrite relation is strongly normalising and confluent, and thus reduction to normal form provides a decision procedure for the associated equational theory.

However η-contractions behave badly when combined with rewrite rules arising from either algebraic rewrite systems or from other type constructors. For instance, the presence of the unit type with η-rewrite rule $t \Rightarrow *$ leads to a loss of confluence [12]. Specifically if f is a variable of type $1 \to 1$ then the following divergence cannot be completed.

$$\lambda x\!:\!1.\, * \Leftarrow \lambda x\!:\!1.fx \;\Rightarrow f \tag{2}$$

Although the combination of a confluent first order rewrite system and a type theory (eg the simply typed λ-calculus, System F) equipped with β-reduction is known to be confluent [], these results cannot be generalised to type theories equipped with η-contraction as confluence is invariably lost. For example, consider a single sort 1 with constants $f : 1 \to 1, * : 1$ and rewrite rule $fx \Rightarrow *$. This relation is confluent, but when taken together with η-contractions confluence is lost as equation 2 demonstrates — for a detailed discussion the reader should consult [3].

These deficiencies in η-contractions have recently led several authors [1, 4, 6, 12] to reconsider the old proposal [11, 13, 14] that η-conversion be interpreted as an expansion

$$f \;\Rightarrow \lambda x\!:\!T.fx \qquad\qquad \text{if } f : T \to T' \text{ and } x \not\in \mathtt{FV}(f)$$

and the resulting rewrite relation has been shown confluent. In these works infinite reduction sequences such as

$$f \;\Rightarrow \lambda x\!:\!T.fx \;\Rightarrow \lambda x\!:\!T.(\lambda y\!:\!T.fy)x \;\Rightarrow \;\ldots$$

are avoided by imposing syntactic restrictions to limit the possibilities for expansion; namely λ-abstractions cannot be expanded, nor can terms which are applied. This restricted expansion relation is strongly normalising, confluent and generates the same equational theory as the unrestricted expansionary rewrite relation. Thus $\beta\eta$-equality can be decided by reduction to normal form in this restricted fragment and, in addition, the normal forms of this restricted rewrite relation are exactly Huet's *long $\beta\eta$-normal forms* [11, 14]. More importantly these properties are maintained when rewrite rules for other type constructors are added, e.g. the rewrite rule $t \Rightarrow *$ for the unit type mentioned above, and when ... **Cite references to Bobby's results**.

This work has been extended to the more difficult problem of providing a decision procedure for $\beta\eta$-equality for negative type constructors such as the coproduct and the tensor of linear logic [7, 8]. The proposed η-rewrite rules for these types are substantially more complex than those for the product and exponentials — not only is there a facility for expanding terms of sum type analogous to that for the product and exponential, but also the ability to permute the order in which different subterms of sum type are eliminated. The reader should consult the above references for further details.

This paper extends the initial results in a different direction by investigating the use of expansionary η-rewrite rules in a polymorphic λ-calculus called System F [10, 9]. This calculus was introduced by Girard over twenty years ago and may be thought of as the simply typed λ-calculus enriched with type variables and a mechanism for forming Π-types by universally quantifying over all other types. Elements of these Π-types are thought of as polymorphic functions and there are introduction and elimination rules which describe how polymorphic functions may be defined and how such polymorphic functions may be used to construct other functions. After presenting System F, we define an equational theory called *$\beta\eta$-equality* on the terms of System F by adding second order β- and η-equations to their usual first order counterparts. Next, the restrictions on the applicability of the first order η-expansions are generalised to the second order η-expansions and we obtain a rewrite relation which we prove to be strongly normalising, confluent, and to have as its reflexive, symmetric and transitive closure $\beta\eta$-equality.

Our proof of strong normalisation of the restricted rewrite relation is a cross between the traditional proof of strong normalisation for the fragment containing only β-reductions [10], and the proof of strong normalisation for the simply typed λ-calculus with expansionary η-rewrite rules [12, 8]. This requires several alterations to the traditional definition a *reducibility candidate* to cope with the presence of expansions. This reducibility candidate method differs from the modular approach of [5] which investigates rewriting in System F but does not consider the second order η-rewrite rule. The reader is encouraged to consult [2] where a similar approach to rewriting in System F to that of this paper is taken. In summary, we prove that η-expansions are robust enough to be applied to System F and regard this as a crucial first step towards a new theory of reduction in type theories which mix algebraic rewrite rules with $\beta\eta$-equality.

2 System F

The formulation of System F presented here is based on that found in [10] and takes advantage of the relatively simple type structure of System F to avoid a context based presentation of the calculus. Of course there are still a few technical details about which we must be careful, but the overall simplification of notation is considerable.

Let $\mathtt{Var}^*$ be an infinite set of *type variables*. The *types* of System F are defined as by the grammar

$$T := X \mid T{\rightarrow}T \mid \Pi X.T$$

where $X \in \mathtt{Var}^*$. The set of all types of System F is denoted $\Lambda(*)$ and a type is called *atomic* iff it is a member of $\mathtt{Var}^*$. We use $T, U, V, ..$ to range over types and $X, Y, Z, ...$ to range over atomic types. The set of free type variables occurring in a type T is denoted $\mathtt{FTV}(T)$ and α-equivalent types are treated as being equal. A type valued substitution is a finite partial function $\theta : \mathtt{Var}^* \rightarrow \Lambda(*)$ and the result of applying such a substitution to a type T is defined as expected and denoted $T\theta$.

