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A Zoo of ℓ_1-embeddable Polytopal Graphs

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Abstract

A simple graph $G = (V, E)$ is called l_1-*graph* if, for some λ,$n \in I\!N$, there exists a vertex-addressing of each vertex v of G by a vertex $a(v)$ of the n-cube H_n preserving, up to the scale λ, the graph distance, i.e. $\lambda d_G(v, v') = d_{H_n}(a(v), a(v'))$ for all $v \in V$. We distinguish l_1-graphs between 1-skeletons of a variety of well known classes of polytopes: semi-regular, regular-faced, zonotopes, Delaunay polytopes of dimension ≤ 4 and several generalizations of prisms and anti-prisms.

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1 Introduction

A connected simple graph G is called an l_1-*graph* if its (shortest path) distance matrix d_G, seen as a metric space (d, X) on the set X of vertices of G, is embeddable isometrically into some l_1^m-space. The l_1^m-space is $\mathbb{R}^m$ with l_1-metric

$$d_{l_1}(x, y) = ||x - y||_{l_1} = \sum_{i=1}^{m} |x_i - y_i|.$$

Equivalently, see [AsDe80], G is an l_1-graph iff d_G is embedded isometrically into the restriction of l_1^n on an n-cube up to a certain *scale* λ . Clearly, $\lambda = 1$ (respectively, $\lambda = 1, 2$) means that G is an isometric subgraph of the n-cube graph (respectively, of the n-half-cube graph). We denote these embeddings by $G \to H_n$ (respectively, $G \to \frac{1}{2}H_n$).

Any metric subspace A of l_1^m is (see [Dez60]) *hypermetric*, i.e. it satisfies

$$\sum_{1 \le i < j \le n} b_i b_j d_{l_1}(x_i, x_j) \le 0 \tag{1}$$

for every distinct $x_1, x_2, ..., x_n \in A$, $b := (b_1, ..., b_n) \in \mathbb{Z}^n$, and $\sum_{i=1}^n b_i = 1$. Clearly, in the special case $b = (1, 1, -1)$ the above inequality is the classical triangle inequality. Call the above inequality *5-gonal* (*2k+1-gonal*, respectively) if $b = (1, 1, 1, -1, -1)$ ($\sum_i |b_i| = 2k + 1$, respectively). Call a graph G *hypermetric* (*5-gonal*, *(2k+1)-gonal*), if d_G is hypermetric (respectively, satisfies 5-gonal, the $(2k + 1)$-gonal inequalities). Any $(2k + 1)$-gonal graph is $(2k - 1)$-gonal, [Dez60], but $K_{2k+1} - C_{k+1}$ is an example of a $(2k - 1)$-gonal which is not $(2k + 1)$-gonal for any $k \ge 2$. So, for a graph G the following chain of implications holds

$$(G \to H_{n_2}) \Rightarrow (G \to \frac{1}{2}H_{n_1}) \Rightarrow l_1\text{-graph} \Rightarrow \text{hypermetric} \Rightarrow (2k + 1)\text{-gonal} \Rightarrow 5\text{-gonal}$$

In this paper we systematically study well known polytopal graphs in connection within the above chain of notions, in particular, looking for l_1-graphs within 1-skeletons of polytopes with a degree of Euclidean regularity. These polytopes include: regular-faced polyhedra, semi-regular polytopes, zonotopes, Delaunay polytopes of dimension ≤ 4, and some generalizations of prisms and anti-prisms. In [DeSt96] one can find a related study of isometric, up to a scale, embeddings of 1-skeletons of *infinite* graphs (of Archimedean and Laves plane tilings) into *cubic lattices*.

Here are other instances of embeddings of graphs which are related to our subject:

1) well known cubical graphs (*induced* subgraphs of hypercubes) and a hypercube embedding with a *small distortion* (of distances) used in computer architecture.

2) isometric (or isometric with a small distortion) embeddings of graphs into other (than l_1-spaces) normed spaces; see [LLR95] and the huge bibliography there.

The following criterion for a graph G to be an l_1-graph is known, [Shp93], [DeGr93]: G is an l_1-graph iff it is an isometric subgraph of the direct product of half-cubes and Cocktail-Party graphs $K_{m\times 2}$ (1-skeleton of the cross-polytope β_m). Even, a good polynomial algorithm for a recognition of l_1-graphs is known [DeSp96]. So the "raisons d'être" for this paper are the following:

1.– To classify l_1-graphs within important *classes* of graphs. Until now this has been achieved only for strongly regular graphs ([DeGr93], [Koo90]), and two classes of planar (mainly non polyhedral) graphs ([PSC90], [DeTu96]).

2.– To find better criteria and algorithms for the recognition of l_1-graphs within special classes of graphs.

3.– Since an l_1-embedding produces a binary matrix, we want to look for new codes, having, in particular, symmetries of the original polytopes. Besides, l_1-embeddings, for example, of chains of hexagons (= benzenoid chains) and some fullerenes (see §4.5) give new ways of an encoding of chemically-relevant polyhedra and counting molecular parameters depending only on graph distances.

4.– To confront metrics of skeletons of polytopes, distinguished by their Euclidean, l_2-properties, to a larger l_1-metric. (We remind that an l_p-space is a metric subspace of an l_1-space for any $1 < p \leq 2$). For example, we shall see below how strong "l_1-ness" occurs in zonotopes and interesting 3-polytopes.

The somewhat surprising but easy observation that "interesting" polyhedra have l_1-skeletons was one of the origins of this work.

$P \to \frac{1}{2}H_{10}$ if P is one of 5 Platonic solids,

$P \to H_6$ if P is one of 5 Voronoi polyhedra (see §6.1)

$P \to \frac{1}{2}H_6$ if P is one of 5 Delaunay polyhedra (α_3, β_3, γ_3, $Prism_3$, Pyr_4),

$P \to \frac{1}{2}H_6$ if P is one of 5 chemically important (for main group elements) coordination polyhedra (α_3, β_3, icosahedron, $Prism_3$, Pyr_4),

$P \to H_6$ if P is one of 5 *isozonohedra* (i.e. zonohedra with the same rhombic faces, see §6.1): γ_3, rhombic dodecahedron, second (Bilinski's) rhombic dodecahedron, rhombic icosahedron and triacontahedron. See [Cox73] for names of those polyhedra.

Also 1-skeletons of 3 regular partitions of a plane are scale 2 isometrically embeddable into the cubic lattice $\mathbb{Z}_6$. But in $\mathbb{R}^4$, "interesting" non 5-gonal polytopes appear; for example, the 24-cell which is regular and is a Voronoi polytope and a coordination polytope (for the root lattice $D_4 = D_4^*$).

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2 Some notation, properties of polytopal graphs and hypermetrics

Here we give definitions and notation of graphs and polytopes that we use below.

A polytope of dimension n is called n-polytope. A 3-polytope is called a *polyhedron*. The polytope P^* is dual of the polytope P, and $(P^*)^* = P$. For a polytope

P, we denote by $G(P)$ its 1-skeleton. In other words, $G(P)$ is a graph on vertices of P such that two vertices are adjacent in $G(P)$ iff they are endpoints of an edge of P.

We use the following usual notation of graphs.

K_n is the complete graph on n vertices,

$K_{k_1,...,k_n}$ is the multipartite graph with n parts of sizes $k_1, ..., k_n$. If $n = 2$ and $k_1 = p, k_2 = q$, we have $K_{p,q}$, the bipartite graph.

If $k_i = 2$ for all i, then we have $K_{n\times 2}$, the Cocktail Party graph.

$T(n)$ is the triangular graph on $\binom{n}{2}$ vertices, the line graph of K_n,

$J(n, k)$, $1 \leq k < n$, is the Johnson graph, in particular $J(n, 1) = K_n$, $J(n, 2) = T(n)$,

$C_n = C_{i_1,...,i_n}$ is the cycle on n vertices $i_1, ..., i_n$ in this order.

$P_m = P_{i_1,...,i_m}$ is the path on the m vertices $i_1, ..., i_m$ in this order,

H_n is the cube-graph, $\frac{1}{2}H_n$ is the half-cube-graph,

$G - e$ is the graph G without its edge e,

∇G denotes the suspension of G, i.e. the graph G with a vertex which is not in G and is adjacent to all vertices of G,

$\nabla^m G = \underbrace{\nabla\nabla...\nabla}_{m} G.$

The first 3 of the following notation of polytopes are used in [Cox73].

α_n is the n-simplex, and $G(\alpha_n) = K_{n+1}$,

β_n is the n-cross-polytope, and $G(\beta_n) = K_{n\times 2}$,

γ_n is the n-cube, and $G(\gamma_n) = H_n$,

$\frac{1}{2}\gamma_n$ is the n-half cube, and $G(\frac{1}{2}\gamma_n) = \frac{1}{2}H_n$,

ambo-α_n is the convex hull of the midpoints of all edges of α_n, and $G($*ambo*-$\alpha_n) = T(n + 1)$.

C_n is a planar n-gon, and $G(C_n) = C_n$.

Note that $\alpha_2 \cong$ *ambo*-$\alpha_2 = C_3$, $\frac{1}{2}\gamma_3 = \alpha_3$, $\frac{1}{2}\gamma_4 = \beta_4$, *ambo*-$\alpha_3 = \beta_3$.

If P is an n-polytope, then $Pyr(P)$ is the $(n+1)$-pyramid with the base P, and $G(Pyr(P)) = \nabla G(P)$. The apex of $Pyr(P)$ is its vertex which does not lie in the n-space spanned by P and is adjacent to all vertices of P. An $(n + 1)$-bipyramid $BPyr(P)$ with the n-base P We have

$\underbrace{Pyr(Pyr(...Pyr}_{m}(P)...)) := Pyr^m(P)$, and $G(Pyr^m(P)) = \nabla^m G(P)$.

$Pyr_n := Pyr(C_n)$ is a 3-pyramid, and $G(Pyr_n) = \nabla C_n$.

$BPyr(P)$ is $(n + 1)$-bipyramid with the n-base P. It has two apexes which are opposite with respect to the hyperplane spanned by P. $G(BPyr(P)) = \nabla^2 G(P) - e$, where e is the edge connected in $\nabla^2 G(P)$ the two apexes.

A *t-capped* polyhedron is one obtained from a polyhedron P as follows. Choose some t faces of P (usually with a maximal number of vertices). To each chosen face add a vertex so that this vertex does not lie in the plane spanned by the face and is connected by an edge to each vertex of the face.

If P_i is an n_i-polytope, $i = 1, 2$, then the direct product $P_1 \times P_2$ is an $(n_1 + n_2)$-polytope, and $G(P_1 \times P_2) = G(P_1) \times G(P_2)$.

$Prism(P) = \alpha_1 \times P$ is the prism with the base P,

$Prism_n = Prism(C_n)$.

$APrism_n$ is the anti-prism with the base C_n. It is a polyhedron having two n-gon faces $(1, 2, ..., n)$, $(1', 2', ..., n')$ and all edges (i, i'), $(i + 1, i')$, $1 \leq i \leq n$ with addition modulo n.

$G_1 \prec G_2$ means that the graph G_1 is an isometric subgraph of G_2.

Remind that

$$G \to \frac{1}{2} H_n$$

means that G is isometrically embeddable into H_n with the scale 2. Similarly, we write $G \to \frac{1}{2m} H_n$ if G is embeddable into H_n with the scale $2m$. Also

$Pyr^m(\alpha_n) = \alpha_{n+m}$, $Pyr^m(\beta_n) \prec \beta_{n+m}$, $BPyr^m(\beta_n) = \beta_{n+m}$, $Pyr^m(Pyr_n) \prec \beta_{m+3}$,

$\alpha_n^* = \alpha_n$, $\gamma_n^* = \beta_n$, $Prism_n^* = BPyr_n$.

We suppose that definitions of Voronoi, Delaunay and coordination (= contact) polytopes are known to the reader; see [CoSl87] for details.

2.1 Vector representations of l_1-metrics and hypermetrics

All l_1-metrics d on n points (or l_1-metric spaces (d, X), $|X| = n$) form a cone in $\binom{n}{2}$-dimensional space of all matrices $\|d(i, j)\|$, $1 \leq i < j \leq n$. This cone is called the cut cone Cut_n. The extreme rays of Cut_n are cut metrics $\delta(S)$ for all $S \subseteq X$ containing a given point of X, where $(\delta(S))(i, j) = 1$ if $|S \cap \{ij\}| = 1$, and $=0$ otherwise. Some of facets of Cut_n are described by the hypermetric inequalities (1). Moreover, for $n \leq 6$, all facets of Cut_n are hypermetric. For $n \geq 7$, the cut cone has non hypermetric facets, and not each hypermetric inequality (1) determines a facet. The cut cone Cut_n lies inside a hypermetric cone Hyp_n, the cone of all metrics which satisfies the hypermetric inequalities (1) for all $b \in Z^n$ with $\sum_{i=1}^n b_i = 1$.

It is known [DGL95] that Hyp_n is polyhedral, i.e. it has a finite number of facets (and extreme rays). A hypermetric lying on an extreme ray is called *extreme*.

Any hypermetric $d(i, j)$ on n points can be represented by a generating n-subset of the set of vertices of a Delaunay polytope, with $d(i, j)$ equal to the squared Euclidean distance between corresponding vertices. This Delaunay polytope is uniquely constructed by the hypermetric d, and its dimension is not greater than $n - 1$.

A *Delaunay polytope* of a lattice is the convex hull of lattice points on the boundary of a locally maximal empty (from lattice points) sphere in this lattice. Voronoi proved that there is a finite number of combinatorial types of Delaunay polytopes in each dimension.

Let P be the Delaunay polytope vertices of which represent a hypermetric $d(i, j)$, $1 \leq i < j \leq n$, and let r be the radius of the empty sphere circumscribed P. If we take the center of P as a new origin, then vertices of P are represented by vectors of the same length r. A subset X of vertices of P is called *generating* if every vertex of P is an integer affine combination of vertices from X.

Let v_i represents a vertex $i \in X$. Then we have

$$d(i,j) = (v_i - v_j)^2 = 2r^2 - 2v_iv_j, \tag{2}$$

where v_iv_j is the inner products of these vectors. If $X = \{1, ..., n\}$ and v represent a vertex of P, we have

$$v = \sum_{i=1}^n b_iv_i, \text{ where } \sum_{i=1}^n b_i = 1 \text{ and } b_i \in \mathbb{Z}.$$

Every integer combination of v_i defines a point of a lattice whose Delaunay polytope is P. Since P is inscribed in the empty sphere of radius r, we have

$$(\sum_{i=1}^n b_iv_i)^2 \geq r^2.$$

This inequality holds as equality if the vector $\sum_{i=1}^n b_iv_i$ represents a vertex of P. Expanding this inequality and using the equality (2), we obtain the hypermetric inequality (1). We see that any vertex of P determines an equality which is satisfied by the hypermetric d. In particular, if $n > k+1$, where k is dimension of P, then there is an affine integral dependency $\sum_{i=1}^n b_iv_i = 0$ with $\sum_{i=1}^n b_i = 0$, $b_i \in \mathbb{Z}$, between vectors v_i, $i \in X$. The expansion of the equality $(\sum_{i=1}^n b_iv_i)^2 = 0$ determines a hypermetric equality which is satisfied by d. This equality is called an *m-gonal equality* for *even* $m = \sum_{i=1}^n |b_i|$.

Now we present a method of an economic recognition of l_1-embeddable metric spaces often used in constructions below. If d is an l_1-metric, then P can be isometrically inscribed into an m-dimensional box such that the set of vertices of P is a subset of all vertices of the box, and $m \geq n-1 \geq k$. If d is an integral l_1-metric then P can be isometrically inscribed into a cube γ_m, and $G(P)$ is either $\to H_m$ or $\to \frac{1}{2}H_m$. Below, we denote by $a(v) \subseteq \{1, 2, ..., m\}$ the vertex of γ_m corresponding to v.

When a graph G is embedded into a hypercube with a scale λ, then endpoints of every edge e of G are mapped into opposite vertices of a λ-dimensional cube. This cube is spanned by a λ-set $N(e)$ of mutually orthogonal vectors f_i, $i \in N(e)$. In other words, each edge e of G is represented by the vector $\sum_{i\in N(e)} f_i$. Conversely, we can say that each vector f_i defines a *zone* $Z_i = \{e : N(e) \ni i\}$ edges of which determine a *cut* of the graph G. Each cut corresponds to a subset S of vertices of G such that edges of the cut have exactly one vertex in S. Hence the indicator vector of the zone Z_i has the form $\delta(S_i)$ for some S_i.

It is shown in [DeSp96] that the amount $p(e,e') = |N(e) \cap N(e')|$ can be computed from the distance matrix d. Moreover, $p(e,e')$ takes only 3 values: 0, $\frac{\lambda}{2}$ and λ. Since $p(e,e')$ is an integer, it takes only 2 values if λ is odd. In the case of odd λ the zones are either disjoint or coincide. Hence any odd λ gives, in a sense, the same embedding as an embedding with $\lambda = 1$, when all zones are disjoint; above we denoted this embedding by $\rightarrow H_n$. This is the case of zonotopes (see §6). For $\lambda = 1$ the zones determine uniquely the expansion $d_G = \sum_i \delta(S_i)$ with 0,1 coefficients.

