The Hilbert Basis of the Cut Cone over the Complete Graph on Six Vertices

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Abstract

Let C be a real polyhedral cone, generated by the integer vectors $x_1, \ldots, x_n$. The set of points of this cone with integer coordinates forms a semi-group whose minimal set of generators (for linear combinations with coefficients in $\mathbb{Z}^+$) is called the *Hilbert basis* of C. The Hilbert basis always contains $x_1, \ldots, x_n$. The integer points of C which are not in the integer cone $\mathbb{Z}^+(x_1, \ldots, x_n)$ are called the *quasi-h points.*

This report presents the Hilbert basis for the cut cone over K_6. Two results are proven:

• The Hilbert basis of the cut cone over K_6 is composed of the 31 cuts and of the 15 vectors d^e defined for each edge e by: $d^e_f = 2$ for $f \neq e$ and $d^e_e = 4$.

• The quasi-h points for K_6 are exactly the $d^e + n\delta(v)$ for v non adjacent to e and $n \in \mathbb{Z}^+$.

This report is the extended version of [Lab95], where Hilbert bases are studied within the general framework of integer programming and polytope theory. Moreover, detailed proofs are provided.

Keywords: Cut cone, Integer Programming, Hilbert basis, Gordan lemma

Résumé

Soit C un cône polyédral réel, engendré par les vecteurs $x_1, \ldots, x_n$ à coordonnées entières. L'ensemble des points à coordonnées entières de C forme un semi-groupe dont l'ensemble générateur minimal (par des combinaisons linéaires à coefficients dans $\mathbb{Z}^+$) est appelé *base de Hilbert* de C. Les points entiers de C qui ne sont pas dans le cone entier $\mathbb{Z}^+(x_1, \ldots, x_n)$ sont appelés les *points quasi-h.*

Ce rapport présente la base de Hilbert pour le cône des coupes sur K_6. Deux resultats sont prouvés:

• La base de Hilbert pour le cône des coupes sur K_6 est formée des 31 coupes et des 15 vecteurs d^e ainsi définis pour chaque arête e: $d^e_f = 2$ pour $f \neq e$ et $d^e_e = 4$.

• Pour K_6, les points quasi-h sont exactement les $d^e + n\delta(v)$ pour v non adjacent à e et $n \in \mathbb{Z}^+$.

Ce rapport est la version complète de [Lab95], où les bases de Hilbert sont étudiées dans le cadre général de la programmation linéaire en nombres entiers et de la théorie des polytopes. De plus, les preuves détaillées sont fournies.

Mots-clés: Cone des Coupes, Programmation Linéaire en Nombres Entiers, Base de Hilbert, Lemme de Gordan

1 Introduction: Cuts on a graph

Let $G = (V, E)$ be a symmetric graph, with vertex-set $V = \{1, \ldots, n\}$, and edge-set $E \subseteq V^2$. When $E = \{(i,j),\ i \neq j \ \&\ 1 \leqslant i,j \leqslant n\}$, G is called the complete graph, denoted K_n.
Any subset $S \subseteq V$ partitions the vertices into two "shores", S and $\overline{S}$. The set of edges $\{(i,j) \in E,\ i \in S,\ j \in \overline{S}\}$ is called the **cut induced by S** (note that a cut can clearly be defined by any of its two shores).

The **cut vector** $\delta(S) \in \{0,1\}^E$ is the binary vector with entries indexed by edges characterizing the cut induced by S :

$$\delta(S)_{ij} = \begin{cases} 1 \text{ if } card(\{i,j\} \cap S) = 1 \\ 0 \text{ otherwise} \end{cases}$$

By abuse of language, the cut vector is also called the cut associated to S.

Since $\delta(S) = \delta(\overline{S})$, the set of cuts can be enumerated by considering all sets of vertices containing a given vertex. The set of all cuts over the complete graph K_n will be denoted by $\mathcal{K}_n$.

We will denote by $\mathbb{R}^+(\mathcal{K}_n)$ (resp. $\mathbb{Z}(\mathcal{K}_n)$ or $\mathbb{Z}^+(\mathcal{K}_n)$) the set of linear combinations of cuts, with coefficients in $\mathbb{R}^+$ (resp. $\mathbb{Z}$ or $\mathbb{Z}^+$). For $x = \sum \lambda_S \delta(S)$ with $\lambda_S \in \mathbb{Z}^+$ for all S, the sequence (λ_S) will be called a $\mathbb{Z}^+$-*realization* of x. The following objects will be considered throughout this report :

• the **cut polytope**, $\mathcal{P}_n = Conv(\{\delta(S),\ S \subseteq V\})$, where $Conv$ denotes the operation of taking the convex hull.

• the **cut cone**, $\mathcal{C}_n = \mathbb{R}^+(\mathcal{K}_n)$, defined as the real conic hull of the cuts.

• the **cut lattice**, $\mathcal{L}_n = \mathbb{Z}(\mathcal{K}_n)$, defined as the set linear combinations of cuts with integer coefficients.

• the **integer cut cone** $\mathcal{I}C_n = \mathbb{Z}^+(\mathcal{K}_n)$, defined as the set linear combinations of cuts with non-negative integer coefficients. Points in the integer cut cone are called h-points.

• the set of **quasi-h points** $\mathcal{C}_n \cap \mathcal{L}_n - \mathcal{I}C_n$.

This study focuses on the complete graph over six vertices, K_6. As a matter of fact, for $n \leqslant 5$ [De 61],

$$\mathcal{C}_n \cap \mathcal{L}_n = \mathcal{I}C_n$$

Hence, any vector admitting a $\mathbb{R}^+$-realization and a $\mathbb{Z}$-realization also admits a $\mathbb{Z}^+$-realization.

This property still holds for any non complete graph over six vertices [La 93], but collapses for K_6. Indeed, for an edge $e \in E$, the point d^e defined as:

$$d^e_f = 2 \text{ for } f \neq e \text{ and } d^e_e = 4$$

is in the cut lattice

$$d^{ij} = \sum_{v \in V} \delta(v) \quad + 2\ (\delta(i) + \delta(j) - \delta(i,j))$$

and in the cut cone

$$d^{ij} = 1/2 \sum_{k \in V - \{i,j\}} \delta(i,k) + \delta(j,k)$$

but not in the integer cut cone.

The first result of this report is to prove that all points that are both in the cut cone and the cut lattice can be represented as linear non-negative integer combinations of the cuts and the d^e. This set $(\mathcal{K}_6 \cup \{d^e,\ e \in E\})$ is minimal for this property.

The second result is to characterize the quasi-h points. It is shown that they are exactly of the form $d^e + n\delta(v)$ for $v \in V$ not incident to e and for $n \in \mathbb{Z}^+$.

This report is organized as follows: Section 2 presents the cut cone and the cut lattice, Section 3 introduces the fundamental notions and theorems on Hilbert bases, Section 4 contains both results of this report, Section 5 explores the mapping of these results on the related boolean quadric cone, Section 6 presents other possible directions of research for finding Hilbert bases for larger n. Finally, Section 7 contains the detailled proof of Theorem 1.

2 The cut cone and the cut lattice

2.1 The cut lattice

Membership to the cut lattice is characterized by a very simple condition [As 82]:

$$x \in \mathcal{L}_n \Leftrightarrow \forall i,j,k \in \{1,\ldots,n\}, x_{ij} + x_{jk} + x_{ik} \equiv 0 \ mod\ 2$$

Proof.
All cuts clearly verify this condition, so it is a necessary condition for belonging to the the cut lattice.
Conversely, let x be an integer vector verifying this condition. We can part the set of vertices in two: $V_n = S \cup T$ such that x_{ij} is odd if and only if $i \in S$ and $j \in T$. We set $x' = x + \delta(S)$. All coordinates of x' are even, so the following decomposition is a $\mathbb{Z}$-realization:

$$x' = \sum_{1 \leqslant i < j \leqslant n} \frac{x'_{ij}}{2} (\delta(i) + \delta(j) - \delta(ij))$$

Hence $x' \in \mathcal{L}_n$ and $x \in \mathcal{L}_n$. □

One of the consequences of this characterization is that $\mathcal{L}_n \subset 2\mathbb{Z}^{\frac{n(n-1)}{2}}$.

2.2 The cut cone

The cut cone C_n is a convex polyhedron of $\mathbb{R}^N$, with $N = \frac{n(n-1)}{2}$ generated by $2^{n-1} - 1$ vectors (there are as many cuts in K_n as subsets of $\{1, \ldots, n-1\}$ and the empty cut $\delta(\emptyset) = (0, \ldots, 0)$ is the apex of the cone). It was shown in [De 73] that the cuts are conically independent, hence, the cuts form exactly a minimal set of generators (each cut defining an extreme ray of the cone). These cuts are spread over several layers by l_1-norm :

$\|\delta(S)\|_1 = k(n-k)$ where $k = card(S)$.

The cone is invariant under all linear applications induced by a permutation on the vertices (if $\sigma \in \mathcal{O}_n$, one can define $u_\sigma : \mathbb{R}^N \mapsto \mathbb{R}^N$ by $u_\sigma(x) = y$ with $y_{ij} = x_{\sigma(i)\sigma(j)}$). Moreover, it is thought that these are the only transformations under which the cone is invariant.

For the cut polytope (which is a finite section of the cut cone), the group of transformations under which it is invariant is composed of the permutations on the vertices u_σ and of some reflections called *switching* [DGL91]. It turns out that $\mathcal{C}_n$ is the support cone of $\mathcal{P}_n$ at each vertex.

A vector plays a central role for the cut cone: the vector $\mathbf{1}_N = (1, 1, \ldots, 1) \in RM^N$. Indeed, $\mathbb{R}\mathbf{1}_N$ is exactly the vector space that is stable under all permutations u_σ on vertices. It can thus be considered as the symmetry-axis of the cone.

No general description of the facets of $\mathcal{C}_n$ is known. They are all known for $n \leqslant 7$ ([De 61] for $n \leqslant 5$, [AM 89] for $n = 6$ and [Gr 90] for $n = 7$), but from $n = 8$ on, only classes of facets are known. Some facets of the cut cone can be described as hypermetric facets :

Definition 2.1 *[De 61] for $b \in \mathbb{Z}^n$ such that $\sum_{i=1}^n b_i = 1$, one can define the **hypermetric inequality** $Hyp(b_1, \ldots, b_n)$ as such:*

$$Hyp(b)(x) := Q(b_1, \ldots, b_n)^T x = \sum_{1 \leqslant i < j \leqslant n} b_i b_j x_{ij} \leqslant 0$$

By abuse of language, the facet of the cone induced by the equality $Hyp(b)(x) = 0$ is called a hypermetric facet (and is also noted $Hyp(b)$) .

Remark, that $Hyp(1, 1, -1)(x) \leqslant 0$ corresponds to the well-known triangle inequality. For the inequalities $Hyp(b_1, \ldots, b_n)$, one has $\sum |b_i| = 2k + 1$, they are denoted by the value of k: for $k = 1, 2$ or 3, the induced inequalities (resp. faces or facets) are called triangular, pentagonal or heptagonal.

Hypermetric facets do entirely define the cut cone for $n \leqslant 6$. From $n = 7$ on, other facets must be taken into account for the definition of the cone.

The cone $\mathcal{C}_6$ is defined by the following facets $Hyp(b_1, \ldots, b_6)$:
• 60 triangular facets $T_{i,jk}(x) = 0$ for $1 \leqslant j < k \leqslant 6$
corresponding to $b_i = -1, \; b_j = b_k = 1$ and $b_l = 0$ otherwise
• 60 pentagonal facets $P_{ij,klm}(x) = 0$ for $1 \leqslant i < j \leqslant 6, \; 1 \leqslant k < l < m \leqslant 6$
corresponding to $b_i = b_j = -1, \; b_k = b_l = b_m = 1$ and $b_n = 0$ otherwise
• 30 heptagonal-I facets $H_{i,j}(x) = 0$ for $i \neq j$
corresponding to $b_i = -2, \; b_j = -1$ and $b_k = 1$ otherwise
• 60 heptagonal-II facets $H'_{i,jk}(x) = 0$ for $1 \leqslant j < k \leqslant 6$
corresponding to $b_i = 2, \; b_j = b_k = 1$ and $b_l = -1$ otherwise

For 6 vertices, the cut cone admits 210 facets (partitioned into 4 orbits) and for 7 vertices, it has 38 780 facets (partitioned into 36 orbits) [DD 95]. Hence for $n = 7$, it becomes very hard to solve linear programs because of the number of facets. For $n \geqslant 8$, it is impossible since the description of the cone is no longer known in terms of facets.

3 Hilbert bases

Hilbert bases were introduced in [GP 79] after the work of Hilbert (in relation with the *Nullenstellensatz*) and Gordan. A clear description of these notions can be found, for example, in [Sc 86].