There is also an infinite set of *term variables*, $\mathtt{Var}$, which is disjoint from $\mathtt{Var}^*$. These term variables are used to construct the *pre-terms* of System F as follows:

$$t := x^T \mid tt \mid \lambda x^T.t \mid tT \mid \Lambda X.t$$

where x is a term variable, T is a type and X is a type variable. The set of pre-terms of System F is denoted Λ and we use $t, u, v, ..$ to range over pre-terms and $x, y, z, ..$ to range over term variables. The following definitions are used throughout the paper:

- A pre-term is an *introduction* pre-term iff it is of the form $\lambda x^T.t$ or $\Lambda X.t$.
- A pre-term is *neutral* if it is not an introduction term.
- A sub-preterm is said to occur *negatively* in a term iff it is either applied to another pre-term or applied to a type.
- In the term x^T, the type T is called the *type annotation* of the variable x

The free term variables of a pre-term t are denoted $\mathtt{FV}(t)$ and contain their respective type annotations, i.e.

$$\begin{aligned}
\mathtt{FV}(x^T) &= \{x^T\} \\
\mathtt{FV}(\lambda x^T.t') &= \{y^U \in \mathtt{FV}(t') \mid x \neq y\} \\
\mathtt{FV}(tu) = \mathtt{FV}(t) \cup \mathtt{FV}(u) & \\
\mathtt{FV}(tT) = \mathtt{FV}(t) & \\
\mathtt{FV}(\Lambda X.t) = \mathtt{FV}(t) &
\end{aligned}$$

The free type variables of a term are denoted $\mathtt{FTV}(t)$ and, as before, terms differing only in bound variables are treated as being equal.**footnote about annotations**. A term-valued substitution is a finite, partial function $\theta : \mathtt{Var} \rightarrow \Lambda$ and the result of applying a type or term-valued substitution θ to a pre-term t is denoted $t\theta$. Two pre-terms t and u are said to be *compatible*, written $t \approx u$ iff whenever $x^T \in \mathtt{FV}(T)$ and $x^U \in \mathtt{FV}(u)$, then $T = U$. This compatibility relation is the price we pay for avoiding a context-based presentation of System F.

The *typing judgements* of System F are of the form $t : T$, where t is a pre-term and T is a type, and are derived by the inference rules in Table 1. A pre-term t is said to be a *term* iff there is a type T and a typing judgement $t : T$, and in this case we shall say that t has type T. The set of terms which have type T is denoted $\Lambda(T)$. Terms satisfy special properties which allow us to simplify their notation further.

Table 1: Typing Judgements for System F

$$\frac{x \in \mathtt{Var}}{x^T : T}$$

$$\frac{t : T' \quad x^U \in \mathtt{FV}(t) \text{ implies } T = U}{\lambda x^T.t : T \to T'} \qquad \frac{t : T \to T' \quad u : T \quad t \approx u}{tu : T'}$$

$$\frac{t : \Pi X.T \quad U \text{ is a type}}{tU : T[U/X]} \qquad \frac{t : T \quad x^U \in \mathtt{FV}(t) \text{ implies } X \notin \mathtt{FTV}(U)}{\Lambda X.t : \Pi X.T}$$

Lemma 2.1 *The following are true*

- *If there is a typing judgement $t : T$, then $t \approx t$.*
- *If there is a typing judgement $t : T$ and $x^T \in \mathtt{FV}(t)$, then $\mathtt{FTV}(T) \subseteq \mathtt{FTV}(t)$*
- *If there are typing judgements $t : T$ and $t : T'$, then $T = T'$.*

Proof The proofs are by induction on the typing judgements. □

Thus if t is a term then all free occurences of a term variable x in t have the same type annotation and henceforth sometimes omit mention of the type annotation of a free variable. Lemma **??** also allows us to define a function # which maps a term to its unique type. A term-valued substitution is *compatible* with a term t iff whenever $x^T \in \mathtt{FV}(t)$ and $\theta(x)$ is defined, then $\#\theta(x) = T$ and $\theta(x) \approx \lambda x^T.t$.
difference between simultaneous and sequential substitution

Lemma 2.2 *The following are true*

- *If θ is a type-valued substitution, and there is a judgement $t : T$, then there is also a judgement $t\theta : T\theta$*
- *If θ is a term valued substitution, there is a term judgement $t : T$ and θ is compatible with t, then there is also a typing judgement $t\theta : T$.*

Proof Induction on the typing judgement $t : T$ □

Later on we shall permit the η-expansion of a term providing the term inhabits a function type or a universally quantified type. Thus type inhabitation will be crucial in enumerating the reducts of a term. Fortunately in our Church style formulation of System F typeability and type inhabitation are decidable.

Lemma 2.3 *Given a pre-term t it is decidable whether there exists a type T such that there is a typing judgement $t : T$. In addition, given a pre-term t and a type T it is decidable whether there is a typing judgement $t : T$.*
Proof Both parts of the lemma are proved by induction on the structure of t.
□

3 $\beta\eta$-equality System F

Equality is usually defined in System F as the least congruence on terms containing the following pair of basic equations

$$\begin{array}{lrcl} \beta_{\rightarrow} & (\lambda x^T.t)u & = & t[u/x] \\ \beta_{\Pi} & (\Lambda X.t)V & = & t[V/X] \end{array} \tag{3}$$

The rewrite relation obtained by orienting these equations from left to right is denoted $\Rightarrow_\beta$ and is well-known to be both strongly normalising, confluent [10] and hence β-equality is decidable in System F. However this is a rather minimal equational theory and in the rest of this paper we consider the effect of adding first and second order η-equations. The equational theory known as *$\beta\eta$-equality* is the least congruence on the terms of System F including the equations of Table 2. We have define $\beta\eta$-equality on the set of all terms. The next lemma shows

