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Abstract

We consider in this report Datalog programs with arithmetical constraints of the following form:

$$\begin{array}{lcll} p(x_0, y_0, z_0). & & & \\ p(x+e, y+\lambda, z+\gamma) & \leftarrow & x \geq 0, & p(x,y,z). \\ p(x+\mu, y+f, z+\delta) & \leftarrow & y \geq 0, & p(x,y,z). \\ p(x+\alpha, y+\beta, z+g) & \leftarrow & z \geq 0, & p(x,y,z). \end{array}$$

where $x_0, y_0, z_0, e, \lambda, \gamma, \mu, f, \delta, \alpha, \beta, g$ denote integer constants.
The problem is to find an arithmetic formula $f(x,y,z)$ equivalent to the relation $p(x,y,z)$ defined by the above program.
This characterization problem has useful applications in several fields, like the generation of lemmas for proving the termination of Prolog programs, the compilation of queries in Temporal Deductive Databases, or the verification of safety properties in parametric concurrent systems.
We show here that programs of the above form are divided into three classes: the hierarchic, periodic and spiralling classes. More than 99% of the programs fall into the hierarchic and periodic classes and can be characterized by a linear arithmetic formula, unlike programs of the spiralling class.

Résumé

On étudie dans ce rapport des programmes Datalog avec contraintes arithmétiques de la forme:

$$\begin{array}{lcll} p(x_0, y_0, z_0). & & & \\ p(x+e, y+\lambda, z+\gamma) & \leftarrow & x \geq 0, & p(x,y,z). \\ p(x+\mu, y+f, z+\delta) & \leftarrow & y \geq 0, & p(x,y,z). \\ p(x+\alpha, y+\beta, z+g) & \leftarrow & z \geq 0, & p(x,y,z). \end{array}$$

où $x_0, y_0, z_0, e, \lambda, \gamma, \mu, f, \delta, \alpha, \beta, g$ désignent des entiers relatifs.
Le problème est de trouver une formule arithmétique $f(x,y,z)$ équivalente à la relation $p(x,y,z)$ définie par le programme ci-dessus.
Ce problème de caractérisation a d'importantes applications dans plusieurs domaines comme la génération automatique de lemmes pour démontrer la terminaison de programmes Prolog, la compilation de requêtes récursives en Bases de Données Déductives Temporelles ou la vérification de propriétés de sûreté dans les systèmes concurrents paramétrés.
Nous montrons que les programmes étudiés se divisent en trois classes: la classe hiérarchique, périodique et en spirale. Plus de 99% des programmes tombent dans les classes hiérarchiques et périodiques, et peuvent être caractérisés par une for mule d'arithmétique linéaire, à la différence des programmes en spirale.

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1 Introduction

We study in this report the least fixpoints associated with Datalog programs with arithmetical constraints with 2 rules of the form:

$$\begin{array}{lcl} p(x_0, y_0). & & \\ p(x+e, y+\lambda) & \leftarrow & x \geq 0, \quad p(x,y). \\ p(x+\mu, y+f) & \leftarrow & y \geq 0, \quad p(x,y). \end{array}$$

and with 3 rules of the form:

$$\begin{array}{lcl} p(x_0, y_0, z_0). & & \\ p(x+e, y+\lambda, z+\gamma) & \leftarrow & x \geq 0, \quad p(x,y,z). \\ p(x+\mu, y+f, z+\delta) & \leftarrow & y \geq 0, \quad p(x,y,z). \\ p(x+\alpha, y+\beta, z+g) & \leftarrow & z \geq 0, \quad p(x,y,z). \end{array}$$

The matrix $\begin{pmatrix} e & \lambda \\ \mu & f \end{pmatrix}$ (resp. $\begin{pmatrix} e & \lambda & \gamma \\ \mu & f & \delta \\ \alpha & \beta & g \end{pmatrix}$) is called the *incrementation matrix* of the 2-rule program (resp. 3-rule program).

The analysis of the least fixpoint will be performed according to the sign of the coefficients of the corresponding incrementation matrix.
As already seen in [2][4], the analysis for 2-rule Datalog programs can be decomposed into:

1. The class of *hierarchic programs*, whose matrices contain a line or a column made of coefficients of the same sign.
2. The class of *periodic programs* whose matrices are of the form $\begin{pmatrix} - & + \\ + & - \end{pmatrix}$ or $\begin{pmatrix} + & - \\ - & + \end{pmatrix}$

The least fixpoint associated to a hierarchic program can be represented as a constant number (≤ 3) of straight lines.

The least fixpoint associated to a periodic program can be represented as a repeated pattern (possibly preceded or followed by one straight line).

In both cases, the least fixpoint can be expressed by a linear arithmetic formula.

We will see in this report that the analysis of 3-rule Datalog programs can be decomposed into:

1. The class of *hierarchic programs*, which are, roughly speaking, characterized by matrices containing a line or a column made of coefficients of the same sign.
2. The class of *periodic programs*, which are, characterized by matrices containing a submatrix $\begin{pmatrix} - & + \\ + & - \end{pmatrix}$ or $\begin{pmatrix} + & - \\ - & + \end{pmatrix}$ as those below:

   $$\begin{pmatrix} - & + & \bullet \\ + & - & \bullet \\ \bullet & \bullet & \bullet \end{pmatrix} \quad \begin{pmatrix} - & \bullet & + \\ \bullet & \bullet & \bullet \\ + & \bullet & - \end{pmatrix} \quad \begin{pmatrix} \bullet & \bullet & \bullet \\ \bullet & - & + \\ \bullet & + & - \end{pmatrix}$$

   or their opposite matrices (that is, matrices with opposite signs of the elements).
3. The class of *spiralling programs*, which are characterized by matrices of the form

   $$\begin{pmatrix} - & - & + \\ + & - & - \\ - & + & - \end{pmatrix} \quad \begin{pmatrix} - & + & - \\ - & - & + \\ + & - & - \end{pmatrix}$$

or their opposite.

The hierarchic Datalog programs for three rules have already been studied in [3][4].

The periodic case was only sketched out in [3][4]. Only the algebraic condition for the existence of a pattern was given. Nothing was said on the applicability of such a pattern, nor on the form of the least fixpoint between the origin and the area where the pattern is applicable. The analysis of such 3-rule periodic programs is carefully done in this report. The analysis relies on new results found for 2-rule periodic programs, that are also given for the first time in this report.

The identification of the 4 "spiralling" programs is also a new contribution of the present report. These cases were overlooked in [3][4]. Although these cases cover less than one per cent of all the cases (4 cases on a total of 512), they are interesting, because they are the only ones for which the least fixpoints cannot be characterized by a linear arithmetic formula: the least fixpoint here has a vortical form spiralling around the negative space $x < 0 \ \wedge \ y < 0 \ \wedge \ z < 0$, before reaching the space $x \geq 0 \ \wedge \ y \geq 0 \ \wedge \ z \geq 0$ where a pattern becomes applicable.

The plan of this report is as follows:

Section 2 gives some preliminaries.
In section 3 we present results for programs with 1 recursive rule.
In section 4 we presnt results for programs with 2 recursive rules.
In section 5 we give a classification of programs with 3 recursive rules.
In section 6 we introduce new mathematical tools based on the pigeon hole principle, to prove the existence of certain patterns.
In section 7 we study the class 1 of hierachic programs with 3 recursive rules.
In section 8 we study a certain class of periodic programs with 3 recursive rules (class 2).
In section 9 we study another class of periodic programs with 3 recursive rules (class 3).
In section 10 we study the last class of periodic programs with 3 recursive rules (class 4).
Section 11 contains graphical plots of the fixpoints of some example programs representative of the different classes of programs mentioned.
Section 12 gives a sketch of a proof that that the "spiralling" programs (class 5) cannot be described in linear arithmetic.
Section 13 recapitulates the main results obtained.

2 Preliminaries

All vectors are column vectors, unless otherwise stated. To save space we often write $\langle x_1, \ldots x_n \rangle^T$ instead of

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

We allow ω and $-\omega$ to be components of a vector, where $n \leq \omega$, $n \geq -\omega$, $n + \omega = \omega$ and $n + (-\omega) = -\omega$ for all integers n. Also $\omega + \omega = \omega$ and $-\omega + (-\omega) = -\omega$ ($\omega - \omega$ is undefined). By 1_i we denote the unit vector whose ith component is 1 and all other components are 0. Let $\overline{x}$ and $\overline{y}$ are any vectors with $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ as components respectively, then $\overline{x} \leq \overline{y}$ is defined to hold iff $x_i \leq y_i$ for all $1 \leq i \leq n$. Also $\overline{x} < \overline{y}$ holds iff $\overline{x} \leq \overline{y}$ and $\overline{x} \neq \overline{y}$.

If $\overline{q}$ is a vector with nonnegative components, $|\overline{q}|$ denotes the sum of its components.

2.1 Program Transformations

We consider programs P of the form

$$\begin{array}{lrcl} & p(\overline{b}). & & \\ d_1: & p(\overline{\chi}+\overline{k}_1) & \leftarrow & (\overline{\chi},\overline{c}_1)+a_1 \geq 0, \quad p(\overline{\chi}). \\ & \vdots & & \\ d_n: & p(\overline{\chi}+\overline{k}_n) & \leftarrow & (\overline{\chi},\overline{c}_n)+a_n \geq 0, \quad p(\overline{\chi}). \end{array}$$

where $(\overline{\chi},\overline{c}_i)$ denotes the inner product. We consider the recursive rules to be labeled by $d_1,\ldots,d_n$. Let K be the matrix with $\overline{k}_1,\ldots,\overline{k}_n$ as row vectors, and let C be the matrix with $\overline{c}_1,\ldots,\overline{c}_n$ as row vectors. We call K the *incrementation matrix* and C the *constraint matrix.*

First notice that any $\overline{\chi}$ that satisfies $p(\overline{\chi})$, must necessarily be of the form

$$\overline{\chi}=\overline{b}+q_1\overline{k}_1+\cdots+q_n\overline{k}_n$$

where $0\leq q_i$. Let $\overline{q}$ be the vector with components q_1 to q_n. Then

$$\overline{\chi}=K^T\overline{q}+\overline{b}$$

The constraint $(\overline{\chi},\overline{c}_i)+a_i\geq 0$ of the program P thus becomes $(K^T\overline{q},\overline{c}_i)+\kappa_i\geq 0$, where $\kappa_i=(\overline{b},\overline{c}_i)+a_i$. But since $(K^T\overline{q},\overline{c}_i)=(\overline{q},K\overline{c}_i)$, by putting $\overline{\varphi}_i=K\overline{c}_i$, we get $(\overline{q},\overline{\varphi}_i)+\kappa_i\geq 0$. Thus we construct the new program P'

$$\begin{array}{lrcl} & p'(\overline{0}). & & \\ d_1: & p'(\overline{q}+1_1) & \leftarrow & (\overline{q},\overline{\varphi}_1)+\kappa_1 \geq 0, \quad p'(\overline{q}). \\ & \vdots & & \\ d_n: & p'(\overline{q}+1_n) & \leftarrow & (\overline{q},\overline{\varphi}_n)+\kappa_n \geq 0, \quad p'(\overline{q}). \end{array}$$

A program of the form P' is said to be on *standard form.* We have

Proposition 1:

$$p(\overline{\chi}) \quad \Leftrightarrow \quad \exists\overline{q}:\overline{\chi}=K^T\overline{q}+\overline{b} \;\; \wedge \;\; p'(\overline{q})$$

$\diamond$

Proof:
Follows easily by fixpoint induction. $\diamond$

Note that the size (the number of components) in $\overline{q}$ equals the number of recursive rules, regardless of the number of variables in the original program.

On the other hand, since

$$\overline{\chi}=\overline{b}+q_1\overline{k}_1+\cdots+q_n\overline{k}_n$$

we have

$$(\overline{\chi},\overline{c}_i)+a_i=(\overline{b},\overline{c}_i)+a_i+q_1(\overline{k}_1,\overline{c}_i)+\cdots+q_n(\overline{k}_n,\overline{c}_i)$$

Define $\overline{\theta}_i$ for $1\leq i\leq n$ by

$$\overline{\theta}_i=C\overline{k}_i$$

and let $\overline{\theta}_0$ be

$$\overline{\theta}_0=C\overline{b}+\overline{a}$$

Let $\overline{x}$ be the vector with $x_1, \ldots, x_n$ as components. We construct the program P''

$$\begin{array}{lll} & p''(\overline{\theta}_0). & \\ d_1: & p''(\overline{x}+\overline{\theta}_1) & \leftarrow \quad x_1 \geq 0, \quad p''(\overline{x}). \\ & \vdots & \\ d_n: & p''(\overline{x}+\overline{\theta}_n) & \leftarrow \quad x_n \geq 0, \quad p''(\overline{x}). \end{array}$$

A program on the form P'' is said to be on *simple form*. The relation between a program and its simple form is not so straightforward as for its standard form. To state the relationship we need some further notions.

As usual, for languages $\mathcal{L}_1$ and $\mathcal{L}_2$, denote by $\mathcal{L}_1 + \mathcal{L}_2$ the union of $\mathcal{L}_1$ and $\mathcal{L}_2$ and by $\mathcal{L}_1\mathcal{L}_2$ the set of strings w_1w_2 where $w_1 \in \mathcal{L}_1$ and $w_2 \in \mathcal{L}_2$. We define $\mathcal{L}_1^n$ as

$$\begin{array}{lll} \mathcal{L}_1^0 & = & \{\epsilon\} \\ \mathcal{L}_1^{n+1} & = & \mathcal{L}_1\mathcal{L}_1^n \end{array}$$

where ϵ is the empty string. $\mathcal{L}_1^{\leq n}$ is defined by

$$\mathcal{L}_1^{\leq n} = \bigcup_{i \leq n} \mathcal{L}_1^i$$

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and finally, $\mathcal{L}_1^*$ is defined by

$$\mathcal{L}_1^* = \bigcup_{i \geq 0} \mathcal{L}_1^i$$

We identify d_i with the singleton language $\{d_i\}$. Note that if $\mathcal{L}_1$ is finite, then $\mathcal{L}_1^{\leq n}$ is finite.

With any program P there is an associated language $\mathcal{L}_P$ constructed as follows: A program

$$\begin{array}{lll} & p(\overline{b}). & \\ d_1: & p(\overline{\chi}+\overline{k}_1) & \leftarrow \quad (\overline{\chi}, \overline{c}_1) + a_1 \geq 0, \quad p(\overline{\chi}). \\ & \vdots & \\ d_n: & p(\overline{\chi}+\overline{k}_n) & \leftarrow \quad (\overline{\chi}, \overline{c}_n) + a_n \geq 0, \quad p(\overline{\chi}). \end{array}$$

is extended with an extra argument:

$$\begin{array}{lll} & p(\epsilon, \overline{b}). & \\ d_1: & p(wd_1, \overline{\chi}+\overline{k}_1) & \leftarrow \quad (\overline{\chi}, \overline{c}_1) + a_1 \geq 0, \quad p(w, \overline{\chi}). \\ & \vdots & \\ d_n: & p(wd_n, \overline{\chi}+\overline{k}_n) & \leftarrow \quad (\overline{\chi}, \overline{c}_n) + a_n \geq 0, \quad p(w, \overline{\chi}). \end{array}$$

The language $\mathcal{L}_P$ is defined by

$$w \in \mathcal{L}_P \;\Leftrightarrow\; \exists \overline{\chi} : p(w, \overline{\chi})$$

Now we have

Proposition 2:
Let P be any program and let P'' be its simple form. Then

$$\mathcal{L}_P = \mathcal{L}_{P''}$$

$\diamond$

Proposition 3:
Let P be any program and let P' be its standard form. Then

$$\mathcal{L}_P = \mathcal{L}_{P'}$$

$\diamond$

Let $w \in (d_1 + \cdots + d_n)^*$ be any string. Denote by $\overline{w}$ the vector whose ith component is the number of occurences in w of the label d_i.

