Two-graphs on 36 Points from the Even Unimodular Lattices $E_8 \oplus E_8$ and D_{16}^{+}

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LIENS - 95 - 9

April 1995

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Abstract

We show in this paper a power of a construction [5], [6] of two-graphs from doubly even lattices. A doubly even lattice is an even lattice multiplied by $\sqrt{2}$. We apply this construction to the even unimodular lattices $E_8 \oplus E_8$ and D_{16}^+ multiplied by $\sqrt{2}$. For the lattice $\sqrt{2}(E_8 \oplus E_8)$, the construction gives one family of regular two-graphs on 36 points. We think that many of two-graphs of this family are new. The lattice $\sqrt{2}D_{16}^+$ provides two families of such two-graphs. One of these families from D_{16}^+ consists of all two-graphs related to Steiner triple systems on 15 points. Another family consists of two-graphs related to 2-(10,4,2) designs. This family was discovered by T.Spence, by use a computer. These two-graphs are briefly described by J.Seidel in [10]. Being distinct all the 3 families are not disjoint. In particular, all contain the 2 two-graph related to the Steiner triple systems having No 1 and No 2 in the extended version of [2]. We do not consider here the hard problem on a number of isomorphism classes of two-graphs in each family, but show that root systems related to two-graphs make possible to distinguish nonisomorphic two-graphs.

1 Introduction

Regular two-graphs were invented by G.Higman as a mean of studying doubly transitive representations of finite groups. A two-graph is a set of *cogerent* triples such that every 4-set contains an even number of cogerent triples. But it is more important for us that each two-graph is represented by a set of equiangular lines. If the acute angle between lines is equal to $arccos\frac{1}{m}$, then we can choose a vector of norm (squared length) m along each line such that inner products of chosen vectors is equal to ± 1. If v_1, v_2, v_3 are vectors along 3 lines, then these 3 lines compose a cogerent triple if $(v_1v_2)(v_2v_3)(v_3v_1) = -1$, where v_iv_j is the inner product of vectors v_i and v_j.

Sets of equiangular lines of sufficiently large size in a space of fixed dimension exist only if m is an odd number. There is a special bound on maximal number $n(m,d)$ of equiangular lines at angle $arccos\frac{1}{m}$ in a space of dimension d:

$$n(m,d) \leq \frac{d(m^2-1)}{m^2-d}.$$

This bound is achieved if and only if the corresponding two-graph is *regular*. A two-graph is regular if every pair of points belongs to the same number of cogerent triples.

This is a great problem to classify all regular two-graphs $\mathcal{T}(m,d)$ with parameters m and d, in particular, to find a number $N_m(d)$ of all nonisomorphic two-graphs with the same parameters.

For $m=3$, a regular two-graph $\mathcal{T}(3,d)$ exists only for $d=5,6$ and 7, and $\mathcal{T}(3,d)$ is unique in each dimension, i.e. $N_3(d)=1$ for $d=5,6,7$.

We are interested here in regular two-graphs $\mathcal{T}(5,15)$ with $m=5$ and $d=15$, when $n(5,15)=36$. A set of equiangular lines at angle $arccos\frac{1}{5}$ representing a regular two-graph $\mathcal{T}(5,d)$ may exist only in dimensions $d=5$, 10, 13, 15, 19, 20, 21, 22 and 23. Regular two-graphs $\mathcal{T}(5,d)$ are known for all these $d \neq 19,20$. It is known also the number $N_5(d)$ of all nonisomorphic two-graphs $\mathcal{T}(5,d)$ for $d=5,10,13,23$, namely, $N_5(5)=N_5(10)=N_5(23)=1$, $N_5(13)=4$.

Seidel asserts in [10] that $N_5(15) \geq 227$. The number 227 is composed of 11 two-graphs from Latin squares of order 6, of 80 two-graphs from Steiner triple systems of order 15, and 136 new ones discovered by Spence, by use of a computer. In [10] Seidel describes a large subclass of the Spence' family. Two-graphs of this subclass are related to 2-(10,4,2) designs.

We show in this paper that the family of two-graphs from Steiner triple systems and two-graphs related to 2-(10,4,2) designs can be obtained from the even unimodular lattice D_{16}^+ multiplied by $\sqrt{2}$ by a construction introduced in [5] and [6].

We show also that every two-graph related to a 2-(10,4,2) design is a special gluing of the unique two-graph $\mathcal{T}(5,10)$ and the unique two-graph $\mathcal{T}(3,5)$. More precisely, a restriction of the set of equiangular lines representing a two-graph related to a 2-(10,4,2) design onto a 10-dimensional space X is a set of equiangular lines representing $\mathcal{T}(5,10)$. A projection of the set of equiangular lines onto the 5-dimensional orthogonal complement to X is a set of equiangular lines representing $\mathcal{T}(3,5)$.

Similarly, any two-graph from $E_8 \oplus E_8$ is another special gluing of the unique two-graphs $\mathcal{T}(3,7)$ and $\mathcal{T}(7,7)$.

Note that the two-graph related to the projective space $PG(3,2)$ belongs to all the 3 families obtained from $E_8 \oplus E_8$ and D_{16}^+. We show that this holds also for the two-graph related to the Steiner triple system having No 2 in the extended version of [2].

2 Odd systems and lattices

A set of vectors of an odd norm m with ± 1-inner products spanning equiangular lines is a special case of an odd system. An *odd system* $\mathcal{V}$ is a set of vectors v such that the inner product vv' of any (may be equal) two vectors of $\mathcal{V}$ is an odd integer. (We denote the inner product of two vectors v and v' by its juxtaposition vv'.) The inner product $v^2 = vv$ of a vector v with itself is called *norm* of v. Hence norms of all vectors of an odd system are odd. An odd system is called *uniform* (of norm m) if norms of all its vectors are equal (to m). As we use here only uniform odd systems, in what follows, we omit sometimes the word uniform.

We call an odd system *regular* if it represents a regular two-graph. We consider also *reduced* odd systems such that from two opposite vectors only one belongs to the odd system. Call an interchanging of a subset of vectors of a reduced odd system by its opposite *switching* of the odd system. Similarly, we call the operation of changing the sign of a vector v by *switching* v.

We call odd systems $\mathcal{V}$ and $\mathcal{V}'$ isomorphic if there is a bijection $\phi : \mathcal{V} \to \mathcal{V}'$ such that $\phi(v_1 v_2) = \phi(v_1)\phi(v_2)$. Obviously, isomorphic odd systems are equicardinal.

Let $\mathcal{U}$ be an odd system related to a two-graph (i.e. spanning equiangular lines). Since $vv' = \pm 1$ for $v, v' \in \mathcal{U}$, $v \neq \pm v'$, we can introduce a graph $G(\mathcal{U})$ with $\mathcal{U}$ as the set of its vertices. Two vertices v, v' of $G(\mathcal{U})$ are adjacent if and only if $vv' = -1$. If $\mathcal{U}$ relates to a regular two-graph, then $G(\mathcal{U})$ is a Taylor distance-regular graph of diameter 3 (see [1]).

Let $\mathcal{U}$ be a reduced odd system. A switching of $\mathcal{U}$ corresponds to a switching of $G(\mathcal{U})$. Fix $v_0 \in \mathcal{U}$. By a switching, we can isolate v_0, i.e. $G(\mathcal{U}^{sw}) = \{v_0\} \cup H_0$, where $H_0 = G(\mathcal{U}_0)$ and $\mathcal{U}_0 = \mathcal{U}^{sw} - \{v_0\}$. If $\mathcal{U}$ relates to a regular two-graph, then H_0 is a strongly regular graph. The (± 1)-adjacency matrix A of H_0 has minimal eigenvalue $-m$. Hence the matrix $mI + A$ is positive semidefinite, and it is the Gram matrix of the set of vectors of $\mathcal{U}_0$. For example, H_0 has parameters (35,16,6,8) for a regular two-graph $\mathcal{T}(5, 15)$ on 36 points.

A lattice L of dimension n is a free Abelian group of rank n of vectors. A lattice is called *integral* if inner products of all its vectors are integral. An integral lattice is called *even* if norms of all its vectors are even. An even lattice L is called *doubly even* if $\frac{1}{\sqrt{2}}L$ is even. Norms of all vectors of a doubly even lattice are multiples of 4, and all inner products are even. Hence minimal norm of a nonzero vector of a doubly even lattice L is not less than 4. The set L_4 of all vectors of norm 4 of L is, up to the multiple $\sqrt{2}$, a *root system*. Hence below we call a vector of norm 4 a *root*.

Each root system is a direct sum of irreducible root systems, called its *components*. A root system is called *irreducible* if it cannot be partitioned into two subsystems such that roots of one of these systems are orthogonal to all roots of other. All irreducible root systems are known. These are A_n, D_n and E_m, where n and m are dimensions of the corresponding root systems, and $m = 6, 7, 8$. Following to [4], we denote a root system consisting of components R_1, R_2,..., R_k by the product $R_1R_2...R_k$. In particular, a sum of k equal components R is denoted by R^k. A lattice generated by a root system is called *root lattice*, but it is denoted by the direct sum of corresponding components. For example, the root lattice $E_8 \oplus E_8$ is generated by the root system E_8^2.

In [5] and [6], we introduce a construction of uniform odd systems from a doubly even lattice. Here we describe this construction for uniform odd systems of norm 5.

Let L be a doubly even lattice, and let L_8 be the set of all $a \in L$ of norm 8. Let $c \in L$ have norm 12. We set

$$\mathcal{A}(c) = \{a \in L_8 : ac = 6\}.$$

It is easy to see that $a \in \mathcal{A}(c)$ implies $a^* = c - a \in \mathcal{A}(c)$, and $aa^* = -2$. Conversely, any two vectors $a, a' \in L_8$ with $aa' = -2$ provide a vector $c = a + a'$ of norm 12.

For $a \in \mathcal{A}(c)$, define

$$v(a) = a - \frac{1}{2}c.$$

Then we have $v(a)v(a') = aa' - 3$. Since inner products of all $a \in L$ are even, the inner product $v(a)v(a')$ is odd. In particular, $v^2(a) = 5$. In other words, the set

$$\mathcal{V}(c) = \{v(a) : a \in \mathcal{A}(c)\}$$

is a uniform odd system of norm 5.

The construction can be reversed. Let c be a vector of norm 12, which is orthogonal to the space spanned by an odd system $\mathcal{V}$ of norm 5. Then the vector $a(v) = v + \frac{1}{2}c$ have norm 8, and $a(v) + a(-v) = c$. Let L be the lattice linearly generated by $a(v)$ for all $v \in \mathcal{V}$. Then $a(v) \in L_8$. Hence the odd system $\mathcal{V}(c)$ from this lattice contains the original odd system $\mathcal{V}$ as a subsystem. It is proved in [6] that $\mathcal{V} = \mathcal{V}(c)$ if and only if $\mathcal{V}$ is *closed*.

Now we define the closure of an odd system. This notion is very useful for to distinguish nonisomorphic odd systems (and two-graphs). Consider the following lattices generated by an odd system $\mathcal{V}$:

$$L^q(\mathcal{V}) = \{u : u = \sum_{v \in \mathcal{V}} z_v v,\ \sum_{v \in \mathcal{V}} z_v \equiv q \pmod 2,\ z_v \in \mathbf{Z}\},\ q = 0, 1.$$

Let $\mathcal{V}$ be uniform and of norm 5. It is proved in [6] that $L^0(\mathcal{V})$ is a doubly even lattice, and the affine lattice $L^1(\mathcal{V}) = v + L^0(\mathcal{V})$ is a translation of $L^0(\mathcal{V})$. $L^1(\mathcal{V})$ is an odd system and $u^2 \equiv 1 \pmod 4$ for all $u \in L^1(\mathcal{V})$. Let $L^1_5(\mathcal{V})$ be the set of all vectors of $L^1(\mathcal{V})$ of norm 5. Obviously, $\mathcal{V} \subseteq L^1_5(\mathcal{V})$.

The convex hull conv$L^1_5(\mathcal{V})$ of endpoints of all vectors from $L^1_5(\mathcal{V})$ is very often a Delaunay polytope of the lattice $L^0(\mathcal{V})$. The conditions when conv$L^1_5(\mathcal{V})$ is a Delaunay polytope is given in [6].

Definition. The uniform odd system $L_5^1(\mathcal{V})$ is called the *closure* of the odd system $\mathcal{V}$. The odd system $\mathcal{V}$ is called *closed* if $\mathcal{V} = L_5^1(\mathcal{V})$. Sometimes we denote the closure of $\mathcal{V}$ by cl$\mathcal{V}$. $\square$

Recall that if $\mathcal{V}$ has the form $\mathcal{V}(c)$ for some c, then $\mathcal{V}$ is closed.

Suppose that $\mathcal{U}$ is a maximal uniform odd system of norm 5 spanning equiangular lines, i.e. $uu' = \pm 1$ for distinct $u, u' \in \mathcal{U}$. If $\mathcal{U}$ is not closed, then, for each $v \in$cl$\mathcal{U} - \mathcal{U}$, there is $u \in \mathcal{U}$ such that $vu = 3$. Then the vector $v - u$ has norm 4, i.e. it is a root. Since all roots belong to $L^0(\mathcal{U})$, they form a root system $R(\mathcal{U})$. If $\mathcal{U}$ and $\mathcal{U}'$ represent the isomorphic two-graphs and are not reduced (reduced), then they are isomorphic (switching equivalent to isomorphic odd systems, respectively).

The following obvious proposition helps to distinguish nonisomorphic odd systems spanning equiangular lines, and therefore non isomorphic two-graphs.

Proposition 1 *Let $\mathcal{U}$ and $\mathcal{U}'$ be d-dimensional odd systems representing two-graphs $\mathcal{T}$ and $\mathcal{T}'$ with the same parameters $(5, d)$. Then $\mathcal{T}$ and $\mathcal{T}'$ are not isomorphic if $R(\mathcal{U}) \neq R(\mathcal{U}')$.* $\square$

3 Two-graphs from the lattice $E_8 \oplus E_8$

Recall that there are two nonisomorphic unimodular lattices in dimension 16, namely D_{16}^+ and $E_8 \oplus E_8$, where E_8 is a 8-dimensional root lattice. The root lattice E_8 is generated by its minimal vectors of norm 2 forming the root system E_8. We use the description of the root system E_8 given in [3]. In fact, the description is given in terms of vectors of norm 4, i.e. it gives $\sqrt{2}E_8$. We continue call the minimal vectors of norm 4 of the doubly even lattice $\sqrt{2}E_8$ by roots.

Let $V_8 = \{0\} \cup V_7$, and $V_7 = \{1, ..., 7\}$. Let h_i, $i \in V_8$, be 8 mutually orthogonal vectors of norm 1. Then roots of $\sqrt{2}E_8$ are

1) $\pm 2h_i$, $i \in V_8$,

2) $\sum_{i \in Q} \varepsilon_i h_i$, $\varepsilon_i \in \{\pm 1\}$, $|Q| = 4$, $Q \in S(3, 4, 8)$.

Here $S(3, 4, 8)$ is the Steiner system that has the following form. Let F_7 be 7 triples of the unique STS $S(2, 3, 7)$ on 7 points. Its triples are lines of the projective Fano plane $PG(2, 2)$. Each quadruple $Q \in S(3, 4, 8)$ has the form $Q = t \cup \{0\}$ or $Q' = V_7 - t = \overline{Q} := V_8 - Q$, where $t \in F_7$. If $Q \neq \overline{Q'}$, then $|Q \cap Q'| = 2$. In this case, $Q \Delta Q' \in S(3, 4, 8)$.

Let f_i, $i \in V_8$, be other 8 mutually orthogonal vectors of norm 1. All f_i are orthogonal to all h_j. Then the roots of the second copy of $\sqrt{2}E_8$ are given by the above expressions 1) and 2) with h_i changed by f_i.

The vectors of norms 8 and 12 in the lattice $\sqrt{2}(E_8 \oplus E_8)$ are sums of two and three, respectively, orthogonal roots of the lattice. Since the automorphism group of the root system E_8 is transitive on pairs of orthogonal roots, there are, up to symmetry, two types of vectors of norm 12: a sum of 3 orthogonal roots of the same copy of $\sqrt{2}E_8$, and a sum of two roots of one copy and of one root of the other copy of $\sqrt{2}E_8$.

A vector c of the first type gives a pillar odd system $\mathcal{V}(c)$. This means that vectors of $\mathcal{V}(c)$ have the form $\pm(e+r)$, where e is a vector of norm 1, and r is a root (of norm 4) which belongs to $E_8 \oplus E_8$ and is orthogonal to e and c (details see in [6]. A maximal reduced pillar odd system $\mathcal{U} \subseteq \mathcal{V}(c)$ spanning equiangular lines (i.e. representing a two-graph) contains less than 36 vectors, the number of points of a regular two-graph $\mathcal{T}(5,15)$.

Hence we consider only the vectors c of the second type. Recall that all vectors c of the same type belong to the same orbit of the automorphism group of the lattice $\sqrt{2}(E_8 \oplus E_8)$.

We take c equal to

$$c_0 = h(Q) + h(\overline{Q}) + 2f_0 = h(V_8) + 2f_0.$$

Here and below we use the following denotation: for any set V, any $X \subseteq V$, any g_k, $k \in V$, and for any $i \in V$, we set

$$g(X) := \sum_{i \in X} g_i, \ \ \epsilon(X,i) = \left\{ \begin{array}{cc} 1 & \text{if } i \in X \\ -1 & \text{otherwise.} \end{array} \right. \tag{1}$$

The set $\mathcal{A}(c_0)$ contains the following vectors:

1) 784 vectors $h(Q) + \sum_{i \in P} \varepsilon_i f_i$, $Q, P \in S(3,4,8)$, $0 \in P$, $\varepsilon_0 = 1$.

2) 56 vectors $a = h(Q) - 2h_i + 2f_0$, $i \in Q$, and 56 vectors $c_0 - a = h(Q) + 2h_i$, $i \notin Q$, $Q \in S(3,4,8)$.

3) 8 vectors $2h_i + 2f_0$, and 8 vectors $h(V_8) - 2h_i$, $i \in V_8$.