Table 2: $\beta\eta$-Equality in System F

$$\begin{array}{llcll} \beta_{\rightarrow} & (\lambda x^T.t)u & =_{\beta\eta} & t[u/x] & \\ \eta_{\rightarrow} & t & =_{\beta\eta} & \lambda x^T.tx^T & x \notin \mathtt{FV}(t) \text{ and } t \in \Lambda(T \rightarrow T') \\ \beta_{\Pi} & (\Lambda X.t)V & =_{\beta\eta} & t[V/X] & \\ \eta_{\Pi} & t & =_{\beta\eta} & (\Lambda X.tX) & X \notin \mathtt{FTV}(t) \text{ and } t \in \Lambda(\Pi X.T) \end{array}$$

that $\beta\eta$-equality is actually a family of equational theories indexed by the types of System F.

Lemma 3.1 *Assume $t : T$ and $u : U$ are terms such that $t =_{\beta\eta} u$. Then $T = U$.*
Proof Induction I guess? $\square$

Bearing in mind the discussion in the introduction concerning the difficulties of combining η-contractions with algebraic rewrite systems, we shall investigate $\beta\eta$-equality in System F using β-contractions and η-expansions of the form

$$\frac{t : T \rightarrow T' \quad x \not\in \mathtt{FV}(t)}{t \Rightarrow \lambda x^T.tx} \qquad \frac{t : \Pi X.T \quad X \not\in \mathtt{FV}(t)}{t \Rightarrow \Lambda X.tX}$$

As with the first order η-rewrite rule, unrestricted use of second order η-expansions permits infinite reduction sequences — in particular there are the following reduction loops:

$$\begin{array}{rcccl} \lambda x^T.t & \Rightarrow & \lambda y^T.(\lambda x^T.t)y^T & \Rightarrow & \lambda y^T.t[y^T/x] \equiv \lambda x^T.t \\ tu & \Rightarrow & (\lambda x^T.tx^T)u & \Rightarrow & tu \end{array} \tag{4}$$

and the second order equivalent

$$\begin{array}{rcccl} \Lambda X.t & \Rightarrow & \Lambda Y.(\Lambda X.t)Y & \Rightarrow & \Lambda Y.t[Y/X] \equiv \Lambda X.t \\ tU & \Rightarrow & (\Lambda X.tX)U & \Rightarrow & tU \end{array} \tag{5}$$

Table 3: The Restricted Rewrite Relation

$$\frac{t \Rightarrow_{\beta} t'}{t \Rightarrow_{\mathcal{I}} t'} \quad \frac{t \text{ is expandable}}{t \Rightarrow_{\mathcal{F}} \eta(t)} \quad \frac{t \Rightarrow_{\mathcal{I}} t'}{t \Rightarrow_{\mathcal{F}} t'}$$

$$\frac{t \Rightarrow_{\mathcal{I}} t'}{tV \Rightarrow_{\mathcal{I}} t'V} \quad \frac{t \Rightarrow_{\mathcal{F}} t'}{\Lambda X.t \Rightarrow_{\mathcal{I}} \Lambda X.t'}$$

$$\frac{t \Rightarrow_{\mathcal{I}} t'}{tu\psi \Rightarrow_{\mathcal{I}} t'u} \qquad \frac{u \Rightarrow_{\mathcal{F}} u'}{tu \Rightarrow_{\mathcal{I}} tu'} \qquad \frac{t \Rightarrow_{\mathcal{F}} t'}{\lambda x^T.t \Rightarrow_{\mathcal{I}} \lambda x^T.t'}$$

We follow [7, 8] in defining a rewrite relation $\Rightarrow_{\mathcal{F}}$ by placing restrictions on when these expansions are permitted. These restrictions are the analogues of those for the first order fragment previously studied, i.e. λ-abstractions and Λ-abstractions may not be η-expanded and nor may terms which are applied to other terms or to types. The relation $\Rightarrow_{\mathcal{F}}$ is proven to be strongly normalising, to have as normal forms the secong oreder long $\beta\eta$-normal forms, to be confluent and to have as its reflexive, symmetric and transitive closure $\beta\eta$-equality. Thus $\beta\eta$-equality may be decided by reducing terms to their $\Rightarrow_{\mathcal{F}}$-normal forms or equivalently their long $\beta\eta$-normal form.

The context sensitive restrictions on expansion are enforced by simultaneously defining a further subrelation $\Rightarrow_{\mathcal{I}}$ of $\Rightarrow_{\mathcal{F}}$ which is guaranteed not to include top-level expansions, i.e. rewrites of the form $t \Rightarrow \lambda x^T.tx^T$ and $t \Rightarrow \Lambda X.tX$. Thus a negatively occurring subterm may be safely $\Rightarrow_{\mathcal{I}}$-rewritten without the risk of creating reduction loops as in equations 4 and 5. Define a function mapping terms to terms

$$\eta(t) = \begin{cases} t & \text{if } t \text{ is of atomic type} \\ \lambda x^T.tx^T & \text{if } t : T \to T', \quad x \not\in \mathtt{FV}(t) \\ \Lambda X.tX & \text{if } t : \Pi X.T, \quad X \not\in \mathtt{FTV}(t) \end{cases}$$

A term is *expandable* iff it is neutral and of non-atomic type. The inference rules in Table 3 simultaneously define a relation $\Rightarrow_{\mathcal{F}}$, called the *restricted rewrite relation*, and another relation $\Rightarrow_{\mathcal{I}}$ which is the 'internal' counterpart of $\Rightarrow_{\mathcal{F}}$. Note that by lemma 2.3, the $\Rightarrow_{\mathcal{I}}$ and $\Rightarrow_{\mathcal{F}}$-reducts of a term are enumerable.

