Once More about 80 Steiner Triple Systems on 15 Points

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Abstract

Subsets of a v-set are in one-to-one correspondence with vertices of a v-dimensional unit cube, a Delaunay polytope of the lattice Z^v. All vertices of the same cardinality k generate a $(v-1)$-dimensional root lattice A_{v-1} and are vertices of the Delaunay polytope $P(v,k)$ of the lattice A_{v-1}. Hence k-blocks of a $t-(v,k,\lambda)$ design, being identified with vertices of $P(v,k)$, generate a sub-lattice of A_{v-1}. We show that 80 Steiner triple systems (STS for short) 2-(15,3,1) are partitioned into 5 families. STS's of the same family generate the same lattice L. Each lattice L is distinguished by a set $R(L)$ of its vectors of norm 2. $R(L)$ is a root system. We find that for the 5 types $R(L)=\emptyset$, A_1^7, $A_2A_3^3$, A_6A_7 and A_{14}. The family with $R(L)=\emptyset$ contains only one STS, which is the projective space $PG(3,2)$. The family with $R(L)=A_1^7$ contains also only one STS. Two-graphs related to both the STS's belong to a family of two-graphs discovered by T.Spence and described by J.Seidel in [12].

1 Introduction

A Steiner triple system (STS for short) on n points is a set T of triples of points such that every pair of points is contained in exactly one triple of the set T. An STS on 15 points contains 35 triples. There are 80 nonisomorphic STS on 15 points. The first complete list of these STSs was given in [13]. We use a very informative description of all 80 STSs given in [10].

Any two triples of an STS are either disjoint or intersect in a point. Hence a graph H relates to an STS as follows. The set of vertices of H is the set T of triples of STS. Two vertices are adjacent if and only if the corresponding triples are disjoint. Note that the adjacency of triples in the graph H is complementary to the adjacency considered in [1]. The graph H related to an STS on 15 points has 35 vertices and is strongly regular. Its parameters are $(v,k,\lambda,\mu)=(35,16,6,8)$. (See [7].)

A graph with these parameters determines uniquely a regular two-graph on 36 points with minimal eigenvalue –5. It is proved in [1] that 80 two-graphs related to 80 STSs are not isomorphic. Hence, the 80 nonisomorphic STSs on 15 points provide 80 nonisomorphic strongly regular graphs with the same parameters (35,16,6,8). But, there are much more nonisomorphic graphs with these parameters, since a regular two-graph on 36 points with minimal eigenvalue –5 determines several nonisomorphic (but pseudo switching equivalent) strongly regular graphs with parameters (35,16,6,8).

Let A be (± 1)-adjacency matrix of the graph H. (–1 of A corresponds to a pairs of adjacent vertices of H.) The minimal eigenvalue of A is equal to –5 of multiplicity 20. Hence the matrix $5I+A$ is positive semidefinite, and it is a Gram matrix of a set $\mathcal{V}$ of 35 vectors of norm (squared length) 5 with mutual inner products ± 1. Since rank of the matrix $5I+A$ is equal to 15=35-20, these vectors span a 15-dimensional space.

Let T be a set of all triples of an STS. Denote by $\mathcal{V}(T)$ the set of vectors corresponding to T and by $v(t)\in\mathcal{V}(T)$ the vector related to a triple $t\in T$. Then the inner product is given by the following expression

$$v(t)v(t')=2|t\cap t'|-1. \qquad (1)$$

Note that this formula works for $t=t'$, when $v^2(t)=5$.

The set $\mathcal{V}(T)$ generates affinely a lattice $L(T)$, which is a sub-lattice of the lattice $\sqrt{2}A_{14}$, the root lattice A_{14} multiplied by $\sqrt{2}$. $L(T)$ is an even lattice, and norms of all its vectors are divided by 4. The set $R(T)$ of vectors of norm 4 of $L(T)$ form a root system multiplied by $\sqrt{2}$. We find that 80 nonisomorphic STS's generate lattices $L(T)$ with only 5 types of $R(T)$, namely, $R(T)=\emptyset$, A_1^7, $A_2A_3^3$, A_6A_7 and A_{14}. Hence the 80 STS's are partitioned into 5 families corresponding to these $R(T)$'s.

The family with $R(T)=\emptyset$ contains only one STS No 1 of [10], triple of which are lines of the 3-dimensional projective space $PG(3,2)$ over GF_2.

The family with $R(T)=A_1^7$ contains also only one STS No 2 of [10]. Both the STS's, No1 and No 2, can be obtained by the Moore's construction from the unique STS on 7 points, the Fano plane $PG(2,2)$. We show that the Moore's construction gives only these two STS's from a STS on 7 points.

We show that there is a vector representation of the STS's No 1 and No 2, which is a mate of the Moore's construction. In Section 4.4 this representation is obtained also from the 16-dimensional Barnes-Wall lattice by the method introduced in [3] and [4].

Ending the paper we show that the two-graphs related to STS's No1 and No 2 are given by the incidence matrices of two nonisomorphic 2-(10,4,2) designs.

2 Lattices related to Steiner triple systems

Any set $\mathcal{V}(T)$ of vectors related to an STS on 15 points can be embedded into an odd system related to all triples on these 15 points as follows.

Let $V_{15} = \{1, 2, ..., 15\}$, and let a_i, $i \in V_{15}$ be 15 mutually orthogonal vectors of norm 2, i.e. $a_i^2 = 2$. Let T_{15} be the set of all triples $t \subset V_{15}$, $|t| = 3$. For $t \in T_{15}$, set $a(t) = \sum_{i \in t} a_i$. Note that $a(t)a(t') = 2|t \cap t'|$ and $a^2(t) = 6$. Let $j_n = (1, 1, ..., 1)$ be an n-dimensional all-one vector. Since $a(t)j_{15} = 3\sqrt{2}$ for any t, endpoints of all $a(t)$ lie on a sphere of squared radius $\frac{24}{5} < 5$ in a 14-dimensional affine hyperplane $\mathcal{H} = \{a \in R^{15} : aj_{15} = 3\sqrt{2}\}$. $\mathcal{H}$ is orthogonal to j_{15}. Let $\gamma = \frac{1}{5}(\sqrt{2} - \frac{1}{\sqrt{3}})$. Set $u(t) = a(t) - \gamma j_{15}$. It is easy to verify that the inner product $u(t)u(t') = 2|t \cap t'| - 1$, i.e. it is given by (1) for any pair of t and t'. In particular, $u^2(t) = 5$.

Denote the set of vectors $u(t)$ for all $\binom{15}{3} = 455$ triples $t \in T_{15}$ by $\mathcal{V}_{15}$. The expression (1) shows that $\mathcal{V}_{15}$ is an odd system and the map $v(t) \to u(t)$ for $t \in T$ is an embedding $\mathcal{V}$ into $\mathcal{V}_{15}$. We preserve below the notation $u(t)$ for this special representation $a(t) - \gamma j_{15}$ of triple $t \in T_{15}$.

Recall that a k-dimensional lattice L is an Abelian discrete group of vectors of R^k. L is called *integral* if inner products of its vectors are integral. The inner product v^2 of a vector $v \in L$ with itself is called *norm* of v. An integral lattice L is called *even* (*doubly even*) if norms of all its vectors are even (divisible by 4, respectively). Obviously, L is doubly even if and only if $\frac{1}{\sqrt{2}}L$ is even.

The set $\mathcal{V}(T)$ of vectors of odd norm 5 with odd inner products ± 1 is a special case of an odd system of vectors studied in [4]. (Recall that an *odd system* is a set of vectors such that the inner product of any two (may be equal) vectors is odd.)

Let $\mathcal{V} = \mathcal{V}(T)$ be an odd system of vectors $v(t)$, $t \in T$, for some set T. Let Z be the set of all integers. We relate to $\mathcal{V}$ the following lattices:

$$L_0(\mathcal{V}) = \{u : u = \sum_{t \in T} z_t v(t), \sum_{t \in T} z_t = 0, \ z_t \in Z\}.$$

$$L_1(\mathcal{V}) = \{u : u = \sum_{t \in T} z_t v(t), \sum_{t \in T} z_t = 1, \ z_t \in Z\}.$$

$L_1(\mathcal{V})$ is, in fact, an affine lattice, since the origin does not belong to it. We have $L_1(\mathcal{V}) = v + L_0(\mathcal{V})$ for any $v \in \mathcal{V}$, i.e. $L_1(\mathcal{V})$ is a translation of $L_0(\mathcal{V})$ by a vector v. Note that $\mathcal{V} \subset L_1(\mathcal{V})$. It is shown in [4] that $L_0(\mathcal{V})$ is a doubly even lattice, i.e. $u^2 \equiv 0 \pmod 4$ for any $u \in L_0(\mathcal{V})$. Similarly, $u^2 \equiv 1 \pmod 4$ for any $u \in L_1(\mathcal{V})$.

Recall that a_i, $i \in V_{15}$, are mutually orthogonal vectors of norm 2. Hence the set of vectors

$$\{w : w = \sum_{i=1}^{15} z_i a_i, \ \sum_{i=1}^{15} z_i = 0, z_i \in Z\}$$

is a 14-dimensional lattice. This lattice is the root lattice A_{14} multiplied by $\sqrt{2}$ (see [2]).