Recall that $v(a) = a - \frac{1}{2}c_0$. Then

$$\mathcal{V}(c_0) = \mathcal{V}_1 \cup \mathcal{V}_2,$$

where

$$\mathcal{V}_1 = \{h(Q) - \frac{1}{2}h(V_8) + \sum_{i \in P - \{0\}} \varepsilon_i f_i, \ Q, P \in S(3,4,8), \ 0 \in P\},$$

$$\mathcal{V}_2 = \pm\{h(Q) - \frac{1}{2}h(V_8) - 2h_i + f_0, i \in Q, \text{ and } 2h_i - \frac{1}{2}h(V_8) + f_0, i \in V_8\}.$$

Recall that if $Q \ni 0$, then $Q = \{0\} \cup s$ with $s \in F_7$. For $s \in F_7$, $s \subset Q$, we define 7 vectors of norm 2 as follows:

$$w_s = h(Q) - \frac{1}{2}h(V_8) = h_0 + h(s) - \frac{1}{2}h(V_8).$$

Note that if Q does not contain 0, then $0 \in \overline{Q} = V_8 - Q$. Hence

$$h(Q) - \frac{1}{2}h(V_8) = -(h(\overline{Q}) - \frac{1}{2}h(V_8)) = -w_s \text{ for } s = \overline{Q} - \{0\}.$$

Similarly, for $P \ni 0$,we have $P = \{0\} \cup t$, $t \in F_7$. We set

$$v_s(t,\varepsilon) = w_s + \sum_{i \in t} \varepsilon_i f_i, \;\; s,t \in F_7.$$

In this notation, the odd system $\mathcal{V}_1$ takes the form

$$\mathcal{V}_1 = \pm\{v_s(t,\varepsilon) : \varepsilon \in \{\pm 1\}^t, s,t \in F_7\}.$$

Using this explicit expression, it is not difficult to find that

$$R(\mathcal{V}_1) = D_7 E_7.$$

The roots of D_7 are $w_s \pm w_{s'}$, $s,s' \in F_7$. The roots of E_7 are $\pm 2f_i$, $\sum_{i \in Q} \varepsilon_i f_i$, $Q \in S(3,4,8)$, $0 \notin Q$. Note that the roots of E_7 are orthogonal to f_0.

Let $W = \frac{1}{2}\sum_{s \in F_7} w_s$. Then $W^2 = \frac{7}{2}$, and $W w_s = 1$ for all $s \in F_7$. It is easy to verify that $w_s w_t = 0$ for $s \neq t$, since $|s \cap t| = 1$ for distinct $s,t \in F_7$. Besides, $w_s h(V_8) = 0$. Hence the 8 vectors $h(V_8)$ and w_s, $s \in F_7$, form an orthogonal basis of the space spanned by h_i, $i \in V_8$. The vectors h_i can be expressed through $h(V_8)$ and w_s:

$$2h_0 = W + \frac{1}{4}h(V_8), \;\; 2h_i = -W + \sum_{s \ni i} w_s + \frac{1}{4}h(V_8).$$

For $t \in F_7$, and $i \in V_7$, we define the following vectors of norm 5:

$$v_0(t) = g - W + w_t, \;\; v_i(t) = g + W - \sum_{s \ni i} w_s + \epsilon(t,i) w_t, \;\; u_0 = g + W, \;\; u_i = g - W + \sum_{s \ni i} w_s,$$

where $g = f_0 - \frac{1}{4}h(V_8)$ is the vector of norm $\frac{3}{2}$ orthogonal to all w_s.

Note that these 64 vectors have the form $g + \frac{1}{2}\sum_{s \in F_7} \varepsilon_s w_s$, where $\varepsilon_s \in \{\pm 1\}$, and there is an even number of minus signs. Hence we can redenote these vectors as

$$u(S) = g + W - w(S),$$

where $w(S) = \sum_{s \in S} w_s$, and $|S| \equiv 0 \pmod 2$. Call a subset $S \subseteq F_7$ *even* if it has an even cardinality.

Now the odd system $\mathcal{V}_2$ takes the form

$$\mathcal{V}_2 = \pm\{u(S) : S \subseteq F_7, \; S \text{ is even}\}.$$

The root system of $\mathcal{V}_2$ is $R(\mathcal{V}_2) = D_7$. Since $R(\mathcal{V}_2) \subseteq R(\mathcal{V}_1)$, we have

$$R(\mathcal{V}(c_0)) = R(\mathcal{V}_1) = D_7 E_7.$$

These are roots of $E_8 \oplus E_8$ that are orthogonal to c_0.

We have $\frac{1}{2}|\mathcal{V}_1| = 392$, $\frac{1}{2}|\mathcal{V}_2| = 64$. Hence

$$\frac{1}{2}|\mathcal{V}(c_0)| = 456. \tag{2}$$

It is easy to verify that $v_s(t,\varepsilon)u(S) = -\epsilon(S,s) = \pm 1$, i.e. $vv' = \pm 1$ for $v \in \mathcal{V}_1$ and $v' \in \mathcal{V}_2$. We seek a maximal odd subsystem $\mathcal{U} \subseteq \mathcal{V}(c_0)$ of vectors with all mutual inner products equal to ± 1. Of course, we have to find separately maximal subsets $\mathcal{U}_1 \subseteq \mathcal{V}_1$ and $\mathcal{U}_2 \subseteq \mathcal{V}_2$ such that $\mathcal{U}_1 \cup \mathcal{U}_2 = \mathcal{U}$. Recall that, for $m = 5$, $d = 15$, the special bound gives $\frac{1}{2}|\mathcal{U}| = 36$.

In what follows in this section, we consider only reduced odd systems $\mathcal{V}_1$ and $\mathcal{V}_2$ in the *canonical* form, when the vectors w_s and g in the vectors $v_s(t,\varepsilon)$ and $u(S)$, respectively, have positive signs. We preserve for canonical systems the same notations $\mathcal{V}_1$ and $\mathcal{V}_2$. Obviously, every reduced subsystem of $\mathcal{V}(c_0)$ can be made canonical by a switching.

3.1 Odd systems $\mathcal{U}_1$

Consider at first $\mathcal{V}_1$. Recall that $w_s^2 = 2$ and $w_s w_{s'} = 0$ for $s \neq s'$. We set $\delta(s,s') = 1$ if $s = s'$, and $\delta(s,s') = 0$ if $s \neq s'$. We have

$$v_s(t,\varepsilon)v_{s'}(t',\varepsilon') = 2\delta(s,s') + \sum_{i \in t \cap t'} \varepsilon_i \varepsilon'_i.$$

So, $v_s(t,\varepsilon)v_{s'}(t,\varepsilon) = 2\delta(s,s') + 3$. Since $vv' = \pm 1$ for distinct $v, v' \in \mathcal{U}_1$, this implies that, for each pair (t,ε), there is at most one s such that $v_s(t,\varepsilon) \in \mathcal{U}_1$. We denote this s by $s(t,\varepsilon)$.

We obtain that a map $s : (t,\varepsilon) \to s(t,\varepsilon) \in F_7$ corresponds to a set $\mathcal{U}_1 \subseteq \mathcal{V}_1$ spanning equiangular lines. Let

$$T_s(\mathcal{U}_1) = \{(t,\varepsilon) : s(t,\varepsilon) = s\}.$$

According to what was said above, the sets T_s are disjoint for distinct s.

Lemma 1 *For any $\mathcal{U}_1 \subseteq \mathcal{V}_1$, spanning equiangular lines, $|T_s(\mathcal{U}_1)| \leq 4$ for all $s \in F_7$.*

Proof. Let $T_s = T_s(\mathcal{U}_1)$. For $(t,\varepsilon), (t',\varepsilon') \in T_s$, we have $v_s(t,\varepsilon)v_s(t',\varepsilon') = 2 + \Sigma$, where

$$\Sigma = \sum_{i \in t \cap t'} \varepsilon_i \varepsilon'_i.$$

Note that Σ takes odd values. This implies that Σ should be equal either to –1 or –3. The case $\Sigma = -3$ is possible only if $t = t'$ and $\varepsilon = -\varepsilon'$. Then $T_s = \{(t,\varepsilon), (t,-\varepsilon)\}$. In fact, if there is another $(t'',\varepsilon'') \in T_s$, then $v_s(t,\pm\varepsilon)v_s(t'',\varepsilon'') = 2 \pm \sum_{i \in t \cap t''} \varepsilon_i \varepsilon''_i$ is equal to 3 for one of the signs $\pm$. So, $|T_s| = 2$ in the case .

Now, let $\Sigma = -1$. Then projections of vectors $v_s(t,\varepsilon)$ for $(t,\varepsilon) \in T_s$ on the space spanned by f_i, $1 \leq i \leq 7$, form an odd system of vectors of norm 3 with mutual inner products –1. But such a system contains at most 4 vectors. In fact, let $v_i^2 = 3$, $v_iv_j = -1$, $1 \le i < j \le k$. Then $0 \le (\sum_i^k v_i)^2 = 3k - k(k-1)$, i.e. $k \le 4$. Hence $|T_s| \le 4$ in this case. $\Box$

Since $|F_7| = 7$ and the sets T_s are disjoint for distinct s, Lemma 1 implies that $\mathcal{U}_1$ contains at most $4 \times 7 = 28$ pairs of opposite vectors.

By Lemma 1, if T_s contains more than two pairs (t, ε), then all vectors $\sum_{i \in t} \varepsilon_i f_i$, corresponding to pairs $(t, \varepsilon) \in T_s$, have mutual inner products –1. There are 3 types of T_s containing 4 pairs (t, ε) such that the corresponding vectors $\sum_{i \in t} \varepsilon_i f_i$ have inner products –1. In order to describe them, we introduce the following definitions.

Let ε, ε', ε'' take values ± 1. We call the triple $(\varepsilon, \varepsilon', \varepsilon'')$ *even* if the product $\varepsilon\varepsilon'\varepsilon'' = 1$. Otherwise the triple is called *odd*. There are 4 even triples and 4 odd triples. We set $\varepsilon^0 = (1,1,1)$, $\varepsilon^1 = (1,-1,-1)$, $\varepsilon^2 = (-1,1,-1)$, $\varepsilon^3 = (-1,-1,1)$. Let $0 \le k \le 3$. Then ε^k is an even ε-triple, and $-\varepsilon^k$ is an odd ε-triple. Besides if we change the sign of one of the units in ε^k, we obtain an odd ε-triple $-\varepsilon^l$ for some l. For two ε-triples ε and ε' with the same support t, let $\varepsilon\varepsilon' = \sum_{i \in t} \varepsilon_i \varepsilon'_i$. Hence ε and ε' are of the same parity if and only if $\varepsilon\varepsilon' = -1$.

We use below sums of the type $\sum_{i \in t} \varepsilon_i^k f_i$. In such a sum, we consider $t \in F_7$ as an ordered triple ijl such that $1 \le i < j < l \le 7$, and the orders of ε^k and t agree. For example, $\varepsilon^2 = (-1,1,-1)$ in $\sum_{i \in 237} \varepsilon_i^2 f_i$ means that $\varepsilon_2^2 = -1$, $\varepsilon_3^2 = 1$, and $\varepsilon_7^2 = -1$.

The 3 types of T_s are as follows.

Type 1. $T_s = \{(t(s), \varepsilon^k) : 0 \le k \le 3\}$ or $T_s = \{(t(s), -\varepsilon^k) : 0 \le k \le 3\}$, where $t(s)$'s are the same for all 4 pairs of T_s, and ε^k are even triples defined above. There are $7 \times 2 = 14$ sets T_s of Type 1.

Type 2. $T_s = \{(t(s), \varepsilon^j), (t(s), \varepsilon^k), (t'(s), \varepsilon^l), (t'(s), \varepsilon^m)\}$, where $t(s) \ne t'(s)$, $\varepsilon^j\varepsilon^k = \varepsilon^l\varepsilon^m = -1$ and $\varepsilon_i^j = \varepsilon_i^k = -\varepsilon_i^l = -\varepsilon_i^m$ with $\{i\} = t(s) \cap t'(s)$. There are $21 \times 8 = 168$ sets of Type 2.

Denote by C_i the set of 4 triples $t \in F_7$ not containing $i \in V_7$. Each point $j \in V_7 - \{i\}$ is contained in exactly two triples of C_i, i.e. there is a one-to-one correspondence between 6 points of $V_7 - \{i\}$ and 6 pairs $t, t' \in C_i$.

Type 3. $T_s := T_s^i = \{(t, \varepsilon(t)) : t \in C_i\}$, where for all pairs $t, t' \in C_i$ with $t \cap t' = \{j\}$ we have $\varepsilon_j(t)\varepsilon_j(t') = -1$. There are $7 \times 2^6 = 448$ sets of Type 3.

Note that if we change the signs of ε in all pairs (t, ε) of some T_s, we obtain a new set T'_s of the same type.

We call two sets $T_s = \{(t, \varepsilon)\}$ and $T_{s'} = \{(t', \varepsilon')\}$ for $s \ne s'$ *consistent* if $(\sum_{i \in t} \varepsilon_i f_i)(\sum_{i \in t'} \varepsilon'_i f_i) = \pm 1$ for all $(t, \varepsilon) \in T_s$ and all $(t', \varepsilon') \in T_{s'}$.

So, we obtain that a reduced odd system $\mathcal{U}_1$ has the form

$$\mathcal{U}_1 = \{v_s(t, \varepsilon) : (t, \varepsilon) \in T_s, s \in F_7\},$$

where T_s for each s is one of the above types, and all T_s are mutually consistent.

Let $\mathcal{U}_1$ contain the maximal number 28 of vectors. Then the projection of $\mathcal{U}_1$ on the 7-dimensional space spanned by f_i, $i \in V_7$, is an odd system consisting 28 vectors

$\sum_{i\in t}\varepsilon_i f_i$ of norm 3 with mutual inner products ± 1. These vectors represent a two-graph $\mathcal{T}(3,7)$. The special bound gives $n(3,7)=28$, i.e. the two-graph is the unique regular two-graph with parameters $(m,d)=(3,7)$. Hence the condition of the consistency is equivalent to the requirement that the 28 vectors $\sum_{i\in t}\varepsilon_i f_i$ for $(t,\varepsilon)\in T_s$, $s\in F_7$, have to form an odd system representing the unique two-graph $\mathcal{T}(3,7)$.

3.2 Odd systems $\mathcal{U}_2$

Let $\mathcal{S}$ be a family of even subsets of F_7. We set

$$\mathcal{U}(\mathcal{S})=\{u(S):S\in\mathcal{S}\}.$$

We want to find a family $\mathcal{S}$ such that $\mathcal{U}(\mathcal{S})$ is a maximal odd subsystem of $\mathcal{V}_2$ spanning equiangular lines.

Let S_0 be even. Since the set of even subsets is closed with respect to symmetric difference, the vector $u'(S):=u(S\Delta S_0)$ belongs to $\mathcal{V}_2$. Besides, $u'(S_1)u'(S_2)=u(S_1)u(S_2)$. Hence the odd systems $\mathcal{U}(\mathcal{S})$ and $\mathcal{U}(\mathcal{S}\Delta S_0):=\{u':u\in\mathcal{U}(\mathcal{S})\}$ are isomorphic.

The Abelian group of even subsets of F_7 consists of the following sets: 0-set $\emptyset$, 2-sets $d_i(t)$, two types of 4-sets C_i and $D_i(t)$, and 6-sets $c(t)$, where

$$c(t)=\overline{\{t\}}=F_7-\{t\},\ d_i(t)=\{s\in F_7:s\ni i,s\neq t\},\ t\ni i,i\in V_7,$$

$$C_i=\{s\in F_7:s\not\ni i\},\ D_i(t)=\{t\}\cup\{s\in F_7:s\ni i\},\ t\not\ni i,i\in V_7.$$

Comparing the definitions of vectors u_i, $v_i(t)$ and sets $d_i(t)$, C_i, $D_i(t)$ and $c(t)$ we see that

$$u_0=u(\emptyset),\ v_0(t)=u(c(t)),\ v_i(t)=u(d_i(t))\text{ for }i\in t,$$

$$u_i=u(C_i),\ v_i(t)=u(D_i(t))\text{ for }i\notin t.$$

According to what was said above, we can consider at first the case $\emptyset\in\mathcal{S}$.

Consider inner products of vectors $u(S)$:

$$u(S)u(T)=5-|S|-|T|+2|S\cap T|. \tag{3}$$

For $S,T\in\mathcal{S}$, we have to have $u(S)u(T)=\pm 1$. For $S=\emptyset$, this condition implies $|T|=4$ or 6.

Recall that maximal reduced odd systems $\mathcal{U}_1$ and $\mathcal{U}$ contain 28 and 36 vectors, respectively. Hence a maximal odd system $\mathcal{U}(\mathcal{S})$ contains 8 vectors. In other words, a maximal family $\mathcal{S}$ contains 8 even sets.

Lemma 2 *A maximal family $\mathcal{S}$ with $\emptyset\in\mathcal{S}$ does not contain sets of cardinality 6.*

Proof. Let S, T be subsets of F_7 of cardinality 6. Since $|F_7| = 7$, we have $|S \cap T| = 5$. Hence, for $|S| = |T| = 6$, (3) takes the form $u(S)u(T) = 3$. This implies that $\mathcal{S}$ contains at most one 6-set.

Suppose $\mathcal{S}$ contains a 6-set S_0. For $|S| = 6$ and $|T| = 4$, (3) gives $u(S)u(T) = 2|S \cap T| - 5$. Hence $|S \cap T| = 2$ or 3. But since $|F_7| = 7$, we have $|S \cap T| \geq 3$. Hence $|S \cap T| = 3$. So $\mathcal{S}$ consists of $\emptyset$, S_0 and some 4-sets S such that the set $T := S_0 \cap S$ has cardinality 3. Since for 4-subsets S and S', (3) implies that $|S \cap S'| = 1$ or 2, we have $|T \cap T'| \leq 1$. If $|T \cap T'| = 0$, then $\mathcal{S}$ contains only four sets: $\emptyset$, S_0, S and S', because any other 3-set T'' has an intersection of cardinality 2 with T or T'. Hence $|T \cap T'| = 1$. But a 6-set contains at most four 3-subsets with mutual intersections of cardinality 1. Hence if $\mathcal{S}$ contains a 6-set, then it contains at most 6 sets. This implies that a maximal family $\mathcal{S}$ does not contain a set of cardinality 6. $\square$

So, a maximal family $\mathcal{S}$ contains, besides $\emptyset$, only 4-sets. For 4-sets S, T, the equality (3) takes the form

$$u(S)u(T) = 2|S \cap T| - 3.$$

For a given $i \in V_7$, there are 4 sets $D_i(t)$ with $t \not\ni i$. Since each $i \in V_7$ is contained in 3 triples of F_7, we have $|D_i(t) \cap D_i(t')| = 3$ for $t \neq t'$. Hence every family $\mathcal{S}$ contains at most one set from 4 sets $D_i(t)$ for given i. Denote t corresponding to $D_i(t) \in \mathcal{S}$ by t_i.