3.1 Definition and fundamental theorems

Let C be a polyhedral cone, integral over a lattice L (C is a generated as a cone, by a *finite* number of vectors of L, $x_1, \ldots, x_m$). The intersection $S = C \cap L$ forms a semi-group (the sum of two vectors in S is also in S) of which we want to find the generators (a set of vectors $y_1, \ldots, y_k$ is said to generate a semi-group S if any element of S can be decomposed as a linear combination of the y_i with coefficents in $\mathbb{Z}^+$). Such a generating set is called a **Hilbert generating set**. In the case where this set is minimal (inclusion-wise), it is called a **Hilbert basis**. There are two short but fundamental theorems concerning this theory, one of existence and the other of unicity.

Gordan lemma *[Go 1873]*
For an integral polyhedral cone, the following set is a ***finite*** *Hilbert generating set :*

$$L \;\cap\; \{\sum_{i=1}^{m} \lambda_i x_i, \; 0 \leqslant \lambda_i \leqslant 1\}$$

Proof.

First, note that only the projection of the cone on the vector space spanned by the lattice L matters, so we can suppose that L is full-dimensional. Moreover, one can suppose, up to a linear application, that $L = \mathbb{Z}^n$. The set of integer points in $\{\sum_{i=1}^m \lambda_i x_i,\ 0 \leqslant \lambda_i \leqslant 1\}$ is finite.
Moreover, let $y \in C\ \cap\ \mathbb{Z}^n$, y can be decomposed as $y = \sum_{i=1}^k \lambda_i x_i$,
so $y = \sum_{i=1}^m \lfloor \lambda_i \rfloor x_i\ +\ z$ with $z = \sum_{i=1}^m (\lambda_i - \lfloor \lambda_i \rfloor) x_i$ where z and the x_i are in the exhibited set. So it forms a Hilbert generating set. □

Composition of the Hilbert basis *[Sc 86]*
Moreover, for a "pointed" cone (containing no vector space other than 0), there exists a unique minimal (for inclusion) Hilbert generating set. It is called the ***Hilbert basis****. It is exactly composed of the points:*

$$\mathcal{H} = \{x \in C \cap L,\ \textit{such that } x = y + z \textit{ with } y, z \in C \cap L \Rightarrow y = 0 \textit{ or } z = 0\}$$

Proof.

First, it is clear that these points are in any Hilbert generating set. So, by Gordan lemma, they are in finite number. It remains to prove that they form a Hilbert generating set.
Since the cone is pointed, there exists a vector h such that for any $x \in C$, $x.h > 0$, Moreover, from Gordan lemma, there exists an $\varepsilon > 0$ such that $\forall x \in C,\ x.h > \varepsilon$ (this scalar product with h will be used as a measure on the cone).
For $x \in C\ \cap\ L$, either $x \in \mathcal{H}$, or $x = y + z$, with $y, z \in C\ \cap\ L$, in which case, the decomposition is performed once again with y and z. Since $y.h, z.h < x.h - \varepsilon$, the process ends in a finite number of decompositions, after which we have a decomposition of x as sum of points of $\mathcal{H}$. So $\mathcal{H}$ is a Hilbert generating set. □

The definition of a Hilbert basis is related to a given cone and lattice, but it can also be seen as an intrinsic property of the set of points. As a matter of fact, a set of points $\mathcal{H}$ is a Hilbert generating set for the cone and the lattice that it generates if and only if

$$\mathbb{Z}(\mathcal{H}) \cap \mathbb{R}^+(\mathcal{H}) = \mathbb{Z}^+(\mathcal{H})$$

(the right-hand side is always included into the left-hand side of the equation). So, being a Hilbert generating set can be seen as a completeness property.

3.2 An algorithm for computing Hilbert bases

This section addresses the problem of computing the Hilbert basis of a cone $\mathcal{C}$ of dimension n, generated by integer vectors $x_1, \ldots, x_m$. The Gordan lemma provides us with a Hilbert genrating set. Finding the Hilbert basis is then just a matter of shrinking this set untill it is minimal. The issue at stake is that of finding the primitive elements in a semi-group. Indeed, the integer cone is an additive semi-group and the points x of the Hilbert basis are primitive: their sole decomposition as the sum of two others points of the integer cone is $x = x + 0$. Another instance of this problem is to find all prime integers up to n : The set S of integers that are products of integers in $[1, n]$ is a multiplicative semi-group and the prime integers p in $[1, n]$ are the primitive elements of S : those which sole decomposition as a product of two elements of S is $p = p \times 1$.

The definition of primitive element is negative (*"elements such that there exist no two other elements such that ..."*), so it takes some computations to test whether an element is primitive or not. Algorithmically, the best solution for testing primitiveness over a large set of points is not to test individually but to proceed by flooding, as in Eratosthen's method for prime integers. Starting from a large set of points, one progressively removes those that can be obtained as sum (resp. product for the case of integers) of two others. The final set generates the same sub semi-group as the initial set, but it is minimal for that property.

Unfortunately, the initial set to examine is very large. By Gordan lemma, points of Hilbert bases are located in the parallelepiped

$$\mathcal{P} = \left\{ \sum_{k=1}^{m} \lambda_k x_k \text{ for } 0 \leqslant \lambda_k \leqslant 1 \right\}$$

When the number of generators of the cone is in large excess compared to the dimension of the space, the volume of $\mathcal{P}$ increases dramatically. Moreover, this domain being defined by real parameters, it is impossible to enumerate its integer points (this is precisely what knowing the Hilbert basis allows to do). A solution is to enumerate integer points with positive coordinates of a bound norm and to test for each one of them whether they are in te cone or not. So, $\mathcal{P}$ is approximated by the intersection of a larger cone $((\mathbb{R}^+)^n)$ and a sphere. The sphere is a sphere of radius $M(\mathcal{C}) = Max(\|x\|,\ x \in \mathcal{P})$. l_1 and l_∞-norms are convenient for enumeration. The number of points of $\mathbb{R}^n$ with integer coordinates and of l_1-norm k is $P(k) = C^n_{n+k-1}$. So the total number of points to be looked at is $Tot(\mathcal{C}) = \sum_{k=1}^{M(\mathcal{C})} P(k)$.

For the case of the cut cone, $M(\mathcal{C}_n) = n(n-1)2^{n-3}$ $(M(\mathcal{C}_4) = 24,\ M(\mathcal{C}_5) = 80$ and $M(\mathcal{C}_6) = 240)$. This leads to

$$Tot(\mathcal{C}_{n+1}) \geqslant \sum_{k=0}^{n^2 2^{n-2}} C^{n^2/2}_{n^2/2+k}.$$

This algorithmic approach is unreasonable for the cut cone from $n = 5$ on.

A few remarks help dimishing this number :

- By Caratheodory's theorem, every point of the cone is $\mathbb{R}^+$-combination of at most n generators. This limits the initial set to

$$\left\{\sum_{k=1}^{m} \lambda_k x_k,\ 0 \leqslant \lambda_k \leqslant 1,\ \text{with at most } n\ \lambda' s > 0\right\}$$

- The initial set can be taken up to the symmetries of the cone. In the case of the cut cone, one can, for example, only consider points with decreasing coordinates $12, \ldots, 1n$.

4 Hilbert bases and cuts

The purpose of this research is to pursue the investigations of Deza, Grishukhin [DG 94] and Laurent [La 93] concerning Hilbert bases and cuts. Let $\mathcal{H}$ be the class of graphs whose cuts form a Hilbert basis. Determining $\mathcal{H}$ is a hard open question.

On the other hand, the dual problem, which consists in characterizing the graphs whose family of circuits is a Hilbert basis is completly solved. The family of circuits of a graph G is a Hilbert basis if and only if G is not contractible to the Petersen graph P_{10} [AGZ90]. This problem is actually easier since the circuit cone is much simpler than the cut cone (its facets are all known: they are defined by the cycle inequalities).

For the cut cone, we know that:

- for graphs K_n with $n \leqslant 5$, the cuts form a Hibert basis [De 61].

- for strict subgraphs of K_6, the cuts form a Hilbert basis [La 93].

- for graphs that are not contractible to K_5, the cuts form a Hilbert basis [FG 94].

- if the cuts of G form a Hilbert basis, then G is not contractible to K_6 [La 93].

So, $\mathcal{K}_6$ is not a Hilbert basis, indeed the d^e are known for long to be quasi-h points. A conjecture, formulated in [DGL91], that the d^e complete the family of cuts into the Hilbert basis of $\mathcal{C}_6$ is proved below in theorem 1.

4.1 The Hilbert base of $\mathcal{C}_6$

Theorem 1 *For K_6, the set $\mathcal{K}_6 \cup \mathcal{E}$ with $\mathcal{E} = \{d^e,\ e \in E\}$ composed of the 31 non-zero cuts and of the 15 d^e forms a Hilbert basis.*

A sketch of the proof follows; the complete proof, a little technical, is given in section 7, at the end of this report.

Outline of the proof

Let $\mathcal{B}$ be the Hilbert basis for the cone $\mathcal{C}_6$ and the lattice $\mathcal{L}_6$, we need to show that $\mathcal{B} = \mathcal{K}_6 \cup \mathcal{E}$.

It can be easily checked that $\mathcal{E} \subseteq \mathcal{B}$ (d^e cannot be decomposed as a sum of two points belonging both to the cut cone and the cut lattice). So, it remains to prove that $\mathcal{B} - \mathcal{K}_6 \subseteq \mathcal{E}$. Let x be a "missing" point (a point in $\mathcal{B} - \mathcal{K}_6$) and let us show that $x \in \mathcal{E}$. We start by locating x "near" some facets with the following proposition:

Proposition 4.0 *For any vertex $i \in \{1, \ldots, 6\}$,*

- *either x is on a triangle facet of vertex $T_{i,jk}(x) = 0$ for some $j, k \in V$,*
- *or x is at distance 2 or 4 of a heptagonal-I facet $H_{ij}(x) \in \{-2, -4\}$ for some $j \neq i$.*

Let us note that the points d^e that we want to obtain are in a special situation: they are at distance 4 of two heptagonal-I facets based on the same edge and they are on four triangle facets all based on that same edge. For example, with d^{56}, one has $T_{1,56}(d^{56}) = T_{2,56}(d^{56}) = T_{3,56}(d^{56}) = T_{4,56}(d^{56}) = 0$
and $H_{56}(d^{56}) = H_{65}(d^{56}) = -4$.

We part the vertices in two groups $\mathcal{T}$ and $\mathcal{H}$ depending in which case of the above proposition they are. (For d^{56}, $\mathcal{T} = \{1, 2, 3, 4\}$ *and* $\mathcal{H} = \{5, 6\}$). The rest of the proof consists in narrowing the possibilities for $\mathcal{T}$ and $\mathcal{H}$ in order to find that x is a d^e.

- Technical lemma forbidding x to fullfill at the same time several triangle equalities -in certain configurations- entail that at most 4 vertices are in $\mathcal{T}$. Hence, at least 2 vertices are in $\mathcal{H}$.
- By enumerating all possible configurations for those two vertices in $\mathcal{H}$, we check that the only possible one is that on a common edge ($H_{ij}(x), H_{ji}(x) \in \{-2, -4\}$).
- Up to permutation, we can suppose $H_{56}(x), H_{65}(x) \in \{-2, -4\}$. This gives $T_{1,56}(x) = T_{2,56}(x) = T_{3,56}(x) = T_{4,56}(x) = 0$.

- This leads to a finite (and relatively small) number of possibilities for x. We check by enumeration on a computer that the only quasi-h point of this form is d^{56}.

Hence, the d^e manage to complete $\mathcal{K}_6$ into a Hilbert basis.

The technical (and hard) part of the proof consist in forbidding x to be on several triangular facets -in certain configurations- at the same time. The proofs all follow the same scheme: a proof ad absurdum is performed, the coordinates of x are described in terms of integer parameters (over small domains) and we check by enumeration on a computer that none of these points is quasi-h.

4.2 Some remarks about the d^e

Historically, the d^e have been the first examples of quasi-h points noticed : they were discovered in [BG 73] and expressed in terms of quasi-h points in [AD 80]. First, we need a definition:

Definition 4.2 *Let $x \in \mathbb{R}^N$ be the metric on K_n, for $t > 0$, its **t-antipodal extension** $ant_t(x)$ is a metric on K_{n+1} defined by $ant_t(x)_{n,n+1} = t$, $ant_t(x)_{i,n+1} = t - x_{i,n}$ for $1 \leqslant i \leqslant n-1$ and $ant_t(x)_{i,j} = x_{i,j}$ for $1 \leqslant i < j \leqslant n+1$*

A number of remarks about these points d^e need to be made to illustrate the particularity of their position which could be relevant for points of Hilbert bases for larger n.

d^e is twice the metric on $K_6 - e$ which can be obtained in many ways:

- It is the closest integer point to the point $\mathbf{1}_n$ which is on the axis of symmetry
- it can be constructed as the 2-antipodal extension of $\mathbf{1}_6 - \psi_e$ where ψ_e is the characteristic vector of the edge e.
- $K_6 - e$ is the minimal non-complete subgraph in the cocktail-party graph $K_{5\times 2}$
- $K_6 - e$ is the ridge graph of the prism with base K_4 (Recall that the ridge graph of a polytope is the skeleton of the dual polytope)

4.3 Quasi-h points and scale

From now on, we only consider cut cones over K_n. The quasi-h points are defined as those that are both in the cut cone and the cut lattice, but not in the integer cut cone. The second result of this study is to give the explicit composition of the set of quasi-h points for $n = 6$.