Since for any $i \in V_{15}$ there are $t, t' \in T_{15}$ such that $a_i = a(t) - a(t') = u(t) - u(t')$, we obtain that $L_0(\mathcal{V}_{15}) = \sqrt{2}A_{14}$. Since the inner product $a_i j_{15}$ does not depend on i, $L_0(\mathcal{V}_{15})$ lies in the hyperplane $\mathcal{H}_0 = \{x \in R^{15} : xj_{15} = 0\}$. Since the inner product $u(t)j_{15} = \sqrt{3}$ does not depend on $t \in T_{15}$, the lattices $L_1(\mathcal{V})$ and $L_1(\mathcal{V}_{15})$ lie in the affine hyperplane $\mathcal{H}_1 = \{x \in R^{15} : xj_{15} = \sqrt{3}\}$. Recall that vectors $u(t) \in L_1(\mathcal{V}_{15})$ have norm 5. We shall see that they are all vectors of norm 5 of $L_1(\mathcal{V}_{15})$.

Let L be a k-dimensional lattice. A full dimensional sphere S is called *empty sphere* of L if there is no point of L inside S and points of L on S affinely generate R^k. The convex hull of all points of L lying on S is called a *Delaunay polytope* of the lattice L.

All Delaunay polytopes of root lattices are known (see, for example, [2] or [6], p.32). The Delaunay polytopes of the root lattice A_n are polytopes $P(n+1, k)$. The polytope $P(n+1, k)$ is congruent to a section of a unit $(n+1)$-dimensional cube by a hyperplane $H_k = \{x \in R^{n+1} : xj_{n+1} = k\}$ for an integer k, $1 \le k \le \frac{n+1}{2}$. Each vertex of $P(n+1, k)$ is in one-to-one correspondence with a k-subset of a $(n+1)$-set.

It is clear that endpoints of vectors $u(t) \in \mathcal{V}_{15}$ are in one-to-one correspondence with vertices of the Delaunay polytope $\sqrt{2}P(15, 3)$ of the lattice $L_0(\mathcal{V}_{15}) = \sqrt{2}A_{14}$. In fact, $\sqrt{2}P(15, 3)$ is congruent to the convex hull of endpoints of all vectors of norm 5 of the affine lattice $L_1(\mathcal{V}_{15})$. We will sometimes identify vertices of the Delaunay polytope $\sqrt{2}P(15, 3)$ of the lattice $L_0(\mathcal{V}_{15})$ with corresponding triples of T_{15}.

Let $\mathcal{V} = \mathcal{V}(T)$ be an odd system corresponding to an STS with a set of triples T. Since $L_0(\mathcal{V}) \subseteq L_0(\mathcal{V}_{15})$, the intersection $L_0(\mathcal{V}) \cap T_{15}$ is the set of vertices of a Delaunay polytope of $L_0(\mathcal{V})$. Denote this polytope by $P(T)$. Obviously, T is a subset of vertices of $P(T)$.

Definition. The set of triples $L_0(\mathcal{V}) \cap T_{15}$, i.e. the set of vertices of the Delaunay polytope $P(T)$, is called *closure* of T and is denoted as cl$T$.

A set of triples T is called *closed* if $T =$cl$T$. An STS is called *closed* if its set of triples is closed.

Similarly we define a closure of an odd system $\mathcal{V}$. Recall that $v^2 = 5$ for all $v \in \mathcal{V}$.

Definition. The set of all vectors of norm 5 of the lattice $L_1(\mathcal{V})$ is called *closure* of $\mathcal{V}$. It is denoted by cl$_5\mathcal{V}$. $\mathcal{V}$ is *closed* if $\mathcal{V} =$cl$_5\mathcal{V}$.

Proposition 1 *$cl_5\mathcal{V}(T) = \mathcal{V}(clT)$. In particular, $cl_5\mathcal{V}_{15} = \mathcal{V}_{15}$.*

Proof. Recall that the lattice $L_1(\mathcal{V}(T))$ is a translation of the lattice $L_0(\mathcal{V}(T))$ such that endpoints of vectors of $\mathcal{V}(T)$ correspond to vertices of the Delaunay polytope $P(T)$. Note that endpoints of all vectors of $\mathrm{cl}_5\mathcal{V}(T)$ lie in the intersection of a sphere of squared radius 5 in R^{15} with the affine hyperplane $\mathcal{H}_1$. But this intersection is a sphere circumscribing a translation of the Delaunay polytope $P(T)$. Hence there is one-to-one correspondence between vertices of $P(T)$, i.e. triples of $\mathrm{cl}T$, and endpoints of vectors of $\mathrm{cl}_5\mathcal{V}(T)$. Therefore $\mathrm{cl}_5\mathcal{V}(T) = \mathcal{V}(\mathrm{cl}T)$. Since $\mathcal{V}(T_{15}) = \mathcal{V}_{15}$ and T_{15} is closed, we have $\mathrm{cl}_5\mathcal{V}_{15} = \mathcal{V}_{15}$.

Obviously $\mathrm{cl}_5\mathcal{V} \subseteq \mathcal{V}_{15}$. Loosely speaking, $\mathrm{cl}_5\mathcal{V}$ is a set of all vertices of a Delaunay polytope $P(\mathcal{V})$ of the lattice $L_1(\mathcal{V})$. $P(\mathcal{V})$ is a translation of $P(T)$.

We give below some conditions for a triple t to belong to $\mathrm{cl}T$ for an STS T.

It is obvious that for any pair of points $i, j \in V_{15}$, there is a unique triple $t \in T$ containing i and j. Let $q(i,j)$ be the third point of the triple t. Let $t_1, t_2 \in T$ be two disjoint triples, and let $t_1 = (i_1 i_2 i_3)$, $t_2 = (j_1 j_2 j_3)$. Call three disjoint pairs $(i_k j_k)$, $1 \le k \le 3$, $i_k \in t_1$, $j_k \in t_2$, a *matching* of t_1 and t_2. A pair of disjoint triples has 6 matchings. The pair $(i_k j_k)$ determines a point $q(i_k, j_k)$ for each k.

Lemma 1 *The points $q(i_k, j_k) \notin t_1 \cup t_2$, $1 \le k \le 3$, for any matching $(i_k j_k)$.*

Proof. Set $q_k = q(i_k, j_k)$. Note that $j_1 = q(i_1, q_1)$. Suppose that $q_1 \in t_1 \cup t_2$. Then w.l.o.g. we can suppose that $q_1 = i_2$. This implies that $i_3 = q(i_1, i_2) = q(i_1, q_1) = j_1$, what contradicts to that $t_1 \cap t_2 = \emptyset$.

If all three points $q_k = q(i_k, j_k)$, $1 \le k \le 3$, are distinct, we call the triple $(q_1 q_2 q_3)$ *derived* (from t_1, t_2).

Proposition 2 *For a Steiner triple system with a set of triples T, a derived triple belongs to $\mathrm{cl}T$.*

Proof. Let $t_1 = (i_1 i_2 i_3), t_2 = (j_1 j_2 j_3)$, $t_1, t_2 \in T$ and $t_1 \cap t_2 = \emptyset$. Let $t = (q_1 q_2 q_3)$ be the derived triple using the matching $(i_k j_k)$. Set $s_k = (i_k j_k q_k)$. Recall that, by definition of q_k, $s_k \in T$. Consider vectors $v(t_1), v(t_2)$, $v(s_k)$, $1 \le k \le 3$. These vectors belong to $\mathcal{V}(T) = \{v(t) : t \in T\}$. Consider the vector $v = v(s_1) + v(s_2) + v(s_3) - v(t_1) - v(t_2)$. Obviously, $v \in L_1(\mathcal{V})$. Note that $v(s_k)v(s_l) = v(t_1)v(t_2) = -1$, for $k \neq l$, and $v(s_k)v(t_l) = 1$. Hence $v^2 = 5$. We can suppose that $\mathcal{V} \subseteq \mathcal{V}_{15}$.

For $t = (q_1 q_2 q_3)$, consider the vector $u(t) \in \mathcal{V}_{15}$. According to (1) and Lemma 1, $u(t)v(s_k) = 1$, $u(t)v(t_i) = -1$. Hence $u(t)v = 5$. Since $u^2(t) = v^2 = 5$, this implies that $v = u(t)$, i.e. $u(t) \in L_1(\mathcal{V})$ and $u(t) \in \mathrm{cl}_5\mathcal{V}$ by Proposition 1. This is equivalent to $t \in \mathrm{cl}T$.

Another condition for to belong to $\mathrm{cl}T$ gives the following

Proposition 3 *Let an STS with a set of triples T has 5 disjoint triples. Let t_i, $1 \le i \le 4$, be 4 arbitrary disjoint triples of T. Then the triple of points of $V_{15} - \cup_{i=1}^4 t_i$ belongs to $\mathrm{cl}T$.*

Proof. Let s_i, $1 \le i \le 5$, be 5 disjoint triples of T. Let $t_5 = V_{15} - \cup_{i=1}^4 t_i$. Obviously, the vector $v = \sum_{i=1}^5 v(s_i) - \sum_{i=1}^4 v(t_i)$ belongs to $L_1(\mathcal{V})$. As above, it is not difficult to verify (using, for example, the vectors $u(t)$) that $v = u(t_5)$. This implies that $t_5 \in$cl$T$.