Note the following pairs of sets $D_i(t)$ and C_k having an intersection of cardinality 3:

$$\text{For } i \neq k,\ |D_i(t_i) \cap C_k| = 3 \text{ if } k \notin t_i, \tag{4}$$

and

$$|D_i(t_i) \cap D_j(t_j)| = 3 \text{ if } i \in t_j \text{ and } j \in t_i. \tag{5}$$

Lemma 3 *If there is $i \in V_7$ such that $C_i, D_i(t_i) \in \mathcal{S}$ then the family $\mathcal{S}$ is not maximal.*

Proof. Suppose that $C_i, D_i(t_i) \in \mathcal{S}$ with $i \notin t_i$. Let $D_j(t_j) \in \mathcal{S}$. Recall that $|S \cap T| = 1$ or 2 for $S, T \in \mathcal{S}$.

Hence, by (4) (for $i = j$ and $k = i$), $i \in t_j$, and, by (5), $j \notin t_i$. Since $j \notin t_i \cup t_j$ and $t_i \neq t_j$, there is only one point $j' \neq j$ in the set $V_7 - (t_i \cup t_j)$. We show that there is at most one set $D_l(t_l) \in \mathcal{S}$ for $l \neq i, j$. As above we have $t_l \ni i$, $l \notin t_i$, $t_l \neq t_i$.

If $t_l = t_j$, then the conditions $l \neq j$ and $l \notin t_i \cup t_l = t_i \cup t_j$, imply that $l = j'$.

If $t_l \neq t_j$, then either $j \in t_l$ or $j' \in t_l$. If $j \in t_l$, then, by (5), $l \notin t_j$, what implies $l = j'$. If $j' \in t_l$, then $l \in t_j$, i.e. l is the third point of t_j distinct from the points of the intersections $t_i \cap t_j$ and $t_l \cap t_j = \{i\}$.

So l is uniquely determined by i and j. If $D_l(t_l) \in \mathcal{S}$ with such l, then $\mathcal{S}$ can contain a set C_k with $k \neq i$, only if $t_l = t_j$ and $\{k\} = t_i \cap t_j$. So, we obtain that, in this case, $\mathcal{S}$ contains at most 6 sets: $\emptyset$, C_i, $D_i(t_i)$, $D_j(t_j)$, $D_l(t_j)$ and C_k, i.e. $\mathcal{S}$ is not maximal.

Suppose now, that $\mathcal{S}$ contains only one set $D_i(t_i)$ from D-sets. Then $\mathcal{S}$ contains at most 3 sets C_k for $k \in t_i$. Hence, in this case, $\mathcal{S}$ also contains at most 6 sets: $\emptyset$, C_i, $D_i(t_i)$, and C_k for $k \in t_i$. $\square$

So we obtain that if $\mathcal{S}$ contains a set $D_i(t_i)$, then $k \in t_i$ for all $C_k \in \mathcal{S}$. This implies that $\mathcal{S}$ has one of the following forms:
$\mathcal{S}^0 = \{\emptyset, C_k, k \in V_7\}$,
$\mathcal{S}(t) = \{\emptyset, D_i(t), i \in V_7 - t, C_k, k \in t\}$.
Denoting $\mathcal{U}(\mathcal{S})$ by $\mathcal{U}_2^0$ and $\mathcal{U}_2(t)$ for $\mathcal{S} = \mathcal{S}^0$ and $\mathcal{S} = \mathcal{S}(t)$, respectively, we obtain

$$\mathcal{U}_2^0 = \{u_i : i \in V_8\},\ \mathcal{U}_2(t) = \{u_0, u_k, v_i(t) : k \in t, i \in V_7 - t\}.$$

Since the group of even sets is generated by 6-sets,we can consider $\mathcal{U}(\mathcal{S}\Delta c(t)\Delta ... \Delta c(t'))$ for different families of t,..., t'. By this way, we obtain only two new systems

$$\mathcal{U}_2^0(t) = \{v_i(t) : i \in V_8\},\ \mathcal{U}_2^1(t) = \{v_0(t), v_k(t), u_i : k \in t, i \in V_7 - t\}.$$

Hence we have

Proposition 2 *Every maximal odd system $\mathcal{U}_2$ has one of the form $\mathcal{U}_2^0$, $\mathcal{U}_2(t)$, $\mathcal{U}_2^0(t)$, $\mathcal{U}_2^1(t)$, $t \in F_7$.* $\square$

It is easy to verify, using the definitions of $v_i(t), u_i$, $i \in V_8$, that

$$v_i(t)v_j(t) = v_i(t)u_j = u_iu_j = 1 \text{ for } i, j \in V_8, i \neq j.$$

Hence all the systems $\mathcal{U}_2^0$, $\mathcal{U}_2(t)$, $\mathcal{U}_2^0(t)$, $\mathcal{U}_2^1(t)$ are isomorphic.

Note that, since the vectors g, w_s, $s \in F_7$, and f_i, $i \in V_7$, are mutually orthogonal, a change of signs of some of these vectors transforms the odd system $\mathcal{V}(c_0)$ and any of its subsystem $\mathcal{U}$ into isomorphic odd systems.

Lemma 4 *The odd systems $\mathcal{U}_1 \cup \mathcal{U}_2^0(t)$ and $\mathcal{U}_1 \cup \mathcal{U}_2^1(t)$ are isomorphic to a switching of the odd systems $\mathcal{U}_1' \cup \mathcal{U}_2^0$ and $\mathcal{U}_1' \cup \mathcal{U}_2(t)$, respectively, where the sets T_s' of the odd system $\mathcal{U}_1'$ are of the same type as the corresponding sets T_s of the odd system $\mathcal{U}_1$.*

Proof. For fixed $t \in F_7$, consider the following transformation: $w_t \to -w_t$, $g \to -g$. This transformation generates the following transformation of vectors v and u: $v_i(t) \to -u_i$, $u_i \to -v_i(t)$, $i \in V_8$. Obviously this transformation, up to switching, permutates $\mathcal{U}_2^0(t)$ with $\mathcal{U}_2^0$, and $\mathcal{U}_2^1(t)$ with $\mathcal{U}_2(t)$.

Additionally, we make the transformation $T_t = \{(s(t), \varepsilon)\} \to T_t' = \{(s(t), -\varepsilon)\}$ in the reduced odd system $\mathcal{U}_1$. Obviously, after this transformation, we obtain, up to a switching, an odd system $\mathcal{U}_1'$ of the same form as $\mathcal{U}_1$. The assertion of the lemma follows. $\square$

According to Lemma 4, for to find all nonisomorphic two-graphs given by $E_8 \oplus E_8$, it is sufficient to consider the odd systems $\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_2^0$ and $\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_2(t)$. We choose just these systems, since they contain the vector u_0 which has the inner product $u_0v = 1$ for all other $v \in \mathcal{U}$. In this case, the vertex u_0 is isolated in the graph $G(\mathcal{U})$, and the graph $G(\mathcal{U} - \{u_0\})$ is a strongly regular graph with parameters (35,16,6,8).

The projection of the vector $u(S)$ on the space spanned by w_s, $s \in F_7$, is the vector $w(S) := W - \sum_{s\in S} w_s = \frac{1}{2}\sum_{s\notin S} w_s - \frac{1}{2}{}_{s\in S} w_s$. The norm of the vector $\sqrt{2}w(S)$ equals 7. Moreover, for $u(S) \in \mathcal{U}_2$, all the corresponding 8 vectors $\sqrt{2}w(S)$ have mutual inner products -1 and span the 7-dimensional space with the basis $(w_s, s \in F_7)$. Hence these vectors represent a two-graph $\mathcal{T}(7,7)$. Since, for $(m,d) = (7,7)$ the special bound gives $n(7,7) = 8$, the two-graph $\mathcal{T}(7,7)$ is the unique regular two-graph with these parameters.

Recall that the projection of $\mathcal{U}_1$ on the 7-dimensional space with the basis $(f_i, i \in V_7)$ represent the unique two-graph $\mathcal{T}(3,7)$. Hence one can say that any two-graph $\mathcal{T}(5,15)$ obtained from the lattice $E_8 \oplus E_8$ is a special gluing of the unique two-graphs $\mathcal{T}(3,7)$ and $\mathcal{T}(7,7)$.

3.3 Two-graphs $\mathcal{T}_0(5,15)$ and $\mathcal{T}_1(5,15)$

We obtain that each two-graph from the even unimodular lattice $E_8 \oplus E_8$ is represented by a reduced odd system which is a union of odd systems $\mathcal{U}_1$ and $\mathcal{U}_2$. We consider two special cases of odd systems of such the form. These two odd systems belong also to other two families of odd systems from the second even unimodular lattice D_{16}^+.

Consider a reduced odd system $\mathcal{U}_1$, where all 7 sets T_s, $s \in F_7$, are of Type 1. In this case the sets T_s determine a bijection between two copies of F_7 consisting of triples s and triples $t(s)$. Recall that F_7 is the projective Fano plane $PG(2,2)$. Let the bijection be an isomorphism of the two planes. Then we can, without loss of generality, identify triples $t(s)$ with s. Now, $\mathcal{U}_1$ consists of vectors $v_s(s,\varepsilon^k) = w_s + \sum_{i\in s}\varepsilon_i^k f_i$, where ε^k for all k is either odd or even. For these vectors, we introduce the special denotation $u_s(\pm\varepsilon^k)$, where $+$ and $-$ correspond to even and odd ε-triples respectively. So

$$u_s(\varepsilon^k) = w_s + \sum_{i\in s}\varepsilon_i^k f_i,\ \ s \in F_7,\ \ 0 \le k \le 3. \tag{6}$$

Let $S \subseteq F_7$ be a set of triples s. We denote by $\mathcal{U}_1(S)$ the odd system $\mathcal{U}_1$ containing $u_s(-\varepsilon^k)$ for $s \in S$, and $u_s(\varepsilon^k)$ for $s \notin S$ with even ε^k. We denote by $\mathcal{U}(S)$ the union of $\mathcal{U}_1$ with $\mathcal{U}_2^0$. We have

$$\mathcal{U}(S) = \{u_i : 0 \le i \le 7;\ u_s(-\varepsilon^k), 0 \le k \le 3, s \in S;\ u_s(\varepsilon^k), 0 \le k \le 3, s \notin S;\ \}.$$

$\mathcal{U}(S)$ represents a regular two-graph $\mathcal{T}(5,15)$. Denote this two-graph by $\mathcal{T}(S)$.

We define a transformation of S. Recall that there are 3 triples $s \in F_7$ containing a given point i. For each i, $1 \le i \le 7$, consider triples $s \in S$ containing i. If there are 3 such triples, then delete them from S. If there are two such triples, then change they by the third triple containing i. If there is one or no triple containing i, then S is not transformed. Obviously, after such transformation for all i, we obtain S with either one or none of the triples. We call S *positive* if it is transformed into empty set, and *negative*, otherwise.

Proposition 3 *Let $S \subseteq F_7$ be a set of triples $s \in F_7$. Then $\mathcal{U}_1(S)$ (and $\mathcal{U}(S)$) are isomorphic to $\mathcal{U}_1(\emptyset)$ (and $\mathcal{U}(\emptyset)$) or $\mathcal{U}_1(\{s\})$ (and $\mathcal{U}(\{s\})$) according to S is positive or negative, respectively.*

Proof. Note that if we change in $\mathcal{U}_1(S)$ vectors f_i for some i by $-f_i$, we obtain an isomorphic odd system. This change of the sign of f_i is equivalent to the change of the sign before ε_i^k for all k, $0 \leq k \leq 3$. In other words, the change f_i by $-f_i$ is equivalent to the change of the even ε-triple in $u_s(\varepsilon^k)$ with $s \ni i$ by the odd ε-triple (and conversely).

So, if the set $S \subseteq F_7$ contains 3 triples containing the point i, we can eliminate these triples from S simultaneously transforming f_i into $-f_i$. If the set S contains two triples with i, we can change the two triples by the third triple containing i and transforming the f_i by $-f_i$. The assertion of the proposition for $\mathcal{U}_1(S)$ follows. Since we do not change vectors w_s, the assertion of the proposition is also true for $\mathcal{U}(S)$. $\Box$

Taylor proves [11] that there is a unique two-graph $\mathcal{T}_0(5,15)$ with a doubly transitive automorphism group. The full automorphism group of $\mathcal{T}_0(5,15)$ is $Sp(6,2)$. It is shown in [2] that the two-graph $\mathcal{T}_0(5,15)$ relates to the unique Steiner triple system, triples of which are lines of a 3-dimensional projective space $PG(3,2)$ over the field GF_2. We show in [7] that $\mathcal{T}(\emptyset) = \mathcal{T}_0(5,15)$.

Obviously, there is a permutation of the set V_7 that transforms any triple $s \in F_7$ in any other $s' \in F_7$. This permutation generates an isomorphism of odd systems $\mathcal{U}(\{s\})$ and $\mathcal{U}(\{s'\})$. Denote by $\mathcal{T}_1(5,15)$ the two-graph represented by any of these isomorphic odd systems. We show in [7] that $\mathcal{T}_1(5,15)$ relates to the Steiner triple system having number 2 in the extended version of [2].

3.4 Odd systems representing $\mathcal{T}_1(5,15)$

In this section we consider in details two odd systems $\mathcal{U}(\{t\}) = \mathcal{U}_1(\{t\}) \cup \mathcal{U}_2^0$ and $\mathcal{U}(t) = \mathcal{U}_1(\emptyset) \cup \mathcal{U}_2(t)$, and show that they are isomorphic, closed and its root system is $R(\mathcal{U}(\{t\}) = A_1^7$, i.e. it consists of 7 mutually orthogonal roots.

Each odd system representing the two-graph $\mathcal{T}_1(5,15)$ is a perturbation of the odd system $\mathcal{U}(\emptyset)$ representing the two-graph $\mathcal{T}_0(5,15)$. In what follows, we redenote the odd system $\mathcal{U}(\emptyset)$ by $\mathcal{W}_0$.

We show in [7] that $\mathcal{W}_0$ is closed, since it can be obtained by our construction from the Barnes-Wall lattice Λ_{16}. Namely, $\mathcal{W}_0 = \mathcal{V}(c)$ for every vector $c \in \sqrt{2}\Lambda_{16}$ of norm 12. Since the odd system $\mathcal{W}_0$ has no pair of vectors with the inner product ± 3, the root system of $\mathcal{W}_0$ is empty, i.e. $R(\mathcal{W}_0) = \emptyset$.

Call minimal by inclusion dependencies of $\mathcal{W}_0$ by *circuits*. Minimal by cardinality circuits of $\mathcal{W}_0$ consist of 6 vectors such that a sum of these vectors or its opposites is equal to 0. In the graph $G(\mathcal{W}_0)$, a circuit induces a switched maximal clique (of size 6). Later, in Section 8.4, we consider the graph $G(\mathcal{W}_0)$ in details.

Any 5 vectors of a circuit compose a *broken circuit*. Obviousely, the sum of 5 vectors of a broken circuit is the 6th vector with an opposite sign. In other words, a broken circuit generates a vector of the lattice $L_1(\mathcal{W}_0)$.

An odd system $\mathcal{U}$ representing $\mathcal{T}_1(5,15)$ is obtained from $\mathcal{W}_0$ by a substitution of some vectors. In this case, if a vector v of a circuit is substituted by v', then this circuit ceases to be a dependency, it becomes a broken circuit. But this broken circuit generates v, i.e. v belongs to cl$\mathcal{U}$. Besides $vv' = \pm 3$, and if $vv' = 3$, then the vector $v - v'$ is a root of $R(\mathcal{U})$.

We use circuits of $\mathcal{W}_0$ of the following form

$$\{u_a, u_b, u_s(\varepsilon(s)) : s \in C_i\},$$

where C_i is the 4-set of triples $s \in F_7$ not containing $i \in V_7$, and the pair $a, b \in V_8$ is such that either $a = 0$, $b = i$ or the triple abi belongs to F_7, i.e. it is one of the triples of F_7 containing i. Each point $j \in V_7 - \{i\}$ belongs to exactly two $s, s' \in C_i$. Hence the ε-triples $\varepsilon(s)$ and $\varepsilon(s')$ are such that $\varepsilon_j(s) = \pm\varepsilon_j(s')$, where the sign – corresponds to all j in the case $(a,b) = (0,i)$, and to $j = a, b$ in the cases $(a,b) \neq (0,i)$, and the sign + corresponds to the other cases. The corresponding dependencies are

$$u_i - u_0 + \sum_{s \in C_i} u_s(\varepsilon(s)) = 0 \text{ and } u_a + \sum_{s \ni a, s \not\ni b} u_s(\varepsilon(s)) = u_b + \sum_{s \ni b, s \not\ni a} u_s(\varepsilon(s)). \quad (7)$$

Consider, at first, the odd system $\mathcal{U}(\{t\})$. The vectors of $\mathcal{U}(\{t\})$ distinct from the vectors of $\mathcal{W}_0$ are $u_t(-\varepsilon^k)$, $0 \leq k \leq 3$. Any circuit of (7) with $C_i \ni t$ generates $u_t(\varepsilon^k)$ for some k. Since there are 4 choices of $\varepsilon(s)$, we obtain $u_t(\varepsilon^k)$ for all k, $0 \leq k \leq 3$.

For $k \neq 0$, we have that the vector

$$u_t(\varepsilon^0) - u_t(-\varepsilon^k) = \sum_{j \in t} f_j + \sum_{j \in t} \varepsilon^k_j f_j = 2f_l$$

is a root. Here l is such that $\varepsilon^k_l = 1$ and $\varepsilon^k_j = -1$ for $j \neq l$. So, we obtain 3 roots $2f_j$ for $j \in t$.