Proposition 4:
Let P' be a program on standard form. Then

$$p'(\overline{q}) \;\Leftrightarrow\; \exists w \in \mathcal{L}_{P'} : \overline{q} = \overline{w}$$

$\diamond$

These propositions follow easily by induction. Thus we may without loss of generality restrict our attention to programs on standard form and simple form.

We refer to $\overline{\kappa}$ and $\overline{\theta}_0$ as *base values* since they depend on the base case of the original program.

We denote by Φ the incrementation matrix of a program on simple form. We have the following relationships between the matrices associated with a program and its standard and simple forms:

Proposition 5:
Let P be any program with incrementation matrix K and constraint matrix C. Let P' be its standard form and P'' its simple form. Let Φ be the incrementation matrix of P''.

1. $\Phi = KC^T$
2. The constraint matrix of P' is Φ^T.
3. The incrementation matrix of P' is the identity matrix (the matrix whose diagonal elements are 1 and all the others are 0)
4. The constraint matrix of P'' is the identity matrix.
5. Let w be a string in $\mathcal{L}_P$, and suppose $p(w, \overline{\chi})$, $p'(w, \overline{q})$ and $p''(w, \overline{x})$ hold. then

$$\overline{\chi} = K^T \overline{q} + \overline{b}$$

$$\overline{x} = \Phi^T \overline{q} + \overline{\theta}_0$$

$\diamond$

Informally, the argument $\overline{q}$ of a program on standard form, counts the number of times each recursive rule has been used, while the argument $\overline{x}$ of a program on simple form says which rules are applicable. Or more graphically, given a string $w \in \mathcal{L}_P$ for some program P, its standard form P' computes the position in space reached by w considered as a path, while its simple form P'' says in wich directions one may continue.

We define $\overline{x}_{\overline{q}}$ by

$$\overline{x}_{\overline{q}} = \Phi^T \overline{q} + \overline{\theta}_0$$

to indicate the value of the simple program corresponding to a point of the standard program.

Furthermore we have the following

Proposition 6:
Let P be any program with incrementation matrix K. Assume K is invertible. For any given $\overline{\chi}$, it is decidable wether $p(\overline{\chi})$ holds. $\diamond$

Proof:
Construct the standard form P' of P. Since

$$p(\overline{\chi}) \;\Leftrightarrow\; \exists \overline{q} : \overline{\chi} = K^T \overline{q} + \overline{b} \;\wedge\; p'(\overline{q})$$

and since K is invertible, solve the equation

$$\overline{\chi} = K^T \overline{q} + \overline{b}$$

in $\overline{q}$ ($\overline{\chi}$ is given) and check if $p'(\overline{q})$ holds. This is done by computing P' bottom-up. Either $\overline{q}$ will be reached, or one generates points $\overline{q}'$ such that $\overline{q}' \not\leq \overline{q}$. The branches in the bottom-up tree starting from such a point $\overline{q}'$ cannot contain $\overline{q}$ since every use of of a recursive rule of P' increments some component of its argument vector and leaves the other ones untouched. Thus, these branches are pruned. After at most $|\overline{q}|$ steps, $\overline{q}$ must be reached or all branches are pruned$\diamond$

Clearly, if K is not invertible, then there may be infinitely many $\overline{q}$ such that $\overline{\chi} = K^T \overline{q} + \overline{b}$ holds for a given $\overline{\chi}$. In this case the argument above does not work. It is not known (to us) if $p(\overline{\chi})$ is decidable in general for this case. It *is* decidable if P has at most two recursive rules. This will be shown later in this report (it is also a consequence of the results in [4]).

2.2 Reachability and Paths

With a program on standard form, we associate a reachability relation $\xrightarrow{w}$ where w is a string of the language $(d_1 + \cdots + d_n)^*$. The reachability relation is defined as follows:

$$\begin{array}{lcl}
\overline{q} \xrightarrow{\epsilon} \overline{q} & & \\
\overline{q} \xrightarrow{w\,d_1} (\overline{q}' + 1_1) & \leftarrow & (\overline{q}', \overline{\varphi}_1) + \kappa_1 \geq 0 \quad \wedge \quad \overline{q} \xrightarrow{w} \overline{q}' \\
& \vdots & \\
\overline{q} \xrightarrow{w\,d_n} (\overline{q}' + 1_n) & \leftarrow & (\overline{q}', \overline{\varphi}_n) + \kappa_n \geq 0 \quad \wedge \quad \overline{q} \xrightarrow{w} \overline{q}'
\end{array}$$

Strictly, the reachability relation $\xrightarrow{w}$ should be parameterized by the program P'. For the sake of readability we suppress the parameter P'. It is always understood that $\xrightarrow{w}$ is associated with a given program.

We have the following simple relation

Proposition 7:

$$\overline{q} \xrightarrow{w} \overline{q}' \;\Rightarrow\; \overline{q}' = \overline{q} + \overline{w}$$

$\diamond$

We sometimes write $\overline{q}w$ instead of $\overline{q} + \overline{w}$. By $|w|$ we denote the length of a string w. Thus

$|w| = |\overline{w}|$.

We extend the definition of $\xrightarrow{w}$ to languages $\mathcal{L} \subseteq (d_1 + \cdots + d_n)^*$ and define

$$\overline{q} \xrightarrow{\mathcal{L}} \overline{q}' \Leftrightarrow \exists w \in \mathcal{L} : \overline{q} \xrightarrow{w} \overline{q}'$$

We write $\overline{q} \xrightarrow{\mathcal{L}_1} \overline{q}' \xrightarrow{\mathcal{L}_2} \overline{q}''$ instead of $\overline{q} \xrightarrow{\mathcal{L}_1} \overline{q}' \wedge \overline{q}' \xrightarrow{\mathcal{L}_2} \overline{q}''$. We have

Proposition 8:

$$\overline{q} \xrightarrow{\mathcal{L}_1+\mathcal{L}_2} \overline{q}' \Leftrightarrow \begin{array}{c} \overline{q} \xrightarrow{\mathcal{L}_1} \overline{q}' \\ \vee \\ \overline{q} \xrightarrow{\mathcal{L}_2} \overline{q}' \end{array}$$

$$\overline{q} \xrightarrow{\mathcal{L}_1\mathcal{L}_2} \overline{q}' \Leftrightarrow \exists \overline{q}'' : \overline{q} \xrightarrow{\mathcal{L}_1} \overline{q}'' \xrightarrow{\mathcal{L}_2} \overline{q}'$$

$\diamond$

Thus, for finite languages a linear arithmetic formula can be given by applying the proposition above together with the definition of $\xrightarrow{\mathcal{L}}$.

The motivation for introducing $\xrightarrow{(d_1+\cdots+d_n)^*}$ is the following fact:

Theorem 1:
Let P' be a program on standard form with recursive rules labeled by $d_1, \ldots, d_n$. Then

$$\overline{0} \xrightarrow{(d_1+\cdots+d_n)^*} \overline{q} \Leftrightarrow p'(\overline{q})$$

$\diamond$

Corollary:
Let P be any program with incrementation matrix K, and let $\xrightarrow{(d_1+\ldots+d_n)^*}$ be the reachability relation associated with its standard form. Then

$$p(\overline{\chi}) \Leftrightarrow \exists \overline{q} : \overline{\chi} = K^T\overline{q} + \overline{b} \wedge \overline{0} \xrightarrow{(d_1+\ldots+d_n)^*} \overline{q}$$

$\diamond$

This report is devoted to giving linear arithmetic formulas characterizing $\xrightarrow{(d_1+\cdots+d_n)^*}$ for some programs of the forms discussed above.

2.3 Paths and Motifs

A string w is refered to as a *path*. If $\overline{q} \xrightarrow{w} \overline{q}'$ we say that $\overline{q}'$ is reachable from $\overline{q}$ by the path w. We introduce $\varrho_w(\overline{q})$ defined by

$$\varrho_w(\overline{q}) \Leftrightarrow \exists \overline{q}' : \overline{q} \xrightarrow{w} \overline{q}'$$

We say that w is *applicable* or *admissible* at $\overline{q}$ iff $\varrho_w(\overline{q})$ holds. Remember that $\overline{q} \xrightarrow{w} \overline{q}' \Rightarrow \overline{q}' = \overline{q} + \overline{w}$, so $\varrho_w(\overline{q})$ could equivalently be defined as

$$\varrho_w(\overline{q}) \Leftrightarrow \overline{q} \xrightarrow{w} \overline{q} + \overline{w}$$

Theorem 2:
For any path w there exists a vector ϕ_w (possibly with some components $-\omega$), such that

$$\varrho_w(\overline{q}) \Leftrightarrow \overline{x_{\overline{q}}} \geq \phi_w$$

$\diamond$

Corollary:

$$\overline{x}_{\overline{q}} \leq \overline{x}_{\overline{q}'} \;\Rightarrow\; (\varrho_w(\overline{q}) \Rightarrow \varrho_w(\overline{q}'))$$

$\diamond$

The definition of $\varrho_w(\overline{q})$ is extended to languages $\mathcal{L}$ as follows

$$\varrho_{\mathcal{L}}(\overline{q}) \;\Leftrightarrow\; \exists w \in \mathcal{L} : \varrho_w(\overline{q})$$

Given any vector $\overline{u}$ with nonnegative components, $\xi_{\overline{u}}$ denotes the (finite) language of strings w such that $\overline{w} = \overline{u}$. The set $\xi_{\overline{u}}$ is called the *motif* of $\overline{u}$. Obviously the following holds

Proposition 9:

$$\overline{q} \xrightarrow{\xi_{\overline{u}}} \overline{q}' \;\Rightarrow\; \overline{q}' = \overline{q} + \overline{u}$$

$\diamond$

Proposition 10:

$$\xi_{\overline{u}_1}\xi_{\overline{u}_2} \subseteq \xi_{\overline{u}_1+\overline{u}_2}$$

The converse does not hold in general. $\diamond$

Let $\overline{w}$ be any vector with nonnegative components, and consider the motif $\xi_{\overline{w}}$. Let $\overline{x} = \langle x, y, z\rangle^T$ be the argument vector of a simple program with labels h, v and t, and let $\overline{q}' = \overline{q} + \overline{w}$. If $x_{\overline{q}} = x_{\overline{q}'}$, then x is said to be *preserved* (or *let invariant* by $\xi_{\overline{w}}$). We say that $\xi_{\overline{w}}$ is a *pattern*. If only two rules is used in $\xi_{\overline{w}}$, the pattern is called *planar*. More specifically, in order to distinguish among planar patterns of different planes, we will refer to $\xi_{\overline{w}}$ more specifically as e.g. a planar vt-pattern if $\xi_{\overline{w}} \subseteq (v+t)^*$.

In particular, for programs with three recursive rules, a pattern that preserves two components of the simple program is called a *co-pattern* (or more simply a *pattern*).

The derivations in this report of linear arithmetic formulas that characterize the reachability relation, turns out to be independent of the base value $\overline{\kappa}$ (or equivalently $\overline{\theta}_0$). Strictly the expressions should thus be parameterized by $\overline{\kappa}$ as $\overline{p} \xrightarrow{(h+v+t)^*}_{\overline{\kappa}} \overline{p}'$. But since $\overline{\kappa}$ will not occur as a coefficient in any of the expressions given (that is $n \cdot \kappa_i$ where n is a varaible does not occur, only subexpressions of the form $\ldots + \kappa_i + \ldots$), if $B(\overline{\kappa})$ is a linear arithmetic relation, then $\exists \overline{\kappa} : B(\overline{\kappa}) \;\wedge\; \overline{p} \xrightarrow{(h+v+t)^*}_{\overline{\kappa}} \overline{p}'$ is also a linear arithmetic relation. Therefore we suppress the parameter.

Note that we do not say that the reachability relation does not depend on the base value, only that our arguments do not assume a fixed value.

3 Programs with One Recursive Rule

To give an expression for $\xrightarrow{d_1^*}$ associated with programs with one recursive rule, is straightforward.

Theorem 3:
Consider the program P' on standard form:

$$\begin{array}{lll} & p'(\overline{0}). & \\ d_1: & p'(\overline{q}+1_1) & \leftarrow \quad (\overline{q},\overline{\varphi}_1)+\kappa_1 \geq 0, \quad p'(\overline{q}). \end{array}$$

and let $\xrightarrow{d_1^*}$ be its associated reachability relation. Then

$$\overline{q} \xrightarrow{d_1^*} \overline{q}' \Leftrightarrow \overline{q}=\overline{q}' \vee \left(\begin{array}{c} \exists n \geq 0 : \overline{q}' = \overline{q} + n \cdot 1_1 \\ \wedge \\ (\overline{q},\overline{\varphi}_1)+\kappa_1 \geq 0 \\ \wedge \\ (\overline{q}'-1_1,\overline{\varphi}_1)+\kappa_1 \geq 0 \end{array} \right)$$

$\diamond$

Proof:
Follows easily by induction and the fact that either

$$(\overline{q},\overline{\varphi}_1)+\kappa_1 \geq 0 \;\Rightarrow\; (\overline{q}+1_1,\overline{\varphi}_1)+\kappa_1 \geq 0$$

or

$$(\overline{q}+1_1,\overline{\varphi}_1)+\kappa_1 \geq 0 \;\Rightarrow\; (\overline{q},\overline{\varphi}_1)+\kappa_1 \geq 0$$

$\diamond$

The expression for $\xrightarrow{d_1^*}$ is general, but if one knows that that

$$(\overline{q},\overline{\varphi}_1)+\kappa_1 \geq 0 \;\Rightarrow\; (\overline{q}+1_1,\overline{\varphi}_1)+\kappa_1 \geq 0$$

it can be simplified to

$$\overline{q} \xrightarrow{d_1^*} \overline{q}' \Leftrightarrow \overline{q}=\overline{q}' \vee \left(\begin{array}{c} \exists n \geq 0 : \overline{q}' = \overline{q} + n \cdot 1_1 \\ \wedge \\ (\overline{q},\overline{\varphi}_1)+\kappa_1 \geq 0 \end{array} \right)$$

and if one knows that

$$(\overline{q}+1_1,\overline{\varphi}_1)+\kappa_1 \geq 0 \;\Rightarrow\; (\overline{q},\overline{\varphi}_1)+\kappa_1 \geq 0$$

it can be simplified to

$$\overline{q} \xrightarrow{d_1^*} \overline{q}' \Leftrightarrow \overline{q}=\overline{q}' \vee \left(\begin{array}{c} \exists n \geq 0 : \overline{q}' = \overline{q} + n \cdot 1_1 \\ \wedge \\ (\overline{q}'-1_1,\overline{\varphi}_1)+\kappa_1 \geq 0 \end{array} \right)$$

For any given program, the direction of the implication above is decidable an can be precomputed to get a simpler expression for $\xrightarrow{d_1^*}$.

Actually, the formula for programs with one recursive rule is a special case of the formula for motifs.

Theorem 4:
Let $\overline{w}$ be any vector with nonnegative components and let $\xi_{\overline{w}}$ be its associated motif. Then

$$\overline{q} \xrightarrow{\xi_{\overline{w}}^*} \overline{q}' \Leftrightarrow \overline{q}=\overline{q}' \vee \exists n \geq 1 : \left(\begin{array}{c} \overline{q}' = \overline{q} + n\overline{w} \\ \wedge \\ \forall 0 \leq n' < n : \varrho_{\xi_{\overline{w}}}(\overline{q}+n'\overline{w}) \end{array} \right)$$

holds. $\diamond$

If

$$\varrho_{\xi_{\overline{w}}}(\overline{q}+n'\overline{w}) \;\Rightarrow\; \varrho_{\xi_{\overline{w}}}(\overline{q}+(n'+1)\overline{w})$$

or

$$\varrho_{\xi_{\overline{w}}}(\overline{q}+(n'+1)\overline{w}) \;\Rightarrow\; \varrho_{\xi_{\overline{w}}}(\overline{q}+n'\overline{w})$$

holds, then

$$\forall 0 \leq n' < n : \varrho_{\xi_{\overline{w}}}(\overline{q}+n'\overline{w})$$

collapses to

$$\varrho_{\xi_{\overline{w}}}(\overline{q})$$

or

$$\varrho_{\xi_{\overline{w}}}(\overline{q}+(n-1)\overline{w})$$

respectively. The formula for programs with one recursive rule can be considered as the formula for reachability by a motif where the motif is given by $\xi = \{d_1\}$.