Most of the embeddings with an even λ are with $\lambda = 2$. In particular, this is the case of this paper; we denoted above this embedding by $\rightarrow \frac{1}{2}H_n$. For $\lambda = 2$, the zone determine the expansion $d_G = \frac{1}{2}\sum_i \delta(S_i)$ with half-integer coefficients. In the case $\lambda = 2$ the function $p(e, e')$ takes the values 0,1,2. Obviously, $p(e, e') = 2$ if and only if $N(e) = N(e')$. Hence the equality $p(e, e') = 2$ determines an equivalence relation on all edges of the considered graph. Clearly, if $p(e, e') = 2$ and $p(e, e'') = 1$, then $p(e', e'') = 1$. Hence we obtain the following criteria of an l_1-embedding of a graph G into $\frac{1}{2}H_n$: the function $p(e, e')$ (computed from the distance matrix d) should have the following properties:

(1) $p(e, e')$ is nonnegative and integral,

(2) the equality $p(e, e') = 2$ determines an equivalence relation,

(3) if $p(e, e') = 2$, then $p(e, e'') = p(e', e'')$ for every edge e''.

If at least one of this conditions is violated, then G is not l_1-embeddable. But it can be hypermetric. We give criteria for a non l_1-graph to be hypermetric in the end of this section.

We seek l_1-embeddings of graphic metric spaces (d, X) of 1-skeletons of polytopes with the set of vertices X. This means that we seek a representation of points of X by vertices v_i, $i \in X$, of a Delaunay polytope P. If $n > k + 1$ ($k = \text{dim}P$), then some vectors v_i can be linearly and integrally expressed through other vectors. Usually, one can always find $k + 1$ affinely independent vectors which compose an affine lattice base of P. (This fact is not proved in general, but we do not know any example of a Delaunay polytope without a lattice base).

So, it is clear that if a metric space (d, X) determines a k-dimensional Delaunay polytope, then it is sufficient to find a representation of $k + 1$ affinely independent points of X. For a graphic metric space (d, X), it is usually not difficult to find some dependencies of points of X.

We give some configurations corresponding to dependencies.

1) The most simple configuration consists of 4 points forming a *square*. Four points of a distance space (d, X) compose a square if we can label the points by the numbers 1,2,3,4 such that $d(1, 2) = d(3, 4)$, $d(1, 4) = d(2, 3)$ and $d(1, 3) = d(2, 4) = d(1, 2) + d(1, 4)$. We see that any 3 points of the square satisfy a triangle equality, and all 4 points satisfy a 4-gonal equality. If d_G is graphic, and $d(i, i + 1) = 1$, $1 \le i \le 4$, (addition modulo 4), then the 4 points induce a square = 4-cycle of the graph G.

2) A *hexagon* dependency. Six points of (d, X) compose a hexagon if they can be labeled by numbers i, $1 \le i \le 6$, such that $d(i, i + 1) = a$ (the addition modulo 6), and $d(i, j) = 2a$ for other pairs of these 6 points. Note that any 5 points of the hexagon satisfy a 5-gonal equality, and all 6 points satisfy a 6-gonal equality.

We formulate the above as the following

Lemma 1 *Let a hypermetric space (d, X) contains i squares and j hexagons. Then, deleting $i + j$ points of X which cuts all squares and hexagons, we obtain a reduced hypermetric space (d', X') which uniquely determines the original metric on the set X.*

Similarly one can define other configurations. If we find a configuration, we can delete any point of the configuration and seek a representation of other points. This method of reducing of an embedding problem works very well if the ground graph of a graphic metric has many squares.

Note that if d' of Lemma 1 is not hypermetric, then d is not hypermetric, too. But if d is not hypermetric, it can occur that d' is hypermetric but d is not. Then the above reduction does not work.

If d_G is a non-decomposable path metric of a graph G, which is a hypermetric but not l_1-graph, then d_G is an extreme hypermetric and G is an isometric subgraph of the Gosset graph $G(3_{21})$ [DeGr93]. The Gosset graph has 56 vertices, its diameter is 3, and for every its vertex v there is exactly one vertex at distance 3. The vertices of the Gosset graph can be labeled by the pairs ij and ij^*, $1 \leq i < j \leq 8$, such that the distances between corresponding vertices are as follows:

$d(ij, kl) = |\{ij\}\Delta\{kl\}|$, $d(ij, kl^*) = 3 - d(ij, kl)$. In particular, $d(ij, ij^*) = 3$.

We have the following criterion of a not l_1-metric d_G to be hypermetric, and therefore extreme (see [DGL95]).

Lemma 2 *Let a graphic metric d_G not be l_1-metric. Then d_G is a hypermetric if and only if*

(i) the graph G has diameter at most 3,

(ii) G contains 7 vertices such that the restriction of d_G on these vertices is one of 26 extreme hypermetrics described by graphs G_i, $1 \leq i \leq 26$, of [DGL95],

(iii) if G has diameter 3 then for every vertex v of G there is at most one vertex at distance 3 from v.

The following result was proved first in [CDG96].

Lemma 3 *Let G be an ℓ_1-graph. Then G is an isometric subgraph of a halved cube, i.e. its scale is ≤ 2 if either*
(i) G is planar,
or (ii) G is ≤ 4-partite.

Proof.

If G is hypermetric, then it generates a Delaunay polytope P(G), and contains an affine basis of P(G). If G is an ℓ_1-graph and its scale > 1, then P(G) is a direct product of halved cubes and cross-polytopes. An affine basis of a direct product is a union of affine bases of the components with one point common. Any affine basis of a cross-polytope β_n contains an (n-1)-dimensional simplex. The skeleton of this simplex is the complete graph K_n. Hence, if G is planar, the corresponding direct product P(G) can contain cross-polytopes β_n only for $n < 5$. The skeletons of β_n=cocktail party graphs are isometric subgraphs of a halved cube if $n < 5$. A direct product of halved cubes $\frac{1}{2}H_{q_i}$ is an isometric subgraph of the half-cube $\frac{1}{2}H_n$ with $n = \sum_i q_i$. The case (1) of lemma 3 follows. The same proof is valid for the case (ii) since $\leq$ 4-partite graph does not contain K_5. $\Box$

We use the opportunity to correct a misprint of the paper [DGL95]. The graph $\overline{G}_{18}$ belongs to the class $q = 11$, but not to the class $q = 12$ as it is shown on Fig. 5.14.

3 Regular-faced polytopes

Convex *regular* polytopes are defined by induction on dimension. In $\mathbb{R}^2$, they are regular polygons. A *regular* n-polytope (of dimension n), for $n \geq 3$, is one having only regular facets and regular vertex figures (the convex hull of all midpoints of edges through a given vertex).

A *regular-faced* n-polytope is one having only regular facets. A *semi-regular* n-polytope is a regular-faced n-polytope with equivalent vertices (i.e. the group of symmetries of the polytope is transitive on vertices). It is *quasi-regular* if, moreover, this group is transitive on edges. All regular-faced polytopes are known.

Regular 3-polytopes (Platonic solids) and semi-regular 3-polytopes (Archimedean solids, prisms, anti-prisms) have been known since antiquity. Archimedean polyhedra (and their dual, Catalan polyhedra) were rediscovered during the Renaissance, and Kepler gave them their modern names.

Semi-regular n-polytopes were found in 1900 by T.Gosset; [Mak88] for $n = 4$ and [BlBl91] for any n proved that the Gosset's list is complete (see [Cox73] and [Grü67] for the historical account).

All 92 regular-faced 3-polytopes was found by the work of many people, especially, of N.W.Johnson and V.A.Zalgaller (see, for example, [Ber71], [Zal69], [KoSu92]). Finally, in [BlBl91], the complete list of regular-faced n-polytopes is given.

Below we report the status of regular-faced n-polytopes (see §4 for 92 not semi-regular ones in $\mathbb{R}^3$) versus l_1-ness of their skeletons. By abuse of language we often denote a polytope by its skeleton.

3.1 Regular polytopes

a) for $n = 2$: $C_n \to \frac{1}{2}H_n$, and moreover $C_{2m} \to H_m$ (including both Delaunay 2-polytopes C_3, C_4, and both Voronoi 2-polytopes C_4, C_6).
b) for $n = 3$: icosahedron $\to \frac{1}{2}H_6$, dodecahedron $\to \frac{1}{2}H_{10}$ (this was proved in [Kel75] in weaker form of the scale 4 embedding).
c) $G(\gamma_n) = H_n$, $G(\alpha_n) = K_{n+1} \to \frac{1}{2}H_{n+1}$, $G(\beta_n) = K_{n\times 2} \to \frac{1}{2m}H_{4m}$ iff $K_n \to \frac{1}{2m}H_{4m}$.

In fact, l_1-embeddings of α_n and β_n are not unique for $n \geq 4$. In particular, $K_n \to \frac{1}{\lceil n/2 \rceil} H_{2\lceil n/2 \rceil -1}$ for $n \equiv 0,3 \pmod 4$ and $K_n \to \frac{1}{2\lceil n/2 \rceil} H_{4\lceil n/2 \rceil -2}$ for $n \equiv 1,2 \pmod 4$. Those embeddings give the minimal value $2 - \frac{1}{\lceil n/2 \rceil}$ of the ratio $\frac{N}{2m}$ for any l_1-embedding $K_n \to \frac{1}{2m} H_N$. Hence $G(\alpha_n) \to \frac{2}{n+1} H_n$ for $n \equiv 3 \pmod 4$. Also $G(\beta_n) \to \frac{1}{\lceil n/2 \rceil} H_{2\lceil n/2 \rceil}$ for $n \equiv 0,3 \pmod 4$ and $G(\beta_n) \to \frac{1}{2\lceil n/2 \rceil} H_{4\lceil n/2 \rceil}$ for $n \equiv 1,2 \pmod 4$.
d) for $n = 4$: 24-cell is non 5-gonal, 600-cell is not 7-gonal, (see [Ass81]), 120-cell is not 5-gonal (see [DeGr96]).

3.2 Semi-regular (not regular) polytopes

They belong to the following Gosset's list (see, for example, [BlBl91]; we use notation of [Cox73]).

a) In dimension 4: 0_{21} with the skeleton $T(5) \to \frac{1}{2} H_5$, snub 24-cell (tetricosahedric) and octicosahedric polytope (last two are undecided),

In dimension 5: 1_{21} with the skeleton $\frac{1}{2} H_5$.

In dimension 6: 2_{21} with hypermetric, but not l_1 skeleton.

In dimension 7: 3_{21} with hypermetric, but not l_1 skeleton.

In dimension 8: 4_{21}, skeleton of which is the root graph of all 240 roots of the root system E_8. It is not l_1-graph, since it contains the skeleton of 2_{21} as an induced subgraph of diameter 2, and therefore as an isometric subgraph.

b) In dimension 3:

$Prism_n \to \frac{1}{2} H_{n+2}$, and, for even n, $\to H_{\frac{n}{2}+1}$,

$$BPyr_n = Prism_n^* \begin{cases} \to \frac{1}{2} H_4 \text{ for } n = 3,4, \\ \text{non 5-gonal for } n \geq 5; \end{cases}$$

$APrism_n \to \frac{1}{2} H_{n+1}$,

In fact, remember that the vertices of $APrism_n$ are i, i', $1 \leq i, i' \leq n$ with edges $(i, i+1)$ (i, i'), $(i+1, i')$ $(i', i'+1)$ (here and below the addition is modulo n). We give embeddings $APrism_n \to \frac{1}{2} H_{n+1}$ (addressing the vertex v to a subset $a(v)$ of $\{1, 2, ..., n, n+1\}$) for 3 cases (below $1 \leq i \leq n$).

1) $n = 2k+1$ is odd:

$a(i) = \{i, i+1, ..., i+k-1\}$, $a(i') = \{n+1\} \cup \{i+k\} \cup a(i)$,

2) $n = 2k+2$ is even:

$a(i) = \{i, i+1, ..., i+k\}$,

$$a(i') = \begin{cases} (\{n+1\} \cup a(i)) - \{i\} & \text{if } i \text{ is odd,} \\ \{n+1\} \cup a(i)) \cup \{i+k+1\} & \text{if } i \text{ is even;} \end{cases}$$

3) $n = 4k$ is even:

$a(i) = \{i, i+1, ..., i+2k-1\}$,

$$a(i') = \begin{cases} (\{n+1\} \cup a(i)) - \{i\} & \text{if either } 1 \leq i \leq 2k \text{ and } i \text{ is odd,} \\ & \text{or } 2k < i \leq 4k \text{ and } i \text{ is even,} \\ \{n+1\} \cup a(i)) \cup \{i+k+1\} & \text{otherwise.} \end{cases}$$

$$APrism_n^* \begin{cases} = H_3 \text{ for } n = 3, \\ \text{non 5-gonal for } n \geq 4. \end{cases}$$

We give in Table 2 the results on l_1-ness of Archimedean polyhedra and their duals from [DeSt96].

Remark 1. "The 14th Archimedean solid" (twisted rhombicuboctahedron, unique new polyhedron if we weaken the vertex transitivity to the uniqueness of the vertex figure) and its dual are non 5-gonal.

Remark 2. [PSC90] gives $P \to \frac{1}{2}H_m$ for any 3-polytope without triangular faces and without vertices of degree 3. The truncated icosahedron above provides an example of a 3-polytope without triangular faces (but with all vertices of degree 3) which is non 5-gonal. Also, non 5-gonal polyhedra 6^*, $(6')^*$, $APrism_4^*$, M_{20}^* (see §5 below) have only square faces.

Table 2

#	polyhedron P	emb. of P	D(P)	emb. of P^*	$D(P^*)$
1	truncated tetrahedron	non 5-gonal	3	$\frac{1}{2}H_7$	2
2	truncated octahedron	H_6	6	non 5-gonal	3
3	truncated cube	non 5-gonal	6	$\frac{1}{2}H_{12}$	3
4	cuboctahedron	non 5-gonal	3	H_4	4
5	truncated cuboctahedron	H_9	9	non 5-gonal	4
6	rhombicuboctahedron	$\frac{1}{2}H_{10}$	5	non 5-gonal	5
7	snub cube	$\frac{1}{2}H_9$	4	non 5-gonal	7
$2'$	truncated icosahedron	non 5-gonal	9	$\frac{1}{2}H_{10}$	5
$3'$	truncated dodecahedron	non 5-gonal	10	$\frac{1}{2}H_{26}$	4
$4'$	icosidodecahedron	non 5-gonal	5	H_6	6
$5'$	truncated icosidodecahedron	H_{15}	15	non 5-gonal	6
$6'$	rhombicosidodecahedron	$\frac{1}{2}H_{16}$	8	non 5-gonal	8
$7'$	snub dodecahedron	$\frac{1}{2}H_{15}$	7	non 5-gonal	15

3.3 Regular-faced (not semi-regular) n-polytopes for $n \geq 4$

Going through the list of those polytopes given in [BlBl91], we obtain

$G(Pyr(\beta_{n-1})) = K_{1,2,\dots,2} \prec K_{n\times 2}$ and $G(BPyr(\alpha_{n-1})) = K_{n+2} - e = K_{2,1,\dots,1} \prec K_{(n+1)\times 2}$;

so, both these graphs are l_1-graphs. Besides, for the following two 4-polytopes we have

Pyr(icosahedron) is not 7-gonal (Fig.1f), $BPyr$(icosahedron) is non 5-gonal (its skeleton contains $K_5 - P_2 - P_3$).

The union of 0_{21} and $Pyr(\beta_3)$ (where β_3 is a facet of 0_{21}) $\to \frac{1}{2}H_5$.

Finally, the set of polytopes arising from the 600-cell by special cuts of vertices, is undecided.

4 Prismatic graphs

In this section we group observations on l_1-status of some generalizations and relatives of prisms and anti-prisms.

4.1 Moscow and Globe graphs

Call $M_n^m := P_m \times C_n$ $(m \geq 2, n \geq 3)$ *Moscow* graph, since its path-metric is called (in computer geometry) *Moscow* metric. This graph is sometimes called *annular city street* graph, and this metric also appears under the name *radar-discrimination* distance. $Prism_n$ is the case $m = 2$ of Moscow graph.

Clearly, $P_m \times C_n \to \frac{1}{2}H_{n+2m-2}$, since $P_m \to H_{m-1}$ (so $P_m \to \frac{1}{2}H_{2m-2}$) and $C_n \to \frac{1}{2}H_n$. Also $P_m \times C_{2k} \to H_{k+m-1}$, since $C_{2k} \to H_k$.

The Moscow graph is polyhedral, since it is planar and 3-connected. This polyhedron can be seen as a tower of $m-1$ copies of $Prism_n$ put consecutively one on the top of other.

Call the skeleton of the dual M_n^m the *Globe* graph and denote it 2-M_n^{m-1}. It can be seen as a 2-capped Moscow graph $M_n^{m-1} = P_{m-1} \times C_n$ and as *(m-1)-elongated* bipyramid. For $m = 1$, it is the usual $BPyr_n$ and, for $m = 2$, it is the usual elongated bipyramid; # #34,35,36 of the list of 112 regular-faced polyhedra are the cases $n = 3, 4, 5$ of 2-M_n^2.