Theorem 2 *The quasi-h points for K_6 are exactly the $d^e + n\delta(v)$ for v not incident to e and $n \in \mathbb{Z}^+$.*

Let us note that these quasi-h points belong to $60 = 15 \times 4$ affine rays and that they form an infinite set. For $n \leqslant 5$ there are no quasi-h points (since the cuts form a Hilbert basis), and that for $n \geqslant 7$ there are infinitely many quasi-h points ([DG 94]). This result fills the gap in between. Hence, when the graph increases from K_5 or $K_6 - e$ to K_6, the number of quasi-h points rises up from 0 to infinitely many, so, for cuts, no situation with a finite (strictly positive) number of quasi-h points is known.

Proof

First, let us prove that the $d^e + n\delta(v)$ are indeed quasi-h points for v not adjacent to e and $n \in \mathbb{Z}^+$. Let us set $d^n = d_{12} + n\ \delta(3)$ and suppose that there exists n such that $d^n \in \mathcal{IC}$. Let n_0 be the smallest such n.

$$d^{n_0} = \sum_{S \subset V_n} \alpha_S \delta(S) \text{ with } \alpha_S \in \mathbb{Z}^+ \qquad (1)$$

Moreover, all d^n are on the face $\mathcal{F}$ defined by $T_{4,12}(x) = T_{5,12}(x) = T_{6,12}(x) = 0$.
Let $Supp(\alpha) = \{S \subset V_n,\ \alpha_S \neq 0)$. We have $\forall S \in Supp(\alpha),\ \delta(S) \in \mathcal{F}$.
$\mathcal{F}$ contains 17 cuts: $\delta(1)$, $\delta(2)$, $\delta(3)$, $\delta(1,i)$, $\delta(2,i)$ for $3 \leqslant i \leqslant 6$ and $\delta(1,i,j)$ for $3 \leqslant i < j \leqslant 6$.
All these cuts but $\delta(3)$ separate vertices 1 and 2. But, $\{3\} \notin Supp(\alpha)$ since n_0 is minimal. So, by projecting (1) on edge 1-2, we get

$$4 = \sum_{S \in Supp(\alpha)} \alpha_S \qquad (1')$$

But for all subsets S of vertices we have the bound $\|\delta(S)\|_1 \leqslant 9$ on the l_1 norm of the cuts, so

$$\|d^{n_0}\|_1 \leqslant \| \sum_S |\alpha_S| \|\delta(S)\|_1 \leqslant 9 \sum_S |\alpha_S| = 36.$$

But $\|d^n\|_1 = \|d_{12}\|_1 + n\|\delta(3)\|_1 = 32 + 5n$, so $\|d^{n_0}\|_1 \geqslant 37$.
So, there exists no $d^n \in \mathcal{IC}$, thus all d^n are quasi-h. $\square$

Let us now prove that these points are the only quasi-h points of $\mathcal{K}_6$
We start by showing that all other "perturbations" of d_{12} do have a $\mathbb{Z}^+$-realization.

(1) $d_{12} + \delta(1) = \delta(2) + \delta(1,3) + \delta(1,4) + \delta(1,5) + \delta(1,6)$
(2) $d_{12} + \delta(1,2) = 2\delta(1) + 2\delta(2) + \delta(3) + \delta(4) + \delta(5) + \delta(6)$

(3) $d_{12} + \delta(1,3) = \delta(2) + \delta(1,3) + \delta(3,4,5) + \delta(3,4,6) + \delta(4,5,6)$
(4) $d_{12} + \delta(3,4) = \delta(1) + \delta(3) + \delta(4) + \delta(2,5) + \delta(2,6) + \delta(2,3,4)$
(5) $d_{12} + \delta(1,2,3) = \delta(1) + \delta(2) + \delta(4) + \delta(5) + \delta(6) + \delta(1,3) + \delta(2,3)$

(6) $d_{12} + \delta(1,3,4) = \delta(1,3) + \delta(1,4) + \delta(2,5) + \delta(2,6) + \delta(1,5,6)$
(7) $d_{12} + d_{23} = \delta(1) + \delta(2,3) + \delta(2,4) + \delta(2,5) + \delta(3,6) + \delta(1,2,6) + \delta(1,3,4) + \delta(1,3,5)$
(8) $d_{12} + d_{34} = \delta(1) + \delta(2,3) + \delta(2,4) + \delta(3,5) + \delta(4,6) + \delta(1,3,4) + \delta(1,3,6) + \delta(1,4,5)$

(9) $d_{12} + \delta(3) + \delta(4) = \delta(1,3) + \delta(2,4) + \delta(3,4) + \delta(1,4,5) + \delta(1,4,6)$

Let x be a quasi-h point, x can be written

$$x = \sum_{S \subset V_n} \alpha_S \delta(S) + \sum_{e \in E} \beta_e d_e \qquad \text{with } \alpha_S,\ \beta_e \in \mathbb{Z}^+$$

Since $\forall e \in E,\ 2d_e \in \mathcal{IC}$, one can rewrite

$$x = \sum_{S \subset V_n} \widehat{\alpha_S} \delta(S) + \sum_{e \in E} \widehat{\beta_e} d_e \qquad \text{with } \widehat{\alpha_S} \in \mathbb{Z}^+ \text{ and } \widehat{\beta_e} \in \{0,1\}$$

Because of decompositions (7) and (8) (and their permutations), one can rewrite

$$x = \sum_{S \subset V_n} \widetilde{\alpha_S} \delta(S) + \widetilde{\beta_{e_0}} d_{e_0} \qquad \text{with } \widetilde{\alpha_S} \in \mathbb{Z}^+ \text{ and } \widetilde{\beta_{e_0}} \in \{0,1\}$$

Since $x \notin \mathcal{IC}$, $\widetilde{\beta_{e_0}} = 1$. Set $e_0 = v_1 v_2$.
Decompositions (2), (3), (4), (5) and (6) imply

$$\forall S \subset V_n,\ card(S) \in \{2,3\} \Rightarrow \widetilde{\alpha_S} = 0.$$

Decomposition (1) implies that $\tilde{\alpha}_{\{v_1\}} = \tilde{\alpha}_{\{v_2\}} = 0$.
Decomposition (9) implies that $card(\{i \in V_n - \{v_1, v_2\},\ \tilde{\alpha}_{\{i\}} > 0\}) \leqslant 1$.
Finally, we get the following decomposition :

$$x = d_{e_0} + n\delta(v) \quad \text{for } v \text{ not adjacent to } e_0$$

So all quasi-h points are necessarily of this form. □

The quasi-h points are not in the integer cone, so they admit only $\mathbb{R}^+$-realizations. In fact, the coefficients can always be taken rational rather than real, and their denominators can be bounded. This fact is grasped by the notion of scale.

Definition 4.3 *Let x be a quasi-h point, its scale $\eta(x)$ is defined as the smallest positive integer k such that $kx \in \mathcal{IC}$.*
(x admits a $\mathbb{Q}^+$-realization, so such a k always exists)

Theorem 3 *[DG 94]*
There exits an integer k such that for all quasi-h points, $kx \in \mathcal{IC}_n$.

This is a surprising theorem since there is infinitely many quasi-h points for $n \geqslant 6$, and for each of them x, the scale $\eta(x)$ divides k.
The smallest such k is called the **scale of the cone** and is denoted η. The scale brings the following estimation for the integer cone:

$$\eta\, \mathcal{C} \cap \mathcal{L} \subset \mathcal{IC} \subset \mathcal{C} \cap \mathcal{L}$$

Proof. The proof is like that of the Gordan lemma. Let $x_1, \ldots, x_n$ be a set of generators of the cone. Let k be the smallest common multiple of all $\eta(x)$ for $x \in \{\sum \lambda_i x_i,\ 0 \leqslant \lambda_i \leqslant 1\} \cap \mathcal{L}$.

For all $y \in \mathcal{C} \cap \mathcal{L}$, let us show that $ky \in \mathcal{IC}$. y can be decomposed as

$$y = \sum \mu_i x_i = \sum \lfloor \mu_i \rfloor x_i \quad + z \text{ with } z = \sum (\mu_i - \lfloor \mu_i \rfloor) x_i$$

hence, $kz \in \mathcal{IC}$, so $ky \in \mathcal{IC}$. $\square$

Corollary 1 *For $\mathcal{C}_6$, we have $\eta = 2$.*

Proof. Let us note that $2d^e \in \mathcal{IC}$:

$$2d^{ij} = \sum \delta(i,k) + \delta(j,k)$$

So, for all quasi-h points y (recall that $y = d^e + n\delta(v)$), $2y \in \mathcal{IC}$. So $\eta \leqslant 2$. But, $\eta > 1$ since there are quasi-h points, hence $\eta = 2$ $\square$

Hence, we have three notions for describing the integer cone:

	quasi-h points	Hibert basis	scale
$n \leqslant 5$	$\emptyset$	$\mathcal{K}_n$	1
$n = 6$	$\{d^e + n\delta(v)\}$	$\mathcal{K}_6 \cup \{d^e,\ e \in E\}$	2
$n \geqslant 7$	some infinite set	$\mathcal{K}_n \cup$ some finite set	some finite integer

On this figure, it appears clearly that $\mathcal{C}_6$ is a treshold case between the simple situation of $n \leqslant 5$ and the unknown cases of $n \geqslant 7$.

The quasi-h points represent the finest notion to describe the integer cone. However, they do not seem to be the most adapted to complex examples. For instance, in our case ($n = 6$), the Hilbert basis is relatively simple (it is composed of the generators of the cone plus one vector and its permutations) and the set of quasi-h points is anyhow infinite. The Hilbert basis is a more tractable information since it is a finite set.

The scale of the cone, although it does not give very precise information, allows to *enumerate* the points of the integer one (at least a superset of it), by taking all linear combinations of the generators with coefficients in $1/\eta\ \mathbb{Z}^+$. It is thus a precious information.

5 Boolean quadric programming

The cut polytope can be seen also as an object attached to boolean theory, rather than graph theory. Indeed,

$$\mathcal{P}_n = Conv((|x_i - x_j|)_{1 \leqslant i < j \leqslant n},\ x \in \{0,1\}^n)$$

Under this formulation, another object is naturally related to the cut polytope, the **boolean quadric polytope** :

$$\mathcal{B}QP_n = Conv((x_i x_j)_{1 \leqslant i \leqslant j \leqslant n},\ x \in \{0,1\}^n)$$

Both these polytopes arise in linear programming. One can also define the boolean quadric cone as the conic hull of the boolean quadric polytope. These boolean quadric objects also have their graph-theory description. Indeed, the boolean quadric cone BQC_n can be seen as the conic hull of the intersection vectors on K_n.

Definition 5.4 *Let S be a subset of V_n. The following $n \times n$ matrix $\pi(S)$ defined by*

$$\pi(S)_{ij} = \begin{cases} 1 \ for\ i,j \in S \\ 0 \ otherwise \end{cases}$$

is called the ***intersection matrix****. As for cuts, this matrix is symmetric, so the* ***intersection vector*** *$\pi(s)$ is defined as the vector of $\{0,1\}^{\frac{n(n+1)}{2}}$ containing the entries of the intersection matrix indexed by the ordered pairs $\{(i,j),\ 1 \leqslant i \leqslant j \leqslant n\}$. Let Π_n be the set of all intersection vectors on K_n.*

It can be easily checked that $BQC_n = \mathbb{R}^+(\Pi_n)$

In fact, the intersection vectors of K_n and the cuts of K_{n+1} are isomorphic by the following applications (this dual vision called the **covariance map** was introduced in [De 73]):

$$\phi_0(x) = p \qquad with\ p_{ij} = \begin{cases} x_{0i} & for\ \ 1 \leqslant i = j \leqslant n \\ \frac{x_{0i}+x_{0j}-x_{ij}}{2} & for\ \ 1 \leqslant i < j \leqslant n \end{cases}$$

$$\psi_0(p) = x \qquad with \qquad \begin{cases} x_{0i} &=& p_{ii} & for\ \ 1 \leqslant i \leqslant n \\ x_{ij} &=& p_{ii} + p_{jj} - 2p_{ij} & for\ \ 1 \leqslant i < j \leqslant n \end{cases}$$

(This actually comes to specializing one vertex, here 0, but any other vertex i could have been chosen, bringing similar aplications ϕ_i and ψ_i).

Let S be a subset of $\{0,1,\dots,n\}$ not containing 0, and let $\delta(S)$ denote the cut of K_{n+1} and $\pi(S)$ the intersection vector of K_n, this correspondance gives:

$$\pi(S) = \phi_0(\delta(S)) \text{ and } \delta(S) = \psi_0(\pi(S))$$

and hence

$$\phi_0(\mathcal{C}_{n+1}) = BQC_n$$

This correspondance is all the more interesting for integer programming that it maps the cut lattice into the integer lattice. Indeed,

$$\phi_0(\mathcal{L}_n) = \mathbb{Z}^N$$

Hence, all results of this study on the cut cone and the cut lattice can be mapped onto the boolean quadric cone and the integer cone.