Note that

$$\big(\forall 0 \leq n' < n : \varrho_{\xi_{\overline{w}}}(\overline{q}+n'\overline{w})\big) \;\Leftrightarrow\; \left(\begin{array}{c} \varrho_{\xi_{\overline{w}}}(\overline{q}) \\ \wedge \\ \varrho_{\xi_{\overline{w}}}(\overline{q}+(n-1)\overline{w}) \end{array}\right)$$

holds in general.

4 Programs with Two Recursive Rules

Programs with two recursive rules was allready treated in [2][4]. We present here a uniform analysis along a different line (the programs need not be divided into different classes) which is slightly more general in the sence that the proof does not depend on linear integer arithmetic, but only on some general properties of the functions and the constraints.

4.1 General Analysis

In this section we derive a linear arithmetic formula defining $\xrightarrow{(h+v)^*}$ associated with the program

$$\begin{array}{lllll} & p(x_0, y_0). & & & \\ h: & p(x+e, y+\lambda) & \leftarrow & x \geq 0, & p(x,y). \\ v: & p(x+\mu, y+f) & \leftarrow & y \geq 0, & p(x,y). \end{array}$$

We will work only with its standard form

$$\begin{array}{lllll} & p'(0,0). & & & \\ h: & p'(\overline{p}+1_h) & \leftarrow & (\overline{p},\overline{\varphi_h})+\kappa_h \geq 0, & p(\overline{p}). \\ v: & p(\overline{p}+1_v) & \leftarrow & (\overline{p},\overline{\varphi_v})+\kappa_v \geq 0, & p(\overline{p}). \end{array}$$

where $\overline{\varphi}_h = \langle e, \mu\rangle^T$, $\overline{\varphi}_v = \langle \lambda, f\rangle^T$. We use the labels h and v for "horizontal" and "vertical" respectively. By abuse of notation we also let h and v be variables denoting positions on the horizontal and vertical axes. Thus a point $\overline{p}$ is a vector $\langle h, v\rangle^T$.

Lemma 1:
The constraints of the program above are monotonic. That is, either

$$(\overline{p}.\overline{\varphi}_i) + \kappa_i \geq 0 \;\Rightarrow\; (\overline{p} + 1_j.\overline{\varphi}_i) + \kappa_i \geq 0$$

or

$$(\overline{p} + 1_j.\overline{\varphi}_i) + \kappa_i \geq 0 \;\Rightarrow\; (\overline{p}, \overline{\varphi}_i) + \kappa_i \geq 0$$

holds, for all $i = h, v$ and $j = h, v$. ◇

we use the convention

$$\overline{p}h = \overline{p} + 1_h$$

$$\overline{p}v = \overline{p} + 1_v$$

We say that when $\overline{p}' = \overline{p}h$, then $\overline{p}'$ is the result of making a horizontal move from (or applying rule h to) $\overline{p}$, and similarly for $\overline{p}' = \overline{p}v$

Lemma 2:
Applications of rules are commutative:

$$\overline{p}hv = \overline{p}vh$$

◇

The construction of a formula for $\xrightarrow{(h+v)^*}$ actually relies only on lemmas 1 and 2. Therefore the construction applies to more general constraints than $(\overline{p}.\overline{\varphi}_i) + \alpha_i \geq 0$. To emphasize this generality (and simplify notation), in this section we write $\varphi_i(\overline{p})$ instead of $(\overline{p}.\overline{\varphi}_i) + \alpha_i \geq 0$.

The intuition behind the formula is illustrated in figures 1, 2 and 3. The path from a

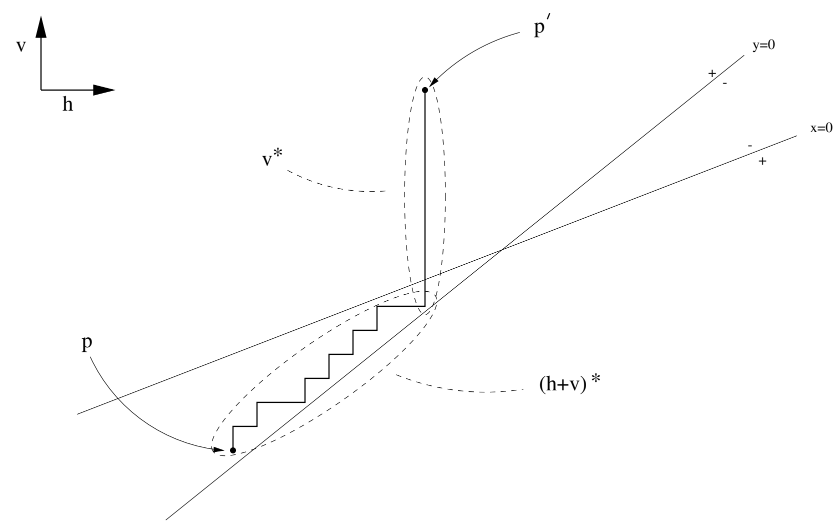


Figure 1

point $\overline{p}$ to $\overline{p}'$ is viewed as consisting of two parts: It begins with a sequence of applications of both rules and ends with a sequence of applications of a single rule, or it begins with a single rule followed by using both rules, or it consists of a sequence of applications of a single rule followed by a sequnce of applications of the other rule. These parts are characterized separately by linear arithmetic formulas, and the formula for the reachability relation is obtained

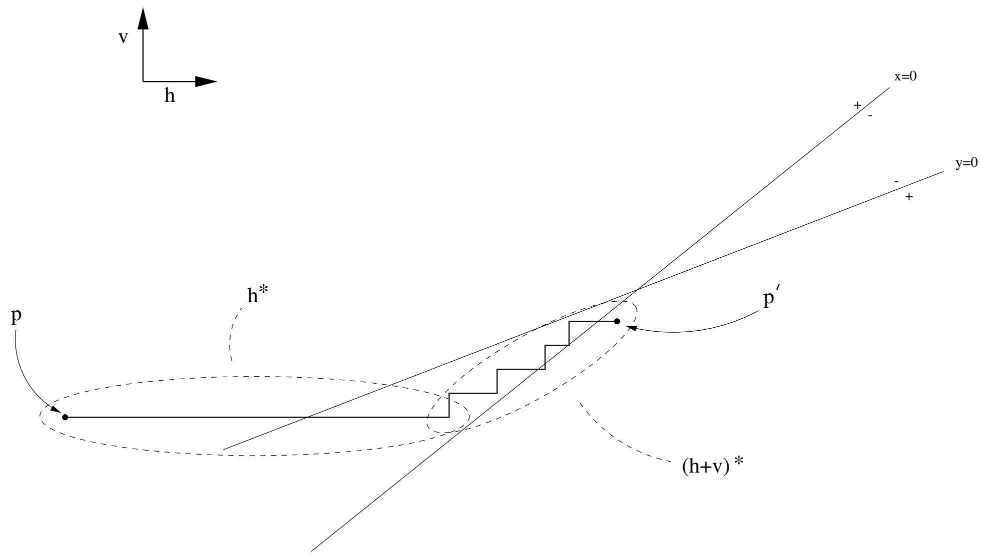


Figure 2

by "concatenating" them. Thus, the formula $R(\overline{p}, \overline{p}')$ for reachability in two dimensions will, in its full generality, be of the form

$$R(\overline{p}, \overline{p}') \;\Leftrightarrow\; \exists \overline{q}, \overline{q}' : \begin{pmatrix} \overline{p} \xrightarrow{h^*} \overline{q} \\ \vee \\ \overline{p} \xrightarrow{v^*} \overline{q} \end{pmatrix} \;\wedge\; r_{hv}(\overline{q}, \overline{q}') \;\wedge\; \begin{pmatrix} \overline{q}' \xrightarrow{h^*} \overline{p}' \\ \vee \\ \overline{q}' \xrightarrow{v^*} \overline{p}' \end{pmatrix}$$

where $r_{hv}(\overline{q}, \overline{q}')$ is a relation characterising reachability in the region where both rules are applicable. The rest of this section will be devoted mostly to the construction of $r_{hv}(\overline{q}, \overline{q}')$ (note that we do not characterize all admissible *paths*, only reachable points). The basic idea behind

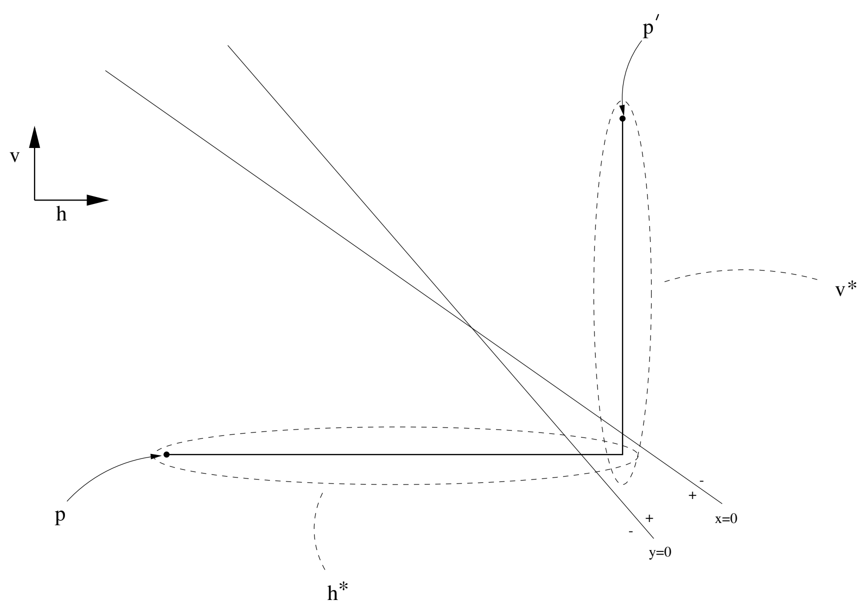


Figure 3

the definition of $r_{hv}(\overline{p}, \overline{p}')$ is based on the observation that the area defined by $\varphi_h(\overline{p}) \wedge \varphi_v(\overline{p})$ is convex in some sence, and that (roughfly speaking) all paths inside this area are admissible. Thus, if two points $\overline{p}$ and $\overline{p}'$ satisfy $\varphi_h(\overline{p}) \wedge \varphi_v(\overline{p})$ and $\varphi_h(\overline{p}') \wedge \varphi_v(\overline{p}')$ (that is, both points lie inside the area) and if $\overline{p} \leq \overline{p}'$, then there exists an admissible path from $\overline{p}$ to $\overline{p}'$. This gives the intuition of the construction, but it is not true as a matter of fact. We will define three predicates $s(\overline{p})$, $e(\overline{p}, \overline{p}')$ and $c(\overline{p}, \overline{p}')$, where the first relation yields an over approximation of the set of reachable points, and the latter two cut away points that are not reachable.

First, the set $\varphi_h(\overline{p}) \wedge \varphi_v(\overline{p})$ is actually too small, so we define an extension of this set as

$$s(\overline{p}) \Leftrightarrow \begin{array}{c} \varphi_h(\overline{p}) \wedge \varphi_v(\overline{p} - 1_v) \\ \vee \\ \varphi_h(\overline{p} - 1_h) \wedge \varphi_v(\overline{p}) \end{array}$$

Figure 4 shows the set defined by $\varphi_h(\overline{p}) \wedge \varphi_v(\overline{p})$ and also illustrates why this set is too small. The

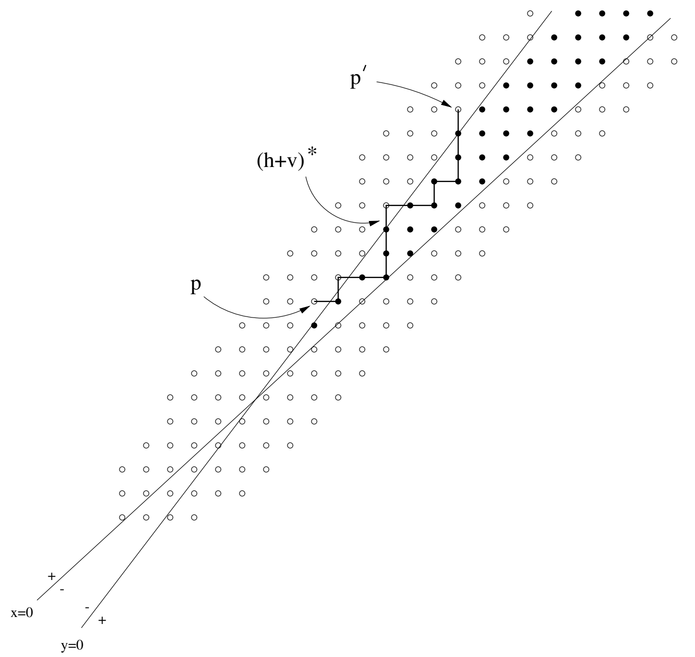


Figure 4

point $\overline{p}'$ is reachable from $\overline{p}$ by a path in $(h+v)^*$ allthough neither $\overline{p}$ nor $\overline{p}'$ belongs to the set. The area $s(\overline{p})$ is illustrated in figure 5 and it is seen that $\overline{p}$ and $\overline{p}'$ belongs to this extended set.

Lemma 3:
Let $s(\overline{p})$ hold. Then

1. If $\varphi_h(\overline{q}) \Rightarrow \varphi_h(\overline{q} + 1_h)$ holds, then $\varphi_h(\overline{p})$ holds.
2. If $\neg\varphi_h(\overline{q}) \Rightarrow \neg\varphi_h(\overline{q} + 1_h)$ holds, then $\varphi_h(\overline{p} - 1_h)$ holds.