Lemma 3.2 *The restricted rewrite relation and its internal counterpart are related as follows:*

$$t \Rightarrow_{\mathcal{F}} t' \textit{ iff } t \Rightarrow_{\mathcal{I}} t' \textit{ or } t' = \eta(t) \textit{ and } t \textit{ is expandable}$$

In addition the least equivalence relation containing $\Rightarrow_{\mathcal{F}}$ is precisely $\beta\eta$-equality.

Proof The first part of the lemma is proved by induction on the rewrite in question. For the second part of the lemma, note that $\Rightarrow_{\mathcal{F}}$ is clearly a subrelation of $\Rightarrow$. In addition, if $t \Rightarrow t'$ but $t \not\Rightarrow_{\mathcal{F}} t'$, then t' must be obtained from t by a looping expansion as in equations 4 and 5. As these equations show, in such circumstances there is a β-, and hence an $\Rightarrow_{\mathcal{F}}$-rewrite $t' \Rightarrow_{\mathcal{F}} t$. Thus the expansionary rewrite relation and its restricted subrelation generate the same equational theory. $\square$

When proving strong normalisation we shall require the following lemmas. Note that, although usually trivial, the fact that $\Rightarrow_{\mathcal{F}}$ is not a congruence means that a proof is required.

Lemma 3.3 *Let tx^T and uX be terms. If $x^T \not\in \mathtt{FV}(t)$ and tx^T is $\Rightarrow_{\mathcal{F}}$-strongly normalising, then t is also $\Rightarrow_{\mathcal{F}}$-strongly normalising. Similarly, if $X \not\in \mathtt{FTV}(t)$ and uX is $\Rightarrow_{\mathcal{F}}$-strongly normalising, then u is also $\Rightarrow_{\mathcal{F}}$-strongly normalising.*

Proof We prove by induction on the normalisation rank of tx^T that all the one-step reducts of t are $\Rightarrow_{\mathcal{F}}$-strongly normalising. This implies that t is $\Rightarrow_{\mathcal{F}}$-strongly normalising. The one-step reducts are

- $t \Rightarrow_{\mathcal{F}} \lambda x^T.tx^T$. The term $\lambda x^T.tx^T$ is $\Rightarrow_{\mathcal{F}}$-strongly normalising because all reduction sequences of this term are induced by reduction sequences of the term tx^T which by assumption is $\Rightarrow_{\mathcal{F}}$-strongly normalising.
- $t \Rightarrow_{\mathcal{I}} t'$. In this case there is a reduction $tx^T \Rightarrow_{\mathcal{I}} t'x^T$ and so $t'x^T$ is $\Rightarrow_{\mathcal{F}}$-strongly normalising. By induction, this means that t' is $\Rightarrow_{\mathcal{F}}$-strongly normalising.

A similar argument holds for the second half of the lemma. $\square$

The restrictions imposed on the applicability of η-expansions prevent the triangular expansions **state explicitly and define this term** appearing in equations **??** and **??**. However it is possible to create "fake" triangular expansions as the following reduction sequence shows:

$$(\lambda x^{T \to U}.x)(\lambda y^T.y) \Rightarrow_{\mathcal{I}} (\lambda x^{T \to U}.\eta(x))(\lambda y^T.y) \Rightarrow_{\mathcal{I}} \eta(\lambda y^T.y)$$

and

$$(\lambda x^{T \to U}.xy)z \Rightarrow_{\mathcal{I}} (\lambda x^{T \to U}.xy)\eta(z) \Rightarrow_{\mathcal{I}} \eta(z)y$$

Even though we seem to have "smuggled in" an expansion of a λ-abstraction or a negatively occuring subterm, this does not introduce infinite reduction sequences as each of these fake triangular expansions has required a β-reduction. Formalising this idea amounts to an analysis of the interaction between substitution and η-expansion. A relation is *substitutive* if whenever there are reductions $t \Rightarrow t'$ and $u \Rightarrow u'$ then there is also a reduction sequence $t[u/x] \Rightarrow^* t'[u'/x]$. Of course the rewrite relation $\Rightarrow_{\mathcal{F}}$ is not a congruence and so $\Rightarrow_{\mathcal{F}}$-reduction is not substitutive. The next lemma characterises when substitutivity fails and, in these instances, exhibits alternative reduction sequences which suffice for our needs:

Lemma 3.4 *Let V be a type and t, t', u, u' be terms such that $t \Rightarrow_{\mathcal{R}} t'$ and $u \Rightarrow_{\mathcal{R}} u'$, where $\mathcal{R} \in \{\mathcal{I}, \mathcal{F}\}$. Then*

- *There is a rewrite $t[u/x] \Rightarrow_{\mathcal{R}} t'[u/x]$ unless u is an introduction term and t' is obtained by expanding an occurrence of x in t. In this case there are reduction sequences $t[\eta(u)/x] \Rightarrow^*_\beta t'[u/x] \Rightarrow^*_\beta t[u/x]$.*
- *There is a rewrite $t[u/x] \Rightarrow^*_{\mathcal{I}} t[u'/x]$ unless $u' = \eta(u)$ and either $t = x$ or there are negative occurrences of x in t . In this case $t[u'/x]$ and $t[u/x]$ have a common $\Rightarrow^*_{\mathcal{I}}$-reduct.*
- *There is a rewrite $t[V/X] \Rightarrow_{\mathcal{R}} t'[V/X]$.*