Notation. For $e \in E_n$, let π^e be

$$\begin{cases} \pi^e_{ij} = 1 \text{ pour } i \neq j \\ \pi^e_{ii} = 2 \text{ pour } i \in \{1,\dots,n\} \\ \pi^e_e = 0 \end{cases}$$

and for $v \in \{1,\dots,n\}$ let π^v be

$$\begin{cases} \pi^v_{ij} = 1 \text{ pour } i \neq j,\ i,j \neq v \\ \pi^v_{ii} = 2 \text{ pour } i \neq v \\ \pi^v_{vi} = 2 \text{ pour } i \neq v \\ \pi^v_{vv} = 4 \end{cases}$$

With these notations, the action of the covariance map on the Hilbert basis can be explicited: $\pi^e = \phi_0(d^e)$ and $\pi^v = \phi_0(d^{0v})$.

Since this mapping is a linear bijection , all properties concerning linear combinations are transported from the cut cone to the boolean quadric cone. Hence, we can directly translate our results in the boolean quadric language.

Corollary 2 *The intersection vectors form a Hilbert basis of the boolean quadric cone BQC_n if and only if $n \leqslant 4$.*

Corollary 3 *The Hilbert basis of the boolean quadric cone BQC_5 for the integer lattice is made of the intersection vectors $\pi(S)$ for $S \subseteq \{1,\dots,5\}$, of the π^e for $e \in E_n$ and of the π^v for $v \in \{1,\dots,5\}$.*

Corollary 4 *The quasi-h points of Π_5 are made of:*

- $\pi^v + n\pi(\{v'\})$ *for* $v, v' \in \{1,\ldots,5\},\ v \neq v',\ n \in \mathbb{Z}^+$
- $\pi^e + n\pi(\{1,2,3,4,5\})$ *for* $e \in E,\ n \in \mathbb{Z}^+$
- $\pi^e + n\pi(\{v\}$ *for* $v \in \{1,\ldots,5\},\ e \in E,$ *e not adjacent to* v, $n \in \mathbb{Z}^+$

Corollary 5 *The scale of the boolean quadric cone BQC_5 is 2.*

6 Possible directions of research

6.1 The Hilbert basis for the lattice $\mathbb{Z}^N$

Up to now, Hibert bases have only been considered for the cone $\mathcal{C}_n$ and the lattice $\mathcal{L}_n$. Another problem would be to consider for the same cone $\mathcal{C}_n$, the lattice $\mathbb{Z}^N$. In fact, these two problems are related. Let us denote by $\mathcal{L}B_n$ the Hilbert basis for $\mathcal{C}_n$ and $\mathcal{L}_n$ and by $\mathcal{I}B_n$ the Hilbert basis for $\mathcal{C}_n$ and $\mathbb{Z}^N$. The following inclusion holds :

$$\mathcal{L}B_n \ \subseteq \ 2\mathcal{I}B_n \ \cup \ \left(\mathcal{L}_n \cap \left\{ \sum_{x_i \in \mathcal{I}B_n} \varepsilon_i x_i,\ \varepsilon_i = 0,1 \right\}\right)$$

Hence, $\mathcal{L}B_n$ can be easily obtained from $\mathcal{I}B_n$. But conversely, $\mathcal{I}B_n$ cannot be easily obtained from $\mathcal{L}B_n$.

Proof.
Let us denote by G_n the set $2\mathcal{I}B_n \ \cup \ (\mathcal{L}_n \cap \{\sum_{x_i \in \mathcal{I}B_n} \varepsilon_i x_i,\ \varepsilon_i = 0,1\})$.
It is clear that the points of G_n are in the cut cone and the cut lattice. Moreover let x be a point of the cut lattice and the cut cone, x is *a fortiori* a point of $\mathbb{Z}^N$, and can hence be decomposed over the Hilbert basis $\mathcal{I}B_n$ as $x = \sum_{x_i \in \mathcal{I}B_n} n_i x_i$, with $\forall i,\ n_i \in \mathbb{Z}_+$. For all i, set $p_i = \left\lfloor \frac{n_i}{2} \right\rfloor$.
Hence, $x = \sum_{x_i \in \mathcal{I}B_n} p_i(2\ x_i) + \sum_{x_i \in \mathcal{I}B_n} \varepsilon_i x_i$.
So, G_n is a Hilbert generating set, so $\mathcal{L}B_n \subseteq G_n$. □

The bases $\mathcal{I}B_n$ seem to be composed of many more vectors than the $\mathcal{L}B_n$, but much smaller ones (since the considered lattice is thinner). Geometrically, they seem to be located closely around the point 1_N (the smallest integer point on the axis of symmetry). Moreover, for all n, $1_N \in \mathcal{I}B_n$.

Proof.
Suppose $1_N \notin \mathcal{I}B_n$. Then, $1_N = y + z$ with $y, z \in \mathcal{C}_n,\ y, z \neq 0$. Hence, the set of edges E could be partitionned into $E = F \cup \overline{F}$, such that for all edges $e \in F,\ y_e = 0$ and $z_e = 1$ and conversely, for all edges $e \in \overline{F},\ y_e = 1$ and $z_e = 0$. Up to switching y with z, one can assume $y_{12} = 1$ and $z_{12} = 0$. If $y_{13} = y_{23} = 0$, y breaks a triangle inequality and subsequently is not in the cone and if $y_{13} =$

1 and $y_{23} = 0$ or $y_{13} = 0$ and $y_{23} = 1$, z breaks a triangle inequality and subsequently is not in the cone. So, $y_{13} = y_{23} = 1$. Repeating this argument brings that for all edges e, $y_e = 1$. Hence, $z = 0$, which is contradictory. So, $1_N \in \mathcal{I}B_n$. $\square$

Below, we denote by $S_1(x,n)$ (resp. $S_\infty(x,n)$) the l_1 (resp. l_∞) sphere of center x and of radius n (in $\mathbb{Z}^N$). We have computed (with the algorithm described in section 3.2) the bases $\mathcal{I}B_n$ for $n = 3, 4$ and did most of the computations for $n = 5$.

- $\mathcal{I}B_3 = \mathcal{K}_3 \ U \ \{1_3\} \ = \ S_1(1_3, 1)$ and $card(\mathcal{I}B_3) = 4$.
- $\mathcal{I}B_4 = \mathcal{K}_4 \ U \ S_1(1_6, 1) \ \subset \ S_1(1_6, 3)$ and $card(\mathcal{I}B_4) = 22$
- If $\mathcal{I}B_5 \subset S_\infty(0,4)$ then $\mathcal{I}B_5 \subseteq \mathcal{K}_5 \ U \ S_1(1_{10}, 5) \ \subset \ S_1(1_10, 6)$ and $card(\mathcal{I}B_5) = 298$

6.2 The growth of the scale with n

Let us denote by η_n the scale of the cut cone $\mathcal{C}_n$. We have $\eta_3 = \eta_4 = \eta_5 = 1$ and $\eta_6 = 2$.

Since the cone $\mathcal{C}_n$ is a projection of the cone $\mathcal{C}_{n+1}$ on a lesser-dimensional vector space, one always has $\eta_{n+1} | \eta_n$. Hence, the function η is nondecreasing.

The only known bound on η is a linear lower bound : $\eta_n \geqslant \left\lceil \frac{n-1}{4} \right\rceil$. This bound comes from the d^e : $\eta(2\ d(K_n - e)) \geqslant \left\lceil \frac{n-1}{4} \right\rceil$. This inequality becomes an equality for $n \leqslant 6$. No upper bound for η_n is known.

Bounds on the scale would be of great interest since they would allow the enumeration of a superset of the integer cone. Actually, even if the scale gives a much weaker information than the actual composition of the Hilbert basis, it can help finding the Hilbert basis. Indeed, knowing η allows to enumerate the integer points in the parallellepiped $\mathcal{P}$, as $\frac{1}{\eta}\mathbb{Z}^+$-combinations of the cuts.

Note that the problem of integer programming can be easily transfered to other instances of semi-groups. Here, the problem can be transfered to integer and become an arithmetic issue :

given integers $x_1, \ldots, x_m$, let $S = \{\Pi_{i=1}^m x_i^{z_i}$ with for all i, $z_i \in \mathbb{Z}\}$, what is the least positive integer η such that for all $k \in S$, k^η can be decomposed as a product of positive powers of the x_i ?

The mapping is achieved as follows : the basis of $\mathbb{R}^N$ is mapped with the N first prime integers and addition in $\mathbb{R}^N$ becomes multiplication in $\mathbb{N}$.

6.3 Chvátal rank

Another concept of integer programming that could be related to Hilbert bases is the iterative procedure of Chvátal. Starting with a given polytope, it consists in finding cutting planes that do not remove any integer point in the polytope and to sharpen the shape of the polytope until it is the convex hull of its integer points. This peeling of the polytope is done in a finite number of steps which is characteristic of the starting polytope. It would be interesting to relate this number to the scale of the cone for instance.

Definition 6.5 *Let P be a polytope in $\mathbb{R}^n$ and let $P^\star = Conv(P \cap \mathbb{Z}^n)$. The sequence of polytopes (P^n) is defined recursively as follows : $P^{(0)} = P$ and $P^{(n+1)} = \{x \in P^{(n)} \mid a^t x \leqslant b$ for all $a \in \mathbb{Z}^n,\ b \in \mathbb{Z}$ such that $max_{x \in P^{(n)}}(a^t x) < b + 1\}$*

(P^n) is a sequence of decreasing polytopes. The fundamental result of Chvátal and Gomory is that there exists an integer r such that for all integers $k \geqslant r,\ P^{(k)} = P^\star$. The smallest such r is called Chvátal's rank of P.

For the case of cuts, it is interesting to consider the metric polytope $MetP_n$ (defined by the sole triangle inequalities), for which $MetP_n^\star$ is the cut polytope $CutP_n$. Let r_n be Chvátal's rank of $MetP_n$. For the same reason as for scale, r_n is an increasing function of n. It is known that $r_3 = r_4 = 0$ and that $r_5 = 1$. The rank is unknown from $n = 6$ on. However, a linear lower bound due to Chvátal, Cook and Hartman (quoted in [PT 94]) states that $r_n \geqslant \frac{1}{4}(n-4)$.

It is intriguing that the lower bound is asymptotically the same for Chvátal rank of the metric polytope and the scale of the cut cone.

Finally, it seems that the general paradigm has not been found yet : the notions of scale, Chvátal rank and Hilbert bases give information relevant to the integral structure of the considered polyhedron but lack an unifying frame.

7 Proof of theorem 1.

7.1 The structure of the proof

This section is devoted to the overall structure of the proof that $\mathcal{K}_6 \cup \{d^e,\ e \in E\}$ is the Hilbert basis for $\mathcal{C}_6$ and $\mathcal{L}_6$. As sketched above on page 10, the proof is performed as follows. Take x a point of the Hilbert basis that is not a cut, we prove succesively that :

- **Step 1.** For all vertices i,
  - (i) either $T_{i,jk}(x) = 0$ for $j, k \neq i$.
  - (ii) or $H_{ij}(x) \in \{-2, -4\}$ for $j \neq i$.

  Let us denote by $\mathcal{T}$ the set of vertices i verifying condition (i) and by $\mathcal{H}$ the set of those verifying condition (ii). $V_6 = \mathcal{T} \cup \mathcal{H}$.

- **Step 2.** $card(\mathcal{T}) \leqslant 4$. Hence $card(\mathcal{H}) \geqslant 2$ and up to permutation, we can take $5, 6 \in \mathcal{H}$.
- **Step 3.** $H_{56}(x), H_{65}(x) \in \{-2, -4\}$.
- **Step 4.** $T_{1,56}(x) = T_{2,56}(x) = T_{3,56}(x) = T_{4,56}(x) = 0$.
- **Step 5.** Finally, $x = d^{56}$.

Proof of step 1.

x is in the Hilbert basis without being a cut, so it is a quasi-h point. In particular, for all $i \in \{1, \ldots, 6\}$, $x - \delta(i) \notin \mathcal{C}_6$ (otherwise, x could be decomposed as $x = \delta(i) + y$ with $\delta(i), y \in \mathcal{C}_6 \cap \mathcal{L}_6 - \{0\}$).
$x - \delta(i)$ being out of the cone, there exists a hypermetric facet $Hyp(b)$ such that $Hyp(b)(x - \delta(i)) = Hyp(b)(x) - Hyp(b)(\delta(i)) > 0$.

Note that one always has $Hyp(b)(\delta(i)) = b_i(1 - b_i)$.