The analogous statement for v is also true. $\diamond$

Proof:
Consider the first property. Since $s(\overline{p})$ holds, $\varphi_h(\overline{p}) \vee \varphi_h(\overline{p} - 1_h)$ must hold. Assume, $\neg\varphi_h(\overline{p})$. Then $\varphi_h(\overline{p} - 1_h)$ must hold. But since $\varphi_h(\overline{q}) \Rightarrow \varphi_h(\overline{q} + 1_h)$, it follows that $\varphi_h(\overline{p})$ holds, which is a contradiction. Thus $\varphi_h(\overline{p})$ must hold. The proof of the second property is similar. $\diamond$

The set defined by $s(\overline{p})$ is not always connected in the sence that there may exist points $\overline{p}$ and $\overline{p}'$ such that $s(\overline{p})$, $s(\overline{p}')$ and $\overline{p} \leq \overline{p}'$ hold, but $\overline{p}'$ is not reachable from $\overline{p}$. Figures 6 and

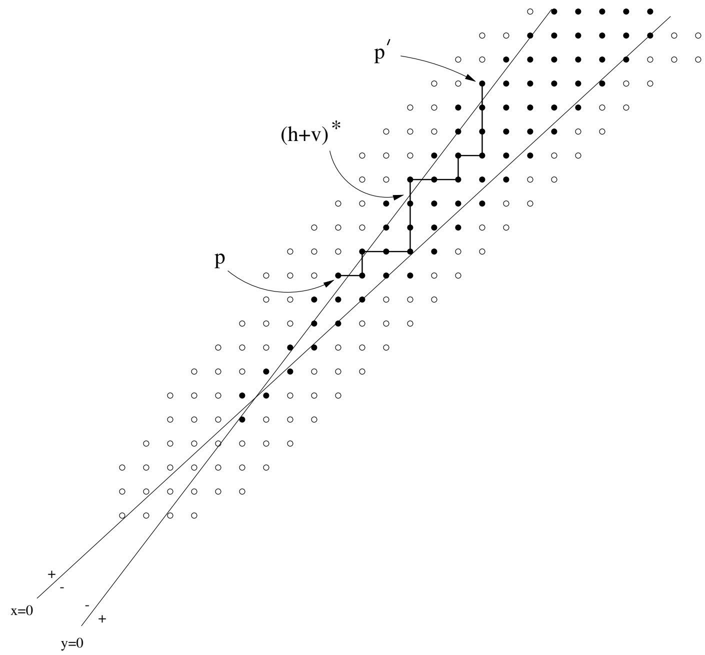


Figure 5

9 illustrates two such situations. We introduce two predicates to take care of this problem. Define

$$e(\overline{p},\overline{p}') \Leftrightarrow \forall \overline{p}'' : \overline{p} \leq \overline{p}'' \leq \overline{p}' \wedge \neg\varphi_h(\overline{p}'') \wedge \neg\varphi_v(\overline{p}'') \Rightarrow \left(\begin{array}{c} \varphi_v(h_{\overline{p}}, v_{\overline{p}''}) \\ \vee \\ \varphi_h(h_{\overline{p}''}, v_{\overline{p}}) \end{array} \right)$$

The intuitive meaning of the predicate $e(\overline{p},\overline{p}')$ is illustrated in figure 6. The idea is that for $\overline{p}'$ to be reachable from $\overline{p}$, there must exist a path u or u' that "goes around" any point $\overline{p}''$ between $\overline{p}$ and $\overline{p}'$ for which $\neg\varphi_h(\overline{p}'') \wedge \neg\varphi_v(\overline{p}'')$ holds. In the figure, the prefixes of w are the only admissible paths starting at $\overline{p}$ and w ends at $\overline{p}''$. Thus, $\overline{p}'$ is not reachable from $\overline{p}$, and it is seen in the figure that $e(\overline{p},\overline{p}')$ is not satisfied since $\varphi_h(h_{\overline{p}''}, v_{\overline{p}}) \vee \varphi_v(h_{\overline{p}}, v_{\overline{p}''})$ is not satisfied.

That a simple predicate as $e(\overline{p},\overline{p}'')$ is sufficient for expressing such an a priori complicated property is due to the fact that the constraints satisfy lemma 1 *and* because we are working in only *two* dimensions. In figure 7 the lines has been slightly shifted so that no point $\overline{p}''$ between $\overline{p}$ and $\overline{p}'$ satisfies $\neg\varphi_h(\overline{p}'') \wedge \neg\varphi_v(\overline{p}'')$ so $e(\overline{p},\overline{p}')$ holds, and clearly $\overline{p}'$ is reachable from $\overline{p}'$. Figure 8 illustrates a situation where $e(\overline{p},\overline{p}')$ is satisfied even though a point $\overline{p}''$ between $\overline{p}$ and $\overline{p}'$ satisfies $\neg\varphi_h(\overline{p}'') \wedge \neg\varphi_v(\overline{p}'')$. In this case $\varphi_h(h_{\overline{p}''}, v_{\overline{p}})$ holds and the path u is admissible.

Lemma 4:
Let $e(\overline{p},\overline{p}')$ and $\neg e(\overline{p}+1_h,\overline{p}')$ hold. Then

$$\exists \overline{p}'' : \overline{p}+1_h \leq \overline{p}'' \leq \overline{p}' \wedge \left(\begin{array}{c} \neg\varphi_h(h_{\overline{p}''}, v_{\overline{p}''}) \wedge \neg\varphi_v(h_{\overline{p}''}, v_{\overline{p}''}) \\ \wedge \\ \neg\varphi_h(h_{\overline{p}''}, v_{\overline{p}}) \\ \wedge \\ \varphi_v(h_{\overline{p}}, v_{\overline{p}''}) \wedge \neg\varphi_v(h_{\overline{p}}+1, v_{\overline{p}''}) \end{array} \right)$$

holds. The analogous statement is also true when $e(\overline{p},\overline{p}')$ and $\neg e(\overline{p}+1_v,\overline{p}')$ hold. ◇

Proof:
Follows immediately from the definition. ◇

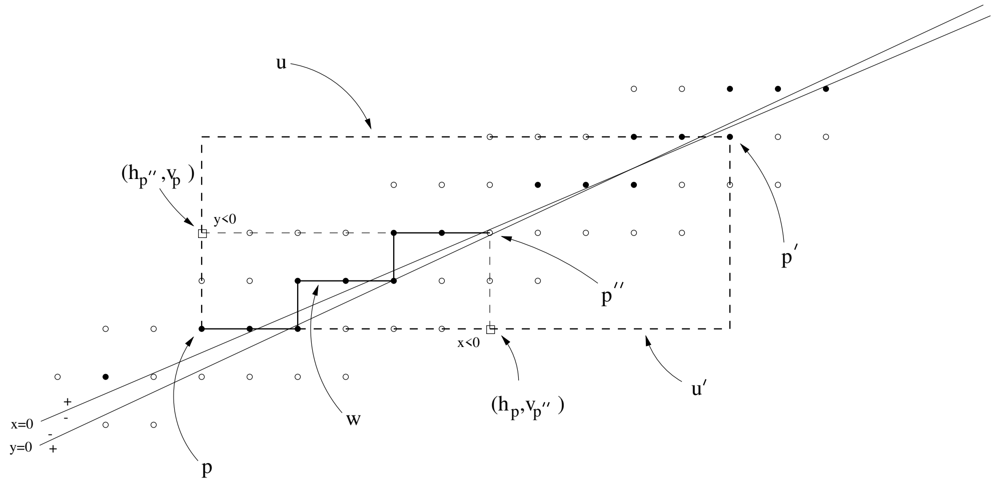


Figure 6

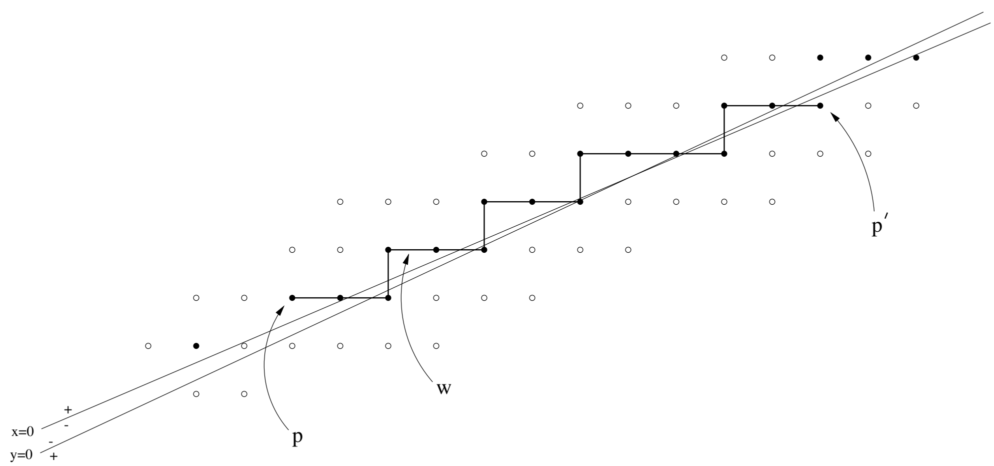


Figure 7

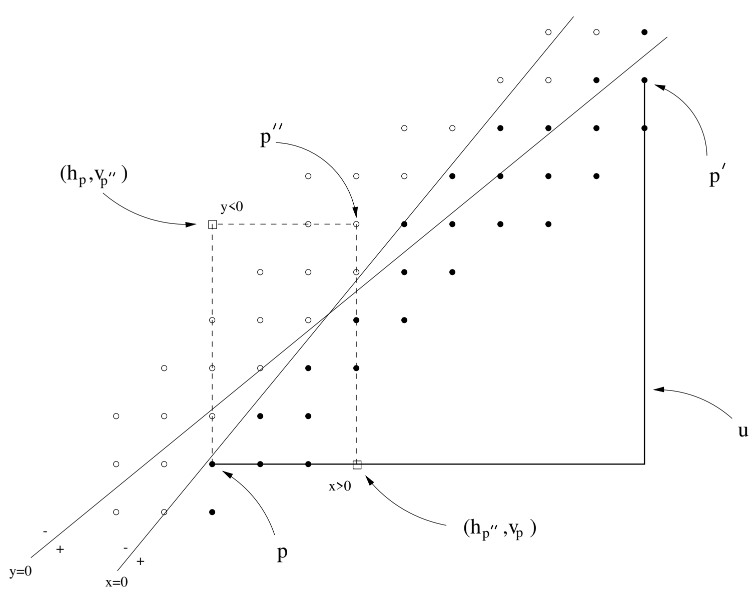


Figure 8

Lemma 5:
Let $e(\overline{p},\overline{p}')$ and $\neg e(\overline{p}+1_h,\overline{p}')$ hold. Then

1. $\neg\varphi_v(\overline{q}) \Rightarrow \neg\varphi_v(\overline{q}+1_h)$
2. If $s(\overline{p}+1_h)$ holds, then $\neg\varphi_v(\overline{q}) \Rightarrow \neg\varphi_v(\overline{q}+1_v)$

$\diamond$

Proof:

1. From lemma 4 we get $\varphi_v(h_{\overline{p}}, v_{\overline{p}''}) \wedge \neg\varphi_v(h_{\overline{p}}+1, v_{\overline{p}''})$. Lemma 1 immediately yields the statement.
2. From $s(\overline{p}+1_h)$ we get $\varphi_v(\overline{p}+1_h)\vee\varphi_v(\overline{p}+1_h-1_v)$. That is, $\varphi_v(h_{\overline{p}}+1, v_{\overline{p}})\vee\varphi_v(h_{\overline{p}}+1, v_{\overline{p}}-1)$. But by lemma 4, $\neg\varphi_v(h_{\overline{p}}+1, v_{\overline{p}''})$ holds, and since $v_{\overline{p}} \leq v_{\overline{p}''}$, lemma 1 yields the statement.

$\diamond$

Lemma 6:
Let $e(\overline{p},\overline{p}')$, $\neg e(\overline{p}+1_h,\overline{p}')$ and $s(\overline{p}+1_h)$ hold. Then $\neg s(\overline{p}')$ must hold. $\diamond$

Proof:
By lemma 4 there exists some $\overline{p}+1_h \leq \overline{p}'' \leq \overline{p}'$ such that $\neg\varphi_v(h_{\overline{p}''}, v_{\overline{p}''}) \wedge \neg\varphi_h(h_{\overline{p}''}, v_{\overline{p}''}) \wedge \neg\varphi_h(h_{\overline{p}''}, v_{\overline{p}})$ holds. By $s(\overline{p}+1_h)$ we have $\varphi_h(h_{\overline{p}}, v_{\overline{p}}) \vee \varphi_h(h_{\overline{p}}+1, v_{\overline{p}})$, so from $\neg\varphi_h(h_{\overline{p}''}, v_{\overline{p}})$ and lemma 1, $\neg\varphi_h(\overline{q}) \Rightarrow \neg\varphi_h(\overline{q}+1_h)$ must be true. By lemma 5, $\neg\varphi_v(\overline{q}) \Rightarrow \neg\varphi_v(\overline{q}+1_h)$ So if $v_{\overline{p}''} = v_{\overline{p}'}$, from the two implications above, and from $\neg\varphi_v(h_{\overline{p}''}, v_{\overline{p}''}) \wedge \neg\varphi_h(h_{\overline{p}''}, v_{\overline{p}''})$ we get $\neg\varphi_v(h_{\overline{p}'}, v_{\overline{p}'}) \wedge \neg\varphi_h(h_{\overline{p}'}, v_{\overline{p}'})$, so $\neg s(\overline{p}')$ must hold. If on the other hand $v_{\overline{p}''} < v_{\overline{p}'}$, since by lemma 5, $\neg\varphi_v(\overline{q}) \Rightarrow \neg\varphi_v(\overline{q}+1_h)$ and $\neg\varphi_v(\overline{q}) \Rightarrow \neg\varphi_v(\overline{q}+1_v)$ hold, from $\neg\varphi_v(h_{\overline{p}''}, v_{\overline{p}''})$ it must follow that $\neg\varphi_v(\overline{p}') \wedge \neg\varphi_v(\overline{p}'-1_v)$, and consequently, $\neg s(\overline{p}')$ must hold. $\diamond$

Define

$$c(\overline{p},\overline{p}') \Leftrightarrow \forall \overline{p}'' : \overline{p} \leq \overline{p}'' \leq \overline{p}' \wedge \varphi_h(\overline{p}'') \wedge \varphi_v(\overline{p}'') \Rightarrow \begin{pmatrix} \varphi_h(\overline{p}''-1_h) \\ \vee \\ \varphi_v(\overline{p}''-1_v) \end{pmatrix}$$

The intuition behind the predicate $c(\overline{p},\overline{p}')$ is illustrated in figure 9. It essentially says that the area defined by $\overline{p} \leq \overline{p}'' \leq \overline{p}' \;\wedge\; s(\overline{p}'')$ must be "wide" enough to allow for a path to be included in it.

Lemma 7:
Consider $\overline{p}$ and $\overline{p}'$ where $\overline{p} \leq \overline{p}'$ and $h_{\overline{p}} \neq h_{\overline{p}'}$. Then

1. $c(\overline{p},\overline{p}') \Rightarrow c(\overline{p}+1_h,\overline{p}')$
2. $c(\overline{p},\overline{p}') \Rightarrow c(\overline{p},\overline{p}'-1_h)$

$\diamond$

Proof:
Follows immediately from the definition. $\diamond$

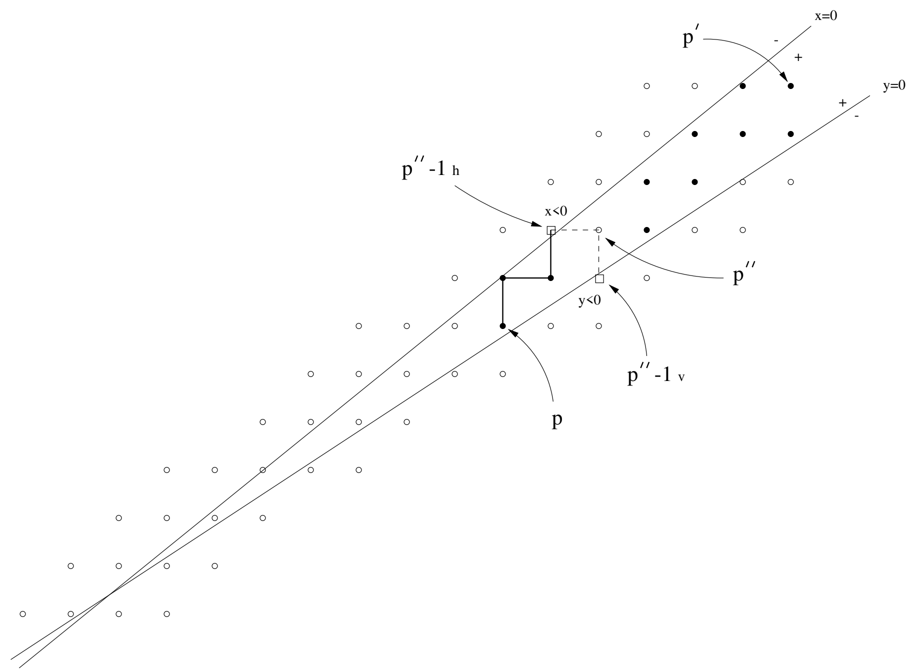


Figure 9

Now we define the reachability relation $r_{hv}(\overline{p}, \overline{p}')$ as

$$r_{hv}(\overline{p}, \overline{p}') \Leftrightarrow \overline{p} = \overline{p}' \vee \left(\begin{array}{c} \overline{p} \leq \overline{p}' \\ \wedge \\ s(\overline{p}) \wedge s(\overline{p}') \\ \wedge \\ e(\overline{p}, \overline{p}') \wedge c(\overline{p}, \overline{p}') \end{array} \right)$$

We first prove soundness of this relation. That is, if $r_{hv}(\overline{p}, \overline{p}')$ holds there endeed exists an admissible path from $\overline{p}$ to $\overline{p}'$:

$$r_{hv}(\overline{p}, \overline{p}') \;\Rightarrow\; \overline{p} \xrightarrow{(h+v)^*} \overline{p}'$$

This is stated in lemma 17 which is preceeded by several technical lemmas. Soundness of $\overline{p} \xrightarrow{(h+v)^*} \overline{p}'$ then immediately follows from lemma 8.