Proposition 1

$$\textit{2-}M_n^m \begin{cases} \to \frac{1}{2}H_{2m+2} & \textit{if } n = 3, 4 \\ \textit{is non 5-gonal} & \textit{if } n \geq 5. \end{cases}$$

Proof. Let 0, ij $(1 \leq i \leq m,\ 1 \leq j \leq n)$, m be the $mn + 2$ vertices of the Globe graph 2-M_n^m such that all its edges are either on meridian paths $P_{0,1j,\ldots,(m-1,j),m}$ $(1 \leq j \leq n)$ or on parallel cycles $C_{i1,\ldots,in}$ $(1 \leq i \leq m-1)$. For $n = 3$, an embedding in $\frac{1}{2}H_{2m+2}$ is given by

$a(0) = \emptyset$, $a(m) = \{1, ..., 2m+2\}$, $a(ij) = \{1, j+1\} \cup \{t \in \mathbb{Z} : 5 \leq t \leq 2i+2\}$.

For $n = 4$ such an embedding is given by

$a(0) = \emptyset$, $a(m) = \{1, ..., 2m+2\}$, $a(ij) = \{j, j+1 \pmod 4\} \cup \{t \in \mathbb{Z} : 5 \leq t \leq 2i+2\}$.

We show non 5-gonality of the Globe graph by exhibiting 5 vertices $[x, y; a, b, c]$ such that

(i) $d(x,y) + (d(a,b) + d(a,c) + d(b,c)) > (d(x,a) + d(x,b) + d(x,c)) + (d(y,a) + d(y,b) + d(y,c))$.

They are

[0,2;11,12,14] for $m = 2$, [0,3;11,22,24] for $m = 3$,

$[0, m; 11, \lceil m/2 \rceil 3, (m-1)5]$ for $m \geq 4$.

(i) above take the form:

for $m=2$: $2+(1+2+2)>3+3$, for $m=3$: $3+(2+3+2)>5+4$, for $m\geq 4$: $m+((1+\lceil m/2\rceil)+(m-2+1)+(m+1-\lceil m/2\rceil)=3m+1>(1+\lceil m/2\rceil+(m-1))+((m-1)+(m-\lceil m/2\rceil)+1)=3m$ if $n=5$, $m+((1+\lceil m/2\rceil)+(m-2+2)+(m+1-\lceil m/2\rceil)=3m+2>(1+\lceil m/2\rceil+(m-1))+((m-1)+(m-\lceil m/2\rceil)+1)=3m$ if $n\geq 6$. □

Finally, we consider the graph 1-M_n^m :=2-M_n^m with the deleted vertex m, i.e.1-capped Moscow graph. It is the usual pyramid for $m=1$. Clearly, it is the skeleton of a self-dual polyhedron.

Corollary 1 *1-M_n^m* $\begin{cases} \to \frac{1}{2}H_{2m+n-2} \text{ if } n=3,4,5 \\ \text{is non 5-gonal for } n\geq 6 \end{cases}$

Proof. In fact, 1-M_n^m is an isometric subgraph of 2-M_n^m if $n\leq 5$. But, the cycle $C_{(m-1)1,\ldots,(m-1)n}$ is not an isometric subgraph (the polar way via 0 is shorter than within this cycle) iff $\lfloor n/2\rfloor > 2(m-1)$. So, embeddings for $n=3,4$ come by deleting the vertex m. For $n=5$, an embedding is given by $a(0)=\emptyset$, $a(ij)=\{j,j+1((\text{mod }5)\}\cup\{t\in\mathbb{Z}:6\leq t\leq 2i+3\}$, for $1\leq i\leq m$, $1\leq j\leq 5$. For $n\geq 6$, one can find (see Fig.4a for $n=6$) 5 points violating 5-gonal inequality. □

Another (than Moscow graph) generalization of $Prism_n=C_2\times C_n$ is the Lee graph $\prod_{i=1}^t C_{n_i}$; its path metric is the Lee distance

$d_{\text{Lee}}(x,y)=\sum_{i=1}^t min(|x_i-y_i|,n_i-|x_i-y_i|)$.

used in the coding theory (a discrete analog of elliptic metric). Remind that $H_4=C_4\times C_4$, since $C_4=H_2$.

Clearly, $\prod_{i=1}^t C_{n_i}\to\frac{1}{2}H_n$ (and moreover, $\to H_{\frac{n}{2}}$ if all n_i are even), where $n=\sum_{i=1}^t n_i$.

$\prod_{i=1}^t P_{n_i}\to H_{n-t}$; it is a *grid* graph with usual *grid* distance, i.e. l_1-distance $\sum_{i=1}^t|x_i-y_i|$. $\prod_{i=1}^t K_{n_i}\to\frac{1}{2}H_{n-t}$; it is the Hamming graph with Hamming metric

$|\{1\leq i\leq t: x_i\neq y_i\}|$.

4.2 Stellated k-gons

Stellated k-gon $Stel_k$ is the $2k$-cycle $C_{1,1',2,2',\ldots,k,k'}$ with the additional edges of the k-cycle $C_{1,2,\ldots,k}$. So, $Stel_k$ is C_k with a triangle on each edge. Besides $Stel_k$ is $APrism_k$ without the edges of the k-cycle $C_{1',2',\ldots,k'}$. Also it is an isometric subgraph of skeletons of many important polyhedra, for example, of icosahedron and 3 others (with 8,9,10 vertices) convex deltahedrons for $k=3$, of cuboctahedron and snub cube for $k=4$ of icosidodecahedron and snub dodecahedron for $k=5$, of Archimedean tiling (3,6,3,6) for $k=6$. For $n=5,6,8$ it is used as a well known symbol (Red Star, Star of David, Muslim Star, respectively).

Denote by Sun_{2k} the isometric subgraph of $Stel_{2k}$ obtained by deleting vertices $(2i-1)'$, $1\leq i\leq k$. It is an isometric subgraph, for example, of truncated tetrahedron, truncated cube, of Archimedean tiling $(3,12^2)$ for $k=3,4,6$, respectively.

Proposition 2 *(i) $Stel_k\to\frac{1}{2}H_{2k}$,*
(ii) $Sun_{2k}\to\frac{1}{2}H_{3k}$.

Proof. In fact, let i, i', $1 \le i \le k$, denote vertices. For odd $k = 2m+1$, an embedding is given by $a(i) = \{i, i+1, ..., i+m-1\}$, $a(i') = \{i, i+1, ..., i+m, 2m+1+i\}$, for $1 \le i \le k$ (addition is modulo k, except for $2m+i+1$).

For even $k = 2m$ we define $a(i)$ inductively. We take $a(m+1) = \{1, ..., k\}$. Then

$a(i) = a(i+1)$ with two largest elements deleted if $1 \le i \le m$,

$a(i) = a(i-1)$ with two smallest elements deleted if $m+2 \le i \le k$.

$a(m') = \{1, ..., k-1, k+m\}$, $a((m+1)') = \{2, ..., k, k+m+1\}$,

$a(i') = \{k+i\} \cup (a((i+1)') - \{k+i+1\}$ with two largest elements deleted) if $1 \le i < m$,

$a(i') = \{k+i\} \cup (a((i+1)') - \{k+i-1\}$ with two smallest elements deleted) if $m+1 \le i \le k$,

4.3 Cupolas

Denote by Cup_n the graph with vertices i, i', i'' $(1 \le i \le n)$ and edges on the inner cycle $C_{1,...,n}$, on the outer cycles $C_{1',1'',...,n',n''}$, and on the paths $P_{i',i,i''}$ for all $1 \le i \le n$. So Cup_n is the skeleton of a polyhedron; for $n = 3, 4, 5$ they are triangular, square and pentagonal cupolas: the polyhedra ##23,24,25 in the list of 112 regular faced polyhedra.

Proposition 3 *(i)* $Cup_n \to \frac{1}{2}H_{2n}$ *for* $n \ge 4$,
(ii) Cup_3 *and* Cup_n^*, $n \ge 3$, *are non 5-gonal.*

Proof. In fact, Cup_3 contains a non 5-gonal isometric subgraph $v(3,4,3,4)$ (see definitions in §5) on 7 points.

We give the following explicit embedding of $Cup_n \to \frac{1}{2}H_{2n}$. We take an embedding of the inner cycle $C_{1,...,n} \to \frac{1}{2}H_n$. Let the vertex i is mapped in this embedding into an subset $a(i)$ of an n-set V. Let $a(1) = \emptyset$, and $V \cap \{1, 2, ..., n\} = \emptyset$. Then we map the vertices i' and i'' into the sets $a(i) \cup \{i, i+1\}$ and $a(i) \cup \{i+1, i+2\}$, respectively (the addition is by mod n). Note that the shortest path between non-adjacent vertices of the outer cycle goes through the inner cycle. Hence, it is not difficult to verify that the obtained embedding is isometric for $n \ge 4$.

Cup_n^* contains $K_{2,3}$ as an isometric subgraph for $n \ge 3$. □

4.4 Antiwebs

An *antiweb* AW_n^k, for $1 \le k \le \lfloor \frac{n}{2} \rfloor$, is the graph with the vertices $0, 1, ..., n-1$ and $i \sim i+1, i+2, ..., i+k \pmod n$. (Here $i \sim j$ denotes that the vertex i is adjacent to the vertex j.) It is a common generalization of the following polyhedral graphs:

$AW_n^1 = C_n \to \frac{1}{2}H_n$ (and $\to H_{\frac{n}{2}}$ if n is even),

$AW_n^2 = G(APrism_{\frac{n}{2}}) \to \frac{1}{2}H_n$ if n is even,

$AW_n^{\lfloor n/2 \rfloor} = K_n = G(\alpha_{n-1}) \to \frac{1}{2}H_n$,

$AW_n^{\frac{n}{2}-1} = K_{\frac{n}{2}\times 2} = G(\beta_{\frac{n}{2}}) \to \frac{1}{2m}H_{4m}$ (for some m) if n is even.

In the next proposition we show that all other antiwebs are not l_1-graphs. It is easy to check that AW_n^k has diameter 2 iff $\lfloor \frac{n+2}{4} \rfloor \le k \le \lfloor \frac{n}{2} \rfloor - 1$.

Proposition 4 *(i) AW_n^2 is non 5-gonal for odd $n \ge 13$, AW_{11}^2 is not 7-gonal.*

(ii) AW_n^k is non 5-gonal in the following cases:

(a) if $\lceil \frac{n+1}{4} \rceil \le k < \frac{n}{2} - 1$,

(b) if $n = 4k+3$ for $k \ge 3$, and if $n = 4k+4$ for $k \ge 4$.

(iii) AW_n^k is not l_1-embeddable if either $k = 3$ and $n \ge 10$ or $4 \le k \le \frac{n-3}{2}$ (and therefore $n \ge 11$). In particular, AW_{13}^3 and AW_{16}^4 are not 7-gonal.

(iv) AW_9^2, AW_{12}^3 and AW_{14}^3 are extreme hypermetrics.

Proof. (i) We show explicitly 5 points $(a,b,c;x,y)$ of AW_n^2, where the 5-gonal inequality (1) with $b_a = b_b = b_c = -b_x = -b_y = 1$ is violated.

If $n = 4t-1 \ge 15$, take the points $(0, 2t-1, 2t; 2t-4, 2t+3)$.

If $n = 4t+1 \ge 13$, take the points $(0, 2t-1, 2t+2; 2t-2, 2t+3)$.

Similarly, for AW_{11}^2 we have the following 7 points (0,5,6,11;3,4,9) violating a 7-gonal inequality.

(ii) a) In fact, any A_n^k with k as in Proposition 4 is of diameter 2, so any its induced subgraph of diameter 2 is isometric. Only connected graphs on 5 vertices which are non 5-gonal are $K_{2,3}$, $K_5 - K_3$ and $K_5 - P_2 - P_3$ (with $P_2 \cap P_3 = \emptyset$). Hence the only way for AW_n^k of diameter 2 to be non 5-gonal is to contain one of these 3 graphs on 5 vertices. But any antiweb is $K_{1,3}$-free graph. Hence the first 2 above subgraphs, $K_{2,3}$ and $K_5 - K_3$, are excluded. On the other hand, we will give subgraph $K_5 - P_2 - P_3$, namely, the complement of

$P_{1,2k-i,4k-2i-1} + P_{k-i,n+1-k}$, for any i with $n+1+i \le 4k \le n+1+2i$ and $0 \le i \le k-2$.

So, it will cover all cases $n+1 \le 4k \le n-3+2k$, i.e. $\lceil \frac{n+1}{4} \rceil \le k \le \lfloor \frac{n-3}{2} \rfloor$ considered in Proposition 4.

Now, $\overline{P_{1,2k-i,4k-2i-1}} \prec AW_n^k$, since $1+k < 2k-i$, $(2k-i)+k < 4k-2i-1$ (because $i \le k+2$) and $4k-2i-1 \le n$, $(4k-2i-1)+k-n \ge 1$.

$\overline{P_{k-i,n+1-k}} \prec AW_n^k$, since $1 \le k-i$, $(k-i)+k < n+1-k \le n$. But $k-i, n+1-k \sim 1, 2k-i, 4k-2i-1$ in AW_n^k.

(ii) b) If $n = 4k+3$, take the 5 points $(0, 2k+1, 2k+2; k+2, 3k+3)$, and if $n = 4k+4$, take the 5 points $(0, 2k+1, 2k+2; k+2, 3k+4)$.

(iii) For $k = 3$, consider the subgraph of AW_n^3 induced by the points i, $0 \le i \le 6$. Here the point 3 is adjacent to all other 6 points. Since $n \ge 10$, the point 6 is not adjacent to the points 0,1,2. Finally we have that the induced subgraph is ∇B_8. For $k \ge 4$, consider the subgraph of AW_n^k induced by the following 7 points: $2, 3, 4, k, k+2, k+3, k+4$. Since $k \le \frac{n-3}{2}$, this graph is $\nabla(K_6 - P_4) = \nabla H_2$, where $P_4 = P_{3,11,2,9}$. The graphs B_8 and H_2 are the graphs on Figs. 6.3 and 6.7 of [DGL95]. Since $\nabla B_8 = G_4$ and $\nabla H_2 = G_2$ are graphs from the list of the 26 extreme hypermetrics on 7 points, we have that AW_n^k is not l_1-embeddable.

The following 7 points violate a 7-gonal inequality: (0,4,5,6;2,3,7) for AW_{13}^3 and (0,2,5,7;3,4,9) for AW_{16}^4.

(iv) The graph AW_9^2 has 3 hexagons (0,2,3,5,6,8), (0,1,3,4,6,7) and (1,2,4,5,7,8) determining 3 6-gonal equalities. Any two of them are linearly independent. Hence representations of two vertices, say, 7 and 8, are determined by representations of other vertices. If we delete the vertices 7 and 8, we obtain the extreme hypermetric G_{19} from Fig.5.14 of [DGL95]. Since AW_9^2 has diameter 2, it is a subgraph of the Shläfli graph. For example, the following labelings the vertices 0,1,...,8 by the pairs 13, 12, 14, 24, 45, 17*, 56, 36, 48* gives an embedding. A propos, $AW_9^2 - e$ is a polyhedral graph (see Fig.11c).

Similarly, we find that the vertices i, $0 \leq i \leq 6$, of the graph AW_{12}^3 induce the extreme hypermetric graph $G_4 = \nabla B_8$. The following labeling shows that AW_{12}^3 is an extreme hypermetric graph: $0 \to 12$, $1 \to 13$, $2 \to 14$, $3 \to 15$, $4 \to 26^*$, $5 \to 45$, $6 \to 57$, and $7 \to 16^*$, $8 \to 18^*$, $9 \to 27$, $10 \to 23$, $11 \to 58^*$.

The graph AW_{14}^3 has diameter 3 and the set of its vertices consists of 7 pairs of vertices at distance 3. The 7 vertices i, $0 \leq i \leq 6$, induce the extreme hypermetric graphs $G_4 = \nabla B_8$. Hence AW_{14}^3 is not l_1-graph. But the following labeling ϕ shows that it is extreme hypermetric: $0 \to 12$, $1 \to 13$, $2 \to 14$, $3 \to 15$, $4 \to 26^*$, $5 \to 23^*$, $6 \to 57$, and $i + 7 \to \phi(i)^*$. □

4.5 Capped anti-prisms, towers and fullerenes

We consider here an anti-prism analog of capped prisms (1-M_n^2, 2-M_n^2 of §4.1). Take at first $APrism_3 = \beta_3$. The i-capped $\beta_3 \to \frac{1}{2}H_{i+4}$, $(0 \leq i \leq 8)$. In fact, they are isometric subgraphs of omni(8)-capped β_3 (= dual truncated γ_3). Denote the vertices of β_3 by i, i', where $1 \leq i \leq 3$ and all (i, i') are not edges, and the caps of faces $(1, 2', 3')$, $(1', 2, 3')$, $(1', 2', 3)$, (1,2,3) by t and the caps of respectively opposite faces by t', $1 \leq t \leq 4$. Put then all $a(i) = \{i, 4\}$, $a(i') = \{1, 2, 3, 4\} - a(i)$, $a(t) = \{t, t+4\}$, $a(t') = (\{1, 2, 3, 4\} - \{t\}) \cup \{8 + t\}$. The dual i-capped $\beta_3 \to \frac{1}{2}H_{i+6}$ for $0 \leq i \leq 2$, but it is non 5-gonal for $i = 8$.

From now on we consider i-capped $APrism_n$, $n \geq 4$, only for $i = 1, 2$ (i.e. only n-cycles are capped). Denote by i-$APrism_n$ i-capped $APrism_n$ for $i = 1, 2$. Remind (§3.2) that $APrism_n \to \frac{1}{2}H_{n+1}$ for $n \geq 3$ and dual $APrism_n$ is non 5-gonal if $n \geq 4$.