Lemma 1 *For all points $x \in \mathcal{L}_6$ and all hypermetric facets $Hyp(b)$, the following holds :*

$$Hyp(b)(x) \in 2\mathbb{Z}$$

Proof. Note that for $S \subset V$, $Hyp(b)(\delta(S)) = \sum_{i \in S} b_i(1 - \sum_{i \in S} b_i)$. Hence, since $\sum_{i \in S} b_i$ et $1 - \sum_{i \in S} b_i$ have opposite parities, $Hyp(b)(\delta(S)) \in 2\mathbb{Z}$. So for each point x of the lattice $\mathcal{L}_6$, $Hyp(b)(x) \in 2\mathbb{Z}$ $\square$

With both these remarks, the following cases can be distinguished for the inequality $Hyp(b)$ such that $Hyp(b)(x) > Hyp(b)(\delta(i))$

$$\begin{array}{lll}
if\ Hyp(b) = T_{i,jk}\ then\ Hyp(b)(\delta(i)) = -2 & \Rightarrow & Hyp(b)(x) \geqslant 0,\\
if\ Hyp(b) = T_{j,kl}\ with\ j \neq i\ then\ Hyp(b)(\delta(i)) = 0 & \Rightarrow & Hyp(b)(x) \geqslant 2,\\
if\ Hyp(b) = P_{kl,mnp}\ then\ Hyp(b)(\delta(i)) \in \{0,-2\} & \Rightarrow & Hyp(b)(x) \geqslant 0,\\
if\ Hyp(b) = H'_{k,lm}\ then\ Hyp(b)(\delta(i)) \in \{0,-2\} & \Rightarrow & Hyp(b)(x) \geqslant 0,\\
if\ Hyp(b) = H_{kl}\ with\ k \neq i\ then\ Hyp(b)(\delta(i)) \in \{0,-2\} & \Rightarrow & Hyp(b)(x) \geqslant 0,\\
if\ Hyp(b) = H_{ij}\ then\ Hyp(b)(\delta(i)) = -6 & \Rightarrow & Hyp(b)(x) \geqslant -4.
\end{array}$$

Lemma 2 *Heptagonal facets I and II contain no quasi-h points.*

This lemma comes from the fact that these facets are simplices (so the decomposition of a point as a linear combination of cuts of this face is unique) and from an argument on linear dependencies in the lattice $\mathcal{L}_6$ [DL 94].

Lemma 3 *Pentagonal facets contain no quasi-h points.*

With lemma 3 (which will be proved in section 7.2) and the fact that for all hypermetric facet, $Hyp(b)(x) \leqslant 0$ (since $x \in \mathcal{C}_6$), the only possibilities left for the facet separating $x - \delta(i)$ from the cone are

- either $T_{i,jk}(x) = 0$ for vertices $j, k \neq i$,
- or $H_{ij}(x) \in \{-2, -4\}$ for $j \neq i$. □

Proof of step 2

The proof fo this step requires a few lemmas which will be proved in section 7.2.

Lemma 4 $\forall e \in E,\ x_e \geqslant 1.$

Lemma T 1 *There exist no distinct i, j, k such that $T_{i,jk}(x) = T_{j,ik}(x) = 0$.*

Lemma T 2 *There exist no distinct i, j, k, l, m such that $T_{i,jk}(x) = T_{j,lm}(x) = 0$.*

Lemma T 3 *There exist no distinct i, j, k, l, n such that $T_{i,kl}(x) = T_{j,ln}(x) = 0$.*

Lemma T 4 *There exist no distinct i, j, l, m, n such that $T_{i,jk}(x) = T_{l,mn}(x) = 0$.*

Suppose $card(\mathcal{T}) \geqslant 5$ and consider two cases :

- **case 1.** There exists $i, j, k \in \mathcal{T}$, such that $T_{i,jk}(x) = 0$.
  Up to permutation, we may assume that $T_{1,23}(x) = 0$ and $4, 5 \in \mathcal{T}$.
  So there exists $e, f, g \in E$ such that $T_{2,e}(x) = T_{3,f}(x) = T_{4,g}(x) = 0$.
  By Lemma T2, $g \notin \{15, 16\}$.
  By Lemma T3, $g \notin \{35, 36, 25, 26\}$.
  By Lemma T4, $g \neq 56$.
  Hence, $g \in \{12, 13, 23\}$.

  - **case 1.1.** $g = 13$ (the case $g = 12$ is symmetrical).
    $T_{1,23}(x) = T_{4,13}(x) = 0$.
    By Lemma T1, $e \neq 13$ and $f \notin \{12, 14\}$.
    By Lemma T2, $e \notin \{45, 46, 56\}$ and $f \notin \{25, 26, 45, 46, 56\}$.
    By Lemma T3, $e \notin \{15, 16, 35, 36\}$.
    Hence, $e \in \{14, 34\}$ and $f \in \{15, 16, 24\}$.

    * **case 1.1.1.** $e = 14$. By Lemma T3, $f \notin \{15, 16\}$, so $f = 24$.
      Thus $T_{1,23}(x) = T_{2,14}(x) = T_{3,24}(x) = T_{4,13}(x) = 0$.
      Adding these four equalities gives $x_{24} = 2(x_{34} + x_{21}) + x_{24}$.
      Hence $x_{34} = x_{21} = 0$, which is forbidden by Lemma 4.
    * **case 1.1.2.** $e = 34$. By Lemma T2, $f \notin \{15, 16\}$, so $f = 24$.
      Thus $T_{2,34}(x) = T_{3,24}(x) = 0$ which is forbidden by Lemma T1.

  - **case 1.2.** $g = 23$. $T_{1,23}(x) = T_{4,23}(x) = 0$.
    Vertices 4 and 5 play similar roles, so by case 1.1, $T_{5,23}(x) = 0$.
    By Lemma T1, $e \notin \{12, 34, 35\}$ and $f \notin \{12, 24, 25\}$.
    By Lemma T2, $e, f \notin \{14, 15, 16, 45, 46, 56\}$.
    So, $e = 36$ and $f = 26$, which leads to
    $T_{2,36}(x) = T_{3,26}(x) = 0$, which is forbidden by Lemma T1.

- **case 2.** There exists no triple $i, j, k \in \mathcal{T}$, such that $T_{i,jk}(x) = 0$. So, atleast one vertex is not in $\mathcal{T}$. Up to permutation, we may assume that $1 \notin \mathcal{T}$, and that $T_{2,13}(x) = 0$.
  Vertices 3,4,5 and 6 being in $\mathcal{T}$, there exists $e, f, g, h \in \{12, 13, 14, 15, 16\}$, such that $T_{4,e}(x) = T_{5,f}(x) = T_{6,g}(x) = T_{3,h}(x) = 0$.
  By Lemma T3, $e, f, g \in \{12, 13\}$.

  - If $e = f = g = 13$ then, Lemma T1 forbids all possibilities for h.
  - Hence, one among e, f, g is 12, say $e = 12$.
    By Lemma T2, $h \notin \{15, 16\}$ and by Lemma T1, $h \neq 12$. So $h = 14$.
    Thus, $T_{2,13}(x) = T_{3,14}(x) = T_{4,12}(x) = 0$.
    Adding these three equalities gives $x_{13} = x_{23} + x_{24} + x_{34} + x_{13}$.
    Hence $x_{23} = x_{24} = x_{34} = 0$, which is forbidden by Lemma 4. $\square$

Proof of step 3

There exist two heptagonal I facets H_{ij} and H_{kl} such that $H_{ij}(x), H_{kl}(x) \in \{-2,-4\}$. The three following lemmas will be proved in the section 7.2.

Lemma H 1 *there exist no distinct i, j, k such that $H_{ik}(x), H_{jk}(x) \in \{-2,-4\}$.*

Lemma H 2 *there exist no distinct i, j, k, l such that $H_{ij}(x), H_{kl}(x) \in \{-2,-4\}$.*

Lemma H 3 *there exist no distinct i, j, k such that $H_{ij}(x), H_{jk}(x) \in \{-2,-4\}$.*

With these three lemmas, the only possibility left for these two facets is to have $H_{ij}(x), H_{ji}(x) \in \{-2,-4\}$ for some pair of vertices (i, j). Up to permutation, we can assume $H_{56}(x), H_{65}(x) \in \{-2,-4\}$. $\square$

Proof of step 4

The purpose of this step is to locate x on its minimal face. We have $H_{56}(x) \geqslant -4$ and $H_{65}(x) \geqslant -4$. The addition of both inequalities gives :

$$4x_{56} - 3\sum_{i=1}^{4}(x_{5i} + x_{6i}) + 2\sum_{1 \leqslant i,j \leqslant 4} x_{ij} \geqslant -8$$

which can be rewritten as $P_{56,\ 123}(x) + P_{56,\ 124}(x) + P_{56,\ 134}(x) + P_{56,\ 234}(x) \geqslant -8$.

By Lemma 3, x is on no pentagonal facet.
So, $P_{56,\ 123}(x) = P_{56,\ 124}(x) = P_{56,\ 134}(x) = P_{56,\ 234}(x) = -2$ and $H_{5,6}(x) = H_{6,5}(x) = -4$.

Let us examine vertex 1 : By Lemmas H1, H2 and H3, $\forall i,\ H_{1,i}(x), H_{i,1}(x) < 4$. So, $1 \in \mathcal{T}$ (and similarly $2, 3$ and 4 are in $\mathcal{T}$). Hence, $T_{1,e}(x) = T_{2,f}(x) = 0$ for some edges e and f.
Up to permutation, we may assume that $e \in \{23, 25, 56\}$.
Suppose $T_{1,56}(x) < 0$. Then, $e \in \{23, 25\}$.

- **case 1:** $e = 23$.
  $-2 = P_{56,123}(x) + T_{1,23}(x) = T_{1,56}(x) + T_{5,23}(x) + T_{6,23}(x)$.
  So $T_{1,56}(x) = -2$ and $T_{5,23}(x) = T_{6,23}(x) = 0$.
  By Lemma T1, $f \notin \{13, 35, 36\}$.
  By Lemma T2, $f \notin \{14, 15, 16, 45, 46, 56\}$.
  Thus, $f = 34$, and consequently $T_{2,34}(x) = 0$. A similar argument for vertex 3 gives $T_{3,24}(x) = 0$, which is contradictory with Lemma T1.

- **case 2:** $e = 25$.

  $-2 = P_{56,123}(x) + T_{1,25}(x) = T_{6,23}(x) + T_{5,13}(x) + T_{1,56}(x)$.
  So $T_{1,56}(x) = -2$ and $T_{5,13}(x) = T_{6,23}(x) = 0$, which is forbidden by Lemma T3.

Hence, $T_{1,56}(x) = 0$ and similarly, $T_{2,56}(x) = T_{3,56}(x) = T_{4,56}(x) = 0$. □

Proof of step 5

This last step of the proof is devoted to finding the coordinates of x from its minimal face.
Substracting to a pentagonal inequality $P_{56,123}, P_{56,124}, P_{56,134}$ or $P_{56,234}$ three of the triangle inequalities $T_{1,56},\ T_{2,56},\ T_{3,56}$ and $T_{4,56}$, gives

$$\begin{cases} -2x_{56} + x_{12} + x_{13} + x_{23} = -2, \\ -2x_{56} + x_{13} + x_{14} + x_{34} = -2, \\ -2x_{56} + x_{23} + x_{24} + x_{34} = -2, \\ -2x_{56} + x_{12} + x_{14} + x_{24} = -2. \end{cases} \quad \text{which implies,} \quad \begin{cases} a := x_{12} = x_{34}, \\ b := x_{13} = x_{24}, \\ c := x_{14} = x_{23}, \\ d := 1 + \frac{a+b+c}{2} = x_{56}. \end{cases}$$

$H_{56}(x) = -4$ implies that $\sum_{i=1}^{4} x_{5i} + 2\sum_{i=1}^{4} x_{6i} = 6d$ and
$H_{65}(x) = -4$ implies that $\sum_{i=1}^{4} x_{6i} + 2\sum_{i=1}^{4} x_{5i} = 6d$.
Hence, $\sum_{i=1}^{4} x_{5i} = \sum_{i=1}^{4} x_{6i} = 2d$, d'où $x_{15} + x_{35} = x_{26} + x_{46}$.

The minimal face of x contains the 16 following cuts :

$$x = \sum_{S \in \mathcal{F}} \alpha_S \delta(S)$$

with $\mathcal{F} = \{5, 6, 15, 16, 25, 26, 35, 36, 45, 46, 125, 126, 135, 136, 145, 146\}$

Set $\beta_1 := \alpha_{15} + \alpha_{16},\ \beta_2 := \alpha_{25} + \alpha_{26},\ \beta_3 := \alpha_{35} + \alpha_{36},\ \beta_4 := \alpha_{45} + \alpha_{46}$,
$\gamma_{12} := \alpha_{125} + \alpha_{126},\ \gamma_{13} := \alpha_{135} + \alpha_{136},\ \gamma_{14} := \alpha_{145} + \alpha_{146}$.