3 Roots

A set of all vectors of norm 2 in an even lattice is called a *root system*. A vector of a root system is called a *root*. A root system generates an even lattice, called a *root lattice*. A set of all minimal vectors of a root lattice is a root system that generates this root lattice.

Each root system is a direct sum of irreducible root systems, called its *components*. A root system is called *irreducible* if it cannot be partitioned into two subsystems such that roots of one of these systems are orthogonal to all roots of the other. All irreducible root systems are known. These are A_n, D_n and E_m, where n is an positive integer and $m = 6, 7, 8$. The subscripts are dimensions of the corresponding root systems. Following to [2], we denote a root system consisting of components R_1, R_2,..., R_k by the product $R_1 R_2 ... R_k$. In particular, a sum of k equal components R is denoted by R^k. A root lattice generated by a root system R is denoted by the same symbol R. Hence a lattice R^k is a direct sum of k lattices R.

Recall that the lattice $L_0(T) = L_0(\mathcal{V}(T))$ related to an STS with a set of triples T is a doubly even lattice. Hence a set of all vectors of norm 4 of $L_0(T)$ is a root system multiplied by $\sqrt{2}$. Denote this root system by $R(T)$. Since $L_0(T) \subseteq L_0(T_{15}) = \sqrt{2}A_{14}$, we have $R(T) \subseteq A_{14}$.

By definition of an STS, for any $t \notin T$, there is $t_1 \in T$ such that $|t \cap t_1| = 2$. In fact, there are 3 such triples $t_1, t_2, t_3 \in T$ with this property. These triples are uniquely determined by 3 pairs of points of t. According to (1), $v(t)v(t_i) = 3$, and therefore $(v(t) - v(t_i))^2 = 4$. Since for any $t, t' \in T$, $(v(t) - v(t'))^2 = 8$ or 12, we obtain

Proposition 4 *Let T be a set of all triples of an STS. Then $R(T) = \emptyset$ if and only if T is closed.*

Recall that $P(\mathcal{V})$ is the Delaunay polytope of the lattice $L_1(\mathcal{V})$ related to an STS with a set of triples T. The squared Euclidean distance between vertices $v(t)$ and $v(t')$ is equal to $(v(t) - v(t'))^2$. We see that this distance take (according to (1)) values 4,8 and 12. We relate to $P(\mathcal{V})$, and therefore to $P(T) \subset L_0(\mathcal{V})$, a graph $G(T)$ vertices of which are vertices of $P(\mathcal{V})$ (or $P(T)$). So we can identify vertices of $G(T)$ with triples $t \in$ cl T. Two triples t and t' are adjacent in $G(T)$

if and only if $|t \cap t'| = 2$, i.e.

if and only if the squared distance between t and t' is equal to 4, i.e.

if and only if the vector $v(t) - v(t')$ is a root multiplied by $\sqrt{2}$.

By Proposition 4, $G(T)$ is empty (i.e. has no edges) if and only if T is closed.

For sets T of triples of STSs given in [10], we find some triples belonging to cl$T$ using Propositions 2 and 3. We label each STS by its number in [10]. Obtained root systems are given in Table 1.

Table 1

Case	$R(T)$	$G(T)$	Number of STS
1	$\emptyset$	$35K_1$	No 1
2	A_1^7	$7K_1 + 7K_2^3$	No 2
3	$A_2A_3^3$	$K_1 + 3(K_3 \otimes T(4)) + K_4^3$	No 3–7
4	A_6A_7	$J(7,3) + K_7 \otimes T(8)$	No 8–22, 67
5	A_{14}	$J(15,3)$	No 23–66, 68–80

Denotations of graphs in Table 1 are as follows. A union of disjoint components is denoted by a sum. K_n denotes the complete graph on n vertices. Hence $35K_1$ denotes 35 nonadjacent vertices. $G \otimes G'$ denotes a direct product of graphs G and G', and K_n^3 denotes the direct product of 3 graphs K_n. $J(n,k)$ is a Johnson graph. We have $K_n = J(n,1)$, and $J(n,2) = T(n)$ is a triangular graph.

Note that 23 STSs of the first 4 cases contain a *head*, that is a subsystem of 7 triples in which 7 symbols occur exactly 3 times, i.e. the projective Fano plane $F_7 = PG(2,2)$.

We give explicit values of roots using the canonical representation $u(t)$ of vectors of $\mathcal{V}(T)$. For this end, let $t = (ikl)$ and $t' = (jkl)$ be two triples that furnish a root $\frac{1}{\sqrt{2}}(u(t) - u(t')) = \frac{1}{\sqrt{2}}(a(t) - a(t'))$. Set $r(ij) = a(ikl) - a(jkl) = a_i - a_j$. Obviously, $r(ij) = -r(ji)$. If $\frac{1}{\sqrt{2}}r(ij) \in R(T)$, then $v(t) + r(ij) \in \mathrm{cl}_5\mathcal{V}$ if and only if $j \in t$ and $i \notin t$. In this case $v(t) + r(ij) = v(t')$, where $t' = t - \{j\} + \{i\}$. We find the following roots:

$\sqrt{2}R(T) = \{r(2k, 2k+1),\ 1 \le k \le 7\}$ in the case 2,

$\sqrt{2}R(T) = \{r(ij) : i, j \in \{1,2,3\},$ or $4k \le i < j \le 4k{+}3,\ k = 1,2,3\}$ in the case 3,

$\sqrt{2}R(T) = \{r(ij) : 1 \le i < j \le 7,$ or $8 \le i < j \le 15\}$ in the case 4.

Recall that these roots are obtained by using operations described in Propositions 2 and 3. Now, for Cases 3, 4 and 5, we prove that there is no other root.

Proposition 5 *Let T be the set of all triples of one of the STS's No 3–80. Then the root system $R(T)$ is given by Table 1.*

Recall that the lattice $L_0(T) = L_0(\mathcal{V}(T))$ is generated by vectors $u(t) - u(t') = a(t) - a(t')$ for $t, t' \in T$. We prove that every vector of $L_0(T)$ has a property which is not satisfied by roots not belonging to $R(T)$ of Table 1.

Case 1. Let $V_0 = \{1, 2, 3\}$, $V_k = \{4k, 4k+1, 4k+2, 4k+3\}$, $k = 1, 2, 3$.The sets V_k, $0 \leq k \leq 3$, partition V_{15}. Denote an element of V_0 (V_1, V_2, V_3, respectively) by a (b, c, d). Then the triples of an STS of Case 3 have one of the following patterns: aaa, abb, acc, add and bcd. Let $\sum_1^{15} x_i a_i$ be a vector of $L_0(T)$. We set $x(V_k) = \sum_{i \in V_k} x_i$. Then the patterns of triples imply that, for a generating vector of $L_0(T)$, values of $x(V_k)$ are either all even or all odd. This implies that this property holds for all vectors of $L_0(T)$. But any root not from $R(T)$ has the form $a_i - a_j$, where i and j belong to distinct V_k. Hence $x(V_k)$ is equal to 1 for two k such that $|\{ij\} \cap V_k| = 1$, and is equal to 0 for other two k. This implies that roots not from $R(T)$ does not belong to $L_0(T)$.

Case 2. Let $V = \{i : 1 \leq i \leq 7\}$, $V' = V_{15} - V$. Defining as above $x(V)$ and $x(V')$, we obtain that $x(V) \equiv x(V') \equiv 0 \pmod 2$ for any vector of $L_0(T)$. This condition is not satisfied by the roots not from $R(T)$.

Case 3. The equality $R(T) = A_{14}$ of the case 5 is obvious, since $R(T) \subseteq A_{14}$ for all STS's.

The equality in the case 1 will be proved in Sections 4.4. The equality in the case 2 will be proved in the forthcoming paper [5]

4 Representations of the STS No 1 related to PG(3,2)

In this section we give different representations of the STS No 1. These representations are useful when we consider STS No 1 from different point of view.

4.1 A representation of the STS No 1 by all triples of a 7-set

This representation makes very visual construction of the STS No 1. It is given by a bijection t between triples of the STS No 1 and all triples of a 7-set. In fact, an isomorphism between the alternating group A_8 and the group $L_4(2)$ of automorphisms of STS No 1 is tsdsfdsaaafdsaadsdsfdsdssa::::::he basis of the bijection t.

Let V_7 be a set of 7 points, say $V_7 = \{1, 2, ..., 7\}$. Let S_3 be the set of all $35 = \binom{7}{3}$ triples of points of V_7. Let T_1 be the set of all 35 triples of the STS No 1. The bijection $t : S_3 \to T_1$ has the following property:

$$|t(s) \cap t(s')| \equiv |s \cap s'| \pmod 2. \qquad (2)$$

Recall that $|t(s) \cap t(s')|$ takes values 0 and 1, and $|s \cap s'|$ takes values 0,1 and 2. Although this bijection can be given a priori, it is convenient sometimes to see it inside the Steiner system $S(24, 8, 5)$ (cf. [1], Section 5).

A Steiner system $S(24,8,5)$ is a set V_{24} of 24 points and a collection of 8-subsets, called *blocks*, such that every 5-subset is contained in one block. It is well known that there exists a unique S(24,8,5). Any pair of its blocks intersects in 4, in 2 or in no points (see [2]).