Now, consider the vectors $u_s(\varepsilon^k) \in \mathcal{U}(\{t\})$ for $s \neq t$. Let $\{j\} = s \cap t$. Then the vector $u_s(\varepsilon^k) - 2\varepsilon^k_j f_j$ of norm 5 belongs to cl$\mathcal{U}(\{t\})$ and has the form $u_s(-\varepsilon^l)$ for some l. Then, as above, we can obtain the roots $2f_j$ for $j \in s$. Continuing in this way, we obtain the 7 mutually orthogonal roots $2f_j$, $j \in V_7$. In this case, cl$\mathcal{U}(\{t\})$ contains vectors $u_s(\varepsilon)$ for all $8 = 2^3$ ε-triples $\varepsilon \in \{\pm 1\}^s$.

We show below that there is no other vector in cl$\mathcal{U}(\{t\})$, i.e. cl$\mathcal{U}(\{t\})$ contains the following 64 vectors: u_i, $i \in V_8$, and $u_s(\varepsilon)$, $\varepsilon \in \{\pm 1\}^s$, $s \in F_7$. We denote this odd system consisting of these 64 vectors by $\mathcal{W}_1$.

Now, we consider the odd system $\mathcal{U}(t) = \mathcal{U}_1(\emptyset) \cup \mathcal{U}_2(t)$, and show that cl$\mathcal{U}(t)$ is isomorphic to $\mathcal{W}_1$. The vectors $v_i(t) \in \mathcal{U}(t)$ substitute the vectors $u_i \in \mathcal{W}_0$ for $i \in V_7 - t$.

Consider the 4-set $V_7 - t$ in details. Since F_7 is a Steiner triple system, each unordered pair ik of points of V_7 belongs to exactly one triple of F_7. In other words, the pair ik determines uniquely a triple of F_7. Since there are 6 distinct pairs of points in $V_7 - t$, we obtain all other 6 triples of F_7 distinct from t. These 6 triples are partitioned into 3 pairs of triples having the same intersection point with t. For $m \in t$, let $p(m)$ and $q(m)$ be the pair of triples with $p(m) \cap t = q(m) \cap t = \{m\}$. Since $p(m) \cap q(m) = \{m\}$, the triples $p(m)$ and $q(m)$ are determined by complementary pairs of points of the 4-set $V_7 - t$. In other words, each point $m \in t$ determines uniquely both a partition of $V_7 - t$ into complementary pairs ij, kl and the corresponding triples $p(m)$ and $q(m)$.

Fix $j \in V_7 - t$, and, for each $m \in t$, the triple containing j take as $p(m)$. So, the 3 triples $p(m)$, $m \in t$, are triples of the form $i_m j m$ where the point i_m is determined uniquely by the pair j, m.

Recall that, by Proposition 3, the odd system $\mathcal{U}_1(\emptyset)$ is isomorphic to $\mathcal{U}_1(S)$ for positive $S \subseteq F_7$. It is easy to see that the set $P_j := \{p(m) : m \in t\}$ of 3 triples containing $j \in V_7$ is a positive set. For what follows, it is convenient to take the isomorphic odd system $\mathcal{U}_1(P_j) \cup \mathcal{U}_2(t)$ instead of $\mathcal{U}_1(\emptyset) \cup \mathcal{U}_2(t)$. We denote the new odd system by $\mathcal{U}(t)$ as before. So, each vector $u_{p(m)}(\varepsilon) \in \mathcal{U}((t)$ has odd triples ε, i.e. $\prod_{i \in p(m)} \varepsilon_i = -1$.

Recall that $u_i \notin \mathcal{U}(t)$ for $i \in V_7 - t$. Since $u_0 \in \mathcal{U}(t)$, the first dependency in (7) generates u_i for $i \in V_7 - t$. If $ijm = p(m)$, then the vector

$$r_t(m) = u_i - v_j(t) = u_j - v_i(t) = \sum_{s \ni i} w_s + \sum_{s \ni j} w_s + w_t - 2W = w_{p(m)} - w_{q(m)}$$

is a root. Now, the vector $u_{q(m)}(\varepsilon)$ and the root $r_t(m)$ give the following new vectors of $\mathrm{cl}\mathcal{U}(t)$:

$$u_{q(m)}(\varepsilon) + r_t(m) = w_{p(m)} + \sum_{i \in q(m)} \varepsilon_i f_i = v_{p(m)}(q(m), \varepsilon).$$

Similarly, we obtain the vectors $v_{q(m)}(p(m), \varepsilon) = u_{p(m)}(\varepsilon) - r_t(m) \in \mathrm{cl}\mathcal{U}(t)$.

The vectors $v_{p(m)}(q(m), \varepsilon)$ and $u_{p(m)}(\varepsilon')$ provide a new root if $\varepsilon_m = \varepsilon'_m$. Let $p(m) = \{ijm\}$ and $q(m) = \{klm\}$. We set $\sigma = (\sigma_i : i \in V_7 - t\}$, where $\sigma_i = \varepsilon'_i$, $\sigma_j = \varepsilon'_j$, $\sigma_k = -\varepsilon_k$, $\sigma_l = -\varepsilon_l$. Recall that $\{ijkl\} = V_7 - t$. Then

$$r_t(\sigma) = u_{p(m)}(\varepsilon') - v_{p(m)}(q(m), \varepsilon) = \sum_{i \in V_7 - t} \sigma_i f_i$$

is a root. Recall that ε' is odd and ε is even and $\varepsilon'_m = \varepsilon_m$. Hence $\varepsilon_k \varepsilon_l = -\varepsilon'_i \varepsilon'_j = \varepsilon_m = \varepsilon'_m$. This implies that $\prod_{j \in V_7 - t} \sigma_j = -1$.

Call a quadruple $\sigma \in \{\pm 1\}^{V_7 - t}$ *odd* if $\prod_{j \in V_7 - t} \sigma_j = -1$, and *even*, otherwise. There are 4 pair of opposite odd quadruples. In each pair of opposite quadruples, we choose the odd quadruple σ^i having only one –1 on the place $i \in V_7 - t$.

Choosing suitable ε and ε', we can obtain the roots $r_t(\pm\sigma^i)$ for all 8 quadruples σ. Obviously, $r_t(-\sigma) = -r_t(\sigma)$. We redenote $r_t(\sigma^i)$ by $r_t(i)$. So, we obtain that the root system $R(\mathcal{U}(t))$ contains the 7 mutually orthogonal roots $r_t(i)$, $i \in V_7$.

Besides, we obtain that cl$\mathcal{U}(t)$ contains, excepting the vectors of $\mathcal{U}(t)$, the vectors u_i for $i \in V_7 - t$, vectors $v_{p(m)}(q(m), \varepsilon)$ for even ε, and $v_{q(m)}(p(m), \varepsilon)$ for odd ε, $m \in t$. So, we have $4 + 6 \times 4 = 28$ additional vectors in cl$\mathcal{U}(t)$.

Now, we show that these 36+28=64 vectors of cl$\mathcal{U}(t)$ form an odd system isomorphic to $\mathcal{W}_1$. To this end, we introduce a new orthogonal basis of the space spanned by $\mathcal{U}(t)$. Recall that the old basis consists of the following 15 mutually orthogonal vectors: g, f_i, $i \in V_7$, w_s, $s \in F_7$.

Let $H = \frac{1}{2}\sum_{i \in V_7 - t} f_i$. We set

$$g' = \frac{1}{2}g + \frac{3}{4}w_t,\ e_m = \frac{1}{2}(w_{p(m)} - w_{q(m)}), m \in t,\ e_i = H - f_i, i \in V_7 - t,$$

$$x_t = g - \frac{1}{2}w_t,\ x_{p(m)} = \frac{1}{2}(w_{p(m)} + w_{q(m)}) + f_m,\ x_{q(m)} = \frac{1}{2}(w_{p(m)} + w_{q(m)}) - f_m, m \in t.$$

It is easy to verify that $g'^2 = \frac{3}{2}$, $e_m^2 = e_i^2 = 1$, $x_t^2 = x_{p(m)}^2 = x_{q(m)}^2 = 2$, and all these vectors are mutually orthogonal. We can express the vectors of the old basis via the vectors of the new basis as follows. (Note that $2H = \sum_{i \in V_7 - t} f_i = \sum_{i \in V_7 - t} e_i$.)

$$g = \frac{1}{2}g' + \frac{3}{4}x_t,\ f_m = \frac{1}{2}(x_{p(m)} - x_{q(m)}), m \in t,\ f_i = H - e_i, i \in V_7 - t,$$

$$w_t = g' - \frac{1}{2}x_t,\ w_{p(m)} = \frac{1}{2}(x_{p(m)} + x_{q(m)}) + e_m,\ w_{q(m)} = \frac{1}{2}(x_{p(m)} + x_{q(m)}) - e_m, m \in t.$$

Now, we introduce the following vectors of norm 5 similar to vectors u_i, $i \in V_8$, and $u_s(\varepsilon)$, $s \in F_7$. We set $X = \frac{1}{2}\sum_{s \in F_7} x_s$ and

$$y_0 = g' + X,\ y_i = g' - X + \sum_{s \ni i} x_s,\ y_s(\varepsilon) = x_s + \sum_{i \in s} \varepsilon_i e_i.$$

Now, we identify the vectors $v \in \mathcal{U}(t)$ with y_i, $i \in V_8$, and $y_s(\varepsilon)$, $s \in F_7$.

For $i \in V_7 - t$ and $m \in t$, we set $\varepsilon_m(i) = 1$ if $i \in p(m)$, and $\varepsilon_m(i) = -1$ if $i \in q(m)$. Recall that we fix $j \in V_7 - t$ such that $j \in p(m)$ for all $m \in t$. Hence each $i \in V_7 - t - \{j\}$ belongs to $p(m)$ exactly for one m, and to $q(m)$ exactly for two other $m \in t$. This implies that the triple $\varepsilon(i) = \{\varepsilon_m(i) : m \in t\}$ is even for all $i \in V_7 - t$.

It is not difficult to verify that, for $i \in V_7 - t$, we have

$$u_i = y_t(\varepsilon(i)),\ v_i(t) = y_t(-\varepsilon(i)),\ u_t(\varepsilon(i)) = y_i.$$

For $m = 0$ or $m \in t$, we obtain $u_m = y_m$.

Now consider the vectors $u_s(\varepsilon)$ for $s=p(m)$ and $q(m)$. Let $s=p(m)=\{ijm\}$. Recall that ε is odd for $u_{p(m)}(\varepsilon)\in\mathcal{U}(t)$, i.e. $\varepsilon_i\varepsilon_j=-\varepsilon_m$. Hence if $\varepsilon_m=1$, then $\varepsilon_i+\varepsilon_j=0$, and if $\varepsilon_m=-1$, then $\varepsilon_i=\varepsilon_j$. Therefore we obtain

$$\text{if } \varepsilon_m=1,\ \text{then } u_{p(m)}(\varepsilon)=y_{p(m)}(\varepsilon') \text{ where } \varepsilon'=(\varepsilon_m,-\varepsilon_i,-\varepsilon_j) \text{ is odd, and}$$

$$\text{if } \varepsilon_m=-1,\ \text{then } u_{p(m)}(\varepsilon)=y_{q(m)}(\varepsilon'') \text{ where } \varepsilon''=(-\varepsilon_m,\varepsilon_i,\varepsilon_j) \text{ is even.}$$

Similarly, if $s=q(m)=\{klm\}$, then ε is even in $u_{q(m)}(\varepsilon)$, and we obtain

$$\text{if } \varepsilon_m=1,\ \text{then } u_{q(m)}(\varepsilon)=y_{p(m)}(\varepsilon'') \text{ where } \varepsilon''=(-\varepsilon_m,\varepsilon_k,\varepsilon_l) \text{ is odd,}$$

$$\text{if } \varepsilon_m=-1,\ \text{then } u_{q(m)}(\varepsilon)=y_{q(m)}(\varepsilon') \text{ where } \varepsilon'=(\varepsilon_m,-\varepsilon_k,-\varepsilon_l) \text{ is even.}$$

So, we obtain that $\mathcal{U}(t)=\mathcal{Y}_1(P_j\cup\{t\})\cup\mathcal{Y}_2^0$, where $\mathcal{Y}_1(S)$ is the set of $y_s(\varepsilon)$ with odd ε for $s\in S$, and even ε for $s\notin S$, and $\mathcal{Y}_2^0=\{y_i: i\in V_8\}$.

Obviously, the natural bijection $\alpha(t)$ of basic vectors:

$$\alpha(t): g\leftrightarrow g',\ f_i\leftrightarrow e_i,\ w_s\leftrightarrow x_s$$

generates an isomorphism of odd systems of types $\mathcal{U}$ and $\mathcal{Y}$. Recall that $\mathcal{U}(t)=\mathcal{U}_1(P_j)\cup\mathcal{U}_2(t)$, where $\mathcal{U}_2(t)=\{u_0,u_m,v_i(t),m\in t,i\in V_7-t\}$, and $\mathcal{U}_1(P_j)$ is isomorphic to $\mathcal{U}_1(\emptyset)$. Hence we obtain for $\mathcal{U}(t)$ the following two isomorphic representations

$$\mathcal{U}(t)=\mathcal{U}_1(P_j)\cup\mathcal{U}_2(t)\cong\mathcal{U}_1(P_j\cup\{t\})\cup\mathcal{U}_2^0\cong\mathcal{U}_1(\{t\})\cup\mathcal{U}_2^0=\mathcal{U}(\{t\}).$$

The same bijection shows that cl$\mathcal{U}(t)$ contains an odd system isomorphic to $\mathcal{W}_1$. Set

$$\mathcal{U}_{ev}(s)=\{u_s(\varepsilon):\varepsilon \text{ is even}\},\mathcal{U}_{od}(s)=\{u_s(\varepsilon):\varepsilon \text{ is odd}\},\mathcal{U}_1(s)=\mathcal{U}_{ev}(s)\cup\mathcal{U}_{od}(s),$$

$$\mathcal{U}_t^0=\{u_0,u_m:m\in t\},\ \mathcal{U}_t^1=\{u_i:i\in V_7-t\},\ \mathcal{U}_t^2=\{v_i(t):i\in V_7-t\}.$$

Note that each set (besides $\mathcal{U}_1(s)$) contains 4 vectors, and $\mathcal{U}_2=\mathcal{U}_t^0\cup\mathcal{U}_t^1$.

Then we have

$$\mathcal{W}_1=\mathcal{U}_t^0\cup\mathcal{U}_t^1\cup_{s\in F_7}\mathcal{U}_1(s).$$

If we use the same representation of $\mathcal{W}_1$ in y-vectors and then substitute y-vectors by equal u-vectors, we obtain that the bijection $\alpha(t)$ makes the following permutations:

$$\mathcal{U}_t^0\leftrightarrow\mathcal{U}_t^0,\ \mathcal{U}_t^1\leftrightarrow\mathcal{U}_{ev}(t),\ \mathcal{U}_{od}(t)\leftrightarrow\mathcal{U}_t^2,$$

$$\mathcal{U}_{ev}(p(m))\leftrightarrow\mathcal{U}_{ev}(p(m)),\ \mathcal{U}_{od}(p(m))\leftrightarrow\mathcal{U}_{ev}(q(m)),\ \mathcal{U}_{od}(q(m))\leftrightarrow\mathcal{U}_{od}(q(m)).$$

Hence we obtain

$$\mathcal{W}_1=\mathcal{U}_t^0\cup\mathcal{U}_t^1\cup\mathcal{U}_t^2\cup\mathcal{U}_{ev}(t)\cup_{s\in F_7-\{t\}}\mathcal{U}_1(s).$$

We can consider these two representations of $\mathcal{W}_1$ as two distinct embeddings of the odd system $\mathcal{W}_1$ into the odd system $\mathcal{V}(c_0)=\mathcal{V}_1\cup\mathcal{V}_2$.

We have $\mathcal{U}_t^0 \cup \mathcal{U}_t^1 = \mathcal{U}_2^0 \subseteq \mathcal{V}_2$ and $\cup_{s\in F_7}\mathcal{U}_1(s) \subseteq \mathcal{V}_1$ in the first embedding. In the second embedding, we have, as in the first embedding, $\mathcal{U}_t^0 \subseteq \mathcal{V}_2$ and $\cup_{s\in F_7, s\neq t}\mathcal{U}_1(s) \subseteq \mathcal{U}_1$. But $\mathcal{U}_1(t)$ is embedded into $\mathcal{V}_2$ and represented there as the set $\mathcal{U}_t^1 \cup \mathcal{U}_t^2$. Similarly, $\mathcal{U}_t^1$ is embedded into $\mathcal{V}_1$ and represented there as the set $\mathcal{U}_{ev}(t)$. Since t is an arbitrary triple of F_7, we have

Proposition 4 *For any $t \in F_7$, there is an embedding of the odd system $\mathcal{W}_1$ into $\mathcal{V}(c_0)$ such that $\mathcal{U}_1(t)$ is embedded into $\mathcal{V}_2$ and $\mathcal{U}_t^1$ is embedded into $\mathcal{V}_1$.* $\Box$

Now we prove very important fact.

Theorem 1 *The odd system $\mathcal{W}_1$ is closed.*

Proof. We saw that the odd system $\mathcal{W}_1 = \mathcal{U}_2^0 \cup_{s\in F_7} \mathcal{U}_1(s)$can be embedded into the odd system $\mathcal{V}(c_0) = \mathcal{V}_1 \cup \mathcal{V}_2$ in different ways. For example, for any $s, t \in F_7$, there exists an embedding such that $\mathcal{U}_1(s) \subseteq \mathcal{V}_1$ and $\mathcal{U}_1(t) \subseteq \mathcal{V}_2$. Call the sets $\mathcal{U}_2^0$ and $\mathcal{U}_1(s)$, $s \in F_7$, *components* of $\mathcal{W}_1$.

Suppose that $\mathcal{W}_1$ is not closed, and let $v \in$cl$\mathcal{W}_1 - \mathcal{W}_1$. Since $\mathcal{W}_1$ contains a maximal system of vectors with mutual inner products ± 1, there is a vector $u \in \mathcal{W}_1$ such that $vu = \pm 3$. Let $\mathcal{W}_3(v) = \{u \in \mathcal{W}_1 : uv = \pm 3\}$. We show that $\mathcal{W}_3(v)$ is contained in one of components. Suppose to the contrary, that there are $u_1, u_2 \in \mathcal{W}_3(v)$ such that u_1 and u_2 belong to distinct components. Consider an embedding of $\mathcal{W}_1$ into $\mathcal{V}(c_0)$ such that the components containing u_1 and u_2 belong to different systems $\mathcal{V}_1$ and $\mathcal{V}_2$. But this contradicts to that $uu' = \pm 1$ for $u \in \mathcal{V}_1$ and $u' \in \mathcal{V}_2$, and v belongs either to $\mathcal{V}_1$ or to $\mathcal{V}_2$.