The next case is $APrism_4 = AW_8^2$; it can be seen as a half of the Shrikhande graph. i-$APrism_4$ is an extreme hypermetric for $i = 1, 2$ (see §8 and Fig.2); they are ##30,37 of the list of 92 regular-faced polyhedra. see their duals on Fig.3 a), b).

i-$APrism_5 \to \frac{1}{2}H_6$ for $0 \leq i \leq 2$. Its dual (dodecahedron)$\to \frac{1}{2}H_{10}$ for $i = 2$, but it is non 5-gonal for $i = 0, 1$ (Fig. 4c).

1-$APrism_n \to \frac{1}{2}H_{n+1}$ for $n = 5, 6, 7$, but non 5-gonal for $n \geq 8$. In fact, remind that vertices of $APrism_n$ are i, i' for $1 \leq i, i' \leq n$ and edges are pairs (i, i'), $(i+1, i')$, $(i, i+1)$, $(i', i'+1)$. Let 0 be the cap of $C_{1,\dots,n}$. Put then $a(0) = \emptyset$, $a(i) = \{i, i+1\}$, $a(i') = \{i, i+1, i+2, n+1\}$. Clearly, for $n = 5, 6, 7$ it will give a scale 2 isometric embedding into $(n + 1)$-cube. For $n \geq 6$ dual 1-$APrism_n$, 2-$APrism_n$ and their duals are not 5-gonal (see Fig.4d,e for case $n = 6$).

Consider now *towers*, generalizing gyroelongation of anti-prisms in a way similar to how the Moscow graph in §4.1 generalizes elongation of prisms. Denote by Tow_n^k the polyhedron defined by adjoining consecutively k copies of $APrism_n$ one on the top of another. It is easy to check that Tow_n^2 is non 5-gonal for $n \geq 3$: see Fig. 4f,g for Tow_3^2 (i.e. gyroelongated β_3, it appears also on Fig.16.1.1 in [Grü67]) and its dual. It implies that Tow_n^k and 2-capped Tow_5^k are non 5-gonal for $k \geq 2$. Remark similarity of 2-Tow_5^k with polar zonotopes PZ_m mentioned in §6. See also Fig.3e for non 5-gonal Koester's graph, [Koe85], similar to $APrism_5 + APrism_5$ and to PZ_5^*; it is 4-regular planar 4-critical graph, i.e. its chromatic number is 4, but every its proper subgraph has a 3-coloring.

Dual 2-Tow_5^k is a *fullerene*, i.e. a simple polyhedron with $20 + 2m$ vertices, 12 pentagonal and m hexagonal faces) F_{20+2m} for $m = 5(k-1)$. Let us call it *strained* (most "antiaromatic", least stable, in chemical terms), since it has (as opposite to *preferable* fullerenes) the *maximum* number of abutting pairs of pentagonal faces. Except the case $k = 1$ (dodecahedron) the strained fullerene $F_{10(k+1)}$ is non 5-gonal: for $k \geq 3$ it contains the graph in Fig. 10b, for the case $k = 2$ see Fig. 10c. Its dual also non 5-gonal since it contains 1-Tow_5^2 depicted on Fig. 10a.

We give below some observations on l_1-status of several fullerenes. In particular, small fullerenes F_{24}, F_{26}, F_{28} are of special interest for chemistry: besides being carbon cages, they are used ([Wel84], p.74 and pp.660–662 on "ice-like" clathrate hydrates) as space co-filling polyhedra. There are [Bal95] 1, 0, 1, 1, 2, 3, 40, 271, 1812 fullerenes F_{20+2m} for m = 0, 1, 2, 3, 4, 5, 10, 15, 20, respectively. F_{20} is the dodecahedron; the unique F_{24} is the dual of 2-$APrism_6$ (it and F_{24}^* are on Fig. 4e,d); the unique F_{26} and its dual are on Fig. 10g,e; $F_{28}(T_d)$ (T_d is its group of symmetry) and its dual are on Fig. 10d,h; $F_{30}(D_{5h})$ is on Fig. 10c. The strained fullerenes F_{20+2m} for $m \leq 25$ are $F_{30}(D_{5h})$, $F_{40}(D_{5d})$, $F_{50}(D_{5h})$, $F_{60}(D_{5d})$, $F_{70}(D_{5h})$; with this symmetry it is unique for $m = 5, 20$ and one of two for $m = 10, 15, 25$, respectively. The truncated icosahedron with its icosahedral symmetry is defined as $F_{60}(I_h)$. As we checked, all above fullerenes and their duals are non 5-gonal, except $F_{20} \to \frac{1}{2}H_{10}$, $F_{26} \to \frac{1}{2}H_{12}$, $F_{20}^* \to \frac{1}{2}H_6$, $F_{28}^*(T_d) \to \frac{1}{2}H_7$, $F_{60}^*(I_h) \to \frac{1}{2}H_{10}$. $F_{60}(I_h)$ admits 7-embedding in $\frac{1}{2}H_{20}$ (see [DeSp96] and §8 here).

Using also the algorithm of [DeSp96], one can see that the unique preferable (i.e. without abutting pairs of pentagons) F_{70} is not l_1-graph. This F_{70} is second one with the symmetry D_{5h} and contains 20-*bowl*, i.e. the graph consisting of a pentagon surrounded by 5 hexagons; 20-bowl $\to \frac{1}{2}H_{15}$ (see Fig.10f). Also the graph consisting of a pentagon surrounded by 5 pentagons (or by 3 pentagons and 2 non-adjacent hexagons) is embeddable into $\frac{1}{2}H_{10}$ ($\to \frac{1}{2}H_{12}$, respectively); a hexagon surrounded by 6 pentagons (resp. 6 hexagons) embeds into $\frac{1}{2}H_{12}$, see Fig. 10g (resp. into H_9).

5 92 regular-faced (not semi-regular) polyhedra

We use names and numbers of those polyhedra from [Ber71], where they are given as # #21–112 of the list of all 112 regular-faced polyhedra; in [Zal69] they numbered by 1–92. In both references they are given as same appropriate joins of 28 basic polyhedra $M_1 - M_{28}$. See also [KoSu92] with nice pictures of those polyhedra. 28 basic regular-faced polyhedra (all but M_{22} are given on Fig.8,9) include regular ones M_1, M_{15}; pyramids M_2, M_3, cupolas $M_4 - M_6$, The 9 Archimedean polyhedra M_i denoted as $1, 3, 3', 2, 5, 5', 2', 7, 7'$ in Table 2 of §3.2 for $i \in \{10-12, 16-19, 26, 27\}$, respectively; M_{13}, M_{14} which are tridiminished, parabidiminished $RIDo$, M_7 which is tridiminished Ico; pentagonal rotunda M_9 and 8 others whose complicate names reflect their high individuality. In this section we denote icosahedron, dodecahedron and rhombicosidodecahedron by Ico, Do and $RIDo$, respectively. Remark that remaining 4 Archimedean polyhedra of Table 2 are in this notation $4{=}Cup_3+\overline{Cup_3}$, $4' = M_9 + \overline{M_9}$, $6' = RIDo = M_{14} + 2Cup_5$, $6 = Cup_4 + Prism_8 + Cup_4$.

Lemma 4 *(i) M_n is non 5-gonal for $n = 4, 8, 10-13, 19-21, 23-25$,*

(ii) M_{28}(= #105, snub square anti-prism) is not 7-gonal,

(iii) M_{22}(= #106, sphenocorona) is extreme hypermetric,

(iv) Remaining 14 polyhedra M_n have l_1-skeletons.

(v) M_n^* has l_1-skeleton for $n = 1-3$, 10–12, 15, 16, 19, 23, 25 and it is non 5-gonal for remaining 17 polyhedra M_n.

Proof. (i),(ii) are clear from Fig.9; (iii) is proved in the proof of the next lemma. 14 l_1-polyhedra are M_1, M_{15} (see §3.1); $M_{16} - M_{18}$ and M_{26}, M_{27} (see §3.2); $M_2 \to \frac{1}{2}H_4$, $M_3 \to \frac{1}{2}H_5$ and $M_5 \to \frac{1}{2}H_8$, $M_6 \to \frac{1}{2}H_{10}$ (see §§4.1,4.3); the skeleton of M_7 is an isometric subgraph of l_1-skeleton of Ico, $M_7 \to \frac{1}{2}H_6$. Finally, $M_9 = \#26 \to \frac{1}{2}H_{11}$, where $M_9 \prec \#41 \to \frac{1}{2}H_{13}$. $\square$

Lemma 5 *Skeletons of the following 4 polyhedra are extreme hypermetrics:*

#30=$APrism_4 + Pyr_4$ (gyroelongated Pyr_4= 1-capped $APrism_4$),

#37=$Pyr_4 + APrism_4 + Pyr_4$ (gyroelongated $BPyr_4$= 2-capped $APrism_4$),

#71=$Prism_3 + 3Pyr_4$ (triaugmented $Prism_3$),

#106=M_{22} (sphenocorona),

#107=$M_{22} + Pyr_4$ (augmented sphenocorona).

Proof. The skeletons of the polyhedra #30 and #71 have each 9 vertices and diameter 2. The skeleton of #37 has 10 vertices, diameter 3 and contains the skeleton of #30 as an isometric subgraph. These 3 graphs contain the induced extreme hypermetrics G_{24} and G_{25} of [DGL95] Fig.5.15. This implies that these skeletons are not l_1-graphs.

We give explicit embeddings of skeletons of #30 and #71 into the Schläfli graph $G(2_{21})$, and the skeleton of #37 into the Gosset graph $G(3_{21})$; it will show that these skeletons are extreme hypermetrics.

Recall that the vertices of $APrism_4$ are i, i', $1 \le i, i' \le 4$, with the edges $(i, i+1)$, $(i', (i+1)')$, (i, i') and $(i+1, i')$ (here and below additions are mod 4). Let 0 be the new vertex of the cap of the capped anti-prism #30, and 0' be the second cap of the 2-capped anti-prism #37. Then the labels of vertices by pairs are as follows: $a(i) = i, i+1$, $1 \le i \le 4$, $a(1') = 15$, $a(2') = 47^*$, $a(3') = 36$, $a(4') = 28^*$, $a(0) = 24$, $a(0') = 24^*$.

Similarly, let i, i', $1 \le i, i' \le 3$ be the vertices of the prism of #71 with edges $(i.i')$, and let i'' be the apex of the Pyr_4 on the square face of $Prism_3$ opposite to the edge (i, i'). Then the labels are: $a(i) = i7$, $a(i') = i8$, $a(i'') = (i, i+3)^*$, $1 \le i \le 3$.

The skeleton of #107 has 10 vertices and diameter 3; it contains the induced extreme hypermetric G_{25}. It is 1-capped $APrism_5$ with an additional edge between two nonadjacent vertices of the noncapped cycle C_5. Let, as above, the vertices of $APrism_5$ are i, i', $1 \le i \le 5$, such that all vertices i are adjacent to the apex 0 of the cap, and the vertices $1'$ and $4'$ are adjacent. We label the vertices by pairs as follows: $a(i) = i, i+1$, $1 \le i \le 5$. Since the vertices $2'$, $3'$ and $5'$ are at distance 3 from the vertices 4, 5 and 2, respectively, we have $a(2') = 45^*$, $a(3') = 15^*$ and $a(5') = 23^*$. The other two adjacent vertices obtain the labels $a(1') = 16$ and $a(4') = 46$.

The skeleton of #106 is $G(\#107)$ without the vertex $5'$, which is the apex of the cap Pyr_4. The labeling of $G(\#106)$ is induced by the above labeling of $G(\#107)$, since $G(\#106)$ is an isometric subgraph of $G(\#107)$. □

Proposition 5 *Besides non 7-gonal #105 and above 5 hypermetrics ##30, 37, 71, 106, 107, remaining 86 polyhedra consist of 50 non 5-gonal ones and 36 with l_1-skeletons: ##21–35 (except 23,30,33), 39, 40, 41, 44, 48, 69–84 (except 71, 77), 92–95, 100.*

Corollary 2 *l_1-status of 8 convex deltahedra is as follows:*

(i) $\alpha_3 \to \frac{1}{2}H_3$, #32=$BPyr_3 \to \frac{1}{2}H_5$, $\beta_3 \to \frac{1}{2}H_4$, Icosahedron $\to \frac{1}{2}H_6$,

(ii) #33=$BPyr_5$ and #104=M_{25} (snub dispensoid) are non 5-gonal,

(iii) #37, #71 are extreme hypermetrics.

Remark that all 5 *dual* truncated P, where P is a Platonic solid, are simplicial and have l_1-skeletons. They $\to \frac{1}{2}H_n$ with $n = 7, 6, 12, 10, 16$ for P being α_3, β_3, γ_3, Ico, Do, respectively. Between other simplicial Catalan polyhedra, namely $Prism_n^*$ and dual truncated Q (Q being cuboctahedron or icosidodecahedron), only $Prism_n^*$ $(n = 3, 4)$ are 5-gonal; between remaining 9 Catalan polyhedra (including $APrism_n^*$, $n \ge 4$) only dual Q are 5-gonal.

Sketch of the proof of Proposition 5.

a) **12 polyhedra ##92–103**

They are all possible adjoinings of $Cup_5 = M_6$ to 10-gonal faces of two of them: $M_{13} = \#103$ and $M_{14} = \#100$ (see Fig. 8). Also Archimedean $RIDo = M_{14} + 2M_6 = M_{13} + 3M_6 \to \frac{1}{2}H_{16}$. One can check that slight modifications of this embedding produce an embedding into the same $\frac{1}{2}H_{16}$ of M_{14} and all 4 polyhedra without 10-faces: ##92–95=$M_{14} + M_6 + \overline{M_6}$, $M_{14} + 2\overline{M_6}$, $M_{13} + M_6 + 2\overline{M_6}$, $M_{13} + 3\overline{M_6}$. The

7 remaining polyhedra are non 5-gonal: see Fig.8 for M_{13}. Slight modifications of these 5 points give, due to symmetries, forbidden 5-points configurations for others.

b) **non 5-gonality**

For 22 polyhedra (# # 43, 45, 49–54, 57–63, 66, 77, 108–112) 5 vertices violating 5-gonality were found ad hoc. ##108–112=M_i for $i = 23, 21, 24, 8, 20$ (see Fig.9); see Fig.5b for #49.

Denote by $v(3, 2k, 3, 2m)$ (respectively, by $e(3, 2k, 3, 2m)$) the graph consisting of a vertex (respectively, of an edge) surrounded by cycles C_3, C_{2k}, C_3, C_{2m} (for positive integers k, m) in this order; each two consecutive cycles intersect exactly in the edge incident (adjacent) to the original vertex (respectively, to the original edge). It is easy to check that both the families of the above graphs are not 5-gonal.

Now $v(3, 4, 3, 4) \prec \#23 = M_4 \prec \#\#38, 42, 47(\text{“}anticuboctahedron''), 55, 56, 64, 85$;

$v(3, 4, 3, 4) \prec \#46$;

$e(3, 8, 3, 8) \prec \#10 = M_{11} \prec \#\#86, 87$;

$e(3, 10, 3, 10) \prec \#12 = M_{12} \prec \#\#88 - 91$;

$\#43 \prec \#65$; $\#45 \prec \#\#67, 68$;

$K_5 - P_2 - P_3 \prec \#\#33, 104 = M_{25}$; $\#36 = 2\text{-}M_5^2$ (so, non 5-gonal, see §4.1).

#77 is easy to check. Remaining 15 polyhedra have many vertices; they are too cumbersome for to present them on figures. So we leave them to the reader and give here only a helpful clarification of them in terms given in Remark 1 below.

##50, 54 are orthobi-Cup_5, -M_9;   #51 is gyrobi-Cup_5;

##58, 62 are elongated orthobi-Cup_5, -M_9;

##57, 59, 63 are elongated gyrobi-Cup_4, -Cup_5, -M_9;

##43, 45 are gyroelongated Cup_4, M_9; #66 is gyroelongated bi-Cup_5;

#52=$Cup_5 + M_9$, #53=$Cup_5 + \overline{M_9}$ and #60=$Cup_5 + P_{10} + M_9$, #61=$Cup_5 + P_9 + \overline{M_9}$.

c) **l_1-embeddings**

For 9 polyhedra (##39–41, 44, 48, 70, 73, 75, 76) l_1-embeddings were found ad hoc. Now

$$\begin{array}{lll} \#24 = M_5 = Cup_4 \prec \#39, & \#25 = M_6 = Cup_5 \prec \#40, & \#26 = M_9 \prec \#41, \\ \#69 \prec \#70, \ \#72 \prec \#73, & \#74 \prec \#75, \ \#\#31, 82, 83 & \prec Ico, \\ \#84 \prec \text{ dual truncated } Do & \to \frac{1}{2}H_{16}; \ \#84 \to \frac{1}{2}H_7. & \end{array}$$

$\#\#78 - 81 \prec$ dual truncated $Ico \to \frac{1}{2}H_{10}$.

More precisely,

α) $\#\#31, 82 \to \frac{1}{2}H_6$, since

$APrism_5 \prec \#31 = APrism_5 + Pyr_5 \prec Pyr_5 + APrism_5 + Pyr_5 = Ico$,

$\#83 = M_7 \prec \#82 = M_7 + Pyr_5 \prec M_7 + 3Pyr_5 = Ico$ and $M_7, APrism_5, Ico \to \frac{1}{2}H_6$;

β) $\#\#78 - 81 \to \frac{1}{2}H_{10}$, since Do, dual truncated $Ico \to \frac{1}{2}H_{10}$,

$Do \prec \#78 = Do + Pyr_5 \prec \#79 = Pyr_5 + Do + Pyr_5 \prec$ dual truncated Ico,

$\#78 \prec \#80 = Do + 2Pyr_5 \prec \#81 = Do + 3Pyr_5 \prec$ dual truncated Ico.