We fix two points of V_{24}, ∞ and 0, and consider a set $\mathcal{B}$ of all blocks containing ∞. Intersections of these blocks with $V_{24} - \{\infty\}$ form a Steiner system S(23,7,4). Let

$$\mathcal{B}_0 = \{B \in \mathcal{B} : B \ni 0\}.$$

Let $B_1 \in \mathcal{B}$ and $0 \notin B_1$, i.e. $B_1 \notin \mathcal{B}_0$. We partition the set V_{24} as follows

$$V_{24} = \{0\} \cup B_1 \cup V_{15}.$$

So, $0, \infty \notin V_{15}$, $\infty \in B_1$ and $|B_1 - \{\infty\}| = 7$. We can identify $B_1 - \{\infty\}$ with V_7. Let $s \in S_3$ be a triple of points of $B_1 - \{\infty\}$. Then the 5-set $\{\infty, 0\} \cup s$ determines uniquely a block $B(s) \in \mathcal{B}_0$. The 35 triples of S_3 provide 35 blocks of $\mathcal{B}_0$. We show that 35 triples

$$t(s) \stackrel{def}{=} B(s) - s - \{0, \infty\} \subset V_{15},\ s \in S_3,$$

form an STS isomorphic to the STS No 1.

We show that (2) holds. Note that

$$B(s) \cap B(s') = (s \cap s') \cup \{0, \infty\} \cup (t(s) \cap t(s')).$$

Case 1. $|s \cap s'| = 2$. In this case $|B(s) \cap B(s')| \geq 4$. Since 4 is the maximal cardinality of an intersection of two blocks, we have $|B(s) \cap B(s')| = 4$ and $t(s) \cap t(s') = \emptyset$.
Case 2. $|s \cap s'| = 1$. In this case $|B(s) \cap B(s')| \geq 3$. Since there are only even cardinalities of intersections of two blocks, we have $|B(s) \cap B(s')| = 4$ and $|t(s) \cap t(s')| = 1$.
Case 3. $|s \cap s'| = 0$. In this case $|B(s) \cap B(s')| \geq 2$. For to show that, in fact, $|B(s) \cap B(s')| = 2$ here, we give an explicit bijection $t : S_3 \to T$, given by Figure 11.17 on page 312 of the book [2].

Figure 11.17 shows 35 *sextets*, i.e. 35 partitions of the set V_{24} into six *tetrad* (i.e. 4-sets) such that any two tetrads of a sextet give an *octet* (an 8-set), i.e. a block of the Steiner system S(24,8,5). Each sextet is partitioned into a left octet and a right 4×4 square containing 4 tetrads. Let us label cells of the left octet and the right square as follows

∞	1	0	1	2	3
2	3	4	5	6	7
4	5	8	9	10	11
6	7	12	13	14	15

In each sextet we consider two tetrads containing ∞ and 0. Other 3 points of these two tetrads form triples $s \in S_3$ and $t(s) \in T_1$, respectively. Hence the left octet is B_1, and the right square is $\{0\} \cup V_{15}$. It is remarkable that the 35 triples $t(s)$ are exactly the 35 triples of the STS No 1 of [10].

Note that Proposition 2(i) of [11] describes a bijection between 35 lines of a 3-dimensional projective space PG(3,2) and 35 triples of a 7-set. The bijection is such that two lines of PG(3,2) intersect if and only if the corresponding triples have precisely one common point. So, Figure 11.17 of [2] describes a bijection between 35 lines of PG(3,2) and 35 triples of the STS No 1.

4.2 Steiner triple systems No 1 and No 2

There is a remarkable construction of an STS on 15 points using a unique STS on 7 points. This construction is a special case of a Moore's recursive method (see [9], Theorem 15.14.2).

Let $F_7 = \{(123), (145), (167), (246), (257), (347), (356)\}$ be 7 triples of the unique STS on 7 points. Its triples are lines of the projective Fano plane $PG(2,2)$.

We distinguish the point 1 of V_{15} and partition other 14 points into 7 pairs $(2i, 2i+1)$, $1 \le i \le 7$. We relate to the even point $2i$ the number $\varepsilon_i = 1$, and to the odd point $2i+1$ the number $\varepsilon_i = -1$.

Let ε, ε', ε'' take values ± 1. We call the triple $(\varepsilon, \varepsilon', \varepsilon'')$ *even* if the product $\varepsilon\varepsilon'\varepsilon'' = 1$. Otherwise the triple is called *odd*. There are 4 even triples and 4 odd triples. We set $\varepsilon^0 = (111)$, $\varepsilon^1 = (1-1-1)$, $\varepsilon^2 = (-11-1)$, $\varepsilon^3 = (-1-11)$. Let $0 \le k \le 3$. Then ε^k is an even ε-triple, and $-\varepsilon^k$ is an odd ε-triple.

Triples of the STS No 1 are constructed as follows. There are 7 triples containing 1:

$$t_i = (1, 2i, 2i+1), \;\; 1 \le i \le 7.$$

One corresponds 4 triples of the STS to each triple $s \in F_7$ as follows. For $s = ijk$, consider 3 pairs $(2i, 2i+1)$, $(2j, 2j+1)$, $(2k, 2k+1)$. We take a point from each pair such that the related to chosen points ε-triple $\varepsilon(s) = (\varepsilon_i, \varepsilon_j, \varepsilon_k)$ is even. In other words, a sum of the chosen points is even. Note that, for given s, we have four even triples $\varepsilon(s)$.

We represent obtained by this way 28 triples t in the array below. In the first row of this array we show triples $s = (ijk) \in F_7$. The following 3 rows show corresponding pairs. The last 4 rows show obtained triples. The first column shows

ε-triples.

	123	145	167	246	257	347	356
	$(2,3)$	$(2,3)$	$(2,3)$	$(4,5)$	$(4,5)$	$(6,7)$	$(6,7)$
	$(4,5)$	$(8,9)$	$(12,13)$	$(8,9)$	$(10,11)$	$(8,9)$	$(10,11)$
	$(6,7)$	$(10,11)$	$(14,15)$	$(12,13)$	$(14,15)$	$(14,15)$	$(12,13)$
$\varepsilon^0 = (111)$	$2,4,6$	$2,8,10$	$2,12,14$	$4,8,12$	$4,10,14$	$6,8,14$	$6,10,12$
$\varepsilon^1 = (1-1-1)$	$2,5,7$	$2,9,11$	$2,13,15$	$4,9,13$	$4,11,15$	$6,9,15$	$6,11,13$
$\varepsilon^2 = (-11-1)$	$3,4,7$	$3,8,11$	$3,12,15$	$5,8,13$	$5,10,15$	$7,8,15$	$7,10,13$
$\varepsilon^3 = (-1-11)$	$3,5,6$	$3,9,10$	$3,13,14$	$5,9,12$	$5,11,14$	$7,9,14$	$7,11,12$

We denote by $t_s(\varepsilon^k)$, $s \in F_7$, $0 \le k \le 3$, the triple obtained from $s \in F_7$ and related to ε-triple ε^k. We consider here s as an ordered triple (ijl) such that $i < j < l$, and the orders of ε^k and s agree. For example, $t_{257}(\varepsilon^2)$, where $\varepsilon^2 = (-1,1,-1)$, means that $\varepsilon^2_2 = -1$, $\varepsilon^2_5 = 1$, $\varepsilon^2_7 = -1$.

Note that

$$\begin{array}{lll} |t_i \cap t_j| & = 1 & 1 \le i < j \le 7, \\ |t_i \cap t_s(\varepsilon^k)| & = |\{i\} \cap s| & = 0 \text{ or } 1, \\ |t_s(\varepsilon^k) \cap t_s(\varepsilon^l)| & = 1 & s \in F_7,\ 0 \le k < l \le 4, \\ |t_s(\varepsilon^k) \cap t_{s'}(\varepsilon^l)| & = \frac{1}{2}(1 \pm 1) & = 0 \text{ or } 1 \text{ if } s \ne s',\ \{i\} = s \cap s', \varepsilon^k_i = \pm\varepsilon^l_i, \\ & & \text{where the signs agree.} \end{array} \qquad (3)$$

The same construction works for STS No 2. In this case, one takes odd ε-triples $\varepsilon(s)$ for triples $s = (347)$ and $s = (356)$ of F_7. For example, the triple $s = (347)$ provides pairs (6,7),(8,9),(14,15). The 4 odd ε-triples $-\varepsilon^k$, $0 \le k \le 3$, form from these pairs the following 4 triples $t = (7,9,15), (7,8,14), (6,9,14), (6,8,15)$.

In general, if, for each $s \in F_7$, one takes either 4 even or 4 odd ε-triples (parity is chosen arbitrary), one obtains an STS. We show below that in either case we obtain an STS which is isomorphic either to STS No 1 or to STS No 2. For this end, let $S \subseteq F_7$ be a set of triples. We define a transformation of S. Recall that there are 3 triples $s \in F_7$ containing a given point i. For each i, $1 \le i \le 7$, consider triples $s \in S$ containing i. If there are 3 such triples, then delete them from S. If there are two such triples, then change it by the third triple containing i. If there is one or no triple containing i, then S is not transformed. Obviously, after such transformation for all i, we obtain S with either one or none of the triples. We call S *positive* if it is transformed into empty set, and *negative*, otherwise.