Proof The three parts of the lemma are proved separately by induction on the rewrite in question. The first part follows because if u is an introduction term, then $\eta(u) \Rightarrow^*_\beta u$, while the reduct mentioned in the second part of the lemma is constructed from $t[u/x]$ by expanding those instances of u in $t[u/x]$ which do not occur negatively. The final part of the lemma holds as type substitutions preserve the neutrality of a term and also the non-atomicity of the type of the term. $\square$

The obvious next step would be to hypothesise that both $\Rightarrow_{\mathcal{I}}$ and $\Rightarrow_{\mathcal{F}}$ are locally confluent. Unfortunately this is not the case, e.g. there are the following counterexamples:

$$
\begin{tikzcd}
(\lambda x^T.t)u \arrow[r, "\mathcal{I}"] \arrow[d, "\mathcal{I}"'] & (\lambda x^T.\eta(t))u \arrow[d, "\mathcal{I}"] \\
t[u/x] \arrow[r, "\mathcal{F}"'] & \eta(t)[u/x]
\end{tikzcd}
\qquad
\begin{tikzcd}
(\Lambda X.t)V \arrow[r, "\mathcal{I}"] \arrow[d, "\mathcal{I}"'] & (\Lambda X.\eta(t))V \arrow[d, "\mathcal{I}"] \\
t[V/X] \arrow[r, "\mathcal{F}"'] & \eta(t)[V/X]
\end{tikzcd}
$$

In these examples the bottom arrow is $\Rightarrow^*_{\mathcal{F}}$, but not $\Rightarrow^*_{\mathcal{I}}$, and so $\Rightarrow_{\mathcal{I}}$ is not locally confluent. However local confluence of $\Rightarrow_{\mathcal{F}}$ can be proved in conjunction with a slight variant for $\Rightarrow_{\mathcal{I}}$.

Lemma 3.5 *The relation $\Rightarrow_{\mathcal{F}}$ is locally confluent and given any divergence $t \Rightarrow_{\mathcal{I}} t_i$ (where $i = 1, 2$), there is a term t' such that $t_1 \Rightarrow^*_{\mathcal{I}} t'$ or $t_1 \Rightarrow_{\mathcal{F}} t'$ and similarly for t_2.*

Proof The proof is by simultaneous induction on the term t, with the tricky cases handled by lemma 3.4. $\square$

4 A proof of Strong Normalisation for $\Rightarrow_{\mathcal{F}}$

Our proof of strong normalisation of the relation $\Rightarrow_{\mathcal{F}}$ is a cross between the traditional proof of strong normalisation for $\Rightarrow_\beta$ [10], and the proof of strong normalisation for the simply typed λ-calculus with expansionary η-rewrite rules [12, 8]. Thus, for every type we shall define a predicate, called a *reducibility candidate*, on sets of terms of that type. The set of $\Rightarrow_{\mathcal{F}}$-strongly normalising terms of an atomic type form a reducibility candidate of that type and are used to construct canonical reducibility candidates of higher types. We prove that these canonical reducibility candidates contain all terms of that type and, as a corollary, conclude that all terms are $\Rightarrow_{\mathcal{F}}$-strongly normalising.

As with the first order case, the definition of reducibility candidate must be altered from that used to prove strong normalisation of $\Rightarrow_{\beta}$-reduction so as to cope with the presence of expansionary rewrites. These alterations come in two parts: (i) the predicate (CR3) is weakened so that the η-expansion of a neutral term need not be considered and; (ii) a new predicate is introduced to ensure that reducibility candidates are closed under η-expansion. Formally, a *reducibility candidate* of type U is a set P of terms of type U which satisfy the following four reducibility predicates:

CR1 If $t \in P$ then t is $\Rightarrow_{\mathcal{F}}$-strongly normalising.

CR2 If $t \in P$ and $t \Rightarrow_{\mathcal{I}} t'$ then $t' \in P$.

CR3 If t is a neutral term and all $\Rightarrow_{\mathcal{I}}$-reducts of t are members of P, then $t \in P$.

CR4 If $t \in P$ then $\eta(t) \in P$.

The set of reducibility candidates of type U is denoted $\mathtt{RC}(U)$ and we also define $\mathtt{RC}$ to be the set of all reducibility candidates. Let $|\_| : \mathtt{RC} \to \Lambda(*)$ be the function which maps a reducibility candidate to the unique type of the terms which belong to it. If $t \in \mathcal{S}$ for some reducibility candidate $\mathcal{S}$, then the term t is called $\mathcal{S}$-reducible — when $\mathcal{S}$ is clear from the context we simply say t is reducible. Note that the alteration to the traditional reducibility predicate (CR3) and the use of a new predicate (CR4) are exactly as for the first order fragment. Define

$$\mathtt{SN}(T) = \{t \in \Lambda(T) \mid \ t \text{ is } \Rightarrow_{\mathcal{F}}\text{-strongly normalising } \}$$

Lemma 4.1 *If X is an atomic type, then $\mathtt{SN}(X) \in \mathtt{RC}(X)$, while if $\mathcal{S} \in \mathtt{RC}(T)$ and x a term variable, then $x^T \in \mathcal{S}$.*

Proof We must establish that the set of terms $\mathtt{SN}(X)$ satisfies the four reducibility predicates. CR1 is a tautology, while if t is $\Rightarrow_{\mathcal{F}}$-strongly normalising and $t \Rightarrow_{\mathcal{I}} t'$ then, because by lemma 3.2 $\Rightarrow_{\mathcal{I}}$ is a subrelation of $\Rightarrow_{\mathcal{F}}$, t' is also $\Rightarrow_{\mathcal{F}}$-strongly normalising. CR3 holds because all $\Rightarrow_{\mathcal{I}}$-reducts of a term t are $\Rightarrow_{\mathcal{F}}$-strongly normalising by assumption and, because t is a term of atomic type, t has no η-expansion and so no other reducts. Finally, CR4 also holds because terms of atomic type have no η-expansion.