Expressing the coordinates of x in terms of the β and the γ gives

$$\begin{aligned} a &= \beta_1 + \beta_2 + \gamma_{13} + \gamma_{14} = \beta_3 + \beta_4 + \gamma_{13} + \gamma_{14}, \\ b &= \beta_1 + \beta_3 + \gamma_{12} + \gamma_{14} = \beta_2 + \beta_4 + \gamma_{12} + \gamma_{14}, \\ c &= \beta_1 + \beta_4 + \gamma_{12} + \gamma_{13} = \beta_2 + \beta_3 + \gamma_{12} + \gamma_{13}. \end{aligned}$$

So, $$\begin{cases} \beta := \beta_1 = \beta_2 = \beta_3 = \beta_4, \\ \gamma_{12} = d - \beta - a - 1, \\ \gamma_{13} = d - \beta - b - 1, \\ \gamma_{14} = d - \beta - c - 1. \end{cases}$$

Moreover, since $d = \sum \alpha_S$, $\qquad \alpha_5 + \alpha_6 + \beta = 1$.

The equality $x_{15} + x_{35} = x_{26} + x_{46}$ implies that $2\alpha_5 + \alpha_{15} + \alpha_{25} + \alpha_{35} + \alpha_{45} = \beta + 1$.

Finally, the evenness condition from section 2.1. ($x \in \mathcal{L}_6$) on triangles (1,2,5), (1,3,5) and (1,4,5) gives :

$$\varepsilon_2 = \alpha_5 + \alpha_{35} + \alpha_{45} - \alpha_{125} - \beta \in \mathbb{Z},$$
$$\varepsilon_3 = \alpha_5 + \alpha_{25} + \alpha_{45} - \alpha_{135} - \beta \in \mathbb{Z},$$
$$\varepsilon_4 = \alpha_5 + \alpha_{25} + \alpha_{35} - \alpha_{145} - \beta \in \mathbb{Z}.$$

x can be represented in terms of these integer parameters.

$$\begin{array}{rcl} x & = & \varepsilon_2\ (\delta(126) - \delta(125)) \\ & + & \varepsilon_3\ (\delta(136) - \delta(135)) \\ & + & \varepsilon_4\ (\delta(146) - \delta(145)) \\ & + & a/2\ (\delta(136) + \delta(146) - \delta(126)) \\ & + & b/2\ (\delta(126) + \delta(146) - \delta(136)) \\ & + & c/2\ (\delta(126) + \delta(136) - \delta(146)) \\ & + & \delta(6) + \delta(15) - \delta(16) \end{array}$$

This decomposition leaves only a few points to examine, since for all i, $\varepsilon_i \in \{-1, 0, 1\}$ and $a, b, c \in \{2, .., 9\}$ with $a + b + c \leqslant 16$. We checked by computer that d^{56} was the only point of the cone of this form that was quasi-h. $\square$

7.2 Proof of technical results.

Proof of Lemma 3

Let x be a point of cone and the lattice, on the facet $Hyp(1,1,1,-1,-1,0)$. This facet is spanned by 19 cuts : $\delta(S)$ for $S \in \mathcal{F}$ with :
$G = \{2,3,24,25,34,35,234,235,2345\}$ and $\mathcal{F} = G \cup \{S \cup \{6\}\ for\ S \in G\} \cup \{6\}$
Thus, $x = \sum_{S\in\mathcal{F}} \lambda_S \delta(S)$. The dimension of this facet is 14, hence the variables $\lambda_{346}, \lambda_{356}, \lambda_{2346}, \lambda_{2356}$ and λ_{23456} can be taken as free variables. For given coordinates of x, all other 14 parameters can be expressed by means of these five variables :

$$B \begin{pmatrix} \lambda_{346} \\ \lambda_{356} \\ \lambda_{2346} \\ \lambda_{2356} \\ \lambda_{23456} \end{pmatrix} = \begin{pmatrix} -\lambda_{246} \\ -\lambda_{256} \\ \lambda_6 \\ -\lambda_2 \\ -\lambda_{36} \end{pmatrix} + N \qquad (\star)$$

where $N = \begin{pmatrix} C^1_{46} \\ C^1_{56} \\ C^1_{23} - C^6_{23} \\ -P_{12,456}(x) \\ C^1_{36} \end{pmatrix}$ (with the notation $C^k_{ij} = -\frac{1}{2} T_{k,ij}(x)$)

and $B = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$

Since x is in the lattice, the right hand vector N is integral. Moreover, the following equations link these coefficients by pairs. Here again, all right hand side terms are integral.
$\lambda_{24} + \lambda_{246} = C^4_{13}$,
$\lambda_{25} + \lambda_{256} = C^5_{13}$,
$\lambda_{34} + \lambda_{346} = C^4_{12}$,
$\lambda_{35} + \lambda_{356} = C^5_{12}$,
$\lambda_{234} + \lambda_{2346} = C^5_{23}$,
$\lambda_{235} + \lambda_{2356} = C^4_{23}$,
$\lambda_{2345} + \lambda_{23456} = C^1_{45}$,
$\lambda_2 + \lambda_{26} = C^2_{45}$,
$\lambda_3 + \lambda_{36} = C^3_{45}$.

These equations determine uniquely the 14 other coefficients from the 5 free variables. Moreover, one can note that if these 5 free variables take integral values, then all λ_S are integers.

The matrix B is totally unimodullar. It is indeed one of the two 5×5 matrices found by Bixby that are totally unimodullar without being a network matrix (It is proved in [Se 80] that any totally unimodullar matrix can always be decomposed as a combination of network matrices and these two special matrices found by Bixby).

Since B is totally unimodullar, so is the matrix $\begin{pmatrix} I \\ -I \\ B \\ -B \end{pmatrix}$. Hence, for all quadruple of integer vectors (a,b,c,d), the polythedron defined by $\begin{cases} a \leqslant x \leqslant b \\ c \leqslant Bx \leqslant d \end{cases}$ is integral (its vertices are integral). Here, let us consider the polytope :

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \leqslant \begin{pmatrix} \lambda_{346} \\ \lambda_{356} \\ \lambda_{2346} \\ \lambda_{2356} \\ \lambda_{23456} \end{pmatrix} \leqslant \begin{pmatrix} C_{12}^4 \\ C_{12}^5 \\ C_{23}^5 \\ C_{23}^4 \\ C_{45}^1 \end{pmatrix}$$

$$and \quad \begin{pmatrix} C_{46}^1 - C_{13}^4 \\ C_{56}^1 - C_{13}^5 \\ C_{23}^1 - C_{23}^6 \\ -P_{12,456}(x) - C_{45}^2 \\ C_{36}^1 - C_{45}^3 \end{pmatrix} \leqslant B \begin{pmatrix} \lambda_{346} \\ \lambda_{356} \\ \lambda_{2346} \\ \lambda_{2356} \\ \lambda_{23456} \end{pmatrix} \leqslant \begin{pmatrix} C_{46}^1 \\ C_{56}^1 \\ M \\ -P_{12,456}(x) \\ C_{36}^1 \end{pmatrix}$$

The first set of lower bounds compels $\lambda_{346}, \lambda_{356}, \lambda_{2346}, \lambda_{2356}$ and λ_{23456} to be positive. Via the equations by pairs, the first set of upper bounds compels $\lambda_{34}, \lambda_{35}, \lambda_{234}, \lambda_{235}$ and λ_{2345} to be positive. Via the equation $(\star)$, the second set of upper bounds compels $\lambda_{246}, \lambda_{256}, \lambda_2$ and λ_{36} to be positive. Finally, via the equation $(\star)$ and the equations by pairs, the second set of lower bounds compels $\lambda_{24}, \lambda_{25}, \lambda_6, \lambda_{23}$ and λ_3 to be positive.
Hence, each point of this polyhedron corresponds exactly to a $\mathbb{R}_+$-realization of x. So, for M large enough, this polyhedron is not empty. If M is a large enough integer, this polyhedron is a non empty integer polyhedron, so it contains a fortiori an integer point. So there exists a $\mathbb{R}_+$-realization of x such that the 5 free coefficients are integer. This corresponds to a $\mathbb{Z}_+$-realization of x. So x is not quasi-h. $\square$

Proof a Lemma 4

Suppose that there exists e such that $x_e = 0$. Up to permutation, suppose that $e = 56$. Then, the triangular inequalities $T_{5,i6}(x)$ and $T_{6,i5}(x)$ respectively imply

that $x_{i5} \leqslant x_{i6}$ and $x_{i6} \leqslant x_{i5}$. Thus, $\forall i \in \{1,2,3,4\},\ x_{i5} = x_{i6}$.
Let $\tilde{x}$ be the projection of x on K_5 (the graph induced by vertices 1 to 5). The set of cuts $\mathcal{K}_5$ forms a Hilbert basis, so $\tilde{x}$ can be decomposed as

$$\tilde{x} = \sum_{S \subseteq V_5} n_S \delta(S)$$

Set $y = \sum_{T \subseteq V_6} n'_T \delta(T)$ with $n'_{S \cup \{6\}} = n_S$ and $n'_S = 0$ for $6 \notin S$. Thus $x = y$ and x is not quasi-h. □

Proof of Lemma T1

$T_{i,jk}(x) = 0$ implies $x_{jk} \leqslant x_{ik} + x_{ij}$. $T_{j,ik}(x) = 0$ implies $x_{ik} \leqslant x_{ij} + x_{jk}$.
Thus $x_{jk} \leqslant 2x_{ij} + x_{jk}$, so $x_{ij} = 0$, which is forbidden by Lemma 4. □

Another technical lemma is necessary to the proof of Lemma T2.

Lemma T 5 *There exist no distinct i, j, k, l and m such that $T_{i,jk}(x) = T_{i,lm}(x) = 0$.*

Proof of Lemma T5

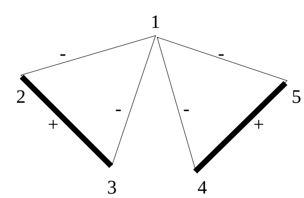

Figure 1: The situation of Lemma T5

Suppose that $T_{1,23}(x) = T_{1,45}(x) = 0$. Then, x is on a face defined by 17 cuts, i.e.

$$x = \sum_{S \in \mathcal{F}} \alpha_S \delta(S)$$

with $\mathcal{F} = \{2,3,4,5,6,24,25,34,35,26,36,46,56,124,125,134,135\}$

Moreover,
$T_{1,24}(x) \in 2\mathbb{Z} \Rightarrow \varepsilon_{24} := \alpha_{24} + \alpha_{135} \in \mathbb{Z}$,
$T_{1,34}(x) \in 2\mathbb{Z} \Rightarrow \varepsilon_{34} := \alpha_{34} + \alpha_{125} \in \mathbb{Z}$,
$T_{1,25}(x) \in 2\mathbb{Z} \Rightarrow \varepsilon_{25} := \alpha_{25} + \alpha_{134} \in \mathbb{Z}$,
$T_{1,35}(x) \in 2\mathbb{Z} \Rightarrow \varepsilon_{35} := \alpha_{35} + \alpha_{124} \in \mathbb{Z}$,
$T_{2,45}(x) \in 2\mathbb{Z} \Rightarrow \varepsilon_{2} := \alpha_{2} + \alpha_{26} \in \mathbb{Z}$,

$T_{3,45}(x) \in 2\mathbb{Z} \Rightarrow \varepsilon_3 := \alpha_3 + \alpha_{36} \in \mathbb{Z},$
$T_{4,23}(x) \in 2\mathbb{Z} \Rightarrow \varepsilon_4 := \alpha_4 + \alpha_{46} \in \mathbb{Z},$
$T_{5,23}(x) \in 2\mathbb{Z} \Rightarrow \varepsilon_5 := \alpha_5 + \alpha_{56} \in \mathbb{Z},$
$T_{2,16}(x) \in 2\mathbb{Z} \Rightarrow \lambda_2 := \alpha_2 + \alpha_{24} + \alpha_{25} \in \mathbb{Z},$
$T_{3,16}(x) \in 2\mathbb{Z} \Rightarrow \lambda_3 := \alpha_3 + \alpha_{34} + \alpha_{35} \in \mathbb{Z},$
$T_{4,16}(x) \in 2\mathbb{Z} \Rightarrow \lambda_4 := \alpha_4 + \alpha_{24} + \alpha_{34} \in \mathbb{Z},$
$T_{5,16}(x) \in 2\mathbb{Z} \Rightarrow \lambda_5 := \alpha_5 + \alpha_{25} + \alpha_{35} \in \mathbb{Z},$
$T_{6,45}(x) \in 2\mathbb{Z} \Rightarrow \gamma := \alpha_6 + \alpha_{26} + \alpha_{36} \in \mathbb{Z}.$