Proposition 6 *Let $S \subseteq F_7$ be a set of triples $s \in F_7$. Let an STS be obtained by Moore's construction using odd $\varepsilon(s)$ for $s \in S$, and even $\varepsilon(s)$ for $s \in F_7 - S$. Then this STS is isomorphic to STS No 1 or No 2 according to S is positive or negative, respectively.*

Proof. Note that a permutation of the set V_{15} transforms an STS into an isomorphic STS. Consider a permutation of V_{15} which is the transposition $(2k, 2k+1)$, for some k, $1 \le k \le 7$. This transposition changes only those triples of an STS which are obtained from triples $s \in F_7$ containing the point k. There are $3 \times 4 = 12$ such triples. It is easy to see that the transformed triples can be obtained by Moore's construction if one takes odd triples $\varepsilon(s)$ for $s \ni k$.

So, if the set $S \subseteq F_7$ contains 3 triples containing the point k, we can eliminate these triples from S simultaneously transforming the STS by the transposition $(2k, 2k+1)$. If the set S contains two triples with k, we can change the two triples by the third triple containing k and performing the transposition $(2k, 2k+1)$. The assertion of the proposition follows.

We finish this section with Table 2 describing the bijection t between all triples $s \in S_3$ of the 7-set V_7 and all triples of the STS No 1.

Table 2.

267	$t_{123}(\varepsilon^1)$	126	$t_{347}(\varepsilon^1)$	127	$t_{145}(\varepsilon^2)$	345	$t_{257}(\varepsilon^0)$
367	$t_{257}(\varepsilon^1)$	136	$t_{145}(\varepsilon^0)$	137	$t_{347}(\varepsilon^3)$	245	$t_{123}(\varepsilon^3)$
467	$t_{145}(\varepsilon^3)$	146	$t_{257}(\varepsilon^3)$	147	$t_{123}(\varepsilon^0)$	235	$t_{347}(\varepsilon^2)$
567	$t_{347}(\varepsilon^0)$	156	$t_{123}(\varepsilon^2)$	157	$t_{257}(\varepsilon^2)$	234	$t_{145}(\varepsilon^1)$
167	t_6	236	$t_{167}(\varepsilon^3)$	237	$t_{356}(\varepsilon^0)$	145	t_4
		246	$t_{246}(\varepsilon^0)$	247	t_7	135	$t_{356}(\varepsilon^1)$
		256	t_5	257	$t_{246}(\varepsilon^1)$	134	$t_{167}(\varepsilon^2)$
		346	t_3	347	$t_{246}(\varepsilon^2)$	125	$t_{167}(\varepsilon^0)$
		356	$t_{246}(\varepsilon^3)$	357	t_1	124	$t_{356}(\varepsilon^2)$
		456	$t_{167}(\varepsilon^1)$	457	$t_{356}(\varepsilon^3)$	123	t_2

Note that the triples of S_3 of the same row laying in the 3th, 5th and 7th columns of Table 2 have the form $ij6$, $ij7$ and $\overline{ij67} = V_7 - \{ij67\}$, respectively.

4.3 A vector representation of STSs No1 and No 2

There is a mate construction of an odd system of vectors of norm 5 related to above obtained STSs.

Let w_s, $s \in F_7$, be 7 mutually orthogonal vectors of norm 2, and let g be a vector of norm $\frac{3}{2}$ orthogonal to all w_s. At first, we construct vectors u_i corresponding to triples $t_i = (1, 2i, 2i+1)$. We set

$$u_i = g - \frac{1}{2}\sum_{s \in F_7} w_s + \sum_{s \ni i} w_s, \ 1 \le i \le 7. \tag{4}$$

Let e_i, $1 \le i \le 7$, be 7 mutually orthogonal unit vectors. All vectors e_i are orthogonal to all w_s and to g, and relate to pairs $(2i, 2i+1)$. Recall that each triple $t_s(\varepsilon^k)$

corresponds to $s \in F_7$ and to the even triple $\varepsilon^k = (\varepsilon_i^k, \varepsilon_j^k, \varepsilon_l^k)$, where $s = (ijl)$. We set

$$u_s(\varepsilon^k) = w_s + \sum_{i \in s} \varepsilon_i^k e_i, \;\; s \in F_7, \;\; 0 \le k \le 3. \tag{5}$$

It can be easy verified, using (3), that if we set

$$v(t_i) = u_i, \;\; v(t_s(\varepsilon^k)) = u_s(\varepsilon^k), \tag{6}$$

then the vectors $v(t)$ satisfy (1).

Note that endpoints of vectors $w_s + \sum_{i \in s} \varepsilon_i e_i$, for fixed s and for $2^3 = 8$ (± 1)-triples ε, form a 3-dimensional cube $Q(s)$ with a center in the endpoint of w_s. We call a vertex v of the cube $Q(s)$ (and corresponding triple of an STS) *even* (*odd*) if v relates to an even (odd, respectively) triple ε. So, 28 even vertices of the 7 cubes $Q(s)$, $s \in F_7$, relate to triples of the STS No 1. If we change even vertices of $Q(347)$ and $Q(356)$ by odd vertices, we obtain the set T_2 of triples of STS No 2. Denote by $\mathcal{V}_1$ and $\mathcal{V}_2$ the sets of 35 vectors u_i, $1 \le i \le 7$, and u_s^k, $s \in F_7$, $0 \le k \le 3$, related to STS's No 1 and No 2, respectively.

Now we show that the vectors $\sqrt{2}e_i$ are roots of the odd system $\mathcal{V}_2$ related to STS No 2. Consider two disjoint triples of STS No 2 $t_4 = (1, 8, 9)$, i.e. t_i for $i = 4$, and $t_{167}(\varepsilon^1) = (2, 13, 15)$, and the matching (1,2), (8,13), (9,15) of these triples. The triples of STS No 2 containing the pairs of this matching are $t_1 = (1, 2, 3)$, $t_{246}(\varepsilon^2) = (5, 8, 13)$, $t_{347}(-\varepsilon^0) = (7, 9, 15)$, respectively. Hence $q(1, 2) = 3$, $q(8, 13) = 5$, $q(9, 15) = 7$. These 3 distinct points form the derived triple $(3, 5, 7) = t_{123}(-\varepsilon^0)$. This triple, as the triple $(7, 9, 15) = t_{347}(-\varepsilon^0)$, is an odd triple. By Proposition 2, the odd triple $t_{123}(-\varepsilon^0) = (3, 5, 7)$ belongs to $\mathrm{cl}T_2$. The corresponding vector $v(t_{123}(-\varepsilon^0)) = u_{123}(-\varepsilon^0)$ satisfies the equality

$$u_1 + u_{246}(\varepsilon^2) + u_{347}(-\varepsilon^0) - u_4 - u_{167}(\varepsilon^1) = u_{123}(-\varepsilon^0),$$

where u-vectors and triples relate according to (6). Hence $u_{123}(-\varepsilon^0) \in L_1(\mathcal{V}_2)$. Since the vectors $u_{123}(\varepsilon^i)$ for even ε^i belong to the odd system $\mathcal{V}_2$, we obtain 3 roots (multiplied by $\sqrt{2}$) (i.e. vectors of $L_0(\mathcal{V}_2)$ of norm 4)

$$2e_i = u_{123}(\varepsilon^i) - u_{123}(-\varepsilon^0), \;\; i = 1, 2, 3.$$

Similarly, we can obtain all 7 roots $\sqrt{2}e_i$. Each root connects an even vertex with an odd vertex of a cube $Q(s)$. Recall that we prove in [5] that there is no other roots in the root system of STS No 2.

4.4 A representation of STS No 1 by minimal vectors of a sub-lattice of the Barnes-Wall lattice

The vector representation of the previous section can be obtained from a doubly even lattice by the following construction [4].

Let L be a doubly even lattice. For the inner product ab of two vectors $a, b \in L$ of norm 8, we have $ab = \pm 4, \pm 2, 0$. Hence a vector $c = a + a^*$ have norm $c^2 = 12$ for any two vectors a and a^* of norm 8 with $aa^* = -2$. Let c be a vector of L of norm $c^2 = 12$. Consider a set $\mathcal{A}(c)$ of vectors $a \in L$ of norm 8 such that $ac = \frac{1}{2}c^2 = 6$. It is easy to show that if $a \in \mathcal{A}(c)$, then the vector $a^* \equiv c - a$ belongs to $\mathcal{A}(c)$, too. Besides, it can be proved that $aa' = 2$ or 4 for $a, a' \in \mathcal{A}(c)$, $a' \neq a^*$ (see [3]). We set

$$v(a) = a - \frac{1}{2}c, \ a \in \mathcal{A}(c).$$

We have

$$v^2(a) = 5, \ v(a^*) = -v(a), \text{ and } v(a)v(b) = ab - 3.$$

Since ab is even, the inner product $v(a)v(b)$ is odd. Hence the set

$$\mathcal{V}(c) = \{v(a) : a \in \mathcal{A}(c)\}$$

is an odd system of vectors of norm 5.

In this section, we apply this construction to the 16-dimensional Barnes-Wall lattice, denoted by Λ_{16} in [2]. We show, in fact, that STS No 1 is closed, that is $P(T_1)$ is a Delaunay polytope of a 14-dimensional sub-lattice of the Barnes-Wall lattice Λ_{16}.