Lemma 8:
Assume $\overline{p} \leq \overline{p}'$, $h_{\overline{p}} < h_{\overline{p}'}$, $r_{hv}(\overline{p}, \overline{p}')$ and $\neg r_{hv}(\overline{p} + 1_h, \overline{p}')$. Then $\neg s(\overline{p} + 1_h)$ must hold. $\diamond$

Proof:
$\neg r_{hv}(\overline{p} + 1_h, \overline{p}')$ implies

$$\overline{p} \neq \overline{p}' \wedge \left(\begin{array}{c} \overline{p} + 1_h \not\leq \overline{p}' \\ \vee \\ \neg s(\overline{p} + 1_h) \vee \neg s(\overline{p}') \\ \vee \\ \neg e(\overline{p} + 1_h, \overline{p}') \vee \neg c(\overline{p} + 1_h, \overline{p}') \end{array} \right)$$

Since $\overline{p} \leq \overline{p}'$ and $h_{\overline{p}} < h_{\overline{p}'}$ hold, $\overline{p} + 1_h \not\leq \overline{p}'$ cannot be true. By $r_{hv}(\overline{p}, \overline{p}')$ we have $s(\overline{p}')$ and $c(\overline{p}, \overline{p}')$, and by lemma 7, $c(\overline{p} + 1_h, \overline{p}')$ holds. So $\neg s(\overline{p} + 1_h) \vee \neg e(\overline{p} + 1_h, \overline{p}')$ must hold. Suppose $s(\overline{p} + 1_h)$ was true, then $\neg e(\overline{p} + 1_h, \overline{p}')$ must hold. By $r_{hv}(\overline{p}, \overline{p}')$ we have $e(\overline{p}, \overline{p}')$. But then, by lemma 6, $\neg s(\overline{p}')$ must hold, which is a contradiction. Thus, $\neg s(\overline{p} + 1_h)$ holds. $\diamond$

Lemma 9:
Assume $s(\overline{p})$ and $\neg\varphi_v(\overline{p})$. Then

1. $\varphi_v(\overline{p} - 1_v)$
2. $\neg\varphi_v(\overline{q}) \Rightarrow \neg\varphi_v(\overline{q} + 1_v)$

$\diamond$

Proof:

1. Follows immediately from the definition of $s(\overline{p})$.
2. Follows from lemma 1 and from $\varphi_v(\overline{p} - 1_v) \wedge \neg\varphi_v(\overline{p})$.

$\diamond$

Lemma 10:
Assume $\neg s(\overline{p} + 1_h)$ and $\varphi_h(\overline{p})$. Then $\neg\varphi_v(\overline{p} + 1_h)$ holds. $\diamond$

Proof:
Follows immediately from the definition of $s(\overline{p})$. $\diamond$

Lemma 11:
Let $e(\overline{p}, \overline{p}')$, $\neg\varphi_h(\overline{p} + 1_h)$ and $\neg\varphi_v(\overline{p} + 1_h)$ hold. Then $\varphi_v(\overline{p})$ must hold. $\diamond$

Proof:
Follows immediately from the definition of $e(\overline{p}, \overline{p}')$. $\diamond$

Lemma 12:
Let $\overline{p} \leq \overline{p}'$, $v_{\overline{p}} < v_{\overline{p}'}$, $s(\overline{p}')$, $\neg\varphi_v(\overline{p})$ and $\neg\varphi_v(\overline{q}) \Rightarrow \neg\varphi_v(\overline{q} + 1_v)$ hold. Then $\varphi_v(\overline{q}) \Rightarrow \varphi_v(\overline{q} + 1_h)$ must hold. $\diamond$

Proof:
By lemma 3, $\varphi_v(\overline{p}' - 1_v)$ must hold. But since $\neg\varphi_v(\overline{q}) \Rightarrow \neg\varphi_v(\overline{q} + 1_v)$ is true, it cannot be the case that $h_{\overline{p}} = h_{\overline{p}'}$, because $\neg\varphi_v(\overline{p})$ would then imply $\neg\varphi_v(\overline{p}' - 1_v)$. Thus $h_{\overline{p}} < h_{\overline{p}'}$. It cannot be the case that $\neg\varphi_v(\overline{q}) \Rightarrow \neg\varphi_v(\overline{q} + 1_h)$, because then $\neg\varphi_v(\overline{p}' - 1_v)$ would hold. Thus, by lemma 1, $\varphi_v(\overline{q}) \Rightarrow \varphi_v(\overline{q} + 1_h)$ must hold. $\diamond$

Lemma 13:
Consider $\overline{p} \leq \overline{p}'$ such that $h_{\overline{p}} < h_{\overline{p}'}$ and $v_{\overline{p}} < v_{\overline{p}'}$, and let $s(\overline{p})$, $s(\overline{p}')$, $\neg s(\overline{p} + 1_h)$, $\varphi_h(\overline{p})$ and $e(\overline{p}, \overline{p}')$ hold. Then $\varphi_v(\overline{p})$ must hold. $\diamond$

Proof:

Assume, by contradiction, that $\neg\varphi_v(\overline{p})$ holds. By lemma 10, $\neg\varphi_v(\overline{p}+1_h)$ holds. Assume $\neg\varphi_h(\overline{p}+1_h)$. By lemma 11, $\varphi_v(\overline{p})$ holds, which contradicts the assumption of $\neg\varphi_v(\overline{p})$. Thus $\varphi_h(\overline{p}+1_h)$ holds. By lemma 9, $\neg\varphi_v(\overline{q}) \Rightarrow \neg\varphi_v(\overline{q}+1_v)$ is true, and then by lemma 12, $\varphi_v(\overline{q}) \Rightarrow \varphi_v(\overline{q}+1_h)$ holds. But by lemma 9, $\varphi_v(\overline{p}-1_v)$ holds, so $\varphi_v(\overline{p}+1_h-1_v)$ is true. But since $\varphi_h(\overline{p}+1_h)$ and $\varphi_v(\overline{p}+1_h-1_v)$ then are true, $s(\overline{p}+1_h)$ must be true, which is a contradiction. Thus, $\varphi_v(\overline{p})$ must be true. $\diamond$

Lemma 14:
Consider $\overline{p} \leq \overline{p}'$ such that $h_{\overline{p}} < h_{\overline{p}'}$ and $v_{\overline{p}} < v_{\overline{p}'}$, and assume $r_{hv}(\overline{p},\overline{p}')$, $\neg r_{hv}(\overline{p}+1_h,\overline{p}')$ and $\varphi_h(\overline{p})$. Then $\varphi_v(\overline{p})$ must be true. $\diamond$

Proof:
By $r_{hv}(\overline{p},\overline{p}')$ we have $s(\overline{p})$, $s(\overline{p}')$ and $e(\overline{p},\overline{p}')$. By lemma 8 we get $\neg s(\overline{p}+1_h)$. The result follows then immediately from lemma 13. $\diamond$

Define

$$d(\overline{p},\overline{p}') = |\overline{p}' - \overline{p}|$$

Thus, $d(\overline{p},\overline{p}')$ is the Manhattan distance between the two points (it is assumed that $\overline{p} \leq \overline{p}'$).

Lemma 15:
Consider $\overline{p} \leq \overline{p}'$, and assume $\varphi_h(\overline{p})$, $\varphi_v(\overline{p})$, $\neg\varphi_h(\overline{p}+1_v)$ and $\neg\varphi_v(\overline{p}+1_h)$. Then

$$\forall \overline{p}' : \overline{p} \leq \overline{p'} \wedge \overline{p} \neq \overline{p}' \wedge c(\overline{p},\overline{p}') \Rightarrow \begin{pmatrix} \neg\varphi_h(\overline{p}') \\ \vee \\ \neg\varphi_v(\overline{p}') \end{pmatrix}$$

$\diamond$

Proof:
Induction over $d(\overline{p},\overline{p}')$. For the base case, assume $d(\overline{p},\overline{p}') = 1$. Then $\overline{p}' = \overline{p}+1_h$, in which case $\neg\varphi_v(\overline{p}+1_h)$ holds, or $\overline{p}' = \overline{p}+1_v$, in which case $\neg\varphi_h(\overline{p}+1_v)$ holds. For the induction hypothesis, assume the statement is true for all $\overline{p}'$ such that $d(\overline{p},\overline{p}') = n$, and consider any $\overline{p}''$ with $d(\overline{p},\overline{p}'') = n+1$ (where $1 \leq n$), and $c(\overline{p},\overline{p}'')$ hold. First note that since $\varphi_h(\overline{p})$ and $\neg\varphi_h(\overline{p}+1_v)$ hold, by lemma 1, $\neg\varphi_h(\overline{q}) \Rightarrow \neg\varphi_h(\overline{q}+1_v)$. Similarly $\neg\varphi_v(\overline{q}) \Rightarrow \neg\varphi_v(\overline{q}+1_h)$ holds. There are three possibilities: $h_{\overline{p}} = h_{\overline{p}''} \wedge v_{\overline{p}} < v_{\overline{p}''}$, $h_{\overline{p}} < h_{\overline{p}''} \wedge v_{\overline{p}} = v_{\overline{p}''}$ or $h_{\overline{p}} < h_{\overline{p}''} \wedge v_{\overline{p}} < v_{\overline{p}''}$. Consider the case when $h_{\overline{p}} = h_{\overline{p}''} \wedge v_{\overline{p}} < v_{\overline{p}''}$. Since $\neg\varphi_h(\overline{p}+1_v)$ and $\neg\varphi_h(\overline{q}) \Rightarrow \neg\varphi_h(\overline{q}+1_v)$ hold, clearly $\neg\varphi_h(\overline{p}'')$ must hold. Analogously, the lemma holds also for the case when $h_{\overline{p}} < h_{\overline{p}''} \wedge v_{\overline{p}} = v_{\overline{p}''}$. Thus we may assume that $h_{\overline{p}} < h_{\overline{p}''} \wedge v_{\overline{p}} < v_{\overline{p}''}$ holds. Suppose, by contradiction, that $\varphi_h(\overline{p}'') \wedge \varphi_v(\overline{p}'')$ was true, and consider $\overline{p}''-1_h$. Since $h_{\overline{p}} < h_{\overline{p}''}$, $d(\overline{p},\overline{p}''-1_h) = n$. By lemma 7, $c(\overline{p},\overline{p}''-1_h)$ holds, and since $\neg\varphi_v(\overline{q}) \Rightarrow \neg\varphi_v(\overline{q}+1_h)$ and $\varphi_v(\overline{p}'')$ hold, $\varphi_v(\overline{p}''-1_h)$ must hold. By the induction hypothesis then, $\neg\varphi_h(\overline{p}''-1_h)$ must hold. Next consider $\overline{p}''-1_v$. By the symmetric reasoning we get that $\neg\varphi_v(\overline{p}''-1_v)$ must hold. Thus $\neg\varphi_h(\overline{p}''-1_h) \wedge \neg\varphi_v(\overline{p}''-1_v)$ holds. But this contradicts the assumption that $c(\overline{p},\overline{p}'')$ holds. Thus it must be the case that $\neg\varphi_h(\overline{p}'') \vee \neg\varphi_v(\overline{p}'')$. $\diamond$

Lemma 16:
Consider $\overline{p} \leq \overline{p}'$ such that $h_{\overline{p}} < h_{\overline{p}'}$ and $v_{\overline{p}} < v_{\overline{p}'}$, and assume $r_{hv}(\overline{p},\overline{p}')$, $\varphi_h(\overline{p})$, $\varphi_v(\overline{p})$ and $\neg r_{hv}(\overline{p}+1_h,\overline{p}')$. Then $r_{hv}(\overline{p}+1_v,\overline{p}')$ is true. $\diamond$

Proof:

Suppose, by contradiction, that $\neg r_{hv}(\overline{p}+1_v,\overline{p}')$ holds. Two applications of lemma 8 yields $\neg s(\overline{p}+1_h)$ and $\neg s(\overline{p}+1_v)$ (by symmetry lemma 8 holds both for $\overline{p}+1_h$ and $\overline{p}+1_v$). $\neg s(\overline{p}+1_h)$ yields (by definition)

$$\begin{array}{c}\neg\varphi_h(\overline{p}+1_h)\vee\neg\varphi_v(\overline{p}+1_h-1_v)\\ \wedge\\ \neg\varphi_h(\overline{p})\vee\neg\varphi_v(\overline{p}+1_h)\end{array}$$

and $\neg s(\overline{p}+1_v)$ yields

$$\begin{array}{c}\neg\varphi_h(\overline{p}+1_v)\vee\neg\varphi_v(\overline{p})\\ \wedge\\ \neg\varphi_h(\overline{p}+1_v-1_h)\vee\neg\varphi_v(\overline{p}+1_v)\end{array}$$

$\varphi_h(\overline{p})$ and $\varphi_v(\overline{p})$ then yields $\neg\varphi_h(\overline{p}+1_v)$ and $\neg\varphi_v(\overline{p}+1_h)$. By lemma 1,

$$\neg\varphi_h(\overline{q})\Rightarrow\neg\varphi_h(\overline{q}+1_v)$$

and

$$\neg\varphi_v(\overline{q})\Rightarrow\neg\varphi_v(\overline{q}+1_h)$$

must then hold. But from $r_{hv}(\overline{p},\overline{p}')$, we get $s(\overline{p}')$, which by definition implies $\varphi_h(\overline{p}')\wedge\varphi_h(\overline{p}'-1_h)$ and $\varphi_v(\overline{p}')\wedge\varphi_v(\overline{p}'-1_v)$. So by lemma 1

$$\varphi_h(\overline{q})\Rightarrow\varphi_h(\overline{q}+1_h)$$

and

$$\varphi_v(\overline{q})\Rightarrow\varphi_v(\overline{q}+1_v)$$

must be true. But by lemma 3 then, $\varphi_h(\overline{p}')\wedge\varphi_v(\overline{p}')$ holds. Since, as noted above, $\varphi_h(\overline{p})\wedge\varphi_v(\overline{p})$ and $\neg\varphi_h(\overline{p}+1_v)\wedge\neg\varphi_v(\overline{p}+1_h)$ hold, and since $r_{hv}(\overline{p},\overline{p}')$ yields $c(\overline{p},\overline{p}')$, lemma 15 applies and says that $\neg\varphi_h(\overline{p}')\vee\neg\varphi_v(\overline{p}')$ holds, which contradicts $\varphi_h(\overline{p}')\wedge\varphi_v(\overline{p}')$. Thus the assumption $\neg r_{hv}(\overline{p}+1_v,\overline{p}')$ must be false. $\diamond$

Lemma 17:
Assume $r_{hv}(\overline{p},\overline{p}')$. Then there exists an admissible path w such that $\overline{p}\xrightarrow{w}\overline{p}'$. $\diamond$