The second part of the lemma follows from the reducibility predicate CR3 because variables are neutral terms and have no $\Rightarrow_{\mathcal{I}}$-reducts. Hence any reducibility candidate must contain all variables of that type. $\square$

By lemma 4.1 we know that for atomic types X, $\mathtt{RC}(X) \neq \emptyset$. These reducibility candidates are now used to construct reducibility candidates of higher type.

4.1 Exponentials

Let $\mathcal{R} \in \mathtt{RC}(U)$ and $\mathcal{S} \in \mathtt{RC}(V)$ be reducibility candidates. Define the set of terms

$$\mathcal{R} \to \mathcal{S} = \{t \in \Lambda(U \to V) \mid \forall u \in \mathcal{R}. \text{ if } t \approx u \text{ then } tu \in \mathcal{S}\}$$

Before proving that $\mathcal{R} \to \mathcal{S}$ is a reducibility candidate, we give an alternate characterisation of which λ-abstractions are members of $R \to \mathcal{S}$:

Lemma 4.2 *Let $\mathcal{R} \in \mathtt{RC}(U)$ and $\mathcal{S} \in \mathtt{RC}(V)$ be reducibility candidates. If for all $u \in \mathcal{R}$, the term $t[u/x] \in \mathcal{S}$, then $\lambda x^U.t \in (\mathcal{R} \to \mathcal{S})$.*

Proof First note that by lemma 4.1 if x is a term variable, then $x^U \in \mathcal{R}$ and hence by assumption $t[x/x] \in \mathcal{S}$. Thus by CR2 for $\mathcal{S}$, t is $\Rightarrow_{\mathcal{F}}$-strongly normalising. We must prove that if $u \in \mathcal{R}$, then $(\lambda x^U.t)u \in \mathcal{S}$ and, by CR3 and the above remark, this may be done by using induction on the sum of the $\Rightarrow_{\mathcal{F}}$-normalisation ranks of t and u to show that all $\Rightarrow_{\mathcal{I}}$-reducts of $(\lambda x^U.t)u$ are reducible. The one-step $\Rightarrow_{\mathcal{I}}$-reducts of $(\lambda x^U.t)u$ induced by $\Rightarrow_{\mathcal{F}}$-rewrites of u are $\mathcal{S}$-reducible by induction, while given a rewrite $t \Rightarrow_{\mathcal{F}} t'$ and any $\mathcal{R}$-reducible term v, by lemma 3.4 there is a reduction of at least one of the following forms

$$t[v/x] \Rightarrow_{\mathcal{F}}^* t'[v/x] \text{ or } t[\eta(v)/x] \Rightarrow_{\mathcal{F}}^* t'[v/x]$$

Thus $t'[v/x] \in \mathcal{S}$ and so $\lambda x^U.t'$ satisfies the induction hypothesis. Thus, by induction, $(\lambda x^U.t')u$ is $\mathcal{S}$-reducible and, as the only other $\Rightarrow_{\mathcal{I}}$-reduct is $t[u/x]$ which is reducible by assumption, the lemma is proved. $\square$

We can now prove that $\mathcal{R} \to \mathcal{S}$ is a reducibility candidate.

Lemma 4.3 *If $\mathcal{R} \in \mathtt{RC}(U)$ and $\mathcal{S} \in \mathtt{RC}(V)$, then the set of terms $R \to \mathcal{S}$ is a reducibility candidate.*

Proof We shall establish the four properties.

CR1 By lemma 4.1, if x is a term variable, then $x^U \in \mathcal{R}$. Thus if $t \in \mathcal{R} \to \mathcal{S}$, then $tx^U \in \mathcal{S}$ and so tx^U is $\Rightarrow_{\mathcal{F}}$-strongly normalising. Thus t is $\Rightarrow_{\mathcal{F}}$-strongly normalising — the proof, which can be found in [8, 12], is delicate as not every reduction sequence from t lifts to one from tx^U.

CR2 Let $t \in \mathcal{R} \to \mathcal{S}$ and $t \Rightarrow_{\mathcal{I}} t'$. Then for any $u \in \mathcal{R}$, $tu \in \mathcal{S}$ and $tu \Rightarrow_{\mathcal{I}} t'u$. Thus $t'u \in \mathcal{S}$ and hence $t' \in \mathcal{R} \to \mathcal{S}$.

CR3 Induction on the $\Rightarrow_{\mathcal{F}}$-normalisation rank of u is used to prove that if $u \in \mathcal{R}$ then $tu \in \mathcal{S}$. Because t is neutral, the one-step $\Rightarrow_{\mathcal{I}}$-reducts of tu are either of the form $t'u$ where $t \Rightarrow_{\mathcal{I}} t'$ or tu' where $u \Rightarrow_{\mathcal{F}} u'$. The first class of terms are $\mathcal{S}$-reducible because by assumption $t' \in \mathcal{R} \to \mathcal{S}$, while terms of the latter form are $\mathcal{S}$-reducible by induction.

CR4 Let $t \in (\mathcal{R} \to \mathcal{S})$. By lemma 4.2 we must prove that if $u \in \mathcal{R}$ then $(tx^U)[u/x] = tu$ is $\mathcal{S}$-reducible. But this is exactly the definition of $\mathcal{R} \to \mathcal{S}$.