It is more convenient to consider the doubly even lattice $\sqrt{2}\Lambda_{16}$. The set of minimal vectors of $\sqrt{2}\Lambda_{16}$ consists of the following vectors of norm 8.

(1) 480 vectors of the form $(\pm 2^2, 0^{14})$, where there are two nonzero components equal to 2 or –2,

(2) 3840 vectors of the form $(\pm 1^8, 0^8)$, where the positions of ± 1's form one of the 30 codewords of weight 8 of the first order Reed-Muller code and there are an even number of minus signs. (These codewords of length 16 are given in Figure 6 of [6]).

There are exactly 36 pairs (a, a^*) of vectors of $\mathcal{A}(c)$ in $\sqrt{2}\Lambda_{16}$ for any $c \in \sqrt{2}\Lambda_{16}$ of norm 12. We give an analytic description of these vectors.

Let $c = (-11^7|20^7)$. There are exactly 7 minimal vectors $a_i = (0^{i-1}20^{8-i}|20^7)$, $2 \leq i \leq 8$, of type $(2^2 0^{14})$, and one vector $a_0 = (-20^7|20^7)$ of type (-220^{14}), such that $a_i c = 6$, i.e. $a_i \in \mathcal{A}(c)$.

Now consider minimal vectors having a codeword as a support. For brevity sake, we call such vectors *codevectors*. We denote a codevector a as a join $a = (b, d)$ of two 8-dimensional vectors $b = (b_0, b_1, ..., b_7)$ and $d = (d_0, d_1, ..., d_7)$.

Note that the vectors $a_0^* = c - a_0 = (1^8|0^8)$, and $a_i^* \in \mathcal{A}(c)$, $1 \leq i \leq 7$, are codevectors (b, d) such that $b_i \neq 0$ and $d_i = 0$, $0 \leq i \leq 7$. All these 8 vectors have the same support. Since any two codewords have exactly 4 common 1's, vectors b and d each have exactly 4 nonzero coordinates for any codevector $a = (b, d) \neq a_i^*$.

For a codevector $a=(b,d)$ to belong to $\mathcal{A}(c)$, it is necessary that $b_0\le 0$, $b_i\ge 0$, $1\le i\le 7$, and $d_0=1$. Since $ac=-b_0+\sum_{i=1}^7 b_i+2d_0=6$, this implies that if $a=(b,d)\in\mathcal{A}(c)$ is a codevector and $a\ne a_i^*$, $0\le i\le 7$, then $(b^*,d^*)\equiv a^*=c-a$ is also a codevector of $\mathcal{A}(c)$. ¿From two vectors b and b^* of a pair (b,d) and (b^*,d^*), exactly one has nonzero first coordinate b_0. We take (b,d) with $b_0=-1$ (and, recall, $d_0=1$) as a representative of the above pair.

Vectors b and d of such a representative have additionally 3 nonzero coordinates among b_i's and d_i's. Let $s=\{i: b_i=1\}$, $t=\{i: 1\le i\le 7, d_i\ne 0\}$. According to this we re-denote representatives as $a(s,t)$, where s and t are two triples of a 7-set. Recall that supports of two codevectors a,a' intersect in 4 elements, and two elements corresponding to $b_0=b'_0=-1$ and $d_0=d'_0=1$ belong necessarily to the intersection. This implies that $|s\cap s'|+|t\cap t'|\le 2$. Since a codevector has even number of minus signs, and $b_i\ge 0$, $1\le i\le 7$, $b_0=-1$, there are exactly 4 codevectors $a^k=(b^k,d^k)\in\mathcal{A}(c)$, $0\le k\le 3$, with the same support such that b^k does not depend on k and d^k has 3 or 1 negative coordinates, i.e. d^k is related to the odd triple $-\varepsilon^k$. Recall that ε^k, $0\le k\le 3$, is an even triple defined in Section 4.2.

Suppose that there are $a=a(s,t)$ and $a'=a(s',t')$ such that $|t\cap t'|=2$. Since there are 4 codevectors of $\mathcal{A}(c)$ with the same support, we can choose $a(s,t)$ with $d=d(t)$ having 3 minus signs, and $a(s',t')$ with $d'=d(t')$ having one minus sign in $t-t\cap t'$. Hence $dd'=-1$. The equality $|t\cap t'|=2$ implies $s\cap s'=\emptyset$. Therefore $bb'=1$, and, for the inner product aa' we have $aa'=bb'+dd'=1-1=0$, what contradicts to $aa'=2$ or 4 for $a,a'\in\mathcal{A}(c)$. This implies that $|t\cap t'|\le 1$ for $t\ne t'$.

Recall that $\mathcal{A}(c)$ is a subset of the set of minimal vectors of the lattice $\sqrt{2}\Lambda_{16}$. An inspection of the set $\mathcal{A}(c)$ shows that one can choose a labeling of coordinates of $b(s)$ and $d(t)$ such that $s=t$, i.e. s and t take positions with the same labeling, for any codevector $a(s,t)$. Hence we can re-denote $a(s,s)$ by $a(s)$. Besides we have that $|s\cap s'|\le 1$, for any two codevectors $a(s),a(s')\in\mathcal{A}(c)$, since we have showed that $|t\cap t'|\le 1$. This means that triples s form a Steiner triple system on 7 points.

So, we obtain that $\mathcal{A}(c)$ has 36 pairs of vectors a,a^*, where a is one of the following 36 vectors:

8 vectors a_i, $0\le 1\le 7$, and

7×4 vectors $a^k(s)$, $s\in F_7$, $0\le k\le 3$.

Note that $a_0a_i=a_0a^k(s)=4$ for all a_i, $i\ne 0$, $a^k(s)$.

Consider vectors $u_i=v(a_i)=a_i-\frac{1}{2}c$, and $u^k(s)=a^k(s)-\frac{1}{2}c$.

Let f_i, e_i, $0\le i\le 7$, be an orthonormal basis of R^{16} such that a codevector $a=(b,d)$ has the form $a=\sum_{i=0}^7(b_if_i+d_ie_i)$. In the basis (f,e) the vectors c and every $a\in\mathcal{A}(c)$ take the form (recall that ε^k, $0\le k\le 3$, are even triples)

$c=-f_0+\sum_1^7 f_i+2e_0$;

$a_0=2(e_0-f_0)$, $a_i=2(e_0+f_i)$, $1\le i\le 7$;

$a^k(s) = (e_0 - f_0) + \sum_{i\in s} f_i - \sum_{i\in s} \varepsilon_i^k e_i,\ s \in F_7,\ 0 \le k \le 3.$

Similarly for vectors $v(a) = a - \frac{1}{2}c \in \mathcal{V}(c)$ we have

$u_0 = v(a_0) = -\frac{3}{2}f_0 - \frac{1}{2}\sum_1^7 f_i + e_0;$

$u_i = v(a_i) = e_0 + \frac{1}{2}f_0 + 2f_i - \frac{1}{2}\sum_1^7 f_i;$

$u_s^k = v(a^k(s)) = -\frac{1}{2}f_0 + \sum_{i\in s} f_i - \frac{1}{2}\sum_1^7 f_i - \sum_{i\in s}\varepsilon_i^k e_i.$

We set

$$w_0 = \frac{1}{2}f_0 - \frac{1}{2}\sum_1^7 f_i, \text{ and } w_s = -\frac{1}{2}f_0 + \sum_{i\in s} f_i - \frac{1}{2}\sum_1^7 f_i,\ s \in F_7.$$

One can verify that

$$w_s^2 = 2,\ w_s w_{s'} = 0 \text{ for } s, s' \in F_7 \cup \{0\}.$$

The vectors w_0, w_s form another basis of the 8-space spanned by f_i, $0 \le i \le 7$. In this basis the vectors u_i and u_s^k take the form

$$u_0 = g + \frac{1}{2}\sum_{s\in F_7} w_s,\ u_i = g - \frac{1}{2}\sum_{s\in F_7} w_s + \sum_{s\ni i} w_s,\ u_s^k = u_s(-w_s - \varepsilon^k) = \sum_{i\in s}\varepsilon_i^k e_i.$$

Here the vector $g = e_0 + \frac{1}{2}w_0$ has norm $\frac{3}{2}$ and is orthogonal to all w_s and e_i.

Note that these vectors u_i and u_s^k form an odd system

$$\mathcal{V}_0(c) = \{v \in \mathcal{V}(c) : vu_0 = 1\}.$$

It is easy to see that each $v \in \mathcal{V}_0(c)$ has the form $v = v(a)$ for $a \in \mathcal{A}(c)$ such that $aa_0 = 4$.

Proposition 7 *The odd system $\mathcal{V}_0(c)$ is closed for every $c \in \sqrt{2}\Lambda_{16}$ of norm 12 and every $a_0 \in \mathcal{A}(c)$. $\mathcal{V}_0(c)$ is isomorphic to the odd system $\mathcal{V}_1$ related to STS No 1.*

Proof. Recall that the set of endpoints of all vectors $v \in \mathcal{V}_0(c)$ is equal to the set of endpoints of all minimal vectors a of the Barnes-Wall lattice $\sqrt{2}\Lambda_{16}$ such that $ac = 6$ and $aa_0 = 4$. Let $\mathcal{H}_{14}(c, a_0)$ be the 14-dimensional affine space of all vectors $x \in R^{16}$ such that $xc = 6$ and $xa_0 = 4$. Let $\mathcal{L}_{14} = \sqrt{2}\Lambda_{16} \cap \mathcal{H}_{14}(c, a_0)$. The above set of endpoints belong to $\mathcal{L}_{14}$. Moreover, as the vectors $a \in \mathcal{A}(c)$ are minimal vectors of $\sqrt{2}\Lambda_{16}$, the set of endpoints of all $v \in \mathcal{V}_0(c)$ is the set of all vertices of a Delaunay polytope of the lattice $\mathcal{L}_{14}$. This implies that $\mathcal{V}_0(c)$ is closed.