We have

$$\begin{array}{lll}
\#21 = M_2 = Pyr_4 \to \frac{1}{2}H_4, & \#22 = M_3 = Pyr_5 \to \frac{1}{2}H_5, & \#32 = BPyr_3 \to \frac{1}{2}H_4, \\
\#27 = 1\text{-}M_3^2 \to \frac{1}{2}H_5, & \#28 = 1\text{-}M_4^2 \to \frac{1}{2}H_6, & \#29 = 1\text{-}M_5^2 \to \frac{1}{2}H_7, \\
\#34 = 2\text{-}\mathrm{M}_3^2 \to \frac{1}{2}H_6, & \#35 = 2\text{-}\mathrm{M}_4^2 \to \frac{1}{2}H_6. &
\end{array}$$

(See §4.1 for those embeddings.)

Now we present 9 embeddings ad hoc. Between them:

$$\begin{array}{ll}
\#39 = Cup_4 + Prism_8 \to \frac{1}{2}H_{10}, & \#40 = Cup_5 + Prism_{10} \to \frac{1}{2}H_{12}, \\
\#44 = Cup_5 + APrism_{10} \to \frac{1}{2}H_{11}, & \#48 = 2Cup_4 \to \frac{1}{2}H_8, \\
\#41 = M_9 + Prism_{10} \to \frac{1}{2}H_{13}. &
\end{array}$$

An embedding $\#48 \to \frac{1}{2}H_8$ is given on Fig.5a.

4 others are obtained by capping:

#70=biaugmented $Prism_3$ (i.e. capped on 2 square faces)$\to \frac{1}{2}H_5$, #73=biaugmented $Prism_5$ (on two non-adjacent square faces)$\to \frac{1}{2}H_7$; ##75,76=parabiaugmented, metabiaugmented $Prism_6$ (on 2 opposite or non-opposite non-adjacent square faces) $\to \frac{1}{2}H_8$. (If we relax definitions of ##73, 74, 76, 77 by permitting to cap adjacent square faces, then we get non-convex polyhedra having same l_1-status of the skeletons as original ones.) Elongated polyhedra ##39, 40, 41 are easy (see Remark 1). We leave #44 and above 4 capped polyhedra to the reader. □

Remark 1. For a polyhedron P (following, for example, pp.349–351 of [Ber71]) call $P+P$, $P+\overline{P}$, $P+Prism$, $P+APrism$, $P+Prism+P$, $P+Prism+\overline{P}$, $P+APrism+P$, respectively: orthobi-P, gyrobi-P, elongated-P, gyroelongated-P, elongated orthobi-P, elongated gyrobi-P, gyroelongated bi-P. Here ortho (gyro, respectively) means that two solids are joined together such that one of two bases is the orthogonal projection of (is turned relative to, respectively) the other. Prisms and anti-prisms above have an appropriate base and are adjoined by it. List of 92 regular-faced polyhedra contains orthobi-P and elongated orthobi-P for exactly $P = M_i$ $(i = 1-6, 9)$; it turns out that both have l_1-skeletons if $i = 1, 2, 5$ and both are non 5-gonal otherwise. All gyrobi-P, gyroelongated bi-P and elongated gyrobi-P in the list are non 5-gonal. Between gyroelongated polyhedra of the list, two (#31=$Pyr_5+APrism_5$ and #44=$Cup_5+APrism_{10}$) have l_1-skeletons and two (#30=$Pyr_4+APrism_4$ and #37=$Pyr_4+APrism_4+Pyr_4$) are extreme hypermetrics. Clearly, elongated P has l_1-skeleton iff P has. We have (see Fig.5) that $\#49 = Cup_4 + \overline{Cup_4}$ is non 5-gonal, while $\#48 = 2Cup_4 \to \frac{1}{2}H_8$ and elongated $\#49 = Cup_4+Prism_8+\overline{Cup_4}$ is not 5-gonal (see Remark 1 in §3.2) while elongated #48 (rhombicuboctahedron) $\to \frac{1}{2}H_{10}$.

Remark 2: duals of regular-faced polyhedra.

Examples of non 5-gonal ones are duals of ##23-25 (cupolas $M_4 - M_6$), #46 (gyrobifastigium); ##26, 83, 100, 103, 105, 109–112= M_i, (i = 9, 7, 14, 13, 28, 21, 24, 8, 20). M_{20}^* is even *strongly* non 5-gonal: its skeleton, as well as $K_{2,3}$, contains 5 vertices x, y, a, b, c with each of x, y being on the same shortest path between vertices of the pairs (a, b), (a, c), (b, c); both M_{20}^* and $K_{2,3}$ have only square faces. M_{28}^* has only (16) pentagonal faces. Other above M_i^* have more than one type of faces; for example, M_{24}^*, dual disphenocingulum, has only square and pentagonal faces.

Examples of l_1-graphs are duals of l_1-graphs ##21–22, 27–29, 32–35 and non 5-gonal ##36, $104 = M_{25}, 108 = M_{23}$. $M_{23}^* \to \frac{1}{2}H_{10}$, $M_{25}^* \to \frac{1}{2}H_8$; see Fig. 3g,h, M_{25}^* ([Wel84], p.75) together with a 17-hedron fills $\mathbb{R}^3$. Duals of ##34–36 are M_n^2 for $n = 3, 4, 5$; see §4.1.

Also, the duals of the extreme hypermetrics $\#\#30, 37 \to \frac{1}{2}H_8, \frac{1}{2}H_9$, respectively (see Fig.3), while duals of ##71, 106=M_{22}, 107 are non 5-gonal.

It is interesting to characterize pairs (P, P^*) of dual polytopes having both l_1-skeletons. Examples: (α_n, α_n), (β_n, γ_n), (1-M_n^m,1-M_n^m) for $n \in \{3, 4, 5\}$, (icosahedron, dodecahedron), (i-capped α_3, its dual) for $i \neq 4$, (i-capped β_3, its dual) for $i \leq 2$, ($P_m \times C_n$, 2-M_n^{m-1}) for $n \in \{3, 4\}$ (notation are from §4.1).

Remark 3. Besides *fullerenes* F_{20+2m} (see §4.5) majority of chemically relevant polyhedra (i.e. most frequent ones as arrangement of nearest neighbours in crystals, molecules or ions) are regular-faced. They include 8 deltahedra (see Fig.1 in [SHDC95] and Corollary 2 above), $Prism_3$ (also its dual and its augmentations ##69–71), α_3, Pyr_4, cuboctahedron, #104* and capped (see §4.5) anti-prisms (i-capped $APrism_n$ for $n = 3, 4$ and $0 \leq i \leq 2$; 2-capped $APrism_5$= icosahedron; 2-capped $APrism_6$, dual of the fullerene F_{24}, see §4.5. Regular-faced polyhedra are also used in large chemical literature on metal clusters; see, for example, [DDMP93] considering ##25, 35, 37, 47, 49 (see it on Fig. 5b), 55, 56, 58.

6 Zonotopes

Let e_i, $1 \leq i \leq n$, be n mutually orthogonal vectors of a same length. Let $e(S) = \sum_{i \in S} e_i$ for $S \subseteq N = \{1, 2, ..., n\}$. Then the convex hull of the vectors $e(S)$ for all $S \subseteq N$ is the n-cube γ_n. The 2^n points $e(S)$ are vertices of γ_n. If we project γ_n onto a k-dimensional space for $k \leq n$, we obtain a zonotope Z_n. The *zonotope* Z_n has the following three equivalent characterizations: it is 1) a projection of an n-cube, 2) a Minkowski sum of line segments, 3) a polytope having only centrally symmetric faces.

Let v_i be the projection of e_i. Without loss of generality we may assume that $v_i \neq 0$ for all $i \in N$. Denote the zonotope Z_n also by $Z(v_1, v_2, ..., v_n) = Z(v_i : i \in N)$ The point $v(S) = \sum_{i \in S} v_i$ is the projection of the vertex $e(S)$. Let all the vectors v_i go out from an origin. Then the origin is the point $v(\emptyset)$. We can take any point $v(S)$ as a new origin. This is equivalent to a change of signs of the vectors v_i for $i \in S$. With the origin in $v(S)$ the zonotope Z_n takes the form $Z(-v_i : i \in S; v_i : i \in N - S)$, and the point $v(T)$ is transformed into the point $v(S \Delta T)$.

The point $v(S)$ is a vertex of Z_n not for every S. There is the following simple criterion when a set S determines a vertex of Z_n. Since every vertex is an extreme point, for every vertex $v(S)$ of Z_n there is a k-vector c such that $cv(S) > cx$ for all

$x \in Z_n$, $x \neq v(S)$. (Here cx is the inner product of vectors c and x). In particular, this implies that $cv_i > 0$ for all $i \in S$, and $cv_i < 0$ for all $i \in N - S$. In other words, the k-dimensional hyperplane $\{x : cx = 0\}$ *supporting* the vertex $v(S)$ separates all vectors v_i for $i \in S$ from all other vectors.

It is easy to see that two vectors v and v' are separated by a hyperplane if and only if they have distinct directions, i.e. $v' \neq \lambda v$ for some real $\lambda \geq 0$. If v_k and v_j have a same direction, then $Z(v_i : i \in N) = Z(v'_k, v_i : i \in N - \{j,k\})$, where $v'_k = v_k + v_j$. Hence without loss of generality we may suppose that all vectors v_i have distinct directions.

We denote the family of all subsets S determining the vertices of Z_n by $\mathcal{S}_n$. If the origin is the vertex $v(\emptyset)$, then all vectors lie in a halfspace determined by the hyperplane supporting $v(\emptyset)$. Without loss of generality, we can suppose that the origin is a vertex of Z_n, i.e. $\emptyset \in \mathcal{S}_n$. Let $V \subseteq N$ be such that $v(\{i\}) = v_i$ is a vertex for all $i \in V$. Then Z_n lies in the conic hull of v_i for $i \in V$.

Lemma 6 *$S \cap V \neq \emptyset$, for every $S \in \mathcal{S}_n$.*

Proof. Let $S \in \mathcal{S}_n$, and let c be a vector such that $cv_k > 0$ for $k \in S$, and $cv_i < 0$ for $i \in N - S$. Every v_k lies in the cone generated by v_i, $i \in V$, i.e. $v_k = \sum_{i \in V} \lambda_i(k) v_i$ with $\lambda_i(k) \geq 0$. If $S \cap V = \emptyset$, then the inequality $cv_i < 0$ for all $i \in V$ implies the inequality $cv_k < 0$ for $k \in S$, a contradiction. $\square$

Proposition 6 *The skeleton $G(Z_n)$ of any zonotope Z_n is isometrically embeddable into the skeleton H_n of γ_n of whose projection Z_n is; n is the diameter of $G(Z_n)$.*

Proof. We prove that the natural map $v(S) \to e(S)$ for $S \in \mathcal{S}_n$ determines an l_1-embedding of the skeleton of Z_n into H_n. For this end, it is sufficient to prove that the distance between vertices $v(S)$ and $v(T)$ is equal to the symmetric difference $|S \Delta T|$. We proceed by induction on $m = |S \Delta T|$.

The assertion is trivially true for $m = 0$. By the construction of Z_n, each its edge which connects the vertex $v(S)$ with another vertex is parallel to the vector v_i for some $i \in N$.

Suppose there is an edge which connects $v(S)$ with $v(S')$ and is parallel to v_i for $i \in S \Delta T$. Then either $S' = S \cup \{i\}$ or $S' = S - \{i\}$, depending on $i \in T$ or $i \in S$, respectively. Clearly, in both the cases $|S' \Delta T| = m - 1$, and we can apply the induction step. Hence, for to prove the proposition, it is sufficient to prove that $v(S)$ is incident to an edge parallel to v_i for $i \in S \Delta T$. But this is implied by Lemma 6 if we take $v(S)$ as a new origin. $\square$

Komei Fukuda kindly permitted to us to see and to refer preliminary version of [Fuk95] containing (within his treatment of oriented matroid) a nice example of a centrally symmetric but non zonohedral polyhedron F with $G(F) \to H_9$. The Fukuda's polyhedron contains 54 vertices, 16 hexagonal and 20 square faces. Fukuda constructed it in the dual form as a nonlinear (non-Pappus) extension of an oriented rank 3 matroid on 8 points. As a linear extension of the same matroid he obtain a dual zonohedron Z_9. (Z_9 has 52 vertices, 18 hexagonal and 14 square faces.) Comparing Fukuda's picture for (duals of) Z_9 and F, we realize that F comes from Z_9 by the following general construction.

Let $C{=}C_{1,...,6}$ and $C'{=}C_{1',...,6'}$ be two opposite hexagonal faces of a centrally symmetric polyhedron P, and $(1,1'),...,(6,6')$ are 6 pairs of antipodal vertices of P. Denote by $Q(P)$ the polyhedron obtained from P by adding two new vertices v, v' and edges $(v,1)$, $(v,3)$, $(v,5)$, $(v',1')$, $(v',3')$, $(v',5')$. Clearly, $G(P)$ is centrally symmetric (v and v' are antipodal) and $G(P) \prec G(Q(P))$, i.e. the skeleton of P is an isometric subgraph of the skeleton of $Q(P)$. If $P \to H_n$ (it is so if P is a zonotope), then $Q(P) \to H_n$ also. (In fact, let $a(1) = \emptyset$, $a(3) = \{ij\}$, $a(5) = \{ik\}$. Then $a(v) = \{i\}$ is uniquely determined. No other edge (v,u) is possible, since otherwise $a(u) = \{it\}$ for $t \neq j,k$, and we get a contradiction with C being a hexagonal face.)

Define by induction $Q^m(P) = Q(Q^{m-1}(P))$. Let $ElDo$ denote the elongated dodecahedron. Using Remark 2 below one can check that (combinatorially) $Q(Prism_6) =$ rhombic dodecahedron, $Q^2(Prism_6) = \gamma_4$, $Q^2(ElDo) =$ rhombic icosahedron, Q^4 (truncated β_3)= triacontahedron. It will be interesting to see whether Fukuda's polyhedron is minimal for parameters of non-zonohedral $Q(P)$ obtained from a zonohedron P.

Another similar operation is as follows. Select two opposite $2m$-faces $C = C_{1,...,2m}$ and $C' = C_{1',...,(2m)'}$. Take a path $P_{1,i_1,...,i_{m-2},m+2}$ of length $m-1$ connecting the vertices 1 and $m+2$ of C. Add new edges $(i_k, k+2)$, $1 \leq k \leq m-2$. Make the same operation with the face C'.

Let P denote the rhombic dodecahedron with a deleted vertex of degree 3, i.e. P is $Prism_6$ with a new vertex connected to 3 nonadjacent vertices of a 6-cycle. Clearly, P is nonzonohedral polyhedron and $G(P) \to H_n$ (for $n = 4$); probably, P is minimal for this property. Also $G(P)$ is (one of the four) smallest non-Hamiltonian polyhedral graphs realizable as Delaunay tesselations, [Dil96], Fig. 10 and 11. A propos, graphs of Fig. 5–12 of [Dil96] are non 5-gonal, except, 5c $\to \frac{1}{2}H_8$, 8a $\to \frac{1}{2}H_6$, 8b $\to \frac{1}{2}H_7$, 10b $\to H_4$.

Remark 1. Examples of embeddings of zonotopes into hypercubes are

1) 5 of Archimedean and their dual (Catalan) polyhedra are zonohedra: truncated $\beta_3 \to H_6$, truncated cuboctahedron $\to H_9$, truncated icosidodecahedron $\to H_{15}$, dual cuboctahedron (= rhombic dodecahedron) $\to H_4$, dual icosidodecahedron (= triacontahedron) $\to H_6$. Between their duals only the dual of the first one has l_1-skeleton.

2) All 5 (combinatorially) Voronoi polyhedra are zonohedra:

3-cube=H_3, rhombic dodecahedron $\to H_4$, $Prism_6 \to H_4$, elongated dodecahedron $\to H_5$, truncated $\beta_3 \to H_6$.

3) All 5 "golden isozonohedra" of Coxeter (zonohedra with all faces being rhombic with the diagonals in golden proportion) are:

2 types of hexahedrons (equivalent to γ_3) $\to H_3$, rhombic dodecahedron $\to H_4$, rhombic icosahedron $\to H_5$, triacontahedron $\to H_6$.

4) Some infinite families of zonotopes are:

$Prism_{2m} \to H_{m+1}$, $\gamma_m = H_m$,

polar zonohedra $PZ_m \to H_m$ (γ_3, rhombic dodecahedron, rhombic icosahedron for $m = 3, 4, 5$, respectively, see [Cox73]); PZ_m^* is non 5-gonal for $m \geq 4$ (see Fig.1e for $m = 5$).

An m-dimensional permutahedron (the Voronoi polytope of the lattice A_m^*) $\to H_{\binom{m+1}{2}}$. It is C_6, truncated β_3 for $m = 2, 3$, respectively.