$\square$

4.2 Universally Quantified types

Let T be a type. A *reducibility parameter* for T consists of a partial function $\theta : \mathtt{Var}^* \to \mathtt{RC}$ such that the free type variables of T are contained in the domain of θ, denoted $\mathbf{dom}(\theta)$. Every reducibility has an underlying type-valued substitution $|\theta| : \mathtt{Var}^* \to \Lambda(*)$ which maps a variable $X \in \mathbf{dom}(\theta)$ to the type underling the reducibility candidate $\theta(X)$. Given a reducibility parameter θ for T, define the set of terms $T\theta$ as (i) if $T = X$ then $T\theta = \theta(X)$, (ii) if $T = U \to V$ then $T\theta = U\theta \to V\theta$, (iii) if $T = \Pi Y.W$ then

$$T\theta = \bigcap_{V \in \Lambda(*)\ \mathcal{S} \in \mathtt{RC}(V)} \{t \in T|\theta| \mid tV \in W\theta[Y \mapsto \mathcal{S}]\}$$

where $\theta[Y \mapsto \mathcal{S}]$ is the function θ, except that Y is mapped to $\mathcal{S}$. Before showing that $T\theta$ is a reducibility candidate, we prove the analogue of lemma 4.2 for universal types.

Lemma 4.4 *Let θ be a reducibility parameter for $\Pi Y.W$ such that for every reducibility candidate $\mathcal{S} \in \mathtt{RC}(V)$, $W\theta[Y \mapsto \mathcal{S}]$ is a reducibility candidate and $w[V/Y] \in W\theta[Y \mapsto \mathcal{S}]$. Then $\Lambda Y.w \in (\Pi Y.W)\theta$.*
Proof We have to show that $(\Lambda Y.w)V \in W\theta[Y \mapsto \mathcal{S}]$ for every type V and reducibility candidate $\mathcal{S}$. By lemma 4.1, $\mathtt{SN}(Y) \in \mathtt{RC}(Y)$ and so $w \in W\theta[Y \mapsto \mathtt{SN}(Y)]$. Thus w is $\Rightarrow_{\mathcal{F}}$-strongly normalising and the lemma may be proved using the reducibility predicate CR3 and by induction on the $\Rightarrow_{\mathcal{F}}$-normalisation rank of w. The one-step $\Rightarrow_{\mathcal{I}}$-reducts of $(\Lambda Y.w)V$ are $w[V/Y]$ and terms of the form $(\Lambda Y.w')V$ where $w \Rightarrow_{\mathcal{F}} w'$. The first term is reducible by assumption while by lemma 3.4 w' satisfies the induction hypothesis and so $\Lambda Y.w'$ is a member of $(\Pi Y.W)\theta$. Hence $\Lambda Y.w \in (\Pi Y.W)\theta$. $\square$

Lemma 4.5 *If θ is a reducibility parameter for T, then $T\theta \in \mathtt{RC}(T|\theta|)$.*
Proof The proof is by induction on the type T. If T is a type variable the lemma is trivial, while if T is an exponential the lemma follows by induction and lemma 4.3. The only case left is where $T = \Pi Y.W$.

CR1 Let $t \in T\theta$. As $\mathtt{SN}(Y) \in \mathtt{RC}(Y)$, $tY \in W\theta[Y \mapsto \mathtt{SN}(Y)]$. Hence tY is $\Rightarrow_{\mathcal{F}}$-strongly normalising and thus so is t.

CR2 If $t \in T\theta$ and $t \Rightarrow_{\mathcal{I}} t'$, then for all reducibility candidates $\mathcal{S} \in \mathtt{RC}(V)$, $tV \Rightarrow_{\mathcal{I}} t'V$ and so, by CR2, $t'V \in W\theta[Y \mapsto \mathcal{S}]$.

CR3 Let t be a neutral term, all of whose $\Rightarrow_{\mathcal{I}}$-reducts are $T\theta$-reducible. The one step $\Rightarrow_{\mathcal{I}}$-reducts of tV are of the form $t'V$ where $t \Rightarrow_{\mathcal{I}} t'$ and so, for any reducibility candidate $\mathcal{S} \in \mathtt{RC}(V)$, $t'V \in W\theta[Y \mapsto \mathcal{S}]$. Thus tV is also $W\theta[Y \mapsto \mathcal{S}]$-reducible and hence $t \in T\theta$.

CR4 Let t be a member of $(\Pi Y.W)\theta$. By lemma 4.4, if we can prove that for all reducibility candidates $\mathcal{S} \in \mathtt{RC}(V)$, $(tY)[V/Y] = tY$ is $W\theta[Y \mapsto \mathcal{S}]$-reducible, then $(\Lambda Y.tY) \in (\Pi Y.W)\theta$. But this follows as we assume $t \in (\Pi Y.W)\theta$.

$\square$

In proving strong normalisation we shall need the following lemma relating the these constructions of reducibility candidates from reducibility parameters to type-valued substitutions.

Lemma 4.6 *Let θ be a reducibility parameter for T and V. Then the reducibility candidates*

$$(T[V/Y])\theta \quad \textit{and} \quad T\theta[Y \mapsto V\theta]$$

are equal.
Proof The proof is by induction on the type T. $\square$

Before proving that all terms are strongly normalising we find alternate criteria for proving a term is a member of a given reducibility candidate.