Proof:
Induction over $d(\overline{p},\overline{p}')$. The base case when $d(\overline{p},\overline{p}')=0$ is trivial. For the induction hypothesis, assume that for all $\overline{p}$ and $\overline{p}'$ such that $d(\overline{p},\overline{p}')=n$ and $r_{hv}(\overline{p},\overline{p}')$, there exists an admissible path w such that $\overline{p}\xrightarrow{w}\overline{p}'$. Consider any $\overline{p}''$ with $d(\overline{p}'',\overline{p}')=n+1$ and $r_{hv}(\overline{p}'',\overline{p}')$. There are three cases: $h_{\overline{p}''}<h_{\overline{p}'}\wedge v_{\overline{p}''}=v_{\overline{p}'}$, $h_{\overline{p}''}=h_{\overline{p}'}\wedge v_{\overline{p}''}<v_{\overline{p}'}$ or $h_{\overline{p}''}<h_{\overline{p}'}\wedge v_{\overline{p}''}<v_{\overline{p}'}$. Consider the first case. By $r_{hv}(\overline{p}'',\overline{p}')$ we have $s(\overline{p}'')$ and $s(\overline{p}')$, which means that $\varphi_h(\overline{p}'')\vee\varphi_h(\overline{p}''-1_h)$ and $\varphi_h(\overline{p}')\vee\varphi_h(\overline{p}'-1_h)$ hold. Now, $\varphi_h(\overline{q})$ must be true for all $\overline{p}''\le\overline{q}\le\overline{p}'-1_h$, otherwise $\neg\varphi_h(\overline{q})\Rightarrow\neg\varphi_h(\overline{q}+1_h)$ would hold by lemma 1, and consequently $\neg\varphi_h(\overline{p}')\wedge\neg\varphi_h(\overline{p}'-1_h)$ would be the case. But then h^{n+1} is an admisible path such that $\overline{p}''\xrightarrow{h^{n+1}}\overline{p}'$. The case when $h_{\overline{p}''}=h_{\overline{p}'}\wedge v_{\overline{p}''}<v_{\overline{p}'}$ is treated analogously. Thus we may assume that $h_{\overline{p}''}<h_{\overline{p}'}\wedge v_{\overline{p}''}<v_{\overline{p}'}$. By $r_{hv}(\overline{p}'',\overline{p}')$ we have $s(\overline{p}'')$, which yields $\varphi_h(\overline{p}'')\vee\varphi_v(\overline{p}'')$. Suppose that $\varphi_h(\overline{p}'')$ holds. Then an h-move is admissible, and since $h_{\overline{p}''}<h_{\overline{p}'}$, $d(\overline{p}''+1_h,\overline{p}')=n$. If $r_{hv}(\overline{p}''+1_h,\overline{p}')$ is true, by the induction hypothesis there exists an admissible path w such that $(\overline{p}''+1_h)\xrightarrow{w}\overline{p}'$ holds, but then hw is an admissible path such that $\overline{p}''\xrightarrow{hw}\overline{p}'$ and we're done. Suppose on the other hand that $\neg r_{hv}(\overline{p}''+1_h,\overline{p}')$ holds. By lemma 14, $\varphi_v(\overline{p}'')$ is true. Thus a v-move is admissible. Since $v_{\overline{p}''}<v_{\overline{p}'}$, $d(\overline{p}''+1_v,\overline{p}')=n$, and by lemma 16, $r_{hv}(\overline{p}''+1_v,\overline{p}')$ is true. By the induction hypothesis then, there exists a admissible path w such that $(\overline{p}''+1_v)\xrightarrow{w}\overline{p}'$ holds, but then

vw is an admissible path such that $\overline{p}'' \xrightarrow{vw} \overline{p}'$. This concludes the proof. $\diamond$

Theorem 5:
Assume $R(\overline{p}, \overline{p}')$. Then there exists an admissible path w such that $\overline{p} \xrightarrow{w} \overline{p}'$. $\diamond$

Proof:
Follows immediately from lemma 17 and proposition 8. $\diamond$

Comleteness of $R(\overline{p}, \overline{p}')$ is mutch simpler. The idea is to show that all admissible paths inside the s-area are captured by the $r_{hv}(\overline{p}, \overline{p}')$-relation. Completeness of $R(\overline{p}, \overline{p}')$ then follows by observing that any point where an admissible path changes direction must lie in the s-area, so $r_{hv}(\overline{p}, \overline{p}')$ connects the points of the first and the last change of direction, and $\overline{p} \xrightarrow{h^*} \overline{p}'$ and $\overline{p} \xrightarrow{v^*} \overline{p}'$ characterises the prefix and suffix of the path.

Lemma 18:
Assume $s(\overline{p})$ and $\overline{p} \xrightarrow{w} \overline{p}'$ for some w. Then $c(\overline{p}, \overline{p}')$ is true. $\diamond$

Proof:
Induction over the length of w. If $w = \epsilon$, the statement is trivially true. For the induction hypothesis, assume the lemma is true for w. That is, For all points $\overline{p} \leq \overline{p}' \leq \overline{p}w$ (with $\overline{p} \neq \overline{p}'$) such that $\varphi_h(\overline{p}') \wedge \varphi_v(\overline{p}')$ is true, $\varphi_h(\overline{p}' - 1_h) \vee \varphi_v(\overline{p}' - 1_v)$ holds. Consider $w' = wh$. When we move one step horizontally, the only new points that must be considered, are those $\overline{p}''$ such that $h_{\overline{p}w'} = h_{\overline{p}''}$ and $v_{\overline{p}} \leq v_{\overline{p}''} \leq v_{\overline{p}w'}$. So it is enough to show that every such point that satisfies $\varphi_h(\overline{p}'') \wedge \varphi_v(\overline{p}'')$ must also satisfy $\varphi_h(\overline{p}'' - 1_h) \vee \varphi_v(\overline{p}'' - 1_v)$. Since w' is admissible, $\varphi_h(\overline{p}w' - 1_h)$ is true. Consider any $\overline{p}''$ such as mentioned above and assume $\neg\varphi_h(\overline{p}'' - 1_h)$. By lemma 1, $\varphi_h(\overline{q}) \Rightarrow \varphi_h(\overline{q} + 1_v)$ must hold, otherwise $\varphi_h(\overline{p}w' - 1_h)$ would not be true. Also $\varphi_h(\overline{q}) \Rightarrow \varphi_h(\overline{q} + 1_h)$ must hold, otherwise $\varphi_h(\overline{p}'')$ would not be true. But from $s(\overline{p})$ we get $\varphi_h(\overline{p}) \vee \varphi_h(\overline{p} - 1_h)$, so for all points $\overline{q}$ such that $\overline{p} \leq \overline{q}$, we have that $\varphi_h(\overline{q})$ holds. This contradicts the assumption that $\neg\varphi_h(\overline{p}'' - 1_h)$ holds. So $\varphi_h(\overline{p}'' - 1_h)$ must be true. The case when $w' = wv$ is treated analogously. $\diamond$

Lemma 19:
Assume $s(\overline{p})$, $s(\overline{p}')$ and $\overline{p} \xrightarrow{w} \overline{p}'$ for some path w. Then $e(\overline{p}, \overline{p}')$ is true. $\diamond$

Proof:
Induction over length of the path. The statement is trivially true when $w = \epsilon$. For the induction hypothesis assume the statement is true for $\overline{p}w$. Consider $w' = wh$ and let $\overline{p} \xrightarrow{wh} \overline{p}'$. When we move one step horizontally, the only new points that must be considered, are those $\overline{p}''$ such that $h_{\overline{p}'} = h_{\overline{p}''}$ and $v_{\overline{p}} \leq v_{\overline{p}''} \leq v_{\overline{p}'}$. So it is enough to show that every such point that satisfies $\neg\varphi_h(\overline{p}'') \wedge \neg\varphi_v(\overline{p}'')$ must also satisfy $\varphi_v(h, v'') \vee \varphi_h(h'', v)$ Consider such a point $\overline{p}''$. Since by $s(\overline{p}')$, $\varphi_h(\overline{p}') \vee \varphi_v(\overline{p}')$, clearly $v_{\overline{p}''} < v_{\overline{p}'}$. But then, by lemma 1, $\varphi_v(\overline{q}) \Rightarrow \varphi_v(\overline{q} + 1_v)$, must hold. By $s(\overline{p})$ and lemma 3 we thus get $\varphi_v(\overline{p})$ and so $\varphi_v(h, v'')$ must hold. The case when $w' = wv$ is treated analogously. $\diamond$

Lemma 20:
Assume $s(\overline{p})$, $s(\overline{p}')$ and $\overline{p} \xrightarrow{w} \overline{p}'$ for some path w. Then $r_{hv}(\overline{p}, \overline{p}')$ is true. $\diamond$

Proof:
If $w = \epsilon$ then $\overline{p} = \overline{p}'$, so $r_{hv}(\overline{p}, \overline{p}')$ is true. If $w \neq \epsilon$, Then claerly $\overline{p} \leq \overline{p}'$. Lemma 18 and lemma 19 together with the assumptions $s(\overline{p})$ and $s(\overline{p}')$ yields the result. $\diamond$

Theorem 6:
Assume $\overline{p} \xrightarrow{w} \overline{p}'$ for some admissible path w. Then $R(\overline{p}, \overline{p}')$ holds. $\diamond$

Proof:
For any path w it is clearly true that $w = u_1u_2u_3$ where $u_1 \in (h^* + v^*)$, $u_2 \in (h+v)^*$ and $u_3 \in (h^* + v^*)$. If $w = \epsilon$ the theorem is trivially true. Let u_1 be the longest prefix belonging to $(h^* + v^*)$. If $w \neq \epsilon$ then $u_1 \neq \epsilon$ since w at least must begin with h, say. Since w is admissible, $\varphi_h(\overline{p})$ must be true, and since u_1 ends with h, $\varphi_h(\overline{p}u_1 - 1_h)$ also must be true. Clearly $\overline{p}u_1 = \overline{p} + n \cdot 1_h$ for some n, so $\overline{p} \xrightarrow{h^*} (\overline{p}u_1)$ holds. If $u_2u_3 = \epsilon$, the theorem follows trivially. Suppose $u_2u_3 \neq \epsilon$. Let u_3 be the longest suffix belonging to $(h^* + v^*)$. If $u_2 = \epsilon$, $u_3 \in v^*$ must hold, and the theorem follows by similar reasoning as above. Suppose $u_2 \neq \epsilon$. Then $u_3 \neq \epsilon$ since u_2 must end with some move, and this belongs to $(h^* + v^*)$. u_2 must begin with v, otherwise u_1 would not be the longest path of h^*. Since w is admissible, $\varphi_v(\overline{p}u_1)$ must hold. This, together with $\varphi_h(\overline{p}u_1 - 1_h)$, yields $s(\overline{p}u_1)$. Suppose that u_2 ends with h, say. Then $\varphi_h(\overline{p}u_1u_2 - 1_h)$ holds, and u_3 must begin with v, so $\varphi_v(\overline{p}u_1u_2)$ holds, and thus $s(\overline{p}u_1u_2)$ holds. By lemma 20, $r_{hv}(\overline{p}u_1, \overline{p}u_1u_2)$ holds. Finaly, since $\varphi_v(\overline{p}u_1u_2)$ and $\varphi_v(\overline{p}u_1u_2u_3 - 1_v)$ holds, $(\overline{p}u_1u_2) \xrightarrow{v^*} (\overline{p}u_1u_2u_3)$ holds. This concludes the proof. $\diamond$

Note that the whole proof only uses the properties stated in lemma 1 and lemma 2. The expression given for the least fixpoint can therefore be addapted to more general situations than mereley linear integer arithmetic with one linear constraint. Conjunctions, nonlinear constraints, reals e.t.c. can be allowed as long as lemma 1 and lemma 2 are satisfied. For instance, any constraints of the form $f_i(h_{\overline{p}}, v_{\overline{p}}) \geq 0$, where $i = h, v$, is allowed for functions f_i that are monotonic in both arguments. The construction is limited, however, to two recursive rules.

4.2 Alternated incrementation matrices

In this section we consider programs of the form

$$\begin{array}{llcll} & p(x_0, y_0). & & & \\ h: & p(x+e, y+\lambda) & \leftarrow & x \geq 0, & p(x,y). \\ v: & p(x+\mu, y+f) & \leftarrow & y \geq 0, & p(x,y). \end{array}$$

That is, we restrict our attention to programs of linear integer arithmetic. More specifically, we focus on programs which have "alternated incrementation matrices", i.e., matrices with a negative diagonal and a positive antidiagonal, or the opposite. For a point $\overline{p} = \langle h, v \rangle^T$, the corresponding variables $x_{\overline{p}}$ and $y_{\overline{p}}$ are defined by $x_{\overline{p}} = eh + \mu v + \kappa_h$ and $y_{\overline{p}} = \lambda h + fv + \kappa_v$.

4.2.1 negative diagonal

Consider a program with the signs of the matrix given by

$$\Phi = \begin{pmatrix} -e & \lambda \\ \mu & -f \end{pmatrix}$$

where $e, f > 0$ and $\lambda, \mu \geq 0$. Depending on the values of the coefficients, this corresponds to one of the situations illustrated in figures 10 and 11. Consider two points $\overline{p}$ and $\overline{p}'$, and denote by $x_{\overline{p}}$, $y_{\overline{p}}$, $x_{\overline{p}'}$ and $y_{\overline{p}'}$ the x and y values associated with the points.