Remark 2. For an isometric m-vertex subgraph G of an n-cube (which is not an isometric subgraph of H_{n-1}) denote by $\tilde{G}$ the subgraph of H_n induced by $2^n - m$ remaining vertices. If G is the skeleton of a zonotope $Z \to H_n$, then $\tilde{G}$ can be :

a) $\cong G$ if $m = 2^{n-1}$ (examples are γ_{n-1} and triacontahedron);

b) an isometric subgraph of H_n (for example, $\tilde{G} = C_{10}$ for rhombic icosahedron and $\tilde{G} = C_{1,\dots,10} + P_{1,11,3,12,5} + P_{6,13,8,14,10}$ for elongated dodecahedron (see Fig 11a);

c) or not isometric subgraph of a hypercube (for example, $\tilde{G} = 2K_1$ for rhombic dodecahedron, $\tilde{G} = 2K_2$ for $Prism_6$ and $\tilde{G}$ is a 40-vertex non isometric subgraph of H_6 for truncated β_3).

Remark 3. See [DeSt96] for similar isometric embeddings of skeletons of *infinite* zonohedra (plane tilings) into *cubic lattices* $\mathbb{Z}_n$. For example, regular hexagonal tiling and dual Archimedean tiling [3,6,3,6] are embeddable into $\mathbb{Z}_3$, Archimedean tiling (4,6,12) is embeddable into $\mathbb{Z}_6$, Penrose aperiodic rhombic tiling is embeddable into $\mathbb{Z}_5$.

7 Delaunay polytopes

Here we consider Delaunay polytopes of dimension at most 4 and some operations on Delaunay polytopes (see a definition of a Delaunay polytope in §2.1).

For $n = 2$, Delaunay polytopes have the skeletons $C_3 \to \frac{1}{2}H_3$ and $C_4 = H_2$.

For $n = 3$, skeletons of all 5 Delaunay polytopes are also l_1-graphs:

$\gamma_3 = H_3$, $\alpha_3 = \frac{1}{2}H_3$, $\beta_3 \to \frac{1}{2}H_4$, $Pyr_4 \to \frac{1}{2}H_4$, $Prism_3 \to \frac{1}{2}H_5$.

All 19 types of Delaunay 4-polytopes are given in [ErRy87]. The # i, $1 \leq i \leq 16$, and the letters A,B,C below are the notation from [ErRy87]. (Elsewhere than this section, # means the number of a polyhedron in the list of 92 regular-faced polyhedra.) We get the following proposition by direct check.

Proposition 7 *1) #16=*$\gamma_4 = \alpha_1 \times \gamma_3 = H_4$,

2) $P \to \frac{1}{2}H_4$ *for the following P:*

#2=$Pyr(Pyr_4)$ *with* $G(\#2) = K_{2,2,1,1}$, *B=*$Pyr(\beta_3)$ *with* $G(B) = K_{2,2,2,1}$,

C=$BPyr(\beta_3) = \beta_4 = \frac{1}{2}H_4$;

3) $P \to \frac{1}{2}H_5$ *for the following P:*

#1=$Pyr(\alpha_3) = \alpha_4$, *#3=*$Pyr(Prism_3)$, *#5, #7=*$\alpha_1 \times \alpha_3 = Prism(\alpha_3)$, *#9, #13 with* $G(\#13) = T(5)$,

4) $P \to \frac{1}{2}H_6$ *for the following P:*

#10= $\alpha_2 \times \alpha_2$, *#11=*$\alpha_1 \times Pyr_4 = Prism(Pyr_4)$, *#15=* $\alpha_1 \times \beta_3 = Prism(\beta_3)$,

A with $G(A) = K_6$ (the cyclic 4-polytope);
5) $P \to \frac{1}{2}H_7$ for $P = \#14 = \alpha_1 \times Prism_3 = Prism(Prism_3)$;
6) the following P are non 5-gonal:
#4, #6= $BPyr(Prism_3)$, #8= $Pyr(\gamma_3)$, #12=$BPyr(\gamma_3)$.

Proof. Recall that a graph G is non 5-gonal if $K_5 - K_3 \prec G$ or $K_5 - P_2 - P_3 \prec G$ with $P_2 \cap P_3 = \emptyset$. Hence 5) above is implied by that the skeletons of #8=$Pyr(\gamma_3)$ and #12=$BPyr(\gamma_3)$ each contains the isometric subgraph $K_5 - K_3$. The skeletons of #4 and #6=$BPyr(Prism_3)$ contain the isometric subgraph $K_5 - P_2 - P_3$. All embeddings in 1)-4) above, except # #3,5,9, either are trivial or come from the direct product construction below. The skeleton $G(\#k)$ of the four polytopes $\#k$ in 3) for $k = 3, 5, 9, 13$ are isometric subgraphs of the triangular graph $T(5)$. In fact $G(\#3) = T(5)-K_3$, $G(\#5) = T(5)-(v, v')$, $G(\#9) = T(5)-v$, and $G(\#13) = T(5)$. Here $T(5)-K_3$ is the graph $T(5)$, where any 3 mutually adjacent vertices are deleted, $T(5) - (v, v')$ is $T(5)$, where any two nonadjacent vertices are deleted, and $T(5) - v$ is $T(5)$ where a vertex is deleted (vertices are deleted with incident edges). In other words, the vertices of the polytopes $\#k$, $k = 3, 5, 9, 13$, are labeled by pairs ij, $1 \leq i < j \leq 5$. For example, #3=$Pyr(Prism_3)$ has the labels #45 and $i4, i5$, $i = 1, 2, 3$. From this embedding of $Pyr(Prism_3)$ we obtain an embedding of $Pyr^2(Prism_3)$ if we label the new apex by the set $\emptyset$. Remark, that #13 is combinatorially equivalent to a semi-regular 4-polytope *ambo*-α_4 (0_{21} in the Coxeter's notation). □

Also #8=$Pyr(\gamma_3)$ is the graph of the unit cell of the body-centered orthorhombic crystal system; the graph of all other systems are also not 5-gonal, except simple one ($\gamma_3 \to H_3$) and end-face centered ($Pyr_4 + \gamma_3 + Pyr_4 \to \frac{1}{2}H_6$). See [DeDeGr96] for more detail on it and other chemical applications.

It is well known that the direct product $P \times P'$ of Delaunay polytopes P and P' is a Delaunay polytope, and $G(P \times P') = G(P) \times G(P')$.

Also, for every Delaunay polytope P there is a pyramid $Pyr(P)$ and (if P is centrally symmetric) a bipyramid $BPyr(P)$ which are Delaunay polytopes (see [DeGr93]). The direct product construction preserves l_1-ness of Delaunay polytopes. For example, $G(P \times P') \to \frac{1}{2}H_{m+m'}$ if $G(P) \to \frac{1}{2}H_m$, $G(P') \to \frac{1}{2}H_{m'}$. But, the pyramid and bipyramid constructions can produce non 5-gonal Delaunay polytopes from those with l_1-skeletons. Consider, for example, the following Delaunay polytopes:

α_n, β_n, γ_n, $\frac{1}{2}\gamma_n$ ($n \geq 5$), *ambo*-α_n ($n \geq 4$), the Johnson n-polytope P_J with $G(P_J) = J(n+1, k)$ ($3 \leq k \leq \lfloor (n+1)/2 \rfloor$), $\alpha_{n-1} \times \alpha_{n-1}$ for $n \geq 3$; $Prism_3, Pyr_4 \subset \mathbb{R}^3$.

All of them have l_1-skeletons: see §3.1 for regular ones, also *ambo*-$\alpha_n \to \frac{1}{2}H_{n+1}$, $P_J \to \frac{1}{2}H_{n+1}$, $Prism_3 \to \frac{1}{2}H_5$, $Pyr_4 \to \frac{1}{2}H_4$. The pyramid and bipyramid constructions produce from them l_1-polytopes in the following cases:

$Pyr^m(\alpha_n)$, $Pyr^m(\beta_n)$, $BPyr^m(\beta_n)$, $Pyr^m(Pyr_4)$ (see §2).

Additionally we have the following l_1-embeddings:

$Pyr^2(Prism_3) \to \frac{1}{2}H_5$, $Pyr(ambo\text{-}\alpha_n) \to \frac{1}{2}H_{n+1}$, $Pyr(\alpha_{n-1} \times \alpha_{n-1}) \to \frac{1}{2}H_{2n}$.

The inclusions (i), (ii) below show that the skeletons of the mentioned there polytopes are non 5-gonal, and therefore they are not l_1-graphs:

(i) $K_5 - K_3 \prec Pyr(\gamma_n)$, $BPyr(\gamma_n)$, $Pyr^2(ambo\text{-}\alpha_n, n \geq 6)$, $Pyr(P_J)$ (since $K_{1,k} \prec J(n+1,k)$), $Pyr(\frac{1}{2}H_n, n \geq 6)$, $Pyr^2(\alpha_{n-1} \times \alpha_{n-1})$, $BPyr(ambo\text{-}\alpha_n, n \geq 6,\ \text{even})$, $BPyr(\frac{1}{2}H_n, n \geq 6\ \text{even})$;

(ii) $K_5 - P_2 - P_3 \prec BPyr_3(Prism_3)$.

Finally, the following skeletons are (extreme) hypermetrics (see Proposition 4.4 of [DeGr93]): $Pyr(\frac{1}{2}\gamma_5)$, $Pyr^2(\frac{1}{2}\gamma_5)$, $Pyr^2(ambo\text{-}\alpha_4)$, $Pyr^3(ambo\text{-}\alpha_4)$, $Pyr^3(Prism_3)$, $Pyr^4(Prism_3)$.

8 Small l_1-polyhedra and examples of polytopal hypermetrics

Combinatorial types of d-polytopes were enumerated for small values of $v-d$, where v is the number of vertices; see, for example, §3.16 and Tables 1,2 on p.424 of [Grü67]. The two propositions below give l_1-status for polyhedra of the first two simple classes.

Proposition 8 *(i) For all 10 combinatorial types of polyhedra with ≤ 6 vertices, we have*

a) 4 skeletons:

$G(\alpha_3) = K_4$, $G(Pyr_4) = \nabla C_4$, $G(BPyr_3 = \text{1-capped } \alpha_3) = K_5 - e$, $G(\beta_3) = K_{3\times 2} \to \frac{1}{2}H_4$;

b) 4 skeletons:

$G(Prism_3) = K_6 - C_6$, $G(Pyr_5) = \nabla C_5$, $K_6 - P_5$, $G(\text{2-capped } \alpha_3) = K_6 - P_4 \to \frac{1}{2}H_5$;

c) 2 skeletons, $K_6 - P_6$ and $K_{3\times 2} - e$, are non 5-gonal (each contains $K_5 - P_2 - P_3$).

(ii) For all 10 combinatorial types of polyhedra with ≤ 6 faces (duals of above), we have:

a) α_3, Pyr_4, Pyr_5 and one with the skeleton $K_6 - P_6$ are self-dual, $Prism_3 = BPyr_3^*$ and $\gamma_3 = \beta_3^*$;

b) $P^* \to \frac{1}{2}H_6$ if $G(P) = K_6 - P_4$;

c) P^* is non 5-gonal if $G(P) = K_6 - P_5$, $K_{3\times 2} - e$ (Fig.6).

One can check that i-capped α_3, dual $(i-1)$-capped $\alpha_3 \to \frac{1}{2}H_{i+3}$ for $1 \leq i \leq 4$, but dual 4-capped α_3 (truncated α_3) is non 5-gonal. Also i-capped $\beta_3 \to \frac{1}{2}H_{i+4}$, $0 \leq i \leq 8$, and dual i-capped $\beta_3 \to \frac{1}{2}H_{i+6}$ for $0 \leq i \leq 2$. Compare with i-capped $\gamma_3 \to \frac{1}{2}H_6$, $0 \leq i \leq 2$ and for $i = 3$ if no 2 opposite faces are capped; it is not 5-gonal otherwise.

Remark that extreme hypermetrics graphs G_1 and G_2 are skeletons of 4-pyramids with bases Pyr_5 and 2-capped α_3, see (i) b) above.

Proposition 9 *(i) For all 5 combinatorial types of simplicial polyhedra with 7 vertices, we have:*

a) 1-capped $\beta_3 \to \frac{1}{2}H_5$*, 3-capped* $\alpha_3 \to \frac{1}{2}H_6$*;*

b) skeletons $G(BPyr_5)$*,* $\nabla^2 P_5$ *are non 5-gonal (they contain* $K_5 - P_2 - P_3$*,* $K_5 - K_3$*, respectively);*

c) skeleton ∇B_8 *of* $Pyr(B_8)$ *is the extreme hypermetric* G_4*.*

(ii) For all 5 combinatorial types of simple polyhedra with 10 vertices (dual of above), we have:

a) $P^* \to \frac{1}{2}H_7$ *if* P *is a 1-capped* β_3*, 3-capped* α_3 *or* $BPyr_5$ *(Fig.7);*

b) P^* *is non 5-gonal if* $G(P) = \nabla^2 P_5$*,* ∇B_8 *(Fig.6).*

There are 34 combinatorial types of polyhedra with 7 vertices: 2,8,11,8,5 having 6,7,8,9,10 faces, respectively. The last 5 are simplicial ones. The first two are both non 5-gonal (Proposition 8 (ii) c)).

One can enumerate isometric subgraphs (and polytopal ones between them) of given $\frac{1}{2}H_n$ which are not isometric subgraphs of a facet of it. For example, the following graphs are such isometric polytopal subgraphs of the Clebsh graph $\frac{1}{2}H_5$:

1) C_5, $K_5 = G(\alpha_4)$ (together with 2) below they are only such subgraphs of $\frac{1}{2}H_5$ on $m \le 6$ vertices);

2)$K_6 - C_6 = G(Prism_3)$, $K_6 - P_5$ (polyhedral, see Fig. 6.2), $K_6 - P_4 = G(\text{2-capped } \alpha_3)$, $\nabla C_5 = G(Pyr_5)$ (between all 10 6-vertex isometric subgraphs of $\frac{1}{2}H_5$);

3) $\ge$ 7-vertex polyhedral; #27 (1-capped $Prism_3$), #60 (augmented $Prism_3$), #70 (biaugmented $Prism_3$), 1-capped β_3, $APrism_4$;

4) 4-polytopal: $Pyr(Prism_3)$, $\alpha_1 \times \alpha_3$, $T(5)$, $T(5) - K_1$, $T(5) - \overline{K_2}$, 1-capped (on a facet β_3) $T(5)$;

5) 5-polytopal: $Pyr^2(Prism_3)$, $\nabla T(5) = G(Pyr(\text{ambo-}\alpha_4))$.

On the other hand, all hypermetric but not l_1-graphs with 7 vertices are known (see §7 of [DGL95]); they are 12 graphic metrics between 26 extreme hypermetrics.

Proposition 10 *Between all the 12 hypermetric graphs on 7 vertices, the polytopal ones are only 3-polytopal* $G_4 = \nabla B_8$ *and 4-polytopal* $G_1 = \nabla^2 C_5 = K_7 - C_5$*,* $G_2 = \nabla H_2 = K_7 - P_4$*.*

Proof. In fact, $G_6 = \nabla H_4$, $G_8 = \nabla B_5$, G_{16}, G_{24}, G_{26} have minimal degree 2. $G_5 = \nabla B_7$, $G_7 = \nabla H_3$ and $H_1 = K_6 - P_3$ have minimal degree 3 and are not planar; so $G_3 = \nabla H_1$ is not polytopal also. Finally, G_{18} is planar and has minimal degree 3, but it can be disconnected by deleting 2 vertices. It is the skeleton of a *skew* (with a non-planar face) 3-polytope. □

Metric of a graph which is hypermetric but not l_1-graph is necessarily an extreme hypermetric. The number of vertices of any extreme hypermetric graph is within [7,56]. Any polytope such that its skeleton is extreme hypermetric, has dimension within [3,7]. Call an extreme hypermetric graph of type I (of type II) if it generates the root lattice E_6 (E_7, respectively). A graph G of type I has diameter 2, since it is an induced subgraph of the Schläfli graph $G(2_{21})$ of diameter 2, and ∇G is an extreme hypermetric of type II.

Clearly, $G(Pyr(P)) = \nabla G(P)$, dim($Pyr(P)$)=dim$P$+1. (Remark that $Pyr^k(C_5) \to \frac{1}{2}H_5$ for $k \in \{0,1\}$, it is extreme hypermetric for $k \in \{2,3\}$, and it is 7- but not 9-gonal for $k = 4$.) Examples of polytopal graphs G which are extreme hypermetric but are not represented as $\nabla G'$ for an extreme hypermetric graph G' are (see §5):

of type I: the polyhedra ## 30, 71, 106 $=M_{22}$ and one with skeleton G_4 (see Fig.2);
the 4-polytopes with skeletons G_1, G_2; the 6-polytopes with skeletons $K_9 - C_6 = G(Pyr^3(Prism_3))$, $\nabla^2 T(5)= G(Pyr^2(\text{ambo-}\ \alpha_4))$, $\nabla \frac{1}{2}H_5$, the Shläfli polytope 2_{21} having 9,9,7; 7,7; 9,12,17,27 vertices, respectively.

of type II: the polyhedra #37, #107= $M_{21} + Pyr_4$ and the 7-polytope 3_{21} with 10, 11 and 56 vertices, respectively.

Some examples of extreme hypermetrics which are not polytopal, but are close to polytopal in some sense:

a) 7-vertex graph G_{18} of type I is a planar graph of a *skew* polyhedron.

b) The skeleton of the stella octangula (the section of γ_3 by β_3 with vertices in the centers of faces of γ_3) which is a *non-convex* polyhedron; it is 14-vertex graph. It contains the induced extreme hypermetric G_x. It is an isometric subgraph of the Goseet graph, and since it has diameter 3 it is of type II.

c) Antiwebs AW_9^2, AW_{12}^3, are of type I, and AW_{13}^3 is of type II (see §4.4(iv)).