These integer parameters entirely define x :
$x_{12} = \varepsilon_2 + \varepsilon_{24} + \varepsilon_{25},$
$x_{13} = \varepsilon_3 + \varepsilon_{34} + \varepsilon_{35},$
$x_{14} = \varepsilon_4 + \varepsilon_{24} + \varepsilon_{34},$
$x_{15} = \varepsilon_5 + \varepsilon_{25} + \varepsilon_{35},$
$x_{16} = \gamma + \varepsilon_4 + \varepsilon_5 - \lambda_4 - \lambda_5 + \varepsilon_{24} + \varepsilon_{25} + \varepsilon_{34} + \varepsilon_{35},$
$x_{23} = \varepsilon_2 + \varepsilon_3 + \varepsilon_{24} + \varepsilon_{25} + \varepsilon_{34} + \varepsilon_{35},$
$x_{24} = \varepsilon_2 + \varepsilon_4 + \varepsilon_{25} + \varepsilon_{34},$
$x_{25} = \varepsilon_2 + \varepsilon_5 + \varepsilon_{24} + \varepsilon_{35},$
$x_{26} = \gamma + \varepsilon_4 + \varepsilon_5 - \varepsilon_2 - \lambda_4 - \lambda_5 + 2\lambda_2 + \varepsilon_{34} + \varepsilon_{35},$
$x_{34} = \varepsilon_3 + \varepsilon_4 + \varepsilon_{35} + \varepsilon_{24},$
$x_{35} = \varepsilon_3 + \varepsilon_5 + \varepsilon_{34} + \varepsilon_{25},$
$x_{36} = \gamma + \varepsilon_4 + \varepsilon_5 - \varepsilon_3 - \lambda_4 - \lambda_5 + 2\lambda_3 + \varepsilon_{24} + \varepsilon_{25},$
$x_{45} = \varepsilon_4 + \varepsilon_5 + \varepsilon_{24} + \varepsilon_{25} + \varepsilon_{34} + \varepsilon_{35},$
$x_{46} = \gamma + \lambda_4 - \lambda_5 + \varepsilon_5 + \varepsilon_{25} + \varepsilon_{35},$
$x_{56} = \gamma + \lambda_5 - \lambda_4 + \varepsilon_4 + \varepsilon_{24} + \varepsilon_{34}.$

This leads to $2^{11}3^5$points, and we checked by enumeration on a computer that none of them was quasi-h. □

Proof of Lemma T2

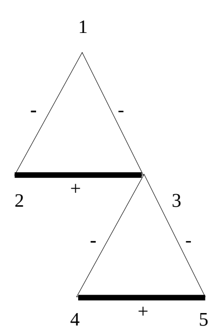

Figure 2: The situation of Lemma T2

Suppose that $T_{1,23}(x) = T_{2,45}(x) = 0$. Then, x is on a face defined by 17 cuts, i.e.

$$x = \sum_{S \in \mathcal{F}} \alpha_S \delta(S)$$

with $\mathcal{F} = \{3, 4, 5, 6, 13, 24, 25, 34, 35, 36, 46, 56, 124, 125, 134, 135, 136\}$
The evenness condition (x is in the cut lattice) on triangles $T_{3,14}(x), T_{3,24}(x)$ and $T_{2,15}(x)$ brings :

$$\begin{cases} \alpha_3 + \alpha_{35} + \alpha_{36} + \alpha_{124} \in \mathbb{Z} \\ \alpha_3 + \alpha_{24} + \alpha_{35} + \alpha_{36} + \alpha_{124} + \alpha_{135} + \alpha_{136} \in \mathbb{Z} \\ \alpha_{24} + \alpha_{135} \in \mathbb{Z} \end{cases}$$

So, $\alpha_{163} \in \mathbb{Z}$, and subsequently $\alpha_{136} = 0$.
Hence, $T_{1,45}(x) = -2(\alpha_{13} + \alpha_{136}) = 0$, which is forbidden by Lemma T5. $\square$

Proof of Lemma T3

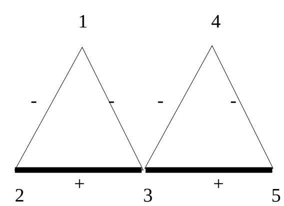

Figure 3: The situation of Lemma T3

Suppose that $T_{1,23}(x) = T_{5,34}(x) = 0$. Then x is on a face defined by 17 cuts, i.e.

$$x = \sum_{S \in \mathcal{F}} \alpha_S \delta(S)$$

with $\mathcal{F} = \{2, 3, 4, 6, 12, 13, 24, 26, 35, 36, 45, 46, 123, 124, 135, 245, 345\}$

Moreover,
$T_{1,25}(x) \in 2\mathbb{Z} \Rightarrow \varepsilon_{13} := \alpha_{13} + \alpha_{245} \in \mathbb{Z}$,
$T_{5,14}(x) \in 2\mathbb{Z} \Rightarrow \varepsilon_{35} := \alpha_{35} + \alpha_{124} \in \mathbb{Z}$,
$T_{1,34}(x) \in 2\mathbb{Z} \Rightarrow \varepsilon_{12} := \alpha_{12} + \alpha_{345} \in \mathbb{Z}$,
$T_{5,23}(x) \in 2\mathbb{Z} \Rightarrow \varepsilon_{45} := \alpha_{45} + \alpha_{123} \in \mathbb{Z}$,
$T_{2,14}(x) \in 2\mathbb{Z} \Rightarrow \varepsilon_{2} := \alpha_{2} + \alpha_{26} \in \mathbb{Z}$,
$T_{3,15}(x) \in 2\mathbb{Z} \Rightarrow \varepsilon_{3} := \alpha_{3} + \alpha_{36} \in \mathbb{Z}$,
$T_{4,25}(x) \in 2\mathbb{Z} \Rightarrow \varepsilon_{4} := \alpha_{4} + \alpha_{46} \in \mathbb{Z}$,
$T_{6,34}(x) \in 2\mathbb{Z} \Rightarrow \mu_{2} := \alpha_{6} + \alpha_{26} + \alpha_{345} \in \mathbb{Z}$,
$T_{6,23}(x) \in 2\mathbb{Z} \Rightarrow \mu_{4} := \alpha_{6} + \alpha_{46} + \alpha_{123} \in \mathbb{Z}$,
$T_{5,36}(x) \in 2\mathbb{Z} \Rightarrow \gamma_{45} := \alpha_{45} + \alpha_{36} + \alpha_{245} \in \mathbb{Z}$,

$T_{5,46}(x) \in 2\mathbb{Z} \Rightarrow \gamma_{35} := \alpha_{35} + \alpha_{46} + \alpha_{135} \in \mathbb{Z}$,
$T_{1,26}(x) \in 2\mathbb{Z} \Rightarrow \gamma_{13} := \alpha_{13} + \alpha_{26} + \alpha_{135} \in \mathbb{Z}$,
$T_{1,36}(x) \in 2\mathbb{Z} \Rightarrow \gamma_{12} := \alpha_{12} + \alpha_{36} + \alpha_{124} \in \mathbb{Z}$,
$P_{15,234}(x) = -2(\alpha_{24} + \alpha_{135}) \in 2\mathbb{Z}_{-}$. So $\alpha_{24} + \alpha_{135} = 1$.

These parameters entirely define x :
$x_{12} = \varepsilon_2 + \varepsilon_{13} + 1$,
$x_{13} = \varepsilon_3 + \varepsilon_{35} + \varepsilon_{12}$,
$x_{14} = \varepsilon_4 + \varepsilon_{12} + \varepsilon_{13} + \varepsilon_{45} + 1$,
$x_{15} = \varepsilon_{12} + \varepsilon_{13} + \varepsilon_{35} + \varepsilon_{45}$,
$x_{16} = \gamma_{12} + \gamma_{13} + \mu_4$,
$x_{23} = \varepsilon_2 + \varepsilon_3 + \varepsilon_{12} + \varepsilon_{13} + \varepsilon_{35} + 1$,
$x_{24} = \varepsilon_2 + \varepsilon_4 + \varepsilon_{12} + \varepsilon_{45}$,
$x_{25} = \varepsilon_2 + \varepsilon_{12} + \varepsilon_{35} + \varepsilon_{45} + 1$,
$x_{26} = \gamma_{12} - \gamma_{13} + \varepsilon_2 + \varepsilon_{13} + \mu_4 + 1$,
$x_{34} = \varepsilon_3 + \varepsilon_4 + \varepsilon_{13} + \varepsilon_{35} + \varepsilon_{45} + 1$,
$x_{35} = \varepsilon_3 + \varepsilon_{13} + \varepsilon_{45}$,
$x_{36} = \gamma_{13} - \gamma_{12} + \varepsilon_3 + \varepsilon_{12} + \varepsilon_{35} + \mu_4$,
$x_{45} = \varepsilon_4 + \varepsilon_{35} + 1$,
$x_{46} = \gamma_{45} - \gamma_{35} + \varepsilon_4 + \varepsilon_{35} + \mu_2 + 1$,
$x_{56} = \gamma_{35} + \gamma_{45} + \mu_2$.

This leaves $2^{13}3^2$ to examine and we checked that none of them was quasi-h. $\square$

Proof of Lemma T4

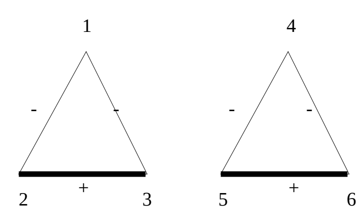

Figure 4: The situation of Lemma T4

Suppose that $T_{1,23}(x) = T_{4,56}(x) = 0$. Then x is on a face defined by 17 cuts, i.e.

$$x = \sum_{S \in \mathcal{F}} \alpha_S \delta(S)$$

with $\mathcal{F} = \{2, 3, 5, 6, 12, 13, 25, 26, 35, 36, 45, 46, 123, 125, 126, 135, 136\}$

Moreover,
$T_{1,24}(x) \in 2\mathbb{Z} \Rightarrow \alpha_{13} + \alpha_{135} + \alpha_{136} \in \mathbb{Z}$,

$T_{1,25}(x) \in 2\mathbb{Z} \Rightarrow \alpha_{13} + \alpha_{25} + \alpha_{136} \in \mathbb{Z}$,
$T_{1,26}(x) \in 2\mathbb{Z} \Rightarrow \alpha_{13} + \alpha_{26} + \alpha_{135} \in \mathbb{Z}$.
Thus $\alpha_{25} = \alpha_{125}$ and $\alpha_{26} = \alpha_1 36$. Set $\varepsilon_2 := \alpha_{13} + \alpha_{25} + \alpha_{26}$,
Similarly, $\alpha_{35} = \alpha_{125}$ and $\alpha_{36} = \alpha_{126}$. Set
$\varepsilon_3 := \alpha_{12} + \alpha_{35} + \alpha_{36} \in \mathbb{Z}$,
$\varepsilon_5 := \alpha_{46} + \alpha_{25} + \alpha_{35} \in \mathbb{Z}$,
$\varepsilon_6 := \alpha_{45} + \alpha_{26} + \alpha_{36} \in \mathbb{Z}$.

$T_{2,14}(x) \in 2\mathbb{Z} \Rightarrow \alpha_2 + \alpha_{25} + \alpha_{26} \in \mathbb{Z}$,
Thus $\alpha_2 = \alpha_{13}$ and similarly $\alpha_3 = \alpha_{12}$, $\alpha_5 = \alpha_{46}$ and $\alpha_6 = \alpha_{45}$.

$T_{1,56} \in 2\mathbb{Z} \Rightarrow \varepsilon_1 := \alpha_{12} + \alpha_{13} + \alpha_{123} \in \mathbb{Z}$.

This leads to the following decomposition :
$x_{12} = 2\varepsilon_2,\ x_{13} = 2\varepsilon_3,\ x_{45} = 2\varepsilon_5,\ x_{46} = 2\varepsilon_6$,
$x_{23} = 2(\varepsilon_2 + \varepsilon_3),\ \ x_{56} = 2(\varepsilon_5 + \varepsilon_6)$,
$x_{14} = x_{15} = x_{16} =_{24}= x_{25} = x_{26} = x_{34} = x_{35} = x_{36} = \varepsilon_1 + \varepsilon_5 + \varepsilon_6$.
Hence, the ε_i fully characterizae x.
$\forall i,\ \varepsilon_i \in \{0, 1, 2\}$ and by Lemma 4, $\varepsilon_2, \varepsilon_3, \varepsilon_5, \varepsilon_6 > 0$. This leaves $2^6 = 64$ points to examine and we checked that none of them was quasi-h. □

The proofs of Lemma H1, H2 and H3 requires a last technical lemma :

Lemma T 6 *There exist no distinct i, j, k and l such that $T_{i,jk}(x) = T_{i,kl}(x) = T_{i,jl}(x) = 0$*

Proof of Lemma T6

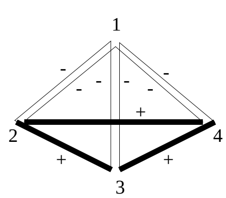

Figure 5: The situation of Lemma T6

Suppose that $T_{1,23}(x) = T_{1,24}(x) = T_{1,34}(x) = 0$. Then x is on a face defined by 15 cuts, i.e.

$$x = \sum_{S \in \mathcal{F}} \alpha_S \delta(S)$$

with $\mathcal{F} = \{2, 3, 4, 5, 6, 25, 26, 35, 36, 45, 46, 56, 256, 356, 456\}$

Moreover,
$T_{2,16}(x) \in 2\mathbb{Z}$, so $\alpha_2 + \alpha_{25} \in \mathbb{Z}$,

$T_{2,15}(x) \in 2\mathbb{Z}$, so $\alpha_2 + \alpha_{26} \in \mathbb{Z}$. Thus, $\alpha_{25} = \alpha_{26}$.
Similarly, $\alpha_{35} = \alpha_{36}$ and $\alpha_{45} = \alpha_{46}$.