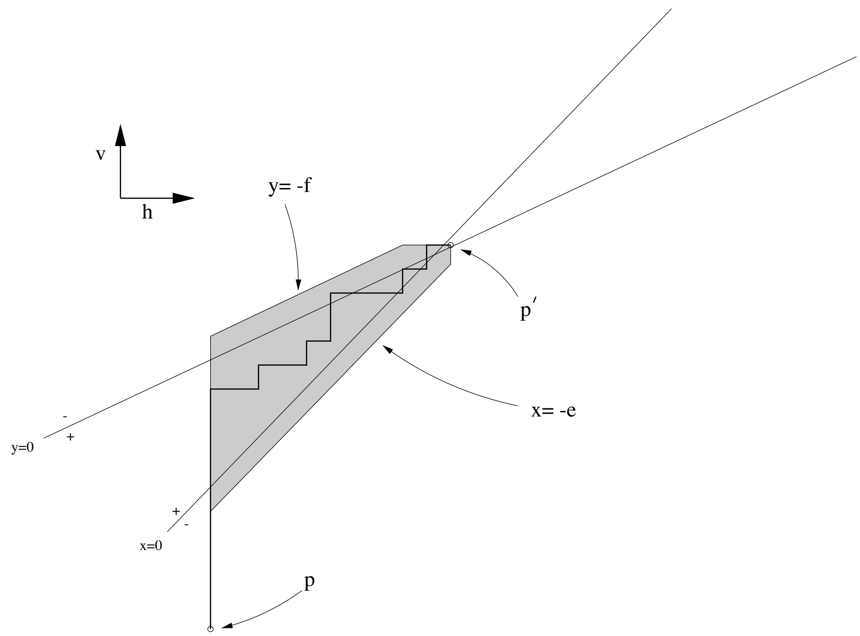


Figure 10

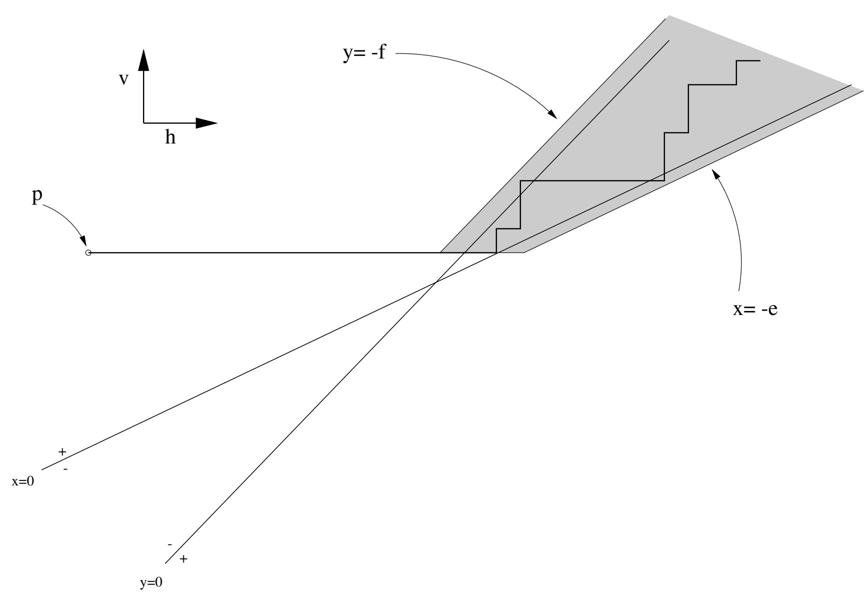


Figure 11

Proposition 11:

$$\forall \overline{p}, \overline{p}', x_{\overline{p}}, y_{\overline{p}}, x_{\overline{p}'}, y_{\overline{p}'} : \left(\begin{array}{c} x_{\overline{p}} \geq -e \\ \wedge \\ y_{\overline{p}} \geq -f \end{array} \right) \wedge \overline{p} \xrightarrow{(h+v)^*} \overline{p}' \quad \Rightarrow \left(\begin{array}{c} x_{\overline{p}'} \geq -e \\ \wedge \\ y_{\overline{p}'} \geq -f \end{array} \right)$$

$\diamond$

Proof:
Follows easily by fixpoint induction. $\diamond$

By looking at figures 10 and 11, this can be interpreted as that one can never move very far away from the cone once it has been reached. Note the special case illustrated in figures 12 and 13 (figure 12 is the same situation as that illustrated in figure 6). The figures 11 and 12

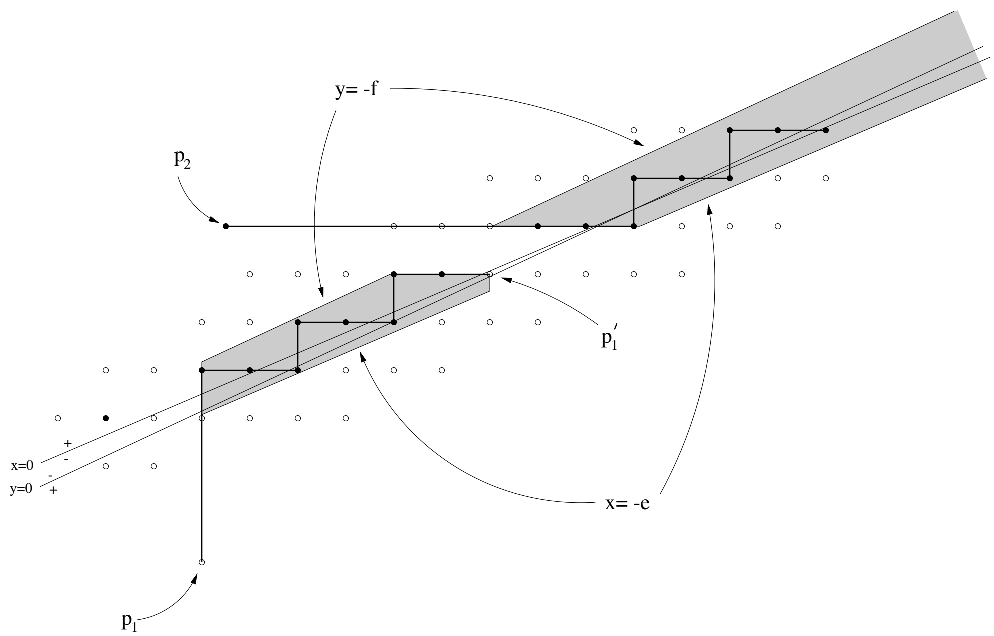


Figure 12

shows the situation when

$$\begin{vmatrix} -e & \lambda \\ \mu & -f \end{vmatrix} < 0$$

That is, the area defined by $\varphi_h(\overline{p}) \wedge \varphi_v(\overline{p})$ diverge. This means that if one starts in a point sufficiently far up to the right, there is an infinite path starting at this point. In figure 12 it is seen that there are only finitely many points reachable from $\overline{p}_1$ since it lies too far to the left. This is formalized in the following proposition.

Proposition 12:
Let

$$\begin{vmatrix} -e & \lambda \\ \mu & -f \end{vmatrix} \leq 0$$

and let $e, f > 0$. Consider the point $\overline{p}$. If $\lambda x_{\overline{p}} + e y_{\overline{p}} \geq 0$ or $f x_{\overline{p}} + \mu y_{\overline{p}} \geq 0$, then there exists an infinite path starting from $\overline{p}$. $\diamond$

Proof:
We give the proof for the case when $\lambda x_{\overline{p}} + e y_{\overline{p}} \geq 0$. The proof of the case when $f x_{\overline{p}} + \mu y_{\overline{p}} \geq 0$ holds is similar. When we apply an h-step we add $-e$ to x and λ to y, and when we apply a v-step we add $-f$ to y and μ to x. It is enough if we can show that for all $w \in (h+v)^*$, if $\langle x', y' \rangle^T = \langle x_{\overline{p}}, y_{\overline{p}} \rangle^T + \Phi^T \overline{w}$ then $x' \geq 0$ or $y' \geq 0$. That is, no matter how we walk, at least one next step can be taken. Assume to the contrary that for some w with h_w horizontal steps and v_w vertical steps,

$$\begin{array}{l} x_{\overline{p}} - e h_{\overline{w}} + \mu v_{\overline{w}} < 0 \\ y_{\overline{p}} + \lambda h_{\overline{w}} - f_{\overline{w}} < 0 \end{array}$$

holds. Then

$$\lambda x_{\overline{p}} + e y_{\overline{p}} - (ef - \mu\lambda) v_{\overline{w}} < 0$$

must hold. But by assumption, $\lambda x_{\overline{p}} + e y_{\overline{p}} \geq 0$ and

$$ef - \mu\lambda = \begin{vmatrix} -e & \lambda \\ \mu & -f \end{vmatrix} \leq 0$$

hold, which is a contradiction (since $v_{\overline{w}} \geq 0$). $\diamond$

The converse of this theorem does not hold. We observe that if $\lambda x + ey < 0$, $0 \leq y$ and $-e \leq x$ hold, then $-e \leq x < 0$ and $0 \leq y < \lambda$. Similarly, $fx + \mu y < 0$, $0 \leq x$ and $-f \leq y$ hold, then $-f \leq y < 0$ and $0 \leq x < \mu$. Thus, there are only finitely many classes of points for which $-e \geq x \ \wedge \ -f \geq y$ and such that there are only finitely many paths starting at these points (two points $\overline{p}$ and $\overline{p}'$ belong to the same class iff $x_{\overline{p}} = x_{\overline{p}'}$ and $y_{\overline{p}} = y_{\overline{p}'}$).

The properties $\lambda x + ey \geq 0$ and $fx + \mu y \geq 0$ are invariant under the assumption that

$$\begin{vmatrix} -e & \lambda \\ \mu & -f \end{vmatrix} \leq 0$$

since then the inequalities says that $v \geq v_0$ and $h \geq h_0$ respectively (for some constants h_0 and v_0), and this property is clearly preserved.

For the special case when the matrix of the program has the form

$$\begin{pmatrix} -e & \lambda \\ \mu & -f \end{pmatrix}$$

where $e, f > 0$ and $\lambda, \mu \geq 0$, the expression for the fixpoint can be given in more "compiled" form. We distinguish the situations when the determinant is strictly positive, and when it is negative or zero.

4.2.2 positive determinant

Consider first the situation when

$$\begin{vmatrix} -e & \lambda \\ \mu & -f \end{vmatrix} > 0$$

and $e, f > 0$, which is illustrated in figures 10 and 13. Consider figure 10. The point $\overline{p}' = \langle h_{\overline{p}'}, v_{\overline{p}'} \rangle^T$ is given by a solution to

$$\begin{array}{ccrrrcl} -e & \leq & -e h_{\overline{p}'} & +\mu v_{\overline{p}'} & +\alpha_h & < & 0 \\ -f & \leq & \lambda h_{\overline{p}'} & -f v_{\overline{p}'} & +\alpha_v & < & 0 \\ & & & & \overline{p}' & \geq & \overline{p} \end{array}$$

or equivalently

$$\begin{array}{rcccl} -e & \leq & x_{\overline{p}'} & < & 0 \\ -f & \leq & y_{\overline{p}'} & < & 0 \\ & & \overline{p}' & \geq & \overline{p} \end{array}$$

(For figure 13 the points $\overline{p}$ and $\overline{p}'$ are $\overline{p}_1$ and $\overline{p}'_1$, or $\overline{p}_2$ and $\overline{p}'_2$ respectively). The set of inequalities above has a finite number of solutions and always has at least one solution when $\overline{p}$ satisfies

$$E(\overline{p}) \Leftrightarrow \exists n : \begin{array}{c} \left(\begin{array}{llcr} x_{\overline{p}} & +n\mu & \geq & 0 \\ y_{\overline{p}} & -nf & \geq & -f \end{array} \right) \\ \vee \\ \left(\begin{array}{llcr} x_{\overline{p}} & -ne & \geq & -e \\ y_{\overline{p}} & +n\lambda & \geq & 0 \end{array} \right) \end{array}$$

or equivalently

$$E(\overline{p}) \Leftrightarrow \exists n : \begin{array}{c} fx_{\overline{p}} \quad +\mu y_{\overline{p}} \quad \geq \quad -fe \quad -f\mu \\ \vee \\ \lambda x_{\overline{p}} \quad +ey_{\overline{p}} \quad \geq \quad -fe \quad -e\lambda \end{array}$$

That is, $\overline{p}$ is to the left of and/or below an area (a set of points) where $-e \leq x < 0$ and $-f \leq y < 0$.

Moreover, all solutions are comparable so there exists a *unique* minimal solution. Therefore the function $\overline{p}'(\overline{p})$, which returns the minimal solution to the inequalities above, is well defined whenever $E(\overline{p})$ holds. The point $\overline{p}'$ must be chosen as $\overline{p}' = \overline{p}'(\overline{p})$. Figure 13 shows that it is essential to choose the *minimal* solution since in this figure both $\overline{p}'_1$ and $\overline{p}'_2$ are solutions allthough $\overline{p}'_2$ is not reachable from $\overline{p}_1$. The set of inequalities consisting of the two first rows has only finitely many solutions since by assumption the determinant is strictly positive. Thus, the solutions may be precomputed (for fixed α_h and α_v). The fact that all solutions are comparable is due to the signs of the matrix

$$\left(\begin{array}{cc} -e & \lambda \\ \mu & -f \end{array} \right)$$

and is not true in general. For $\overline{p}$ the set $\mathcal{Q}_{\overline{p}}(\overline{q})$ defined by

$$\mathcal{Q}_{\overline{p}}(\overline{q}) \Leftrightarrow x_{\overline{q}} \geq -e \ \wedge \ y_{\overline{q}} \geq -f \ \wedge \ \overline{p} \leq \overline{q} \leq \overline{p}'(\overline{p})$$

characterizes all reachable points $\overline{q}$ above $\overline{p}$ that lies "close to" the positive cone ($\overline{q}$ is to the left of and below the intersection $x = -e$, $y = -f$. Note that there does not necessarily exist an *integer* point $\overline{p}'$ such that $x_{\overline{p}'} = -e$ and $y_{\overline{p}'} = -f$.). This is illustrated by the shaded areas in figures 10 and 13. The set of points reachable from $\overline{p}$ is characterised by the formula

$$R_{\overline{p}}(\overline{q}) \Leftrightarrow \begin{array}{c} (E(\overline{p}) \ \wedge \ \mathcal{Q}_{\overline{p}}(\overline{q})) \\ \vee \\ \exists n : \left(\begin{array}{c} 0 \leq x_{\overline{p}} \ \wedge \ -e \leq x_{\overline{q}} \ \wedge \ \overline{q} = \overline{p} + n \cdot 1_h \\ \vee \\ 0 \leq y_{\overline{p}} \ \wedge \ -f \leq y_{\overline{q}} \ \wedge \ \overline{q} = \overline{p} + n \cdot 1_v \end{array} \right) \\ \vee \\ \overline{q} = \overline{p} \end{array}$$

The second disjunct covers the points on a horizontal or vertical line starting at $\overline{p}$. $\exists n : \overline{q} = \overline{p} + n \cdot 1_h$ can also be expressed as

$$\begin{array}{llcll} fx_{\overline{q}} & +\mu y_{\overline{q}} & \leq & fx_{\overline{p}} & +\mu y_{\overline{p}} \\ \lambda x_{\overline{q}} & +ey_{\overline{q}} & = & \lambda x_{\overline{p}} & +ey_{\overline{p}} \end{array}$$

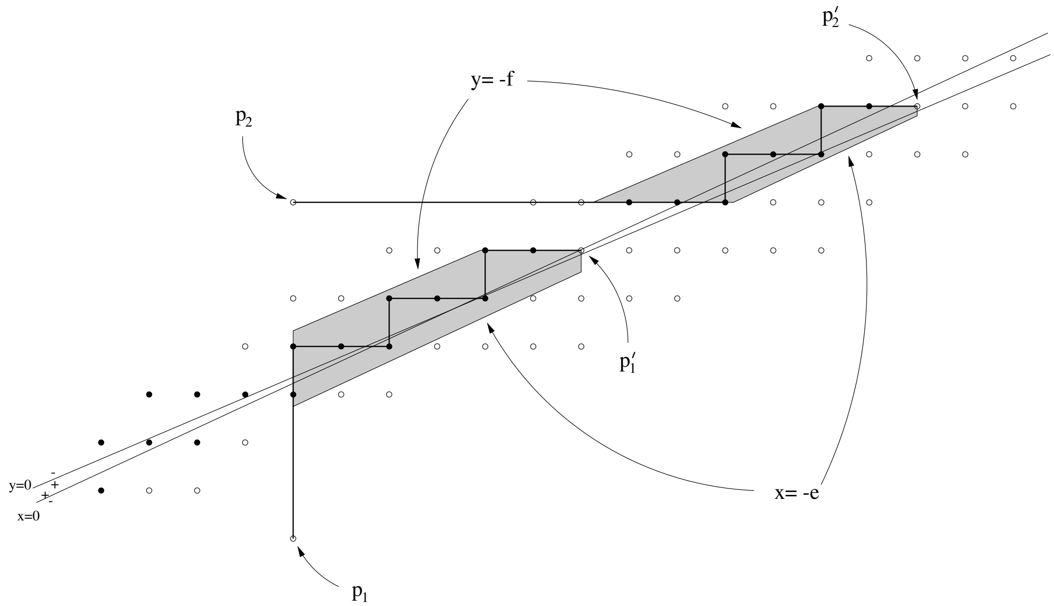


Figure 13

and $\exists n : \overline{q} = \overline{p} + n \cdot 1_v$ as

$$\begin{array}{llcll} fx_{\overline{q}} & +\mu y_{\overline{q}} & = & fx_{\overline{p}} & +\mu y_{\overline{p}} \\ \lambda x_{\overline{q}} & +ey_{\overline{q}} & \leq & \lambda x_{\overline{p}} & +ey_{\overline{p}} \end{array}$$