9 t-embeddings in l_1

A *t-embedding* of a distance $\|d_{ij}\|$ is an embedding of the distance $\|d'_{ij}\| = \|min(t, d_{ij}\|$ in an l_1^n-space.

In what follows, describing a t-embedding of a polyhedron P, we associate to every its vertex v a subset $a(v)$ of a set N. Usually we take as N the set of all k-gonal faces. We say that a face F is *reachable* by an m-path from a vertex v if there is an m-path of length m from the vertex v to a vertex of the face F.

Truncated icosahedron (of diameter 9) has *unique* ([DeSp96]) 7-embedding into $\frac{1}{2}H_{20}$: associate each vertex to 2+2+3 hexagons (from all 20) reachable by 0-,1-,2-paths, respectively. It is also the unique 3-embedding, but not unique 2-embedding: for example, associate every vertex to 2 its hexagons.

Icosidodecahedron (of diameter 5) has *unique* ([DeSp96]) 4-embedding in $\frac{1}{2}H_{12}$: associate every vertex to 1+1+2 pentagons (from all 12) reachable by 0-,1-,2-paths. It is also a unique 3-embedding, but there is another 2-embedding: associate every vertex to 2 its triangular faces.

Cuboctahedron (of diameter 3) has at least two following 2-embeddings:

in $\frac{1}{2}H_6$: associate every vertex to its two square faces,

in $\frac{1}{2}H_8$: associate every vertex to its two triangular faces; this one, as well as two above ones are t-embeddings into $\frac{1}{2}H_{2D(P)+2}$.

Any simple polyhedron has a 2-embedding in the tetrahedral graph $J(n,3)$ (so in $\frac{1}{2}H_n$): associate every vertex to its 3 faces. If the diameter of a simple polyhedron is at least 3, then it is 3-embeddable iff sizes of its faces are from $\{3,4,5\}$. Examples of this procedure are:

1) dodecahedron (of diameter 5) has a 3-embedding into $J(12,3)$; also it has a 5-embedding into $\frac{1}{2}H_{10}$,

2) $\alpha_3 \to J(4,3) \to \frac{1}{2}H_4$ (not unique), $M^*_{25} \to J(8,3) \to \frac{1}{2}H_8$ (see Fig. 3g),

3) it gives a 2-embedding of $Prism_n$ which turns out to be $Prism_n \to \frac{1}{2}H_{n+2}$ ($\to H_{\frac{n+2}{2}}$ for even n).

Another procedure: fix a 5-wheel ∇C_5 in the skeleton of an icosahedron and associate every vertex v to the set of all vertices of the 5-wheel at distances 0 and 1 from v; we get a (unique) 3-embedding of the icosahedron.

Another interesting relaxation of our embeddings will be to consider scale-isometric embedding into hypercubes of 1-skeletons of simplicial and cubical *complexes* more general than the boundary complexes of polytopes. For example, the simplicial complex on $\{1,2,3,4,5\}$ with the facets $\{1,2,3\}$, $\{1,2,4\}$, $\{1,2,5\}$ has the skeleton $K_5 - K_3$; the cubical complex on $\{1,2,3,4,5,6\}$ with the facets $\{1,2,3,4\}$, $\{2,3,5,6\}$, $\{1,4,5,6\}$ has the skeleton $K_{3,3}$. So, both are non 5-gonal.

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APPENDIX

Here we give figures of skeletons of polyhedra (except the 4-polytope on Fig1f). For non hypermetric graphs we indicate on the skeleton points violating a $(2k+1)$-gonal inequality. Actually, it is always 5-gonal, except Fig.1c,1f). When figures show l_1-polyhedra, the embeddings are into $\frac{1}{2}H_n$ (except the embedding in H_5 on Fig.11a). They are shown as follows: we label a vertex by the sequence $i_1, ..., i_m$ (or $\overline{i_1, ..., i_m}$) if this vertex is addressed to the set $\{i_1, ..., i_m\}$ (or $\overline{\{i_1, ..., i_m\}}$, respectively. Since $m \leq 15$ on our figures, we use 1,...,9,0,a,...,e as addresses.

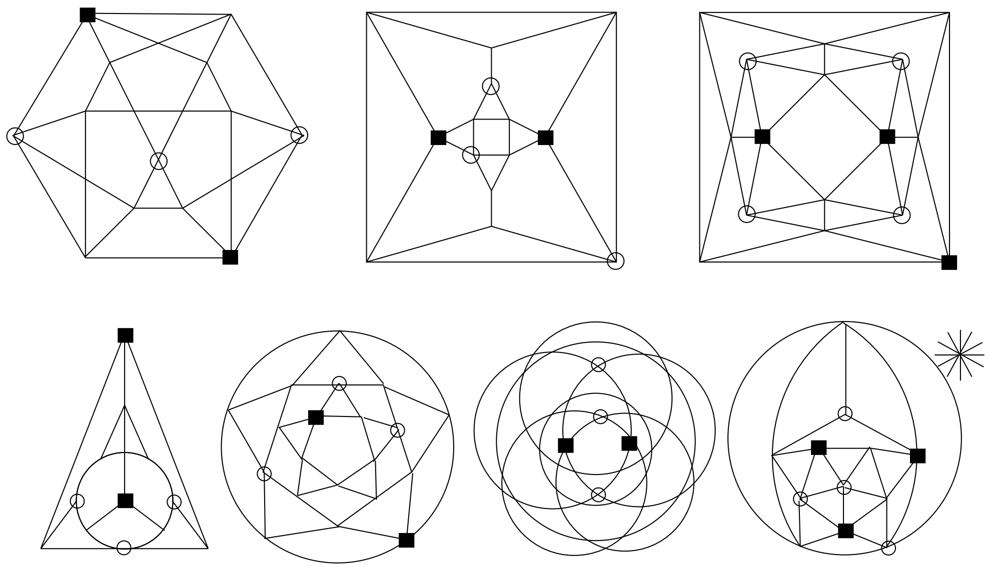

Figure 1: Non hypermetric polytopes: a) M_{20} = #112, triangular hebesphenorotunda, b) M_8 = #111, bilunabirotunda, c) M_{28} = #105, snub $APrism_4$, d) #71*, e) PZ_5^*, dual rhombic icosahedron, f) Koester's graph, g) 4-polytope Pyr(icosahedron)

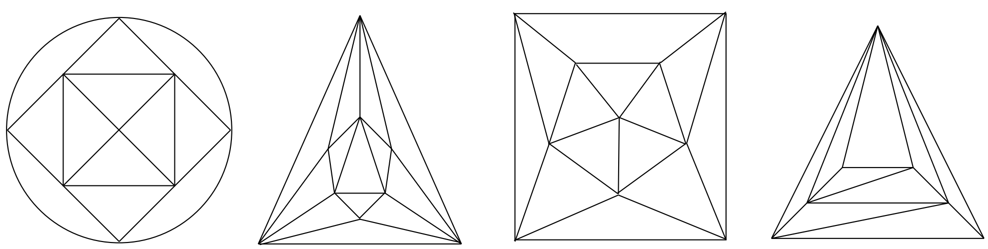

Figure 2: Hypermetric non l_1- polyhedra: a) #30 = 1–$APrism_4$, b) #71, triaugmented $Prism_3$, c) M_{22} = #106, sphenocorona,

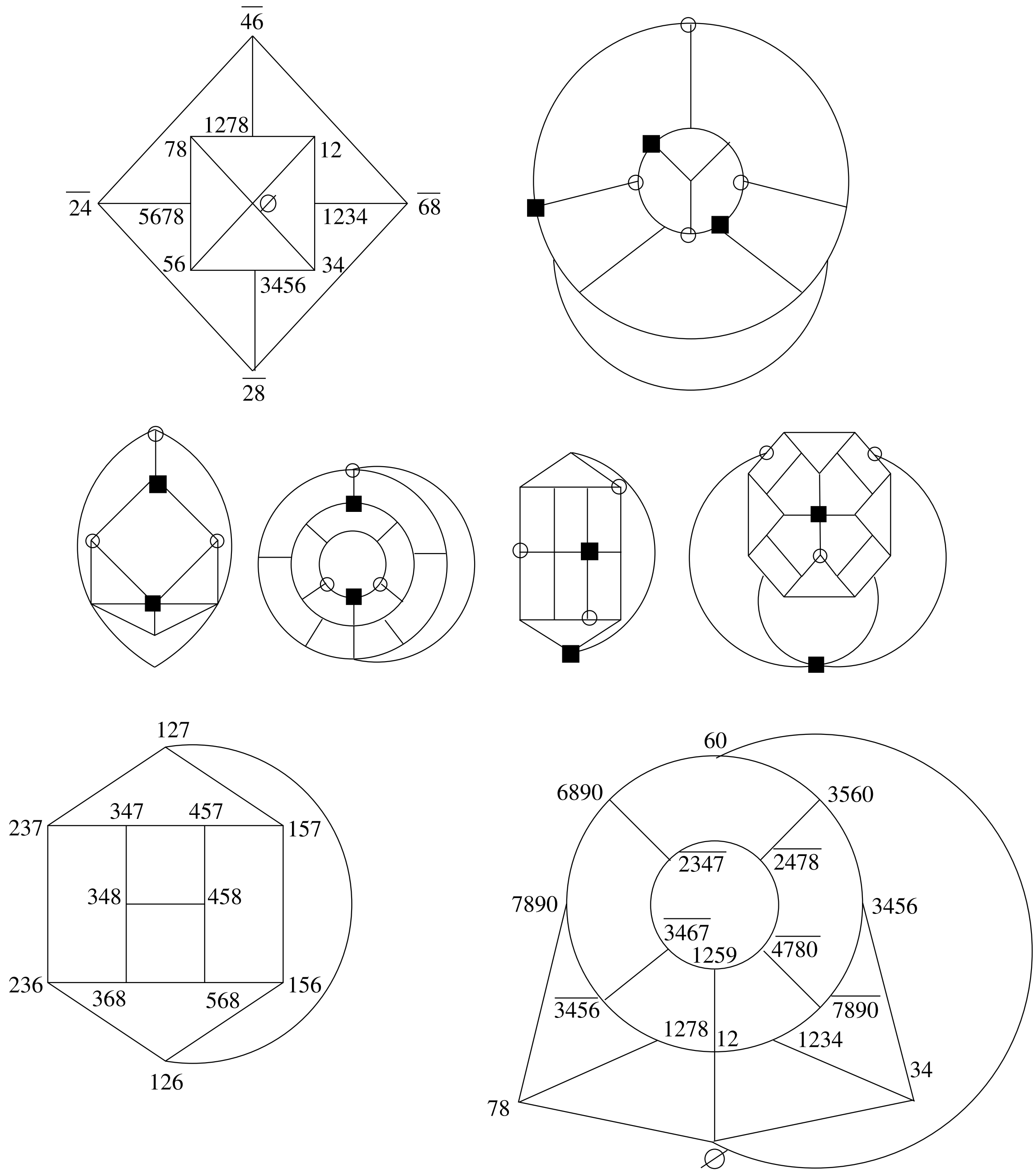

Figure 3: a) Dual $1{-}APrism_4 \to \frac{1}{2}H_8$, b) dual $2{-}APrism_4$, c) M_7^*, dual tridiminished icosahedron, d) M_{21}^*, dual hebesphenomegacorona, e) M_{22}^*, dual sphenocorona, f) M_{28}^*, dual snub $APrism_4$, g)M_{25}^*, dual snub dispensoid$\to\frac{1}{2}H_8$, h) M_{23}^*,dual sphenomegacorona$\to\frac{1}{2}H_{10}$

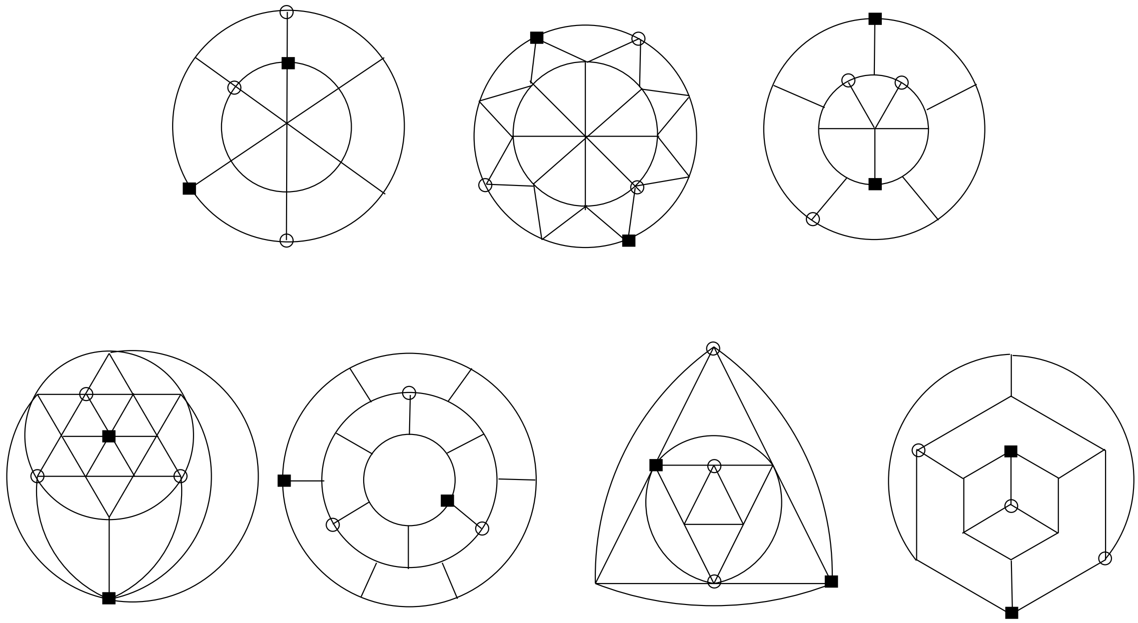

Figure 4: Prismatic non-l_1 graphs with minimal n: a) 1–M_6^2, b) 1–$APrism_8$, c) dual 1–$APrism_5$, d) 2–$APrism_6$, e) dual 2–$APrism_6 = F_{24}$, f) $Tow_3^2 = \beta_3 + \beta_3$, g) dual Tow_3^2

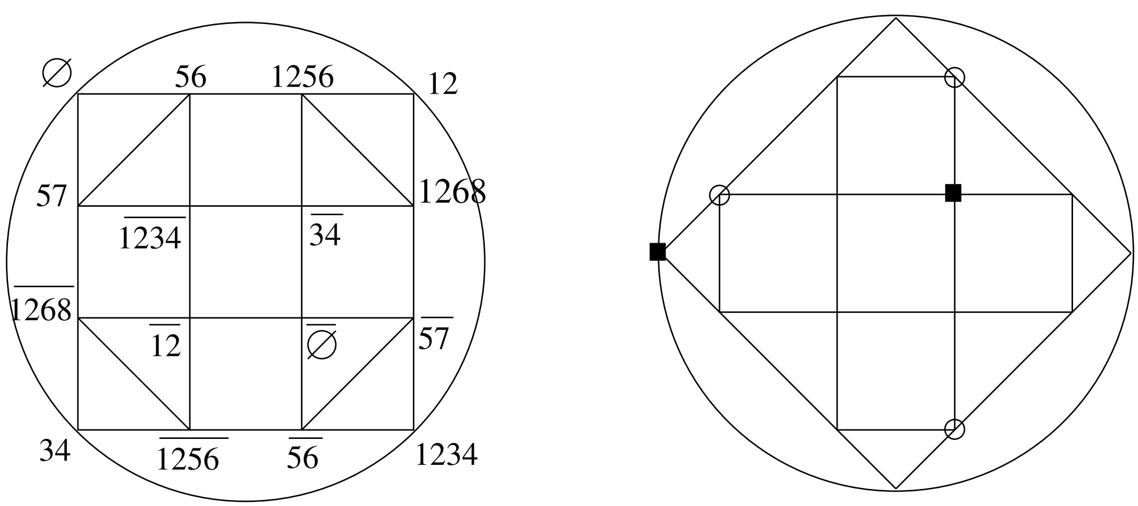

Figure 5: Square ortho- and gyro-bicupolas: $\#48 = Cup_4 + Cup_4 \to \frac{1}{2}H_8$ and $\#49 = Cup_4 + \overline{Cup_4}$, non 5-gonal.