$T_{2,34}(x) - T_{2,16}(x) \in 2\mathbb{Z}$ so $\alpha_{26} + \alpha_{256} \in \mathbb{Z}$. Thus, $\alpha_{256} = \alpha_2$.
Similarly, $\alpha_{356} = \alpha_3$ and $\alpha_{456} = \alpha_4$.

$T_{6,34}(x) \in \mathbb{Z}$, so $\alpha_6 + \alpha_{26} + \alpha_{56} + \alpha_{256} \in \mathbb{Z}$, so $\alpha_6 + \alpha_{56} \in \mathbb{Z}$
Similarly, $\alpha_5 + \alpha_{56} \in \mathbb{Z}$ and so, $\alpha_5 = \alpha_6$.

$T_{5,16}(x) \in 2\mathbb{Z}$ so $\varepsilon_5 + \varepsilon_{25} + \varepsilon_{35} + \varepsilon_{45} \in \mathbb{Z}$.

Set $\varepsilon_2 = \alpha_2 + \alpha_{26},\ \varepsilon_3 = \alpha_3 + \alpha_{36},\ \varepsilon_4 = \alpha_4 + \alpha_{46}$,
$\varepsilon = \alpha_5 + \alpha_{56}$ and $\Delta = \varepsilon_5 + \varepsilon_{25} + \varepsilon_{35} + \varepsilon_{45}$.

The coordinates of x can be expressed in terms of these integer parameters :
$x_{12} = 2\varepsilon_2,\ x_{13} = 2\varepsilon_3,\ x_{14} = 2\varepsilon_4$.
Since the coordinates of x are strictly positive (Lemma 4), $\varepsilon_2 = \varepsilon_3 = \varepsilon_4 = 1$.
$x_{15} = x_{16} = x_{25} = x_{26} = x_{35} = x_{36} = x_{45} = x_{46} = 3 + \varepsilon$,
$x_{23} = x_{24} = x_{34} = 4$,
$x_{56} = 2\Delta$.

There are six possibilities since $\Delta \in \{1, 2, 3\}$ and $\varepsilon \in \{0, 1\}$.

- if $\Delta = 1,\ x = \delta(25) + \delta(26) + \delta(3) + \delta(356) + \delta(4) + \delta(456) + \varepsilon\delta(56)$.
- if $\Delta = 2,\ x = \delta(25) + \delta(26) + \delta(35) + \delta(36) + \delta(4) + \delta(456) + \varepsilon\delta(56)$.
- if $\Delta = 3,\ x = \delta(25) + \delta(26) + \delta(35) + \delta(36) + \delta(45) + \delta(46) + \varepsilon\delta(56)$.

So, in all cases, x has a $\mathbb{Z}_+$ realization, and consequently, x is not quasi-h. $\square$

Proof of Lemma H1

Suppose that $H_{1,2}(x), H_{6,1}(x) \in \{-2, -4\}$. By addition,
$H_{1,2}(x) + H_{6,1}(x) = P_{16,234}(x) + T_{1,34}(x) + T_{1,45}(x) + T_{1,35}(x) + T_{6,12}(x) - 2x_{26} \geqslant -8$
By Lemma 4, $-2x_{26} \leqslant -2$ and by Lemma 3, $P_{16,234}(x) \leqslant -2$.
So, for atleast two of the triangles appearing in this inequality, the equality holds
.

- $T_{6,12}(x) = T_{1,ij}(x) = 0$, with $i, j \in \{3, 4, 5\}$ is forbidden by Lemma T2.
- $T_{1,34}(x) = T_{1,45}(x) = T_{1,35}(x) = 0$ is forbidden by Lemma T6.
- Hence, up to a permutation of vertices 3,4 and 5, $T_{1,34}(x) = T_{1,45}(x) = 0$; $T_{1,35}(x) = T_{6,12}(x) = -2$ and $x_{26} = 1$.
  Lemma T3 forbids $T_{6,25}(x) = 0$ or $T_{2,56}(x) = 0$. Thus, $T_{2,56}(x), T_{6,25}(x) \leqslant -2$.
  Hence $x_{56} \leqslant x_{25} - 1$ and $x_{25} \leqslant x_{56} - 1$ which is incompatible. $\square$

Proof of Lemma H2

Suppose that $H_{2,1}(x), H_{6,1}(x) \in \{-2,-4\}$. So

$-8 \leqslant H_{2,1}(x) + H_{6,1}(x) = P_{12,\ 345}(x) + P_{16,\ 345}(x) - 4x_{26} \leqslant -2-2-4 = -8$

Thus, $P_{12,345}(x) = P_{16,345}(x) = -2$ and $x_{26} = 1$.
This edge x_{26} constrains certain triangular inequalities. Indeed, for all $i \in \{1,3,4,5\}$, $x_{i2}+1 \geqslant x_{i6}$ and $x_{i6}+1 \geqslant x_{i2}$.
The evenness condition implies that $x_{i6} - x_{i2} = \varepsilon_i \in \{-1,+1\}$
$P_{16,\ 345}(x) = P_{12,\ 345}(x)$ impies that $\varepsilon_1 = \varepsilon_3 + \varepsilon_4 + \varepsilon_5$.
Up to permutation, we can set $\varepsilon_1 = \varepsilon_3 = \varepsilon_4 = 1$ and $\varepsilon_5 = -1$. Hence, $T_{6,12}(x) = T_{6,23}(x) = T_{6,24}(x) = T_{2,56}(x) = 0$

Let us examine vertex 1: suppose that $1 \in \mathcal{T}$. Then $T_{1,e}(x) = 0$ for an edge $e \in E$.
By Lemma T5, $e \notin \{23, 24, 25, 26, 36\}$,
By Lemma T1, $e \notin \{34, 35, 45\}$,
By Lemma T6, $e \notin \{46, 56\}$,
By Lemma T3, $e \neq 26$.
There are no possibilities left for e, so $1 \notin \mathcal{T}$.
By Lemma H1 implies $H_{1,i}(x) < -4$, $\forall i \in \{2,3,4,5,6\}$, so $1 \notin \mathcal{H}$. Hence, x is neither in $\mathcal{T}$ nor in $\mathcal{H}$, which is impossible. $\square$

Proof of Lemma H3

Supposons that $H_{1,2}(x)$, $H_{6,5}(x) \in \{-2,-4\}$. The proof will contain three steps: We will first prove that one of the two equalities $T_{6,12}(x) = 0$ and $T_{1,56}(x) = 0$ holds. Then, we will deduce that vertices 3 and 4 are in $\mathcal{T}$. Finally, with the lemmas about triangles, a contradiction will be raised.

step 1. The following decomposition holds :

$$H_{1,2}(x) + H_{6,5}(x) = T_{1,34}(x) + T_{6,34}(x) + T_{6,12}(x) + T_{1,56}(x) + 2X \geqslant -8$$

$$\text{with} \qquad \begin{aligned} X &= T_{6,12}(x) + T_{2,56}(x) + x_{26} - x_{15} \\ &= T_{5,12}(x) + T_{1,56}(x) + x_{15} - x_{26} \end{aligned}$$

Without loss of generality, we can suppose that $x_{15} \geqslant x_{26}$.
Suppose that $T_{6,12}(x) < 0$. Thus, since $H_{1,2}(x) + H_{6,5}(x) \geqslant -8$, at least two among the triangle equalities $T_{1,34}(x)$, $T_{6,34}(x)$ and $T_{1,56}(x)$ hold.
Lemma T2 forbids $T_{6,34}(x) = T_{1,56}(x) = 0$.
Lemma T5 forbids $T_{1,34}(x) = T_{1,56}(x) = 0$.
Thus, $T_{1,56}(x) = T_{6,12}(x) = -2$ and $T_{1,34}(x) = T_{6,34}(x) = T_{2,56}(x) = 0$ which is forbidden by Lemma T2.
So $T_{6,12}(x) = 0$ or (the symmetric case, when $x_{15} \leqslant x_{26}$), $T_{1,56}(x) = 0$.

Step 2. Suppose that $3 \in \mathcal{H}$.
By Lemma H1, $H_{3,2}(x),\ H_{3,5}(x) < -4$.
By Lemma H2, $H_{3,1}(x),\ H_{3,6}(x) < -4$. Thus, $H_{3,4}(x) \in \{-2,-4\}$. The argument used in step 1 on $H_{1,2}$ and $H_{5,6}$ or $H_{5,6}$ and $H_{3,4}$ (instead of $H_{1,2}$ and $H_{3,4}$) brings respectively that $T_{1,34}(x) = 0$ or $T_{3,12}(x) = 0$, $T_{1,56}(x) = 0$ or $T_{5,12}(x) = 0$ and $T_{3,56}(x) = 0$ or $T_{5,34}(x) = 0$.

Lemma T2 forbids $T_{1,56}(x) = T_{3,12}(x) = 0$.
Lemma T5 forbids $T_{1,56}(x) = T_{1,34}(x) = 0$,
Thus, $T_{1,56}(x) < 0$ and $T_{5,12}(x) = 0$
Lemma T2 forbids $T_{5,12}(x) = T_{3,56}(x) = 0$,
Lemma T5 forbids $T_{5,12}(x) = T_{5,34}(x) = 0$.
There are no possibilities left. Hence, $3 \in \mathcal{T}$, and for the same reasons, $4 \in \mathcal{T}$.

Step 3. Since 3, $4 \in \mathcal{T}$, there exist edges e and f such that $T_{3,e}(x) = T_{4,f}(x) = 0$
Suppose (as in step 1) that $x_{15} \geqslant x_{26}$ and thus, $T_{6,12}(x) = 0$.
By Lemma T2, $T_{1,34}(x) < 0$ and Lemma T4 implies that $T_{5,34}(x) < 0$.
By Lemma T2, $e \notin \{46, 56\}$.
By Lemma T3, $e \notin \{14, 15, 24, 25\}$.
By Lemma T4, $e \neq 45$. Thus, $e \in \{12, 16, 26\}$. Similarly, $f \in \{12, 16, 26\}$.
Moreover, Lemma T3 forbids $e \neq f$. Hence, only three cases remain:

- if $e = f = 12$. By Lemma T2, $T_{1,56}(x), T_{2,56}(x) < 0$.
  Thus, $H_{1,2}(x) + H_{5,6}(x) \leqslant -2 - 2 + 0 - 2 + 2(0 - 2 + 0) = -10$ which is impossible.

- if $e = f = 16$. By Lemma T3, $T_{2,56}(x) < 0$. if $T_{1,56}(x) < 0$, the same argument as above would lead to a contradiction, so $T_{1,56}(x) = 0$.
  Let us now consider vertices 2 and 5 and let us prove that either 2 or 5 belongs to $\mathcal{T}$. Suppose that $2, 5 \in \mathcal{H}$.
  By Lemma H1, $H_{5,1}(x), H_{5,2}(x), H_{5,3}(x), H_{5,4}(x) < -4$.
  Thus, $H_{5,6}(x) \in \{-2,-4\}$. Similarly, $H_{2,1}(x) \in \{-2,-4\}$.
  The Step 1 implies that $T_{2,56}(x) = 0$ or $T_{5,12}(x) = 0$.

  But $T_{2,56}(x) < 0$ and Lemma T3 forbids $T_{5,12}(x) = 0$ (since $T_{3,16}(x) = 0$).
  So either 2 or 5 belongs to$\mathcal{T}$. Thus $T_{2,g}(x) = 0$ for an edge $g \in E$ or $T_{5,h}(x) = 0$ for an edge $h \in E$.
  Lemma T3 forbids $g \in \{13, 14, 15, 36, 46, 56\}$ and $h \notin \{24, 34, 36, 46\}$.
  Lemma T2 forbids $g \in \{34, 35, 45\}$ and $h \notin \{12, 13, 14, 23, 26\}$.
  Lemma T1 forbids that $g = 16$ and $h = 16$.
  There are no possiblities left neither for g nor for h.

- If $e = f = 26$. By Lemma T3, $T_{1,56}(x) < 0$.
  If $T_{2,56}(x) < 0$, then $H_{1,2}(x) + H_{5,6}(x) \leqslant -2 - 2 + 0 - 2 + 2(0 - 2 + 0) = -10$ which is impossible. Thus, $T_{2,56}(x) = 0$.

Consider vertex 5. Suppose $5 \in \mathcal{H}$.
By Lemma H1, $H_{5,1}(x), H_{5,2}(x), H_{5,3}(x), H_{5,4}(x) < -4$. Thus, $H_{5,6}(x) \in \{-2, -4\}$. Step 1 implies that either $T_{1,56}(x) = 0$ or $T_{5,12}(x) = 0$. We already know that $T_{1,56}(x) < 0$, and Lemma T3 forbids $T_{5,12}(x) = 0$ (since $T_{3,26}(x) = 0$). Thus, $5 \in \mathcal{T}$. Hence, $T_{5,g}(x) = 0$ for some edge $g \in E$.
By Lemma T3, $g \notin \{12, 13, 14, 16, 23, 24\}$.
By Lemma T2, $g \notin \{34, 36, 46\}$.
By Lemma T1, $g \neq 26$.
There are no possibilities left for g. $\square$

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