Lemma 4.7 *If $t \in (\Pi Y.W)\theta$, then if θ is a reducibility parameter for V, $tV \in (W[V/Y])\theta$.*
Proof By hypothesis $tV \in W\theta[Y \mapsto \mathcal{S}]$ for every reducibility candidate $\mathcal{S} \in \mathtt{RC}(V)$. The lemma follows by taking $\mathcal{S}$ to be $V\theta$ and using lemma 4.6. $\square$

Strong normalisation is a corollary to proving that the substitution of reducible terms into a term produces a reducible term. As there are two types of variable quantified over in System F this substitution must be defined on two levels.

Theorem 4.8 *Let t be a term of type T. Suppose the free term variables of t are amongst $\vec{x}$ which have types $\vec{U}$ and that θ is a reducibility parameter for T. If $u_1, \ldots, u_n$ are $U_1\theta, \ldots, U_n\theta$-reducible, then $t|\theta|[\vec{u}/\vec{x}] \in T\theta$. Thus all terms are $\Rightarrow_{\mathcal{F}}$-strongly normalising.*
Proof The proof of the first part of the lemma is by induction over the structure of t and follows the standard procedure. The second part is proved by instantiating the first part with $\theta(X) = \mathtt{SN}(X)$ and $u_i = x_i$ so as to obtain t is reducible and hence by CR1 $\Rightarrow_{\mathcal{F}}$-strongly normalising. $\square$

5 Further Results

We collect here some further results which are of potential use to anyone wishing to use η-expansions in System F. The first states that one can calculate the $\Rightarrow_{\mathcal{F}}$-normal form of a term by contracting all β-redexes and then performing any remaining η-expansions

Lemma 5.1 *If t is a β-normal form and $t \Rightarrow_{\mathcal{F}} t'$ then t' is a β-normal form.*
Proof The proof is by induction on the rewrite $t \Rightarrow_{\mathcal{F}} t'$ and uses the fact that if $t \Rightarrow_{\mathcal{I}} t'$ then t and t' have the same outer term constructor. $\square$

Infact one can go further and give an explicit function which defines the η-normal form of a term. The first step is to characterise the reducts of a variable and this is done by the function Δ

$$\begin{array}{rcl}
\Delta(z^X) & = & \{z\} \\
\Delta(z^{T \to U}) & = & \{z\} \cup \{\lambda x^T.v[zu/y] \mid u \in \Delta(x^T) \text{ and } v \in \Delta(y^U)\} \\
\Delta(z^{\Pi X.T}) & = & \{z\} \cup \{\lambda X.v[zX/y] \mid v \in \Delta(y^T)\}
\end{array}$$

Lemma 5.2 *If z is a term variable, then $z^T \Rightarrow^*_{\mathcal{F}} \alpha$ iff $\alpha \in \Delta(z^T)$*
Proof lkjsfd $\square$

Now let $\Delta^m(z^T)$ be the largest member of $\Delta(z)$. By lemma **??** $\Delta^m(z^T)$ is the long $\beta\eta$-normal form of z^T. Define functions η^I and η^F which map terms to terms as follows

Table 4: Definition of The Eta-long Form Of A Term

$$\begin{array}{lcll}
\eta^I(x^T) & = & x^T & \\
\eta^I(tu) & = & \eta^I(t)\eta^F(u) & \\
\eta^I(tU) & = & \eta^I(t)U & \\
\eta^I(\lambda x : A.t) & = & \lambda x : A.\eta^F(t) & \\
\eta^I(\Lambda X.t) & = & \Lambda X.\eta^F(t) & \\
\textit{and} & & & \\
\eta^F(t) & = & \Delta(z^T)[\eta^I(t)/z] & \text{if } t : T
\end{array}$$

Lemma 5.3 *There are reduction sequences $t \Rightarrow^*_{\mathcal{I}} \eta^I(t)$ and $t \Rightarrow^*_{\mathcal{F}} \eta^F(t)$. In addition $\eta^I(t)$ is an $\Rightarrow_{\mathcal{I}}$-normal form while $\eta^F(t)$ is an $\Rightarrow_{\mathcal{F}}$-normal form.*
Proof Induction I suppose $\square$

All reduction sequences on a term are essentailly internal reductions followed by envelopes. A reduction sequence $t \Rightarrow^*_{\mathcal{F}} t'$ is called an *envelope* if there is a term $\alpha \in \Delta(z^T)$ (where T is the type of t and $z \notin \mathtt{FV}(t)$) such that $t' = \alpha[t/Z]$.

Lemma 5.4 *Every reduction sequence $t \Rightarrow^*_{\mathcal{F}} u$ factorises into a $\Rightarrow_{\mathcal{I}}$-reduction sequence followed by an envelope.*

$$t \Rightarrow^*_{\mathcal{I}} t' \Rightarrow^*_{\mathcal{F}} u$$

Finally we can prove the rather strange fact that all reduction sequences of a term to another term are of the same length.

Lemma 5.5 *If $t \Rightarrow^*_{\mathcal{F}}$ and $t \Rightarrow^*_{\mathcal{F}} t'$ are reduction sequences between the same term, then they are of the same length.*

Proof Consider the function $S : \Lambda \rightarrow N$ which gives the size of a pre-term

$$\begin{array}{lll} S(x^T) = 1 & S(\lambda x : A.t) = 1 + S(t) & S(\Lambda X.t) = 1 + S(t) \\ S(tu) = 1 + S(t) + S(u) & S(tU) = 1 + S(t) & \end{array}$$

Then clearly if $t \Rightarrow_{\mathcal{F}} t'$, then $S(t') = S(t) + 3$. Hence the length of any reduction sequence $t \Rightarrow^*_{\mathcal{F}} t'$ must be $(S(t') - S(t))/3$ and this is independant of the order in which different expansions are applied. $\square$

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