Comparing $v \in \mathcal{V}_0(c)$ with u_i from (4) and $u_s(\varepsilon^k)$ from (5), we see that both u_i's coincide and $u_s^k = u_s(-\varepsilon^k)$. Hence the odd system $\mathcal{V}_0(c)$ can be obtained from the odd system $\mathcal{V}_1$, related to STS No 1, by interchanging the even vertices of $Q(s)$ by odd vertices for all $s \in F_7$. According to Proposition 6, as the set of all triples of F_7 is a positive set, the odd system $\mathcal{V}_0(c)$ represents STS No 1, too.

5 The two-graph related to STS No 2

Recall that the inner product $v(t)v(t')$ for two triples t and t' of an STS takes the values ± 1. This means that vectors $v(t)$, $t \in T$, of norm 5 span equiangular lines at angle $arccos\frac{1}{5}$.

Sets of equiangular lines and two-graphs are in one-to-one correspondence. We do not need here in an exact description of this correspondence. Details see, for example, in [1]. It is important for us only that each line is spanned by a pair of opposite vectors (of norm 5), and the set of all lines is represented by an odd system of these vectors, taken one from each pair. A change of vectors of a subset by its opposite is called *switching* of the odd system. Obviously, the switched odd system represents the same two-graph as original one.

A maximal set of equiangular lines at angle $arccos\frac{1}{5}$ in a 15-dimensional space contains 36 lines and corresponds to a regular two-graph.

The odd systems described above and related to Steiner triple systems contain 35 vectors. The 36th vector can be obtained as follows. Let an odd system $\mathcal{V}$ contains 5 vectors $v_i, 1 \leq i \leq 5$, with all mutual inner products –1. These 5 vectors correspond to 5 mutually disjoint triples. Then the vector $v = \sum_{i=1}^{5} v_i$ has norm 5 and inner products ± 1 with all other vectors of $\mathcal{V}$. For the odd system described in the previous section, the 36th vector is

$$u_0 = g + \frac{1}{2} \sum_{s \in F_7} w_s.$$

It can be obtained, for example, as the sum of $v(t)$ for t=(1,2,3), (4,8,12), (5,10,15), (6,11,13), (7,9,14). It is easy to verify that v_0 has the inner product $+1$ with all other vectors of $\mathcal{V}_1$ (and $\mathcal{V}_2$), i.e.

$$u_0 u_i = u_0 u_s(\pm\varepsilon^k) = 1, \ 1 \leq i \leq 7, \ s \in F_7, \ 0 \leq k \leq 3.$$

In general, let $\mathcal{V}$ be an odd system of 36 pairs of opposite vectors related to a regular two-graph. Then $vv' = \pm 1$ for any $v, v' \in \mathcal{V}$ such that $v \neq \pm v'$.

Obviously, for any $v_0 \in \mathcal{V}$, we can choose a vector v from each pair of opposite vectors such that $vv_0 = -1$. Then the chosen 35 vectors form a set of vertices of a strongly regular graph $H(v_0) = H(v_0, \mathcal{V})$. Two vectors v and v' are adjacent in H if and only if $vv' = -1$. Obviously the graphs $H(v_0)$ and $H(-v_0)$ are isomorphic. But if $v_0 \neq \pm v_0'$, then the graphs $H(v_0)$ and $H(v_0')$ may be not isomorphic. But they are "pseudoswitching" equivalent. Note that the graph $H(v_0) = H_1$ related to STS No 1 does not depend on v_0, since the corresponding two-graph have a doubly transitive automorphism group, and H_1 is the unique rank 3 graph with parameters (35,18,6,8). (See [1] and [7] for details.)

Any clique of $H(v_0)$ has size at most 5. $H(v_0)$ contains a clique of maximal size not for every v_0. For example, graphs H corresponding to 10 Steiner triple systems No 14, 16, 24, 26, 29, 35, 40, 52, 59, 62 do not contain cliques of maximal size 5. But in the corresponding odd system there is a vector v'_0 such that $H(v'_0)$ has a clique of size 5.

Let $H(v_0)$ contains a clique C of maximal size. Denote vertices of C (and corresponding vectors) by v_i, $1 \leq i \leq 5$. By the construction, $v_iv_j = -1$ for $0 \leq i < j \leq 5$. One can easy to verify that $\sum_{i=0}^{5} v_i = 0$. Let v be a vertex of $H(v_0)$ not belonging to C. Since $vv_0 = -1$, and $\sum_0^5 vv_i = 0$ there are exactly 2 vertices of C, say v_i and v_j, $i, j > 0$, with $vv_i = vv_j = -1$, which are adjacent to v. Since $H(v_0)$ is strongly regular with parameters (35,16,6,8), there are exactly 6 vertices adjacent to given two adjacent vertices. Hence there are exactly 3 vertices of $H(v_0) - C$ adjacent to v_i and v_j. We denote these vertices by (ij), $(ij)_a$ and $(ij)_b$. So, if $H(v_0)$ contains a clique C of size 5, then 35 vertices of $H(v_0)$ can be labeled as v_i, $1 \leq i \leq 5$, (vertices of the clique C), and $(ij), (ij)_a, (ij)_b$, $1 \leq i < j \leq 5$ (other vertices).

Recall that the vertices of the graph H_1 related to STS No 1 are naturally labeled by triples ijk such that $1 \leq i < j < k \leq 7$ (see Table 2). It is easy to see that $H_1 = H(v_0)$, where $v_0 = -u_0$, and u_0 is defined at the beginning of this section, and Table 2 shows that two labeling relates as follows:

$$v_i = i67,\ (ij) = \overline{ij67},\ (ij)_a = ij6,\ (ij)_b = ij7,\ 1 \leq i < j \leq 5,$$

where $\overline{ij67} = V_7 - \{ij67\}$ is a complement of the set $\{ij67\}$ in V_7.

Seidel in [12] describes a family of two-graphs discovered by T.Spence, by use of a computer. Using the above labeling of vertices of the graph $H(v_0)$, we give a detailed description of this family and show that two-graphs related to STS No 1 and No 2 belong to this family.

It is easy to see that the subgraph of H_1 induced on the set of vertices $\{v_i : 1 \leq i \leq 5\} \cup \{ij, 1 \leq i < j \leq 5\}$ is the triangular graph $T(6)$. Similarly, the subgraphs of H_1 induced by vertices $(ij)_a$ and $(ij)_b$, $1 \leq i < j \leq 5$ are $T(5)$'s.

Let Q be the graph induced on 20 vertices $(ij)_a$, $(ij)_b$, $1 \leq i < j \leq 5$. We saw that Q is a "union" of two $T(5)$'s. Now we show that Q is similar "union" of two Petersen graphs. In fact, Q can be partitioned into 4 induced circuits of length 5. Denote these circuits by $C(\varepsilon, \beta)$, where $\beta \in \{a, b\}$ and $\varepsilon \in \{0, 1\}$. The circuit $C(0, \beta)$ contains the vertices $(12)_\beta$, $(23)_\beta$, $(34)_\beta$, $(45)_\beta$, $(15)_\beta$. The vertices of $C(1, \beta)$ are $(13)_\beta$, $(35)_\beta$, $(25)_\beta$, $(24)_\beta$, $(14)_\beta$. Of course, this partition is not unique. It is easy to see that the circuits $C(0, a)$ and $C(1, a)$ (as the circuits $C(0, b)$ and $C(1, b)$) induce the $T(5)$ graph. Similarly, the circuits $C(0, a)$ and $C(1, b)$ (as the circuits $C(1, a)$ and $C(0, b)$) induce the Petersen graph. Just this last partition of the graph $H(v_0)$ into $T(6)$ and two Petersen graphs was used in the considerations of Seidel [12].

Call an edge of the graph Q of the form $((ij)_a, (ij)_b)$ *basic*. Let E_{10} be the set of all 10 basic edges. We denote basic edges as $e(ij)$. Consider, at first, adjacencies of vertices of $T(6)$ and Q in the graph H_1 of STS No 1. If a vertex v of $T(6)$ is adjacent to one endvertex of a basic edge, then it is adjacent to the second endvertex of this edge. Each vertex v of $T(6)$ is adjacent to 4 pairs of endpoints of 4 basic edges. Let $E(v) \subseteq E_{10}$ be the set of basic edges whose endpoints are adjacent to v. It is easy to verify that these 15 subsets $E(v)$ of cardinality 4, where v is a vertex of $T(6)$, form a set $\mathcal{E}_0$ of blocks of a 2-(10,4,2) design.

We give the incidence matrix of this design in Table 3.