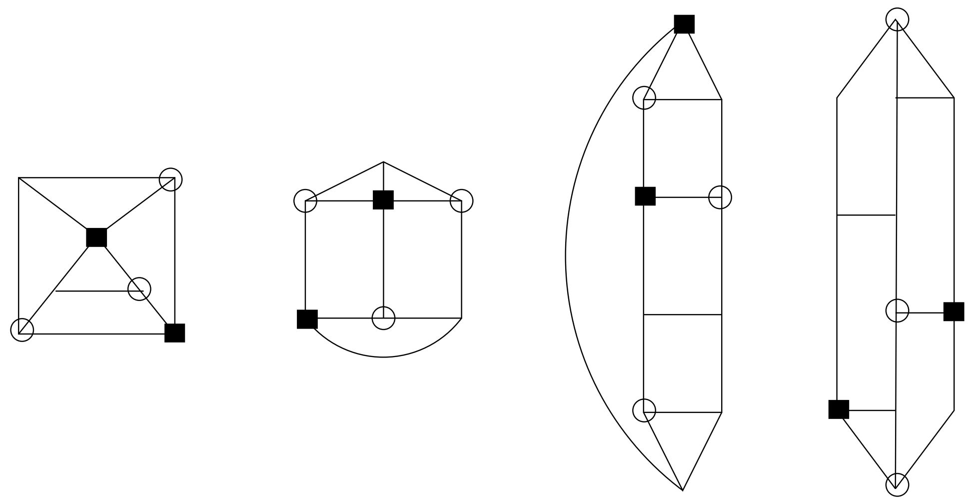

Figure 6: Dual small non 5-gonal polyhedra P with $G(P^*) = K_{3\times 2} - e$, $K_6 - P_5$, $\nabla^2 P_5$, ∇B_8

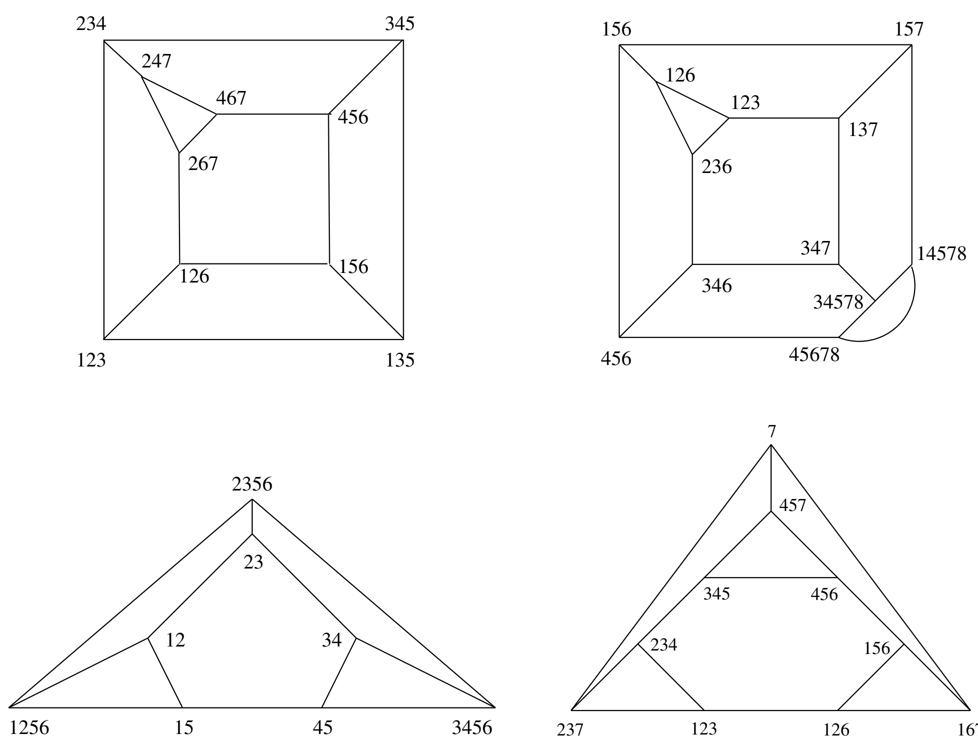

Figure 7: Dual l_1-polyhedra: dual 1-, 2-capped β_3, dual 2-, 3-capped $\alpha_3 \rightarrow \frac{1}{2}H_7, \frac{1}{2}H_8, \frac{1}{2}H_6, \frac{1}{2}H_7$ resp.

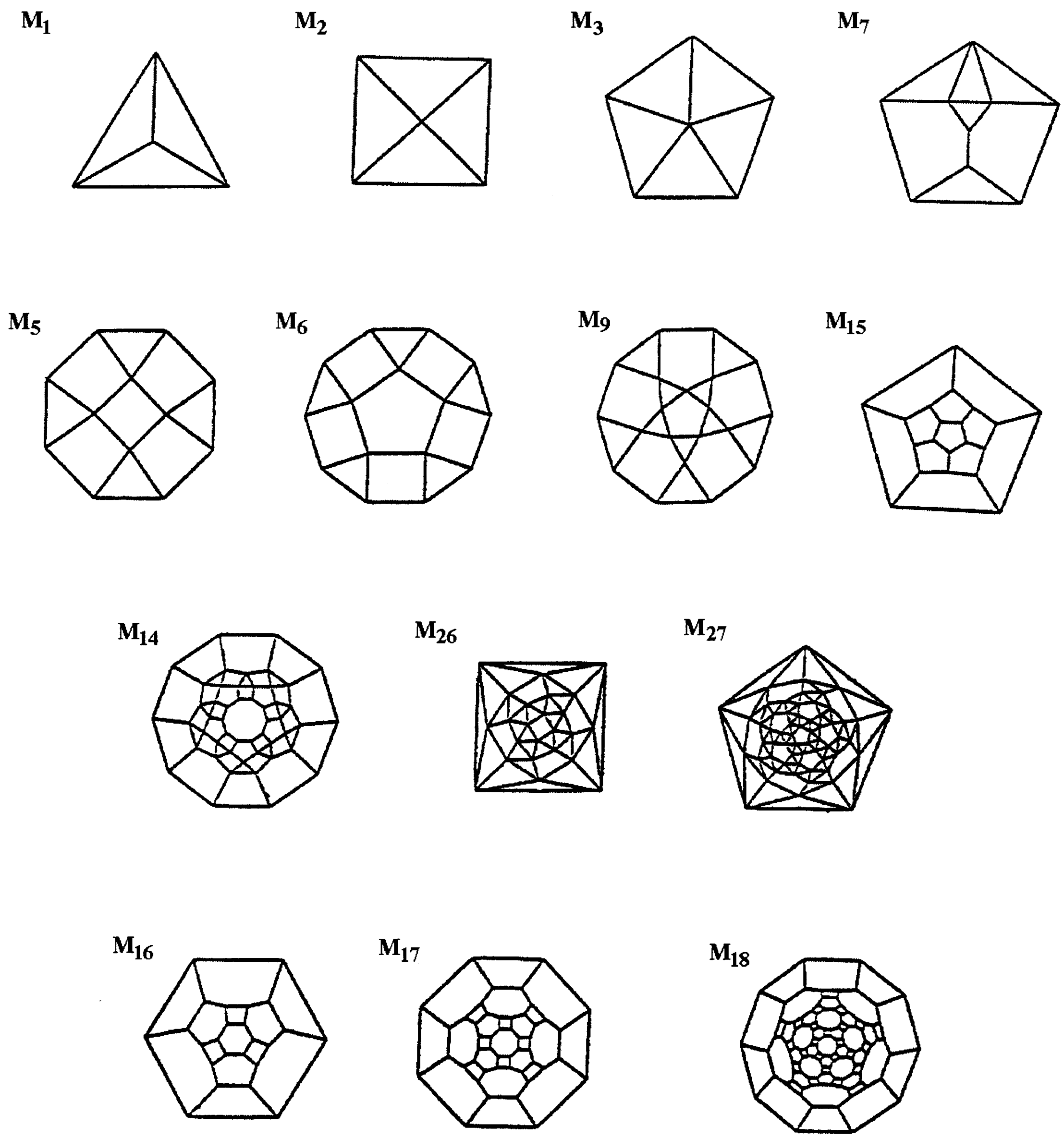

Figure 8: All l_1-skeletons of basic regular-faced polyhedra $M_i, 1 \leq i \leq 28$.

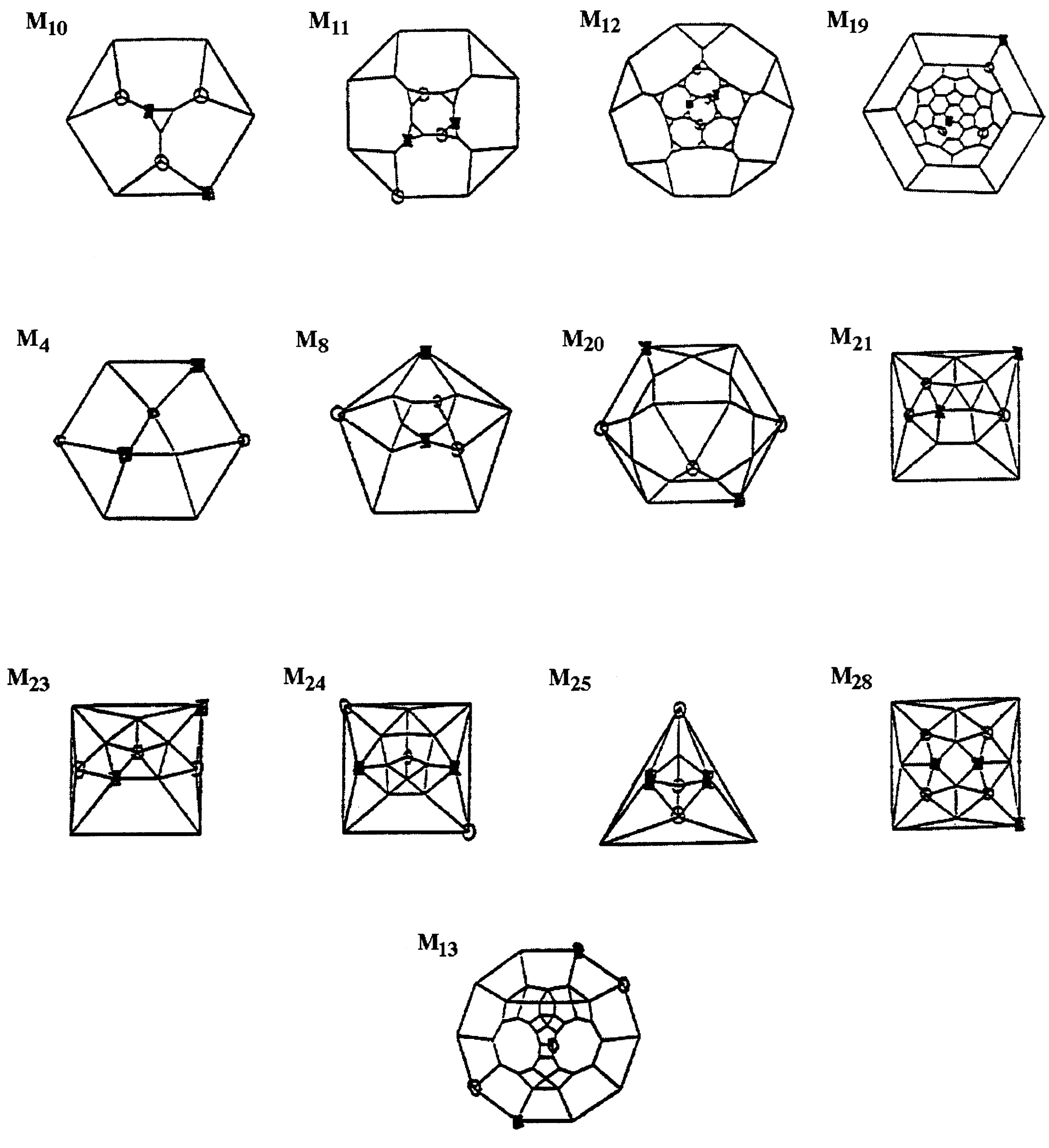

Figure 9: All not hypermetric skeletons of basic regular-faced polyhedra $M_i, l \leq i \leq 28$.

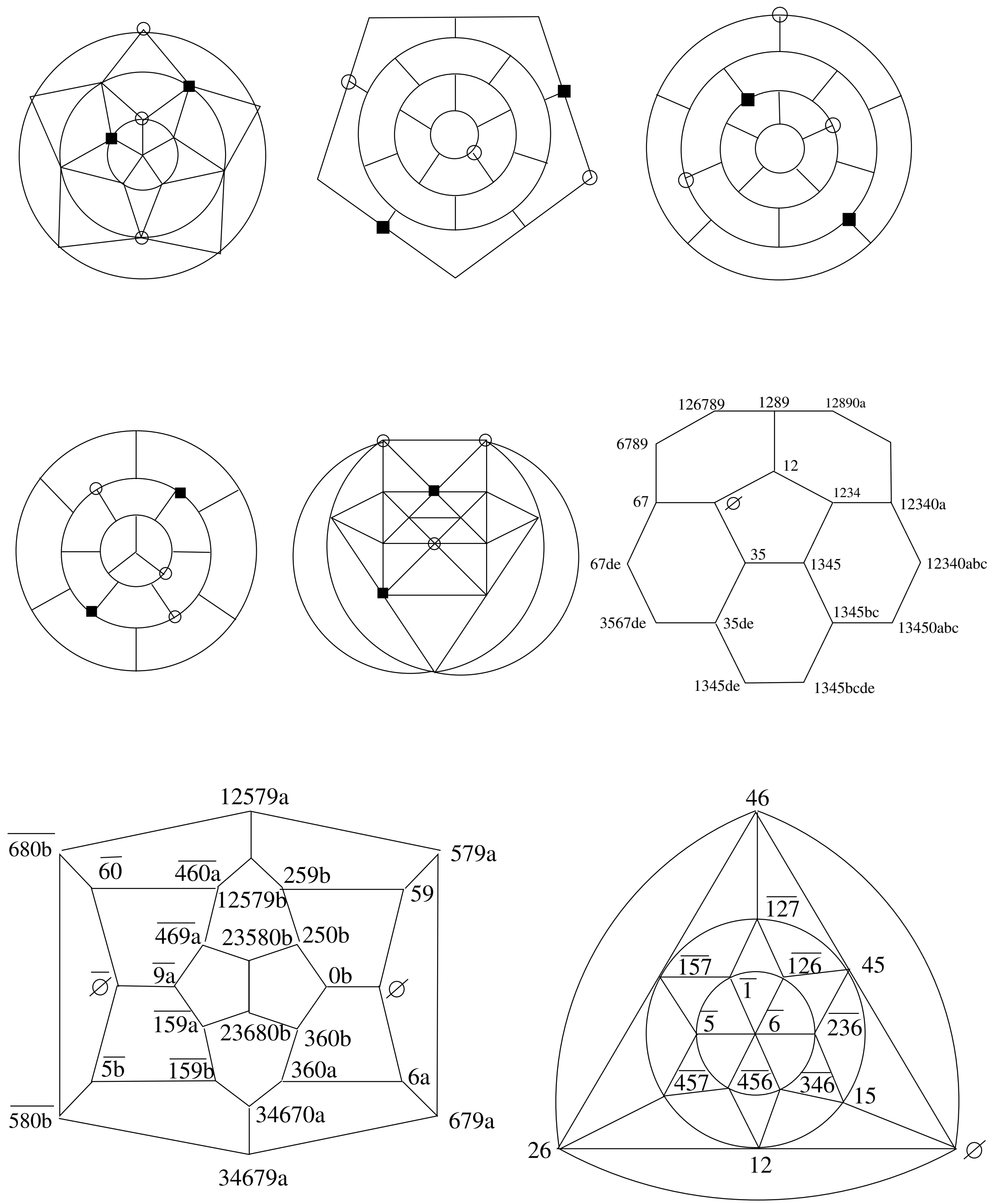

Figure 10: Fullerens: a) 1–Tow_5^2, b) dual 2–Tow_5^3, c) strained $F_{30}(D_{5h})$, d) $F_{28}(T_d)$, e) dual F_{26}, f) 20-bowl$\to\frac{1}{2}H_{15}$, g) $F_{26}\to\frac{1}{2}H_{12}$, h) dual $F_{28}(T_d)\to\frac{1}{2}H_7$

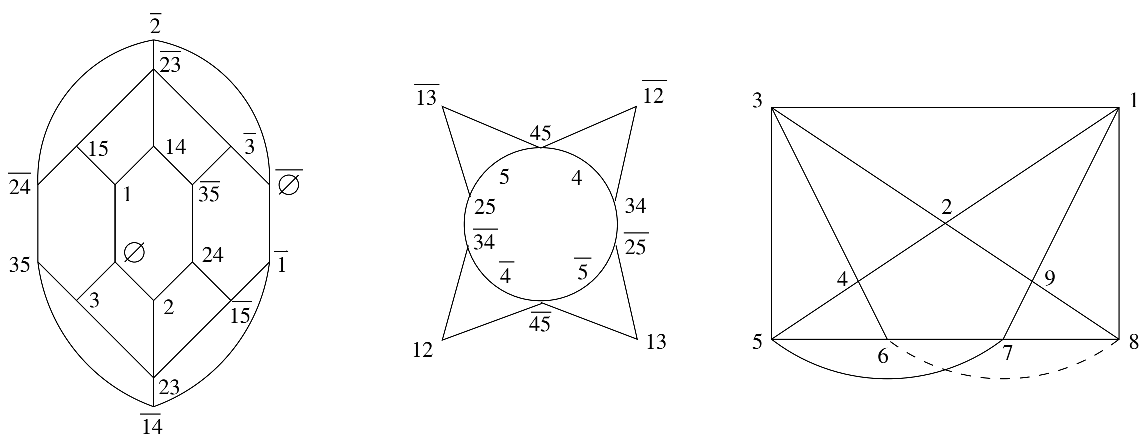

Figure 11: a) ElDo (elongated dodecahedron)$\hookrightarrow H_5$, b) the graph on $H_5 \setminus$ ElDo, c) $AW_9^2 - (6,8)$; antiweb AW_9^2 (a hypermetric, non-l_1 graph)