Now, suppose that $\mathcal{W}_1$ is naturally embedded into $\mathcal{V}(c_0)$, i.e. $\mathcal{U}_2^0 \subseteq \mathcal{V}_2$ and $\cup_{s\in F_7}\mathcal{U}(s) \subseteq \mathcal{V}_1$. Obviously, cl$\mathcal{W}_1 \subseteq \mathcal{V}(c_0)$. Hence each vector of $\mathcal{V}_1\cap$cl$\mathcal{W}_1$ has the form $v_s(t,\varepsilon)$ for $s, t \in F_7$, $s \neq t$, and $\varepsilon \in \{\pm 1\}^t$. Similarly, each vector of $\mathcal{V}_2\cap$cl$\mathcal{W}_1$ has the form $u(S)$ for some even $S \subseteq F_7$.

For $v_s(t,\varepsilon) \neq u_{s'}(\varepsilon')$ we have $|v_s(t,\varepsilon)u_{s'}(\varepsilon')| = 3$ in the following two cases:

1) $s = s'$, $t \neq s'$ and $\varepsilon_i\varepsilon'_i = 1$, where $i \in t \cap s'$, what implies $u_{s'}(\varepsilon') \in \mathcal{U}(s)$,

2) $s \neq s'$, $t = s'$ and $\varepsilon = \varepsilon'$ what implies $u_{s'}(\varepsilon') \in \mathcal{U}(t)$.

In other words, the vector $v_s(t,\varepsilon)$ has the inner product ± 3 with some vectors of distinct components $\mathcal{U}(s)$ and $\mathcal{U}(t)$. Hence $v_s(t,\varepsilon) \notin$cl$\mathcal{W}_1$.

We saw that, for any $t \in F_7$, there is an embedding $\mathcal{W}_1$ into $\mathcal{V}(c_0)$ such that $\mathcal{U}_t^1 \subseteq \mathcal{V}_1$ and $\mathcal{U}_t^0 \subseteq \mathcal{V}_2$. Since for each two points $i, j \in V_7$ there is $t \in F_7$ such that $u_i \in \mathcal{U}_t^1$, $u_j \in \mathcal{U}_t^0$, we conclude as above that each $v \in$cl$\mathcal{W}_1$ has the inner product ± 3 with only one $u_i \in \mathcal{U}_2^0$. Recall that $u_0 = u(\emptyset)$ and $u_i = u(C_i)$, $i \in V_7$. It is not difficult to verify that for any even set $S \subseteq F_7$, $S \neq \emptyset, C_i$, there is at least 3 points $i \in V_8$ such that $|u(S)u_i| = 3$. So, $u(S) \notin$cl$\mathcal{W}_1$ for $S \neq \emptyset, C_i$. This implies that cl$\mathcal{W}_1 = \mathcal{W}_1$, i.e. $\mathcal{W}_1$ is closed. $\Box$

Corollary 1 $R(\mathcal{U}(\{s\})) = R(\mathcal{U}(t)) = R(\mathcal{W}_1) = A_1^7$.

3.5 Two-graphs from $E_8 \oplus E_8$ distinct from $\mathcal{T}_{1,2}(5,15)$

As an example, we give an odd system $\mathcal{U}(i) := \mathcal{U}_1(i) \cup \mathcal{U}_2^0$ with a large root system. Recall that C_i is the set of $s \in F_7$ not containing i, and any $v \in \mathcal{U}_1$ has the form $v = v_s(t,\varepsilon)$, where $(t,\varepsilon) \in T_s$. We take the sets T_s as follows. For $s \notin C_i$, i.e. for $s \ni i$, we take $T_s = \{(s,\varepsilon)\}$ with all odd ε-triples for one $s \ni i$, and with all even ε-triples for the other two $s \ni i$. For $s \in C_i$, we take $T_s = \{(t, \varepsilon(s,t)) : t \in C_i\}$, where $\varepsilon(s,t) \in \{\pm 1\}^t$ satisfies the following conditions. For all $s, t \in C_i$,

$\varepsilon(s,t)$ is even, $\varepsilon(s,t) = \varepsilon(t,s)$, $\varepsilon(s,s) = \varepsilon^0 = (1,1,1)$,

$\varepsilon(s,t)\varepsilon(s',t) = \pm 1$, and $\varepsilon_j(s,t) + \varepsilon_j(s,t') = 0$ for $j \in t \cap t'$.

$\{\varepsilon(s,t) : t \in C_i\} = \{\varepsilon(s,t) : s \in C_i\} = \{\varepsilon^k : 0 \le k \le 3\}$.

It can be verified that these T_s are mutually consistent. By the method as above, we obtain that

$$R(\mathcal{U}(i)) = A_1 D_3 D_4 D_6,$$

where $A_1 = \{2f_i\}$, $D_3 = \{w_s \pm w_{s'} : s, s' \ni i\}$, $D_4 = \{w_s \pm w_{s'} : s, s' \in C_i\}$, $D_6 = \{2f_j : j \neq i\} \cup \{\sum_{j \in Q} \varepsilon_j f_j : \varepsilon_j \in \{\pm 1\}, Q = V_7 - s, s \ni i.\}$

We end this section with a two-graph represented by an odd system $\mathcal{U}$ with $R(\mathcal{U}) = R(\mathcal{V}(c_0)) = D_7 E_7$.

Let $\phi : F_7 \to V_7$ be an arbitrary bijection between two the 7-sets. Using this bijection, we consider an odd system $\mathcal{U}_1(\phi)$ with the sets T_s taken as follows. For $s \in F_7$, we set $T_s = \{(t, \varepsilon_t) : t \in C_{\phi(s)}\}$, and choose ε_t such that the corresponding family of T_s will be consistent. Besides, we take all ε_t even in all T_s with $s \neq s_0$, and ε_t is odd for $(t, \varepsilon_t) \in T_{s_0}$. It is not difficult to verify that $R(\mathcal{U}(\phi)) = D_7 E_7$, where $\mathcal{U}(\phi) = \mathcal{U}_1(\phi) \cup \mathcal{U}_2^0$.

Note that the root systems $A_1 D_3 D_4 D_6$ and $D_7 E_7$ are not contained in the root systems A_{14} and $D_{10} A_5$ of the two odd systems obtained by our construction from the lattice D_{16}^+. Hence the odd systems $\mathcal{U}(i)$ and $\mathcal{U}(\phi)$ represent two-graphs which are not contained in both the families of two-graphs from D_{16}^+. It seems to us that these two-graphs were not known early. This assertion is true up to not known for us the two-graphs announced by T.Spence in the unpublished work referenced in [10].

4 The even unimodular lattice D_{16}^+

Now, we apply our construction to $\sqrt{2} D_{16}^+$, to the 16-dimensional even unimodular lattice D_{16}^+ multiplied by $\sqrt{2}$. A description of D_{16}^+ can be found in [4].

Let $V_{16} = \{0\} \cup V_{15}$ and $V_{15} = \{i : 1 \le i \le 15\}$. Let e_i, $i \in V_{16}$, be a frame of 16 mutually orthogonal vectors of norm $e_i^2 = 2$. The minimal vectors of the lattice $\sqrt{2} D_{16}^+$ (of norm 4) are roots $\pm e_i \pm e_j$, $0 \le i < j \le 15$, of the root system $\sqrt{2} D_{16}$. The lattice $\sqrt{2} D_{16}^+$ is generated by its roots and by the vector $\frac{1}{2}\sum_{i=0}^{15} e_i$ of norm 8.

Any vector of $\sqrt{2} D_{16}^+$ has the form $\sum_0^{15} x_i e_i$ (see [4]) such that

1) $x_0 \equiv x_1 \equiv ... \equiv x_{15} \pmod{\mathbf{Z}}$,
2) $2x_i \in \mathbf{Z},\ 0 \le i \le 15$,
3) $\sum_0^{15} x_i \equiv 0 \pmod 2$.

Using this, we obtain that any vector of norm 8 is one of the following 3 types.
(1) $\pm 2e_i$, $0 \le i \le 15$,
(2) $\sum_{i\in Q} \varepsilon_i e_i$, $Q \subseteq V_{16}$, $|Q| = 4$, $\varepsilon_i \in \{\pm 1\}$, $\sum_{i\in Q} \varepsilon_i \equiv 0 \pmod 2$,
(3) $\frac{1}{2}\sum_{i=0}^{15} \varepsilon_i e_i$, $\varepsilon_i \in \{\pm 1\}$, $\sum_0^{15} \varepsilon_i \equiv 0 \pmod 4$.

There are 3 types (1)(3); (2)(2); (2)(3) of pairs of vectors of norm 8 with the inner product -2. Summing the vectors of each pair, we obtain a vector c of norm 12. It turns out that there are vectors c of the following two types:
(I) $\frac{1}{2}\sum_0^{15} \varepsilon_i e_i - 2\varepsilon_k e_k = -\frac{3}{2}\varepsilon_k e_k + \frac{1}{2}\sum_{i\ne k} \varepsilon_i e_i$, $\sum_0^{15} \varepsilon_i \equiv 0 \pmod 4$,
(II) $\sum_{i\in S} \varepsilon_i e_i$, $|S| = 6$, $S \subseteq V_{16}$.

Note that vectors c of the same type belong to the same orbit of the automorphism group of the lattice $\sqrt{2}D_{16}^+$. Hence, up to the symmetry, we can consider the following representatives of these two types:

$$c_I = \frac{3}{2}e_0 + \frac{1}{2}e(V_{15}),\ \text{and}\ c_{II} = e(S_0),\ S_0 = \{0,1,2,3,4,5\}.$$

5 Case I: two-graphs related to Steiner triple systems

Let T_{15} be the set of all 3-subsets $t \subseteq V_{15}$, $|t| = 3$. For $t \in T_{15}$, we set $a(t) = e_0 + e(t)$.

Then using the definition of the set $\mathcal{A}(c)$, we obtain

$$\mathcal{A}(c_I) = \{a(t), c_I - a(t) : t \in T_{15}\} \cup \{2e_0, c_I - 2e_0\}.$$

Recall that $v(a) = a - \frac{1}{2}c_I$. We set

$$v_0 = v(2e_0) = e_0 + \frac{1}{4}(e_0 - e(V_{15})),$$

$$v(t) = v(a(t)) = e(t) + \frac{1}{4}(e_0 - e(V_{15})),\ \text{and}$$
$$\mathcal{V}(T) = \{v(t) : t \in T\}.$$

Then
$$\mathcal{V}(c_I) = \pm(\{v_0\} \cup \mathcal{V}(T_{15})).$$

It is easy to see that

$$R(\mathcal{V}(c_I)) = R(\mathcal{V}(T_{15})) = A_{14}.$$

Since there are $|T_{15}| = \binom{15}{3} = 455$ triples t, we have

$$\frac{1}{2}|\mathcal{V}(c_I)| = 456. \tag{8}$$

As $v_0 v(t) = 1$ for all $t \in T_{15}$, we have

$$\mathcal{V}(T_{15}) = \{v \in \mathcal{V}(c_I) : vv_0 = 1\}.$$

Note that, for $t, t' \in T_{15}$,

$$v(t)v(t') = 2|t \cap t'| - 1. \tag{9}$$

Hence $v(t)v(t') = \pm 1$ if and only if $|t \cap t'| \leq 1$.

Proposition 5 *Any maximal reduced odd subsystem of $\mathcal{V}(c_I)$ spanning equiangular lines is switching equivalent to one of the 80 regular odd systems corresponding to 80 nonisomorphic Steiner triple systems on 15 points.*

Proof. Obviously, any reduced odd subsystem $\mathcal{U} \subseteq \mathcal{V}(c_I)$ can be switched to the form $\mathcal{U}^{sw} = \{v_0\} \cup \mathcal{V}(T)$ for some $T \subseteq T_{15}$.

Consider an odd system $\mathcal{V}(T)$ for a maximal by inclusion set $T \subseteq T_{15}$ such that vectors $v(t)$ of it have all mutual inner products ± 1. Then, according to (9), for each pair ij of points of V_{15}, there exists at most one triple $t \in T$ containing it. If, for each pair ij, there exists a triple $t \in T$ containing it, then T is maximal and called a *Steiner triple system* (briefly STS). It is well known that there exist 80 nonisomorphic STS on 15 points. $\square$

Each STS with a set of triples T contains $|T| = 35$ triples. Hence the odd system $\{v_0\} \cup \mathcal{V}(T)$ contains 36 vectors and spans 36 equiangular lines at angle $arccos\frac{1}{5}$. Since the special bound is achieved, the odd system $\{v_0\} \cup \mathcal{V}(T)$ with T as a set of triples of an STS corresponds to a regular two-graph on 36 points. It is proved in [2] that all these two-graphs are not isomorphic. The odd systems of norm 5 related to STS's on 15 points are studied in [7].

6 Case II: Two-graphs related to 2-(10,4,2) designs

6.1 The odd system $\mathcal{V}(c_{II})$

Recall that

$$c_{II} = e(S_0), \;\; S_0 = \{0, 1, 2, 3, 4, 5\} \;.$$

We set $V_{10} = V_{16} - S_0$, $T_6 = \{t \subseteq S_0 : |t| = 3\}$, and $E = \{\varepsilon \in \{\pm 1\}^{V_{10}} : \sum_{i \in V_{10}} \varepsilon_i \equiv 2 \pmod 4\}$. Let

$$a(\varepsilon) = \frac{1}{2}e(S_0) + \frac{1}{2}\sum_{i \in V_{10}} \varepsilon_i e_i,\ \varepsilon \in E,$$

$$a_i^\sigma(t) = e(t) + \sigma e_i,\ i \in V_{10}, t \in T_6, \sigma \in \{\pm 1\}.$$

Using the definition of $\mathcal{A}(c_{II})$ we obtain

$$\mathcal{A}(c_{II}) = \{a(\varepsilon), a_i^\sigma(t) : \varepsilon \in E,\ t \in T_6,\ i \in V_{10},\ \sigma \in \{\pm 1\}\}.$$

The corresponding v-vectors $v(a) = a - \frac{1}{2}c_{II}$ are

$$v(\varepsilon) = v(a(\varepsilon)) = \frac{1}{2}\sum_{i \in V_{10}} \varepsilon_i e_i,\ \varepsilon \in E\ ,$$

$$v_i^\sigma(t) = v(a_i^\sigma(t)) = \frac{1}{2}(e(t) - e(\overline{t})) + \sigma e_i,\ \overline{t} = S_0 - t.$$

Consider a 10-dimensional cube Q_{10} with edges of norm 2. Let its edges be parallel to vectors of the frame $\{e_i : i \in V_{10}\}$, and its center be in the origin. Then vertices of the cube are given by the vectors $v(\varepsilon)$ for all 2^{10} values of $\varepsilon \in \{\pm 1\}^{V_{10}}$. The condition $\sum_{i \in V_{10}} \varepsilon_i \equiv 2 \pmod 4$ chooses *even* vertices of the cube Q_{10}.

Now we introduce a new denotation for $v(\varepsilon)$. For $\varepsilon \in \{\pm 1\}^{V_{10}}$, we set $p(\varepsilon) = \{i \in V_{10} : \varepsilon_i = -1\}$. Then $v(\varepsilon) = \frac{1}{2}e(V_{10} - p(\varepsilon)) - \frac{1}{2}e(p(\varepsilon)) = \frac{1}{2}e(V_{10}) - e(p(\varepsilon))$. For $\varepsilon \in E$, the set $p(\varepsilon)$ has an even cardinality. Let P be the family of all subsets $p \subseteq V_{10}$ of even cardinality. Redenoting $v(\varepsilon)$ by corresponding $v(p)$, we rewrite $v(\varepsilon)$ and $v_i^\sigma(t)$ as follows:

$$v(p) = \frac{1}{2}e(V_{10}) - e(p), \tag{10}$$

$$v_i^\sigma(t) = e(t) + \sigma e_i - \frac{1}{2}e(S_0).$$

Hence

$$\mathcal{V}(c_{II}) = \{v(p), v_i^\sigma(t) : p \in P, t \in T_6, i \in V_{10}, \sigma \in \{\pm 1\}\}.$$

For what follows, we need explicit expressions for inner products of vectors from $\mathcal{V}(c_{II})$. Recall that $e_i e_j = 2\delta_{ij}$, where $\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$, otherwise. Since $t \cap p = \emptyset$ for all $t \in T_6$ and $p \subseteq V_{10}$, we have

$$\begin{aligned} v(p)v(p') &= 2|p \cap p'| - |p| - |p'| + 5, && (11)\\ v(p)v_i^\sigma(t) &= \mp\sigma\epsilon(p, i), && (12)\\ v_i^\sigma(t)v_j^{\sigma'}(t') &= 2|t \cap t'| - 3 + 2\sigma\sigma'\delta_{ij}, && (13) \end{aligned}$$

where $\epsilon(p, i)$ is defined in (1).

We introduce the following odd subsystems of $\mathcal{V}(c_{II})$

$$\mathcal{V}_P = \{v(p) : p \in P\}, \ \mathcal{V}_T^\sigma = \{v_i^\sigma(t) : t \in T_6, i \in V_{10}\}, \ \mathcal{V}_T = \mathcal{V}_T^{+1} \cup \mathcal{V}_T^{-1}.$$

Since $v_i^{-1}(t) = -v_i^{+1}(\overline{t})$, where $\overline{t} = S_0 - t$, we have $\mathcal{V}_T^{+1} = -\mathcal{V}_T^{-1}$, and

$$\mathcal{V}(c_{II}) = \mathcal{V}_P \cup \mathcal{V}_T.$$

Note that $vv' = \pm 1$ for $v \in \mathcal{V}_P$, $v' \in \mathcal{V}_T$.

It is not difficult to verify that $R(\mathcal{V}_P) = D_{10}$ and $R(\mathcal{V}_T) = A_5 D_{10}$. Hence

$$R(\mathcal{V}(c_{II})) = A_5 D_{10}.$$

These are roots of D_{16} orthogonal to $c_{II} = e(S_0)$.