Table 3

	v_1	v_2	v_3	v_4	v_5	12	13	14	15	23	24	25	34	35	45
$e(12)$	1	1				1							1	1	1
$e(13)$	1		1				1				1	1			1
$e(14)$	1			1				1		1		1		1	
$e(15)$	1				1				1	1	1		1		
$e(23)$		1	1					1	1	1					1
$e(24)$		1		1			1		1		1			1	
$e(25)$		1			1		1	1				1	1		
$e(34)$			1	1		1			1			1	1		
$e(35)$			1		1	1		1			1			1	
$e(45)$				1	1	1	1			1					1

Gronau [8] shows that there exist exactly 3 nonisomorphic 2-(10,4,2) designs. One can take from each isomorphism class three designs such that its incidence matrices differ only in the right lower 6×6 sub-matrix. This sub-matrix is an adjacency matrix of vertices (ij), $2 \le i < j \le 5$, and $(kl)_a$, $(kl)_b$, $2 \le k < j \le 5$. We give below these 3 sub-matrices.

	23	24	25	34	35	45
$e(23)$	1					1
$e(24)$		1			1	
$e(25)$			1	1		
$e(34)$			1	1		
$e(35)$		1			1	
$e(45)$	1					1

	23	24	25	34	35	45
$e(23)$			1	1		
$e(24)$		1			1	
$e(25)$	1					1
$e(34)$	1					1
$e(35)$		1			1	
$e(45)$			1	1		

	23	24	25	34	35	45
$e(23)$		1			1	
$e(24)$			1	1		
$e(25)$	1					1
$e(34)$	1					1
$e(35)$			1	1		
$e(45)$		1			1	

We denote 2-(10,4,2) designs corresponding to these matrices by $\mathcal{D}_0$, $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively.

Now we describe graphs $H(v_0)$ of the above mentioned family of two-graphs. Let $\mathcal{E}$ be an arbitrary set of 4-subsets of E_{10} forming blocks of a 2-(10,4,2) design $\mathcal{D}(\mathcal{E})$.

Then $|\mathcal{E}| = 15$. Let $G(\mathcal{E})$ be a graph constructed on the set $\mathcal{E}$ as follows. Two vertices E and E' are adjacent in $G(\mathcal{E})$ if and only if $|E \cap E'| = 1$. (Recall that, for any two blocks E, E' of a 2-(10,4,2) design, we have $|E \cap E'| = 1$ or 2.) Using the incidence matrices of the 3 nonisomorphic designs $\mathcal{D}_0$, $\mathcal{D}_1$ and $\mathcal{D}_2$, it is easy to verify that $G(\mathcal{E})$ is isomorphic to the triangular graph $T(6)$ for all these 3 designs.

We form a graph $H(Q, \mathcal{E})$ as follows. We take a union of the graphs $G(\mathcal{E})$ and Q, i.e. the set of vertices of $H(Q, \mathcal{E})$ is the union of the set $\mathcal{E}$ and the set of vertices of Q, and each of these sets induces the corresponding graph. We define an adjacency of a vertex E of $G(\mathcal{E})$ and a vertex v of Q as follows. Recall that E is a set of basic edges of Q. We set the vertex E of $G(\mathcal{E})$ is adjacent to a vertex v of Q if and only if v is an endvertex of a basic edge from E.

Proposition 8 *The graph $H(Q, \mathcal{E})$ is a strongly regular graph with parameters (35,16,6,8) for any 2-(10,4,2) design $\mathcal{D}(\mathcal{E})$.*

Proposition 8 can be proved by inspection. We shall prove it in the forthcoming paper [5], where we show that for each $H(Q, \mathcal{E})$ there is a two-graph having it as $H(v_0)$. Moreover, we show there that all these two-graphs can be obtained from the even unimodular lattice D_{16}^{+} by the method of [3] and [4] described here in Section 4.4.

Call a two-graph of *$\mathcal{D}_i$-type* if the two-graph has $H(Q, \mathcal{E})$ as $H(v_0)$, where $\mathcal{E}$ is the set of blocks of a design isomorphic to $\mathcal{D}_i$.

We saw above that the following proposition holds.

Proposition 9 *The two-graph related to STS No 1 is of $\mathcal{D}_0$-type.*

Now we prove

Proposition 10 *The two-graph related to STS No 2 is of $\mathcal{D}_1$-type.*

Proof. Consider at first the two-graph related to STS No 1. Since $H(v_0)$ does not depend on v_0 for this two-graph, we can take $v_0 = -u_{123}(\varepsilon^0)$. Let the clique C consists of the following 5 vectors:

$$v_1 = u_{123}(\varepsilon^3), v_2 = u_{246}(\varepsilon^0), v_3 = -u_{145}(\varepsilon^2), v_4 = -u_{246}(\varepsilon^3), v_5 = u_{145}(\varepsilon^1).$$

The other vectors obtain labeling as follows:

Table 4.

$(12) =$	$-u_{246}(\varepsilon^2)$	$(12)_a =$	$-u_{167}(\varepsilon^2)$	$(12)_b =$	$u_{167}(\varepsilon^1)$
$(13) =$	$u_{145}(\varepsilon^0)$	$(13)_a =$	$u_{257}(\varepsilon^1)$	$(13)_b =$	$-u_{257}(\varepsilon^2)$
$(14) =$	$u_{246}(\varepsilon^1)$	$(14)_a =$	$u_{167}(\varepsilon^0)$	$(14)_b =$	$-u_{167}(\varepsilon^3)$
$(15) =$	$-u_{145}(\varepsilon^3)$	$(15)_a =$	$-u_{257}(\varepsilon^3)$	$(15)_b =$	$u_{257}(\varepsilon^0)$
$(23) =$	$u_{356}(\varepsilon^1)$	$(23)_a =$	u_1	$(23)_b =$	$-u_6$
$(24) =$	$u_{123}(\varepsilon^1)$	$(24)_a =$	$u_{347}(\varepsilon^1)$	$(24)_b =$	$-u_{347}(\varepsilon^2)$
$(25) =$	$-u_{356}(\varepsilon^3)$	$(25)_a =$	$-u_4$	$(25)_b =$	u_3
$(34) =$	$-u_{356}(\varepsilon^2)$	$(34)_a =$	$-u_7$	$(34)_b =$	u_0
$(35) =$	$u_{123}(\varepsilon^2)$	$(35)_a =$	$u_{347}(\varepsilon^0)$	$(35)_b =$	$-u_{347}(\varepsilon^3)$
$(45) =$	$u_{356}(\varepsilon^0)$	$(45)_a =$	u_2	$(45)_b =$	$-u_5$
$v_0 =$	$-u_{123}(\varepsilon^0)$	$v_1 =$	$u_{123}(\varepsilon^3)$	$v_2 =$	$u_{246}(\varepsilon^0)$
$v_3 =$	$-u_{145}(\varepsilon^2)$	$v_4 =$	$-u_{246}(\varepsilon^3)$	$v_5 =$	$u_{145}(\varepsilon^1)$

The adjacency matrix of these vectors considered as vertices of $H(v_0)$ is given by the incidence matrix of the design $\mathcal{D}_0$. Note that the incidence matrices of designs $\mathcal{D}_0$ and $\mathcal{D}_1$ differ only in a 4×4 sub-matrix A composed by intersections of the columns (23),(25),(34),(45) and rows $e(23)$, $e(25)$,$e(34)$,$e(45)$. These sub-matrices $A_i \subset \mathcal{D}_i$, $i=0,1$, are complemented.

Note now, that if we change the signs before ε^k in $u_{356}(\varepsilon^k)$ for all 4 k, then the vectors of Table 4 will form the switched odd system $\mathcal{V}_2$. Denote the switched odd system $\mathcal{V}_2^{sw}$.

All the vectors $u_{356}(\varepsilon^k)$ stay in rows of Table 4 labeled by (23), (25), (34), (45). Call these rows *special*. Other vectors in the special rows have the form $\pm u_i$, $0 \le i \le 7$. Recall that $u_s(\varepsilon^k)$ is adjacent to u_i (i.e. $u_s(\varepsilon^k)u_i = -1$) if and only if $i \notin s$, i.e. this adjacency does not depend on k. Besides, we have for $s \ne t$, $u_s(\varepsilon^k)u_t(\varepsilon^l) = -1$ if and only if $\varepsilon^k_i = -\varepsilon^l_i$ for $\{i\} = s\cap t$. Hence, when we change the sign before ε^k in $u_{356}(\varepsilon^k)$, we change adjacencies of $u_{356}(\varepsilon^k)$ with all vectors excluding vectors in the special rows. Now we switch 4 vectors $u_{356}(-\varepsilon^k)$, $0\le k\le 3$, of the odd system $\mathcal{V}_2^{sw}$ (i.e. change the signs before $u_{356}(-\varepsilon^k)$). Then the adjacencies of switched vectors with all vectors of not special rows will be as in H_1, i.e. this adjacency will be given by the incidence matrix of $\mathcal{D}_0$. But the adjacencies of vectors $u_{356}(-\varepsilon^k)$ with all vectors of the form u_i will be changed, i.e. they will be given by the matrix A_1 instead of the matrix A_0. All the adjacencies will be given by the incidence matrix of the design $\mathcal{D}_1$. Hence, after switching, we obtain an odd system representing a two-graph of the $\mathcal{D}_1$-type.

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