There are $\sum_{k=0}^{5} \binom{10}{2k} = 2^9 = 512$ vectors $v(p)$, $p \in P$, and $2 \times 10 \times \binom{6}{3} = 400$ vectors $v_i^\sigma(t)$. Hence

$$\frac{1}{2}|\mathcal{V}(c_{II})| = 456.$$

Comparing this number with (2) and (8), we see that $\mathcal{V}(c_0)$, $\mathcal{V}(c_I)$ and $\mathcal{V}(c_{II})$ are equicardinal. One can think that they are isomorphic. But we have

Proposition 6 *The odd systems $\mathcal{V}(c_0)$, $\mathcal{V}(c_I)$ and $\mathcal{V}(c_{II})$ are not isomorphic.*

Proof. Recall that each of the odd systems $\mathcal{V}(c_i)$, $i = 0, I, II$, is partitioned into two odd subsystems $\mathcal{V}_1$ and $\mathcal{V}_2$ such that $v_1 v_2 = \pm 1$, for $v_i \in \mathcal{V}_i$, $i = 1, 2$. But, the table below shows that for the 3 odd systems these partitions are distinct.

	$\mathcal{V}(c_0)$	$\mathcal{V}(c_I)$	$\mathcal{V}(c_{II})$
$\frac{1}{2}\mathcal{V}_1$	392	1	200
$\frac{1}{2}\mathcal{V}_2$	64	455	256

$\square$

Note that the odd systems $\mathcal{V}(c_i)$, $i = 0, I, II$, span sets of lines with only two angles: $arccos\frac{1}{5}$ and $arccos\frac{3}{5}$.

Conjecture *The maximal number of lines spanning a 15-dimensional space and having angles $arccos\frac{1}{5}$ and $arccos\frac{3}{5}$ is equal to 456.*

In what follows, we consider *canonical* reduced odd subsystems of $\mathcal{V}(c_{II})$ of vectors $v(p)$ with $|p| \leq 4$ and $v_i^\sigma(t)$ with $\sigma = +1$. Since $v(V_{10} - p) = -v(p)$, every reduced odd subsystem can be made canonical by a switching.

We seek a maximal odd subsystem $\mathcal{U} \subseteq \mathcal{V}(c_{II})$ of vectors spanning equiangular lines. Hence $vv' = \pm 1$ for all $v, v' \in \mathcal{U}$. Since $vv' = \pm 1$ for $v \in \mathcal{V}_P$ and $v' \in \mathcal{V}_T$, we have to find separately maximal sets $\mathcal{U}_P \subseteq \mathcal{V}_P$ and $\mathcal{U}_T \subseteq \mathcal{V}_T$ spanning equiangular lines, $\mathcal{U}_P \cup \mathcal{U}_T = \mathcal{U}$.

6.2 Odd systems $\mathcal{U}_T$ related to $\mathcal{T}(3,5)$

Now we consider a canonical subsystem $\mathcal{U}_T \subseteq \mathcal{V}_T$ spanning equiangular lines. Recall that $v_i^\sigma(t) \in \mathcal{U}_T$ only for $\sigma = +1$. We redenote $v_i^{+1}(t)$ by $v_i(t)$.

Fix $k \in V_{10}$. The equality (13) gives

$$v_k(t)v_k(t') = 2|t \cap t'| - 1.$$

For $v_k(t), v_k(t') \in \mathcal{U}_T$, this equality implies

$$|t \cap t'| \leq 1. \tag{14}$$

Hence $t \neq t'$, and we set

$$T^k = \{t : v_k(t) \in \mathcal{U}_T\}.$$

Now, for $i \neq j$, (13) and the condition $v_i(t)v_j(t') = \pm 1$ imply

$$|t \cap t'| = 1 \text{ or } 2, \tag{15}$$

i.e. $t \neq t'$. Hence, for each $t \in T_6$, there is at most one index $k \in V_{10}$ such that $v_k(t) \in \mathcal{U}_T$, i.e. the sets T^k are disjoint. Besides, since $|T_6| = 20$, we obtain that $|\mathcal{U}_T| \leq 20$.

Since $|t \cap t'| \leq 1$ for $t, t' \in T^k$, there are possibilities of the following two types:

(i) $T^k = \{t, \bar{t}\}$ for some $t \in T_6$,

(ii) T^k is such that $|t \cap t'| = 1$ for all $t, t' \in T^k$.

T^k contains at most 4 triples $t \in T_6$ in the case (ii). Let T^{k_0} contains only intersecting triples. According to (15), triples from distinct T^k are not disjoint. Hence the triples complementary to triples of T^{k_0} does not belong to no T^k. In this case $\mathcal{U}_T$ contains strictly less than 20 vectors. Now we show that the case (i) gives $\mathcal{U}_T$ with exactly 20 vectors.

There are at most 10 mutually intersecting triples in T_6, since T_6 contains exactly 10 pairs $(t, \bar{t})$ of disjoint triples. Note that $|\bar{t} \cap t'| = 3 - |t \cap t'|$. Hence if the condition (15) holds for t and t', then it holds for all 4 pairs (t, t'), $(\bar{t}, t')$, $(t, \overline{t'})$, $(\bar{t}, \overline{t'})$. Let we have 10 mutually intersecting triples. Then we can add to these triples its complements such that the conditions (14) and (15) still hold. In this case, the set T^k is of type (i), i.e. $T^k = \{t, \bar{t}\}$, for every $k \in V_{10}$. So, distinct partitions $(t, \bar{t})$ of the set S_0 correspond to distinct $k \in V_{10}$. Hence we obtain that there is a one-to-one correspondence between choices of $\mathcal{U}_T \subseteq \mathcal{V}_T$ and bijections between a 10-set and 10 distinct partitions of a 6-set.

We obtain 20 vectors $v_k(t), v_k(\bar{t})$, $k \in V_{10}$, of $\mathcal{U}_T$. Fix the point $0 \in S_0$. In each T^k we choose a triple t_k containing 0. Then the 10 triples t_k mutually intersect (in 0), are distinct and have the form $t_k = \{0ij\}$, where $i, j \in S_0 - \{0\}$. Therefore we obtain a bijection $\phi : E_5 \to V_{10}$ between the set E_5 of 10 pairs ij, $1 \leq i < j \leq 5$, and 10 points of the set V_{10}. Hence we can label points of V_{10} by pairs ij and redenote $v_k(t)$ and $v_k(\overline{t})$ for $t=\{0ij\}$ and for $k=\phi(ij)$, $1\le i<j\le 5$, as

$$v^a_{ij}=e_{\phi(ij)}+e_i+e_j+e_0-\frac{1}{2}e(S_0),\ v^b_{ij}=e_{\phi(ij)}-e_i-e_j-e_0+\frac{1}{2}e(S_0). \qquad (16)$$

In these notations we have

$$\begin{array}{rcl} & v^a_{ij}v^b_{ij}= & -1, \\ v^a_{ij}v^b_{il}= & v^a_{ij}v^a_{kl}= & -1, \\ v^a_{ij}v^b_{kl}= & v^a_{ij}v^a_{il}= & 1. \end{array}$$

Here the indices i,j,k,l are all distinct.

We denote by $\mathcal{U}_T(\phi)$ the odd system $\mathcal{U}_T$ corresponding to the bijection ϕ.

The explicit expression of vectors $v^{a,b}_{ij}\in\mathcal{U}_T(\phi)$ shows that the projection of the odd system $\mathcal{U}_T(\phi)$ onto the space spanned by e_i, $i\in S_0$, is an odd system $\mathcal{V}_3(5)$ of vectors of norm 3 spanning equiangular lines. (Note that $\mathrm{pr}v^a_{ij}=-\mathrm{pr}v^b_{ij}$). Since $ve(S_0)=0$ for all $v\in\mathcal{U}_T(\phi)$, the odd system $\mathcal{V}_3(5)$ spans, in fact, a 5-dimensional space. But there is a unique two-graph $\mathcal{T}(3,5)$ with parameters $m=3$ and $d=5$. Just this two-graph is represented by the odd system $\mathcal{V}_3(5)$.

6.3 Odd systems $\mathcal{U}_P$ related to two-graphs $\mathcal{T}(5,10)$

Now we consider canonical subsystems $\mathcal{U}_P\subseteq\mathcal{V}_P$ spanning equiangular lines.

Note that $\mathcal{V}_P$ spans a 10-dimensional space. But there is a unique (up to an isomorphism) maximal set of equiangular lines at angle $arccos\frac{1}{5}$ spanning a 10-dimensional space, since, recall, $N_5(10)=1$. This set contains 16 lines. Described below odd systems $\mathcal{U}_P$ are, in fact, different but isomorphic representations of this unique regular two-graph $\mathcal{T}(5,10)$ with parameters $m=5$ and $d=10$.

We obtain that the odd system $\mathcal{U}_P\cup\mathcal{U}_T$ is a special gluing of the odd system $\mathcal{U}_P$ representing the unique two-graph $\mathcal{T}(5,10)$, and of the odd system $\mathcal{V}_3(5)$ representing the unique two-graph $\mathcal{T}(3,5)$. This gluing depends on the bijection ϕ and is as follows. We add to each pair of opposite vectors $\pm\mathrm{pr}v^a_{ij}\in\mathcal{V}_3(5)$, spanning the same line, a vector e_k, $k\in V_{10}$, $k=\phi(ij)$, of norm 2 which is orthogonal to the space spanned by $\mathcal{V}_3(5)$. All vectors e_k are mutually orthogonal, and form a basis of the space spanned by $\mathcal{U}_P$. We obtain a 15-dimensional odd system of vectors of norm 5. Then we unite the obtained odd system with $\mathcal{U}_P$.

Let $Q\subseteq P$ be a family of even subsets of V_{10} with $|p|\le 4$. We set

$$\mathcal{U}_P(Q)=\{v(p):p\in Q\}.$$

We want to find all families Q such that $\mathcal{U}_P(Q)$ is a maximal odd subsystem of $\mathcal{V}_P$ spanning equiangular lines. Since $\mathcal{U}_P(Q)$ represents the same two-graph $\mathcal{T}(5,10)$ for any Q, the odd systems $\mathcal{U}_P(Q)$ are isomorphic for all Q. But we seek Q and Q' such that $\mathcal{U}_P(Q)$ and $\mathcal{U}_P(Q')$ are not switching equivalent.

Let $p_0 \in P$. Since the set P of even subsets is closed with respect to symmetric difference, the vector $v'(p) := v(p\Delta p_0)$ belongs to $\mathcal{V}_P$ for $p, p_0 \in P$. Besides $v'(p_1)v'(p_2) = v(p_1)v(p_2)$. Hence the odd systems $\mathcal{U}_P(Q)$ and $\mathcal{U}_P(Q\Delta p_0) := \{v' : v \in \mathcal{U}_P(Q)\}$ are isomorphic.

Obviously, $\mathcal{U}_P(Q\Delta p_0)$ can be switched to the canonical form. If $p_0 \in Q$, then $\emptyset \in Q\Delta p_0$. Hence, at first, we find sets Q containing $\emptyset$, and after that consider $Q\Delta p_0$ for all $p_0 \in P$.

Let $\emptyset \in Q$. We set $Q = \{\emptyset\} \cup B$, and $v_0 := v(\emptyset) = \frac{1}{2}e(V_{10})$. Hence $\mathcal{U}_P(Q) = \{v_0\} \cup \mathcal{U}_P(B)$. The condition $v(p)v(p') = \pm 1$ for $p, p' \in Q$ and (11) for $p' = \emptyset$ implies that $|p| = 4$ for all $p \in B$. Of course, p belongs to B not for all p of cardinality 4. At first, for $|p| = |p'| = 4$, (11) implies

$$v(p)v(p') = 2|p \cap p'| - 3. \tag{17}$$

Now the condition $v(p)v(p') = \pm 1$ implies

$$|p \cap p'| = 1 \text{ or } 2. \tag{18}$$

So, we have to find a maximal family B of 4-subsets of the 10-set V_{10} such that (18) holds. We know that a maximal family contains 15 4-subsets. Such families are well known. These are 2-(10,4,2) designs.

Recall that for any odd system $\mathcal{U}$ with ± 1 inner products, $G(\mathcal{U})$ is a graph on $\mathcal{U}$, where $v, v' \in \mathcal{U}$ are adjacent if and only if $vv' = -1$. Since $v_0 v(p) = 1$ for all $p \in B$, the vertex v_0 is isolated in the graph $G(\mathcal{U}_P(Q))$, and the graph $G(\mathcal{U}_P(B)) = H_0$ is the same for all B, since all odd systems $\mathcal{U}_P(B)$ are isomorphic. It is not difficult to verify that H_0 is the triangular graph $T(6)$ (see, e.g., [8]). It relates to the two-graph $\mathcal{T}(5, 10)$. The equation (17) says that $v(p)$ and $v(p')$, $p, p' \in B$ are adjacent in H_0 if $|p \cap p'| = 1$.

Now we consider the sets $Q = Q_0\Delta p_0$, where $Q_0 \ni \emptyset$.

Proposition 7 *Let $Q_0 \ni \emptyset$ and $p_0 \in P$. Then the odd system $\mathcal{U}_P(Q_0\Delta p_0) \cup \mathcal{U}_T(\phi)$ is isomorphic to a switching of the odd system $\mathcal{U}_P(Q_0) \cup \mathcal{U}_T(\phi)$.*

Proof. Note that if we change e_i by $-e_i$ for $i \in p_0$ in all vectors of an odd system $\mathcal{U} \subseteq \mathcal{V}(c_{II})$, we obtain an isomorphic odd system. We show that this change of signs of e_i for $i \in p_0$ transforms $\mathcal{U}_P(Q_0\Delta p_0) \cup \mathcal{U}_T(\phi)$ into a switching of $\mathcal{U}_P(Q_0) \cup \mathcal{U}_T(\phi)$.

Recall the definition (10) of vectors $v(p)$. We see that the vector $v(p\Delta p_0)$ can be obtained from $v(p)$ by the change of signs of all e_i for $i \in p_0$. Hence the odd system $\mathcal{U}_P(Q_0)$ can be obtained from $\mathcal{U}_P(Q_0\Delta p_0)$ by the map $e_i \to -e_i$. Obviously, $\mathcal{U}_P(Q_0\Delta p_0)$ and $\mathcal{U}_P(Q_0)$ are isomorphic.

Now, recall the definition (16) of vectors $v_{ij}^{a,b} \in \mathcal{U}_T(\phi)$. The change of signs of $e_{\phi(ij)}$ for $\phi(ij) \in p_0$ transforms v_{ij}^a into $-v_{ij}^b$ and v_{ij}^b into $-v_{ij}^a$ for $ij \in \phi^{-1}(p_0)$. Now we switch all vectors $-v_{ij}^{a,b}$. The result follows. $\square$

Proposition 7 implies that we can consider the odd systems $\mathcal{U}_P$ only in the form $\mathcal{U}_P = \{v_0\} \cup \mathcal{U}_P(B)$ with B as the set of blocks of a 2-(10,4,2) design.

Now we move the bijection $\phi : E_5 \to V_{10}$ of previous section into the odd system $\mathcal{U}_P(B)$. Let $(e_{ij} : ij \in E_5)$ be a fixed set of mutually orthogonal vectors of norm 2. Denote by $\mathcal{U}_T$ the odd system of vectors $v_{ij}^{a,b}$, $ij \in E_5$ given by (16), where $e_{\phi(ij)}$ is substituted by e_{ij}. We fix a bijection ϕ_0 and identify V_{10} with $E_5 = \phi_0^{-1}(V_{10})$. Now, we can identify any subset $p \subseteq V_{10}$ with the subset $\phi_0^{-1}(p_0) \subseteq E_5$, and consider every bijection $\phi : E_5 \to \phi_0(E_5)$ as a permutation $\pi = \phi_0^{-1}\phi$ of the 10-set E_5.

We preserve the denotation B for the set $\{\phi_0^{-1}(p) : p \in B\}$. We saw that B is a set of blocks of a 2-(10,4,2) design with the ground set E_5.

Two designs on a set V of points are isomorphic if there is a permutation of V which transforms blocks of one design into blocks of another. Gronau [9] shows that there exist exactly 3 nonisomorphic 2-(10,4,2) designs. Let D_0, D_1 and D_2 be fixed representatives of the 3 isomorphism classes of 2-(10,4,2) designs defined on the set E_5. Let B_i be the set of blocks of the design D_i, $i = 0, 1, 2$. Obviously, a set B defining the odd system $\mathcal{U}_P(B)$ has the form $B = \pi B_i = \{\pi(b) : b \in B_i\}$ for some $i \in \{0, 1, 2\}$ and a permutation π of E_5. Hence we can redenote the odd system $\mathcal{U}_P(\{\emptyset\} \cup B) \cup \mathcal{U}_T(\phi)$ as $\mathcal{U}(\pi B_i)$ and set $\mathcal{U}(\pi B_i) = \{v_0\} \cup \mathcal{U}_0(\pi B_i)$. Recall that $v_0 = v(\emptyset) = \frac{1}{2}e(V_{10})$, and $vv_0 = 1$ for all $v \in \mathcal{U}_0(\pi B_i)$.

Denote the two-graph represented by the odd system $\mathcal{U}(\pi B_i)$ as $\mathcal{T}(\pi B_i)$. In [10] Seidel describes briefly two-graphs of the type $\mathcal{T}(\pi B_i)$, and says that they was discovered by T.Spence by use of a computer.

We formulate the following

Theorem 2 *Every two-graph obtained from the lattice D_{16}^+ is either one of 80 two-graphs related to Steiner triple systems or a two-graph of the type $\mathcal{T}(\pi B_i)$, $i = 0, 1, 2$, where i corresponds to the 2-(10,4,2) design D_i, and π is a permutation of the ground set of D_i.* $\square$

6.4 Combinatorics of the graph $G_T = G(\mathcal{U}_T)$

Now we consider the graph $G_T = G(\mathcal{U}_T)$. It is complementary Johnson graph $J(6, 3)$, where v_{ij}^a corresponds to the triple $0ij$, and v_{ij}^b corresponds to the complementary triple $\overline{0ij} = S_0 - \{0ij\}$. Two vertices v and v' are adjacent if and only if the corresponding triples t and t' satisfy the inequality $|t \cap t'| \leq 1$. Call the edges connecting v_{ij}^a and v_{ij}^b, i.e. connecting complementary triples in $J(6, 3)$, *basic*, and denote them e_{ij} (i.e. we identify the basic edges with the vectors e_{ij}).

The graph G_T has very remarkable properties. It can be partitioned into 4 induced circuits of length 5 (recall that G_T has 20 vertices). Of course, this partition is not unique. We take the following circuits (the neighbouring vertices are adjacent):

$$C(0, s) = \{v_{12}^s, v_{34}^s, v_{25}^s, v_{14}^s, v_{35}^s\}, \text{ and } C(1, s) = \{v_{13}^s, v_{24}^s, v_{15}^s, v_{23}^s, v_{45}^s\}, \ s = a, b.$$

The circuits $C(0,a)$ and $C(1,a)$ (so as $C(0,b)$ and $C(1,b)$) induce the Petersen graph Pe. Similarly, the circuits $C(0,a)$ and $C(1,b)$ (so as $C(0,b)$ and $C(1,a)$) induce the triangular graph $T(5)$. Hence G_T is a union both of the two Petersen graphs, and of the two $T(5)$'s. Note here that just the partition of G_T into two Petersen graphs was used by Seidel [10] for the description of the corresponding two-graph.

In both the partitions, the 10 edges connecting related vertices of the two copies of $T(5)$ (and of Pe) are the 10 basic edges e_{ij}, $1 \le i < j \le 5$. The set of all the basic edges forms a maximal matching in G_T. Since the triangular graph $T(5)$ and the Petersen graph Pe are complementary, we obtain that G_T is also the Johnson graph with added basic edges connecting complementary triples. Any maximal clique of G_T has size 4. There are exactly 30 maximal cliques. The 30 cliques can be partitioned into 5 groups with 6 cliques in each group. The 6 cliques of a group are partitioned into 3 couples of matched cliques. The cliques of a couple are matched by basic edges.

Each group is uniquely determined by an element i of a 5-set. Denote this 5-set $V_5 = \{i,j,k,l,m\}$. The cliques of the i-th group are as follows. There are 3 partitions of the 4-set $V_5 - \{i\}$ into pairs. So if $\{jklm\} = V_5 - \{i\}$, then the partitions into pairs are (jk, lm), (jl, km), (jm, kl). Couples of the partitions are in one-to-one correspondence with couples of cliques.

Let A_i be the set of the 3 couples of partitions of the set $V_5 - \{i\}$. Denote by M_α the couple of cliques corresponding to a couple $\alpha \in A_i$ of partitions. For example, the couple M_α of cliques corresponding to the couple of partitions $\alpha = [(jk, lm), (jl, km)]$ contains the following vertices $(v^a_{jk}, v^a_{lm}, v^b_{jl}, v^b_{km})$ and $(v^b_{jk}, v^b_{lm}, v^a_{jl}, v^a_{km})$.

Denote by $b(\alpha)$ the 4-set of basic edges matching two cliques of the couple M_α, $\alpha \in A_i$. Let $B_0 = \{b(\alpha) : \alpha \in A_i, i \in V_5\}$. It is easy to verify by inspection that the following proposition holds.

Proposition 8 *B_0 is a set of blocks of a 2-(10,4,2) design.* $\square$

We take the design of Proposition 8 as D_0, i.e. as one of fixed representatives of 3 nonisomorphic designs.

The automorphism group of the graph G_T is $S_6 \times \mathbf{Z}_2$, the automorphism group of the Johnson graph $J(6,3)$ (see [1]). This group maps the set of basic edges into itself, and a couple of cliques into a couple of cliques. Its subgroup $\mathbf{Z}_2$ inverts the basic edges. Hence the symmetric group S_6 is contained in the automorphism group of the design D_0. It can be shown that S_6 is the automorphism group of D_0.

6.5 Two-graphs $\mathcal{T}(\pi B_i)$

Recall that $\mathcal{U}(\pi B_i) = \{v_0\} \cup \mathcal{U}_0(\pi B_i)$. In particular, for the identity permutation, and for $i = 0$, we have $\mathcal{U}(B_0) = \{v_0\} \cup \mathcal{U}_0(B_0)$.

Theorem 3 *The two-graph $\mathcal{T}(B_0)$ is isomorphic to $\mathcal{T}_0(5, 15)$.*

Proof. Recall that $\mathcal{T}_0(5,15)$ is represented by the odd system $\{v_0\}\cup\mathcal{V}(T_0)$, where T_0 is the set of triples of the Steiner triple system related to $PG(3,2)$. We show that the odd systems $\mathcal{U}_0(B_0)$ and $\mathcal{V}(T_0)$ are isomorphic.

Recall that vectors $v(t)\in\mathcal{V}(T_0)$ satisfy (9). It is proved in [7] that one can label vectors $v\in\mathcal{V}(T_0)$ by all $35=\binom{7}{3}$ triples s of a 7-set such that for labeled vectors $v(s)$ the following holds:

$$v(s)v(s')=\left\{\begin{array}{cc} 2|s\cap s'|-1 & \text{if } |s\cap s'| \text{ is odd,}\\ -1 & \text{otherwise.}\end{array}\right. \tag{19}$$

We prove our theorem if we can label vectors from $\mathcal{U}(B_0)$ by triples s such that (19) holds. As a labeling 7-set, we take the set $V_7=V_5\cup\{6,7\}$.

Note that the graph G_T is a union of two triangular graphs $T(5)$. Recall that two vertices v,v' are adjacent in $G_T=G(\mathcal{U}_T)$ if and only if $vv'=-1$. There is a natural labeling of vertices of $T(5)$ by pairs ij, $1\le i<j\le 5$, such that two vertices are adjacent if and only if their labels intersect in a point. Recall that vertices of the two copies of $T(5)$ are matched by basic edges. If matched vertices of G_T are labeled by the same pair ij, then not matched vertices from distinct copies are adjacent if and only if their labels does not intersect.

Hence, we label vertices of one copy of $T(5)$ by triples $ij6$ and matched vertices of the other copy by $ij7$, such that the pairs ij correspond to above discussed labeling of $T(5)$. If we denote the corresponding vectors as $v(ij6)$ and $v(ij7)$, then the condition (19) holds.

For this labeling, the cliques of G_T are of two types: 5 couples of cliques $q_{ir}=\{ijr: j\in V_5, j\neq i\}$, $r\in\{6,7\}$, and 10 couples of cliques $q_{ijr}=\{ijr;klr': k,l\in V_5-\{ij\}\}$, $r'=\{6,7\}-r$, $r=6,7$.

Recall that 4 edges matching vertices of a couple of cliques form a block from B_0. We label the block related to the couple (q_{i6},q_{i7}) by the triple $i67$, $i\in V_5$. The block related to the couple (q_{ij6},q_{ij7}) is labeled by the triple $V_5-\{ij\}$. It is easy to verify that obtained labeling of the whole odd system $\mathcal{U}_0(B_0)$ is consistent with (19). $\square$

Note that the labeling by triples of vertices of G_T considered as the complementary Johnson graph is distinct from the labeling given by Theorem 3.

Consider lattices $L^q(\mathcal{U}(\pi B_i))$, $q=1,2$. In particular, we are interested in the set cl$\mathcal{U}(\pi B_i)$ Recall that $v(b)\in$cl$\mathcal{U}(\pi B_i)$ for all $b\in\pi B_i$, where $v(b)=v_0-e(b)$ is defined in (10), and $v_0=\frac{1}{2}e(V_{10})$.

Lemma 5 *For any permutation π, any $b\in B_0$, we have $v(b)\in$cl$\mathcal{U}(\pi B_i)$, $i=0,1,2$.*

Proof. Recall that every block $b=b(\alpha)\in B_0$ corresponds to a pair M_α, $\alpha\in A_i$, of cliques of G_T. The vertices of these cliques are: $v^r_{jk}, v^r_{lm}, v^s_{jl}, v^s_{km}$, $(r,s)=(a,b)$, where v^r_{ij} is given in (16). Since $e_{jk}, e_{lm}, e_{jl}, e_{km}$ are the basic edges matching these cliques, the vector $v(\alpha):=v(b(\alpha))$ has the form $v_0-(e_{jk}+e_{lm}+e_{jl}+e_{km})$.

Each clique of the couple M_α and the vectors $v(\alpha)$ and v_0 form the following minimal dependency, i.e. circuit,

$$v^r_{jk} + v^r_{lm} + v^s_{jl} + v^s_{km} + v(\alpha) - v_0 = 0.$$

If $v(\alpha) \notin \mathcal{U}(\pi B_i)$, then we obtain a broken circuit. Since the norm of $v(\alpha)$ is 5, this broken circuit gives $v(\alpha) \in \mathrm{cl}\mathcal{U}(\pi B_i)$. $\square$

Theorem 4 *The odd system $\mathcal{U}(\pi B_i)$ is closed if and only if $i = 0$, and π is the identity map.*

Proof. By Theorem 3, the odd system $\mathcal{U}(B_0)$ is related to the STS triples of which are lines of $PG(3,2)$. It is proved in [7] that $\mathcal{U}(B_0) = \{v_0\} \cup \mathcal{V}(T_0)$ is closed, since $\mathcal{U}(B_0) = \mathcal{V}(c)$, where $\mathcal{V}(c)$ is the odd system obtained from the Barnes-Wall lattice by the construction described in section 2. Now, by Lemma 5, $\mathcal{U}(B)$ is not closed if $B \neq B_0$. $\square$

6.6 The two-graph $\mathcal{T}_1(5,15)$

We saw that among the two-graphs $\mathcal{T}(\pi B_i)$ there is the unique two-graph $\mathcal{T}(B_0) = \mathcal{T}_0(5,15)$ isomorphic to the two-graph related to the STS triples of which are lines of $PG(3,2)$. This STS has No 1 in the extended version of [2]. In this section we show that many of two-graphs $\mathcal{T}(\pi B_i)$ are isomorphic to the two-graph $\mathcal{T}_1(5,15)$ related to the STS No 2 in the extended version of [2].

At first, we represent explicitly the odd system $\mathcal{U}(B_0)$ as the odd system $\mathcal{W}_0$ representing $\mathcal{T}_0(5,15)$. In fact, this is another proof that $\mathcal{T}(B_0) = \mathcal{T}_0(5,15)$.

We take unit vectors f_i, $0 \le i \le 7$, and vectors g, w_s, $s \in F_7$ as follows.

$$\begin{array}{lll}
f_1 = \frac{1}{2}(e_{23}+e_{45}) & f_2 = \frac{1}{2}(e_{24}+e_{35}) & f_3 = \frac{1}{2}(e_{25}+e_{34}) \\
f_4 = \frac{1}{2}(e_{25}-e_{34}) & f_5 = \frac{1}{2}(e_{24}-e_{35}) & f_6 = \frac{1}{2}(e_{23}-e_{45}) \\
f_7 = \frac{1}{2}(e_0-e_1) & & g = \frac{1}{2}(e_0+e_1) - \frac{1}{4}(e_2+e_3+e_4+e_5)
\end{array}$$

$$\begin{array}{ll}
w_{123} = \frac{1}{2}(e_{12}+e_{13}+e_{14}+e_{15}) & w_{145} = \frac{1}{2}(e_{12}+e_{13}-e_{14}-e_{15}) \\
w_{246} = \frac{1}{2}(e_{12}-e_{13}+e_{14}-e_{15}) & w_{356} = \frac{1}{2}(e_{12}-e_{13}-e_{14}+e_{15}) \\
w_{167} = \frac{1}{2}(e_2+e_3-e_4-e_5) & w_{257} = \frac{1}{2}(e_2-e_3+e_4-e_5) \\
w_{347} = \frac{1}{2}(e_2-e_3-e_4+e_5) &
\end{array}$$

The identification of $v(\alpha)$ and $v^{a,b}_{ij}$ with u_i and $u_s(\varepsilon^k)$ is as follows:

$$\begin{array}{lll}
v(23,45;24,35) = u_{123}(\varepsilon^3) & v(23,45;25,34) = u_{123}(\varepsilon^2) & v(25,34;24,35) = u_{123}(\varepsilon^1) \\
v_0 = u_{123}(\varepsilon^0) & & \\
v(15,34;14,35) = u_{145}(\varepsilon^0) & v(13,45;15,34) = u_{246}(\varepsilon^0) & v(13,45;14,35) = u_{356}(\varepsilon^0) \\
v(15,24;14,25) = u_{145}(\varepsilon^1) & v(12,45;14,25) = -u_{246}(\varepsilon^2) & v(12,45;15,24) = -u_{356}(\varepsilon^2) \\
v(12,35;13,25) = -u_{145}(\varepsilon^2) & v(13,25;15,23) = u_{246}(\varepsilon^1) & v(12,35;15,23) = -u_{356}(\varepsilon^3) \\
v(12,34;13,24) = -u_{145}(\varepsilon^3) & v(12,34;14,23) = -u_{246}(\varepsilon^3) & v(13,24;14,23) = u_{356}(\varepsilon^1)
\end{array}$$

$$\begin{array}{llll} v^a_{23}=u_{167}(\varepsilon^0) & v^b_{23}=-u_{167}(\varepsilon^3) & v^a_{45}=-u_{167}(\varepsilon^2) & v^b_{45}=u_{167}(\varepsilon^1) \\ v^a_{24}=u_{257}(\varepsilon^0) & v^b_{24}=-u_{257}(\varepsilon^3) & v^a_{35}=-u_{257}(\varepsilon^2) & v^b_{35}=u_{257}(\varepsilon^1) \\ v^a_{25}=u_{347}(\varepsilon^0) & v^b_{25}=-u_{347}(\varepsilon^3) & v^a_{34}=-u_{347}(\varepsilon^2) & v^b_{34}=u_{347}(\varepsilon^1) \end{array}$$

$$\begin{array}{llll} v^a_{12}=u_0 & v^a_{13}=u_1 & v^a_{14}=u_2 & v^a_{15}=u_3 \\ v^b_{12}=-u_7 & v^b_{13}=-u_6 & v^b_{14}=-u_5 & v^b_{15}=-u_4 \end{array}$$

We see that $\mathcal{U}(B_0)$ is, up to a switching, the odd system $\mathcal{W}_0$. Now we show that $\mathcal{U}(\pi B_0)$, for some π, and $\mathcal{U}(B_1)$ are, up to a switching, the odd systems $\mathcal{U}(S)$ for some S (see Section 3.2).

Recall that D_0 is the 2-(10,4,2) design with B_0 as the set of its blocks. Among isomorphic to D_i designs, $i=1,2$, one can choose a design such that it has a maximal number of common with D_0 blocks, i.e. with maximal $|B_i\cap B_0|$. Just such designs we take as fixed representatives D_1 and D_2. Then $|B_1\cap B_0|=11$, and $|B_2\cap B_0|=9$, i.e. D_1 and D_2 have 4 and 6 blocks distinct from blocks of D_0.

We take the following representative D_1 (see [9] and [7]). The set B_1 of blocks of D_1 differs from B_0 by the following 4 blocks. Instead of blocks

(13,45;15,34), (12,45;14,25), (13,25;15,23), (12,34;14,23)

of B_0, the design D_1 contains the blocks

$$b_2=(12,45;14,34), b_3=(13,45;15,25), b_4=(12,25;14,23), b_5=(13,34;15,23).$$

Using the definitions (10) of $v(b)$ and (6) of $u_s(\varepsilon^k)$, we obtain

$$v(b_2)=-u_{246}(-\varepsilon^0), v(b_3)=u_{246}(-\varepsilon^2), v(b_4)=-u_{246}(-\varepsilon^1), v(b_5)=u_{246}(-\varepsilon^3).$$

This shows that $\mathcal{U}(B_1)$ is, up to switching, $\mathcal{U}(S)$ for $S=\{246\}$. Hence, by Proposition 3, we obtain (cf. Proposition 8 of [7])

Proposition 9 *The two-graph $\mathcal{T}(B_1)$ is isomorphic to $\mathcal{T}_1(5,15)$.* $\square$

The following lemma is very useful for to find a root system $R(\mathcal{U})$ for $\mathcal{U}=\mathcal{U}(\pi B_i)$. Let $V_5=\{ijklm\}$. We define the following root systems isomorphic to A_1^7:

$$R_i=\{\pm(e_0-e_i),\pm(e_{jk}\pm e_{lm}),\pm(e_{jl}\pm e_{km}),\pm(e_{jm}\pm e_{kl})\},$$

$$R_{ij}=\{\pm(e_i-e_j),\pm(e_{ik}\pm e_{jk}),\pm(e_{il}\pm e_{jl}),\pm(e_{im}\pm e_{jm})\}.$$

Lemma 6 *Let $\mathcal{U}$ be an odd system of the type $\mathcal{U}(\pi B_i)$, $i=0,1,2$. Then*

(i) if one of the vectors of R_i is a root of $R(\mathcal{U})$, then $R_i\subseteq R(\mathcal{U})$,

(ii) if one of the vectors of R_{ij} is a root of $R(\mathcal{U})$, then $R_{ij}\subseteq R(\mathcal{U})$.

Proof. Recall that $v^a_{ij}=e_{ij}+w_{ij}$, $v^b_{ij}=e_{ij}-w_{ij}$, where $w_{ij}=e_0+e_i+e_j-\frac{1}{2}e(S_0)=\frac{1}{2}(e_0+e_i+e_j-e_k-e_l-e_m)$ with $S_0=\{0ijklm\}$.

Using this, we obtain the following identities:

$$v^a_{jk} + v^a_{lm} = e_{jk} + e_{lm} + e_0 - e_i;\ v^a_{jk} - v^b_{lm} = e_{jk} - e_{lm} + e_0 - e_i;$$

$$v^a_{ik} + v^a_{jk} = e_{ik} + e_{jk} + e_i - e_j;\ v^a_{ik} - v^b_{jk} = e_{ik} - e_{jk} + e_i - e_j.$$

By definition of the lattice $L^0(\mathcal{U})$, the sum of any two vectors of $\mathcal{U}$ belongs to $L^0(\mathcal{U})$. Since any root of $R(\mathcal{U})$ is a vector of $L^0(\mathcal{U})$ of norm 4, the assertion of this lemma is implied by the above identities. $\square$

Note that if $e_\alpha - e_\beta$ and $e_\beta - e_\gamma$ are roots of $R(\mathcal{U})$, then $e_\alpha - e_\gamma$ is also a root of $R(\mathcal{U})$. Using this and Lemma 6, we obtain

Proposition 10 *Let $\mathcal{U} \subseteq \mathcal{V}(c_{II})$ represents a two-graph $\mathcal{T}(\pi B_i)$. Then the root system $R(\mathcal{U})$ is one of the following 6 root systems:*

(1) $\emptyset$,

(2) $R_i \cong R_{ij} \cong A_1^7$,

(3) $R_i \cup R_j \cup R_{ij} \cong R_{ij} \cup R_{ik} \cup R_{jk} \cong A_2 D_3^3$,

(4) $R_i \cup R_{jk} \cup R_{lm} \cong A_1^9 D_4$,

(5) $R_i \cup R_j \cup R_k \cup R_{ij} \cup R_{ik} \cup R_{jk} \cup R_{lm} \cong A_1 A_3 D_4 D_6$,

(6) $R(\mathcal{V}(c_{II})) = A_5 D_{10}$. $\square$

Now we consider sets B which are obtained from blocks of B_0 by a permutation of elements of E_5. Supposing that all indices are distinct, we set $\Pi_i = \{(jk, lm), (jl, km), (jm, kl)\}$ and $\Pi_{ij} = \{(ik, jk), (il, jl), (im, jm)\}$. Note that Π_i and Π_{ij} consist of commuting transpositions. Let

$$\pi_i = (jk, lm)(jl, km)(jm, kl), \text{ and } \pi_{ij} = (ik, jk)(il, jl)(im, jm). \qquad (20)$$

be the permutations which are the product of all the 3 transpositions of Π_i and Π_{ij}, respectively. Then π_i and π_{ij} belong to the automorphism group of B_0.

Proposition 11 *Let π be one of the permutations τ and $\tau\tau'$, where τ, τ' both belong to either Π_i or Π_{ij}, $\tau \neq \tau'$. Then the two-graph $\mathcal{T}(\pi B_0)$ is isomorphic to $\mathcal{T}_1(5, 15)$.*

Besides,

(i) if $\tau = (jk, lm) \in \Pi_i$, then $R(\mathcal{U}(\tau B_0)) = R_i$,

(ii) if $\tau = (ik, jk) \in \Pi_{ij}$, then $R(\mathcal{U}(\tau B_0)) = R_{ij}$.

Proof. Note that in the above identification of $\mathcal{U}(B_0)$ with $\mathcal{W}_0$ the index 1 is special, and the vectors $2f_i$ are roots of R_1.

If we set $\pi e_{ij} = e_{\pi(ij)}$, $\pi_{ij} e_i = e_j$, $\pi_{ij} e_0 = e_0$, $i, j \in V_5$, then we have $\pi_{i1} R_1 = R_j$, $\pi_i R_j = \pi_i \pi_{1j} R_1 = R_{ij}$. Using these permutations, we can identify $\mathcal{U}(B_0)$ with $\mathcal{W}_0$ taking f_i equal to the roots of R_i or of R_{ij}. Moreover, $\pi_{ij}\Pi_i\pi_{ij} = \Pi_j$, $\pi_j\Pi_i\pi_j = \Pi_{ij}$.

Hence we can prove this proposition only for $\Pi_1 = \{(23, 45), (24, 35)(25, 34)\}$. The transformation of B_0 by one of the transpositions (23,45), (24,35), (25,34) corresponds to the change of f_i by $-f_i$ for $i = 6, 5, 4$ (respectively) in the vectors $u_s(\varepsilon^k)$ related to $v(b)$, $b \in B_0$. All other vectors f_i and w_s remain unchanged. Similarly, a product of two transpositions corresponds to a change of signs before two f_i's. It easy to verify that so obtained $\mathcal{U}(\pi B_0)$ is, up to a switching, $\mathcal{U}(S)$ for some negative $S \subseteq F_7$. For example, the permutation $\pi = (23, 45)(25, 34)$ corresponds to the change of the signs before f_6 and f_4 such that $S = \{145, 356\}$. By Proposition 3, $\mathcal{U}(S)$ represents the two-graph $\mathcal{T}_1(5, 15)$. □

Proposition 11 implies

Corollary 2 *The closure of the odd system $\mathcal{U}(\pi B_0)$ with $\pi = \tau$ or $\pi = \tau\tau'$ depends only on the set Π_i or Π_{ij} to what τ and τ' belong, but not on particulars τ and τ'.*

Note that the permutation $\pi = \tau\tau'$ is the product of two commuting transpositions from the same set Π_i or Π_{ij}. We have the following similar result for two commuting transpositions from distinct sets Π_i and Π_{ij}. But compare this with Proposition 14 below.

Proposition 12 *Let $\tau = (ik, jk) \in \Pi_{ij}$, $\tau' = (ij, lm) \in \Pi_k$ be the two commuting transpositions. Then $R(\mathcal{U}(\tau\tau' B_0)) = R_{lm} \cong A_1^7$.* □

6.7 Two-graphs $\mathcal{T}(\pi B_i)$ distinct from $\mathcal{T}_{0,1}(5, 15)$

Now consider the design D_2. We take the following representative of D_2. The set B_2 of blocks of D_2 differs from B_0 by the following 6 blocks. Instead of blocks

$$(12, 34; 14, 23), (24, 35; 25, 34), (13, 25; 15, 23), (13, 45; 15, 34), (23, 45; 24, 35), (12, 45; 14, 25)$$

of B_0, the design D_2 contains the blocks

$$(12, 34; 14, 25), (24, 35; 45, 34), (13, 34; 15, 23), (13, 45; 15, 25), (23, 25; 24, 35), (12, 45; 14, 23).$$

Using Lemmas 6 and 5, it is not difficult to verify that the following proposition is true.

Proposition 13 *The root system $R(\mathcal{U}(B_2))$ contains the root system $R_1 \cup R_{24} \cup R_{35} \cong A_1^9 D_4$, where 9 orthogonal roots $e_0 - e_1$, $e_2 - e_4$, $e_3 - e_5$, $e_{12} \pm e_{14}$, $e_{13} \pm e_{15}$, $e_{24} \pm e_{35}$ form the root system A_1^9, and the root system D_4 consists of 6 roots on the set e_{23},e_{25}, e_{34}, e_{45}. Hence $R(\mathcal{U}(B_2)) \supseteq A_1^9 D_4$.* □

It is very remarkable that the same root system is contained in $R(\mathcal{U}(\tau\tau' B_0))$, where $\tau = (12, 14)$ and $\tau' = (23, 45)$. This is implied by the following proposition (cf., with Proposition 12 above).

Proposition 14 *Let $\tau = (ik, jk) \in \Pi_{ij}$, $\tau' = (il, jm) \in \Pi_k$ be the two commuting transpositions. Then $R(\mathcal{U}(\tau\tau' B_0)) \supseteq R_k \cup R_{ij} \cup R_{lm} \cong A_1^9 D_4$.* □

Recall that $R(\mathcal{V}(c_{II})) = A_5D_{10}$. Now we give a permutation π_0 such that $R(\mathcal{U}(\pi_0 B_0)) = A_5D_{10}$. The permutation π_0 is the product of 5 commuting transpositions, namely,

$$\pi_0 = (12,45)(13,25)(14,23)(15,34)(24,35).$$

Note that $\pi_0^{-1} = \pi_0$. The permutation π_0 transforms the original labeling of vertices v_{ij}^a and v_{ij}^b of the graph G_T into the labeling by triples $(\pi_0(ij)6)$ and $(\pi_0(ij)7)$ given in Theorem 3. Note that $B_0 \cap \pi_0 B_0 = \emptyset$, in other words $\pi_0 B_0$ is one of the most far families from B_0.

Proposition 15 $R(\mathcal{U}(\pi B_0)) = A_5D_{10}$.

Proof. By using Lemmas 5 and 6, we find $R(\mathcal{U}(\pi_0 B_0)) \supseteq A_5D_{10}$. But since $R(\mathcal{U}(\pi_0 B_0)) \subseteq R(\mathcal{V}(c_{II})) = A_5D_{10}$, we have the equality. □

7 Two-graphs related to Latin squares

Recall that there are regular two-graphs $\mathcal{T}(5,15)$ related to Latin squares of order 6. A Latin square of order 6 consists of 36 ordered triples ijk of symbols i,j,k taken from a 6-set V_6 such that for each pair of coordinates every pair of symbols occurs exactly once. We represent the set $T_{6,3}$ of all $6^3 = 216$ ordered triples by a 15-dimensional odd system of vectors of norm 5 as follows.

Let e_{ia}, $i \in V_6$, $a = 1,2,3$, be 18 mutually orthogonal vectors of norm 2. As before, we set $e_a(V_6) := \sum_{i \in V_6} e_{ia}$. Let

$$u_{ia} = e_{ia} - \frac{1}{6}e_a(V_6), \;\; i \in V_6, a = 1,2,3.$$

Note that $u_{ia}e_a(V_6) = 0$, i.e. the 6 vectors u_{ia} for fixed a span a 5-dimensional space. Besides, we have $u_{ia}u_{jb} = 0$ if $a \neq b$, $u_{ia}u_{ja} = -\frac{1}{3}$ if $i \neq j$, and $u_{ia}^2 = \frac{5}{3}$.

We represent the ordered triple s by the vector

$$v(s) = u_{i1} + u_{j2} + u_{k3} \text{ if } s = ijk.$$

It is easy to verify that $v(s)v(s') = 2|s \cap s'| - 1$, where $|s \cap s'|$ denotes the number of coordinates where equal symbols stay. In particular, we have $v(s)^2 = 5$. Since, by the above definition of a Latin square, $|s \cap s'| \leq 1$, we have $v(s)v(s') = \pm 1$ for s, s' belonging to the same Latin square.

Since $v(s)v(s')$ is odd for all $s, s' \in T_{6,3}$, the vectors $v(s)$ form an odd system $\mathcal{V}(T_{6,3})$. It is not difficult to see that the lattice $L^0(\mathcal{V}(T_{6,3}))$ is, up the multiple $\sqrt{2}$, the 15-dimensional root lattice $A_5 \oplus A_5 \oplus A_5$. Hence the odd system $\mathcal{V}(T_{6,3})$ has the root system

$$R(\mathcal{V}(T_{6,3})) = A_5^3.$$

As in Section 7, we obtain that odd subsystems $\mathcal{U} \subseteq \mathcal{V}(T_{6,3})$ spanning equiangular lines correspond to Latin squares. It is well known (see, e.g., [2]) that there are 12 nonisomorphic Latin squares. It is proved in [2] that these 12 Latin squares generate 11 nonisomorphic two-graphs. Two Latin squares having No 87 and No 89 in the extended version of [2] are switching equivalent. We find that $R(\mathcal{U}) \supseteq A_1^9$ for the odd system $\mathcal{U}$ corresponding to these two Latin squares and representing the corresponding two-graph.

For the odd systems $\mathcal{U}$ representing other 10 Latin squares No 81–86, 88, 90–92, we find $R(\mathcal{U}) = R(\mathcal{V}(T_{6,3})) = A_5^3$.

Proposition 16 *Let $\mathcal{T}$ be a two-graph related to one of the Latin squares No 81–86, 88, 90–92, and represented by an odd system $\mathcal{U}$ with $R(\mathcal{U}) = A_5^3$. Then $\mathcal{T}$ is not isomorphic to any two-graph from the lattices $E_8 \oplus E_8$ and D_{16}^+.*

Proof. Recall that $R(\mathcal{U}') \subseteq D_7E_7$ for any two-graph $\mathcal{T}(\mathcal{U}')$ from $E_8 \oplus E_8$. It is easy to see that $A_5^3 \not\subseteq D_7E_7$. This implies that $\mathcal{T}$ is not isomorphic to any two-graph from $E_8 \oplus E_8$. Similarly, $\mathcal{T}$ is not isomorphic to any two-graph related to a Steiner triple system, since $A_5^3 \not\subseteq A_{14}$.

Now let $\mathcal{U}'$ represent a two-graph of the type $\mathcal{T}(\pi B_i)$, $i = 0, 1, 2$. Then $R(\mathcal{U}') \subseteq R(\mathcal{V}(c_{II})) = A_5D_{10}$. Moreover, the roots of D_{10} are of the form $\pm e_{ik} \pm e_{jk}$, $\pm e_{jk} \pm e_{lm}$, and the roots of A_5 are of the form $\pm(e_i - e_j)$, $0 \le i < j \le 5$.

If $\mathcal{T}$ is isomorphic to a two-graph of the type $\mathcal{T}(\pi B_i)$, then $\mathcal{U}$ is switching equivalent to $\mathcal{U}'$, and $R(\mathcal{U}) \subseteq A_5D_{10}$. Hence some of roots of $R(\mathcal{U})$ have one of the form $e_{ik} - e_{jk}$ or $e_{jk} - e_{lm}$, and $R(\mathcal{U})$ has at least two nonorthogonal roots of these forms. Then Lemma 6 implies that $R(\mathcal{U})$ has also roots of the form $e_{ik} + e_{jk}$ and $e_{jk} + e_{lm}$. In other words, $R(\mathcal{U})$ contains a root system D_k for some $k > 1$. This contradicts to the equality $R(\mathcal{U}) = A_5^3$. $\square$

8 Problems

1. Prove Conjecture in Section 6.1. Note that the unit vectors $\frac{1}{\sqrt{5}}v$ of a reduced odd system $\mathcal{V}(c_i)$, $i = 0, I, II$, multiplied by $\frac{1}{\sqrt{5}}$, form a *spherical code* of dimension 15 and size $|\mathcal{V}(c_i)| = 456$ with the 4 inner products $\frac{1}{5}vv' = \pm\frac{1}{5}, \pm\frac{3}{5}$. There exists a linear programming bound on size of a spherical code. For to find this bound on $|\mathcal{V}(c_i)|$ it is sufficient to find a real polynomial $f(t)$ of degree N such that $f(t) \le 0$ for $t \in \{\pm\frac{1}{5}, \pm\frac{3}{5}\}$, and coefficients f_i in the expansion of $f(t)$ in terms of Gegenbauer polynomials satisfy $f_0 > 0$, $f_i \ge 0$, $1 \le i \le N$. Then $|\mathcal{V}(c_i)| \le \frac{f(1)}{f_0}$. (Details see in [4]).

2. Prove that in Proposition 13 is equality, i.e. $R(\mathcal{U}(B_2)) = A_1^9D_4$. Similarly, prove the equality in Proposition 14, i.e. $R(\mathcal{U}(\pi B_0)) = A_1^9D_4$, where $\pi = (24, 34)(13, 25)$.

3. Let $\mathcal{R}(\mathcal{V})$ be the complete list of all root systems $R(\mathcal{U})$ for odd subsystems $\mathcal{U} \subseteq \mathcal{V}$ spanning equiangular lines.We show in [7] that $\mathcal{R}(\mathcal{V}(c_I)) = \{\emptyset, A_1^7, A_2A_3^3, A_6A_7, A_{14}\}$.

Find $\mathcal{R}(\mathcal{V}(c_0))$ and $\mathcal{R}(\mathcal{V}(c_{II}))$. We show $\emptyset, A_1^7, A_1D_3D_4D_6, D_7E_7 \in \mathcal{R}(\mathcal{V}(c_0))$, and $\emptyset, A_1^7, A_5D_{10} \in \mathcal{R}(\mathcal{V}(c_{II}))$. In particular, prove that all the 6 root systems of Proposition 10 belong to $\mathcal{R}(\mathcal{V}(c_{II}))$, i.e. prove that $\mathcal{R}(\mathcal{V}(c_{II})) = \{\emptyset, A_1^7, A_2D_3^3, A_1^9D_4, A_1A_3D_4D_6, A_5D_{10}\}$.

4. Find all nonisomorphic two-graphs given by $\mathcal{U} \subseteq \mathcal{V}(c_0)$ and $\mathcal{U} \subseteq \mathcal{V}(c_{II})$. Recall that there are exactly 80 nonisomorphic two-graphs obtained from $\mathcal{V}(c_I)$. They relate to 80 nonisomorphic Steiner triple systems on 15 points.

5. Prove that circuits (7) generate all dependencies of the odd system $\mathcal{V}(c_0)$. More general, prove that minimal by cardinality circuits defined in Section 3.4 generate all dependencies in any odd system of norm 5 representing a regular two-graph.

References

[1] A.E.Brower, A.M.Cohen, A.Neumaier, *Distance-Regular Graphs*, Springer-Verlag, Berlin,1989.

[2] F.C.Bussemaker, J.J.Seidel, *Symmetric Hadamard matrices of order 36*, Annals of the New York Acad. Sci. **175** (1970) 66–79. Extended version: *Symmetric Hadamard matrices of order 36*, Report 70-WSK-02, 68pp.

[3] P.J.Cameron, J.M.Goethals, J.J.Seidel, E.E.Shult, *Line graphs, root systems, and elliptic geometry*, J. of Algebra **43** (1976) 305–327.

[4] J.H.Conway, N.J.A.Sloane, *Sphere Packings, Lattices and Groups*, vol 290 of Grundlagen der mathematischen Wissenshaften Springer-Verlag, Berlin, 1987.

[5] M.Deza, V.P.Grishukhin, *L-polytopes and equiangular lines*, Discrete Appl. Math **56** (1995) 181–214.

[6] M.Deza, V.P.Grishukhin, *Odd systems of vectors and related lattices*, Rapport de Recherche du LIENS **LIENS-94-5**, 1994, 15pp.

[7] M.Deza, V.P.Grishukhin, *Once more about 80 Steiner triple systems on 15 points*, Rapport de Recherche du LIENS **LIENS-95-8**, 1995, 22pp.

[8] V.P.Grishukhin, *Strongly regular graphs related to a regular two-graph*, Rapport de Recherche du LIENS **LIENS-92-27**, 1992, 29pp.

[9] H.-D.O.F.Gronau, *The 2-(10,4,2) designs*, Rostock.Math.Kolloq. **16** (1981) 5–10.

[10] J.J.Seidel, *More about two-graphs*, Fourth Czechoslovakian symposium on Combinatorics, Graphs and Complexity (eds. J.Nešetřil and M.Fiedler) 1992, Elsevier Science Publisher B.V., 287–308.

[11] D.E.Taylor, *Two-graphs and doubly transitive groups*, J. of Combinatorial Theory, Series A **61** (1992) 113–122.