4.2.3 negative determinant

Consider the second case when

$$\begin{vmatrix} -e & \lambda \\ \mu & -f \end{vmatrix} \leq 0$$

and $e, f > 0$, which is illustrated in figures 11 and 12. Consider figure 11. There is an infinite path starting at $\overline{p}$ iff there are no solutions to

$$\begin{array}{rcrrrcl} -e & \leq & -eh_{\overline{p}'} & +\mu v_{\overline{p}'} & +\alpha_h & < & 0 \\ -f & \leq & \lambda h_{\overline{p}'} & -fv_{\overline{p}'} & +\alpha_v & < & 0 \\ & & & & \overline{p}' & \geq & \overline{p} \end{array}$$

Otherwise $\overline{p}'$ is the minimal solution (see figure 12: There is an infinite path leaving $\overline{p}_2$ but only finite paths leaving $\overline{p}_1$, since there is a solution $\overline{p}'_1 \geq \overline{p}_1$ to the inequalities above, but no solution $\overline{p}'_2 \geq \overline{p}_2$). If the determinant is strictly negative, the inequalities of the first two rows have only finitely many solutions (for fixed α_h and α_v). If a solution exists, the set $\mathcal{Q}_{\overline{p}}(\overline{q})$ is defined exactly as above and if no solution exists it is defined by

$$\mathcal{Q}_{\overline{p}}(\overline{q}) \;\Leftrightarrow\; x_{\overline{q}} \geq -e \;\wedge\; y_{\overline{q}} \geq -f \;\wedge\; \overline{p} \leq \overline{q}$$

(which may be seen as though the point $\overline{p}'$ lies at infinity: $\overline{p}' = \langle \infty, \infty \rangle^T$). If

$$\begin{vmatrix} -e & \lambda \\ \mu & -f \end{vmatrix} < 0$$

$\overline{p} \leq \overline{q}$ may alternatively be expressed as

$$\begin{array}{llcll} fx_{\overline{p}} & +\mu y_{\overline{p}} & \leq & fx_{\overline{q}} & +\mu y_{\overline{q}} \\ \lambda x_{\overline{p}} & +ey_{\overline{p}} & \leq & \lambda x_{\overline{q}} & +ey_{\overline{q}} \end{array}$$

The set $\mathcal{Q}_{\overline{p}}(\overline{q})$ for the case when no solution $\overline{p}'$ exists is illustrated by the shaded area in figures 11 (In figure 12 there is a solution $\overline{p}'_1 \geq \overline{p}_1$ but no solution $\overline{p}'_2 \geq \overline{p}_2$. The shaded areas shows $\mathcal{Q}_{\overline{p}}(\overline{q})$ for the two cases.) The set of points $R_{\overline{p}}(\overline{q})$ reachable from $\overline{p}$ is defined as before.

4.2.4 positive diagonal

Let us now consider programs with the signs of the matrix given by

$$\Phi = \begin{pmatrix} e & -\lambda \\ -\mu & f \end{pmatrix}$$

where $\lambda, \mu > 0$ and $e, f \geq 0$. Depending on the values of the coefficients, this corresponds to one of situations illustrated in figures 14 and 16. As before we consider two cases depending on the sign of the determinant.

4.2.5 negative determinant

Consider the case when

$$\begin{vmatrix} e & -\lambda \\ -\mu & f \end{vmatrix} < 0$$

which is illustrated in figure 14. Consider the inequations

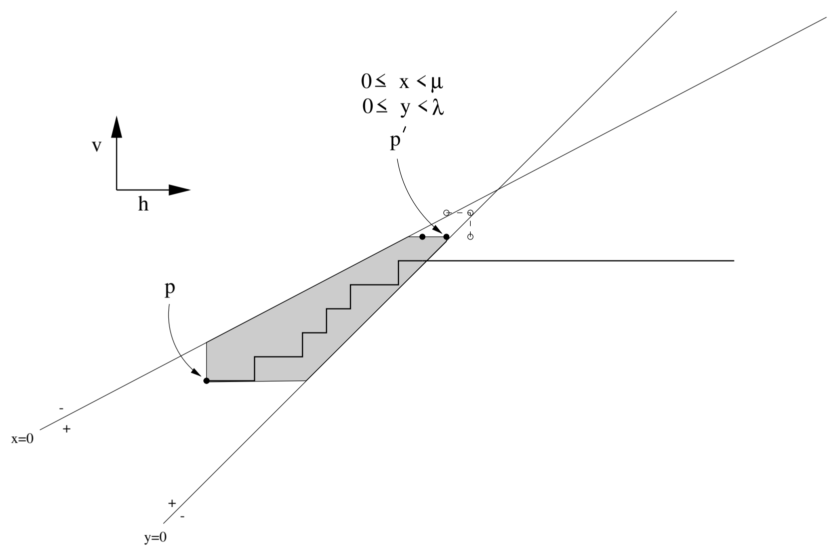


Figure 14

$$\begin{array}{ccrrrcl} 0 & \leq & eh_{\overline{p}'} & -\mu v_{\overline{p}'} & +\alpha_h & < & \mu \\ 0 & \leq & -\lambda h_{\overline{p}'} & +f v_{\overline{p}'} & +\alpha_v & < & \lambda \\ & & & & \overline{p}' & \geq & \overline{p} \end{array}$$

or equivalently

$$\begin{array}{cccll} 0 & \leq & x_{\overline{p}'} & < & \mu \\ 0 & \leq & y_{\overline{p}'} & < & \lambda \\ & & \overline{p}' & \geq & \overline{p} \end{array}$$

A unique minimal solution $\overline{p}'$ always exists when $\overline{p}$ satisfies

$$E(\overline{p}) \;\Leftrightarrow\; x_{\overline{p}} \geq 0 \;\wedge\; y_{\overline{p}} \geq 0$$

Thus, the function $\overline{p}'(\overline{p})$, which returns the minimal solution to the above set of inequations, is well defined for $\overline{p}$ such that $E(\overline{p})$. Since the determinant is strictly negative, the inequations of the first two rows have finitely many solutions. The set

$$\mathcal{Q}_{\overline{p}}(\overline{q}) \;\Leftrightarrow\; x_{\overline{q}} \geq 0 \;\wedge\; y_{\overline{q}} \geq 0 \;\wedge\; \overline{p} \leq \overline{q} \leq \overline{p}'(\overline{p})$$

characterizes all points $\overline{q}$ below $\overline{p}'(\overline{p})$ that are reachable from $\overline{p}$. This set is illustrated by the shaded area in figure 14. When $\overline{p}$ satisfies $E(\overline{p})$, all points reachable from $\overline{p}$ can be reached by starting from some point in $\mathcal{Q}_{\overline{p}}(\overline{q})$ and walking any number of steps horizontally or vertically. Thus, $\overline{p}''$ is reachable from $\overline{p}$ iff

$$\exists \overline{q}, n : \mathcal{Q}_{\overline{p}}(\overline{q}) \ \wedge \ \begin{pmatrix} \overline{p}'' = \overline{q} + n \cdot 1_h \\ \vee \\ \overline{p}'' = \overline{q} + n \cdot 1_v \end{pmatrix}$$

It is easily seen that this is equivalent to

$$\overline{p} \leq \overline{p}'' \ \wedge \ \begin{pmatrix} h_{\overline{p}''} \leq h_{\overline{p}'(\overline{p})} \\ \vee \\ v_{\overline{p}''} \leq v_{\overline{p}'(\overline{p})} \end{pmatrix}$$

which is illustrated by the shaded area in figure 15. Alternatively, $h_{\overline{p}''} \leq h_{\overline{p}'(\overline{p})}$ may be expressed as

$$f x_{\overline{p}'(\overline{p})} \quad + \mu y_{\overline{p}'(\overline{p})} \quad \leq \quad f x_{\overline{p}''} \quad + \mu y_{\overline{p}''}$$

and $v_{\overline{p}''} \leq v_{\overline{p}'(\overline{p})}$ as

$$\lambda x_{\overline{p}'(\overline{p})} \quad + e y_{\overline{p}'(\overline{p})} \quad \leq \quad \lambda x_{\overline{p}''} \quad + e y_{\overline{p}''}$$

It means that $\overline{p}''$ is in the complementary region located at the right and above $\overline{p}'$. The set of

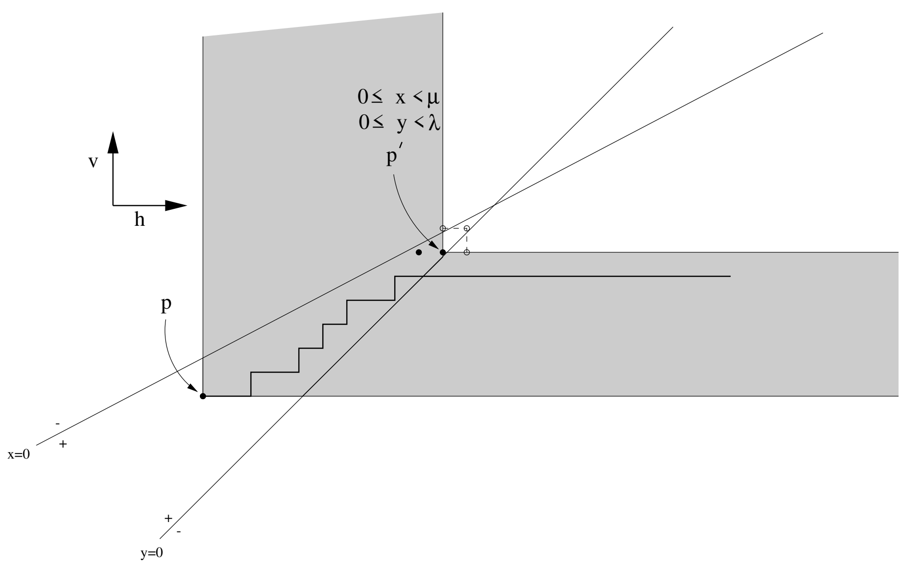


Figure 15

points reachable from $\overline{p}$ is characterised by the formula

$$R_{\overline{p}}(\overline{q}) \quad \Leftrightarrow \quad \begin{array}{c} \left(E(\overline{p}) \ \wedge \ \overline{p} \leq \overline{q} \ \wedge \ \begin{pmatrix} h_{\overline{q}} \leq h_{\overline{p}'(\overline{p})} \\ \vee \\ v_{\overline{q}} \leq v_{\overline{p}'(\overline{p})} \end{pmatrix} \right) \\ \vee \\ \exists n : \begin{pmatrix} 0 \leq x_{\overline{p}} \ \wedge \ \overline{q} = \overline{p} + n \cdot 1_h \\ \vee \\ 0 \leq y_{\overline{p}} \ \wedge \ \overline{q} = \overline{p} + n \cdot 1_v \end{pmatrix} \\ \vee \\ \overline{q} = \overline{p} \end{array}$$

where $\exists n : \overline{q} = \overline{p} + n \cdot 1_h$ may also be expressed as

$$\begin{array}{llcll} fx_{\overline{q}} & +\mu y_{\overline{q}} & \leq & fx_{\overline{p}} & +\mu y_{\overline{p}} \\ \lambda x_{\overline{q}} & +e y_{\overline{q}} & = & \lambda x_{\overline{p}} & +e y_{\overline{p}} \end{array}$$

and $\exists n : \overline{q} = \overline{p} + n \cdot 1_v$ as

$$\begin{array}{llcll} fx_{\overline{q}} & +\mu y_{\overline{q}} & = & fx_{\overline{p}} & +\mu y_{\overline{p}} \\ \lambda x_{\overline{q}} & +e y_{\overline{q}} & \leq & \lambda x_{\overline{p}} & +e y_{\overline{p}} \end{array}$$

4.2.6 positive determinant

Consider the second case when

$$\begin{vmatrix} e & -\lambda \\ -\mu & f \end{vmatrix} \geq 0$$

which is illustrated in figure 14. The inequations

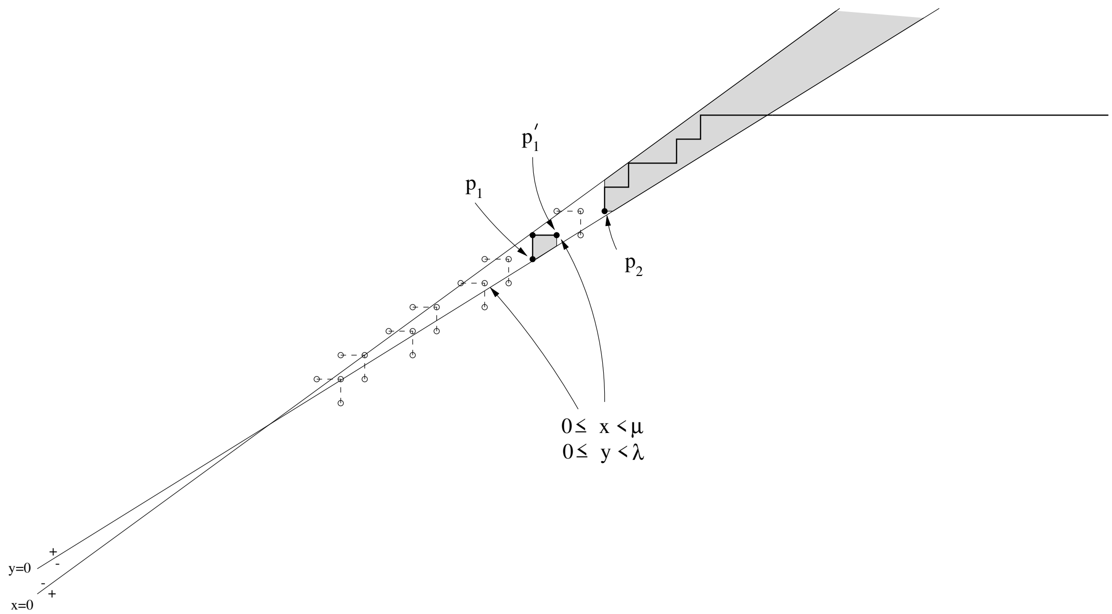


Figure 16

$$\begin{array}{ccrrrcl} 0 & \leq & eh_{\overline{p}'} & -\mu v_{\overline{p}'} & +\alpha_h & < & \mu \\ 0 & \leq & -\lambda h_{\overline{p}'} & +f v_{\overline{p}'} & +\alpha_v & < & \lambda \\ & & & & \overline{p}' & \geq & \overline{p} \end{array}$$

may or may not have solutions. In figure 16 there is a solution $\overline{p}'_1 \geq \overline{p}_1$, but no solution $\overline{p}'_2 \geq \overline{p}_2$. When a solution exists there exists a unique minimal solution so $\overline{p}'(\overline{p})$ is well defined as before. When no solution exists we define $\overline{p}'(\overline{p}) = \langle \infty, \infty \rangle^T$. Thus, with $\mathcal{Q}_{\overline{p}}(\overline{q})$ defined as before, the shaded areas in figure 17 illustrates this set for the cases when a solution exists and when it does not exist. As before, the set of points $\overline{p}''$ reachable from $\overline{p}$ when $E(\overline{p})$ holds is given by

$$\overline{p} \leq \overline{p}'' \ \wedge \ \left(\begin{array}{c} h_{\overline{p}''} \leq h_{\overline{p}'(\overline{p})} \\ \vee \\ v_{\overline{p}''} \leq v_{\overline{p}'(\overline{p})} \end{array} \right)$$

which reduces to $\overline{p} \leq \overline{p}''$ when $\overline{p}'(\overline{p}) = \langle \infty, \infty \rangle^T$. That is, the whole plane above $\overline{p}$ is filled when the inequalities above have no solutions. This is illustrated by the shaded areas in figure 17. With the convention that $\overline{p}'(\overline{p}) = \langle \infty, \infty \rangle^T$ when no solutions exist, the set of points reachable from any point $\overline{p}$ is given by $R_{\overline{p}}(\overline{q})$ as defined for the case when the determinant is negative.

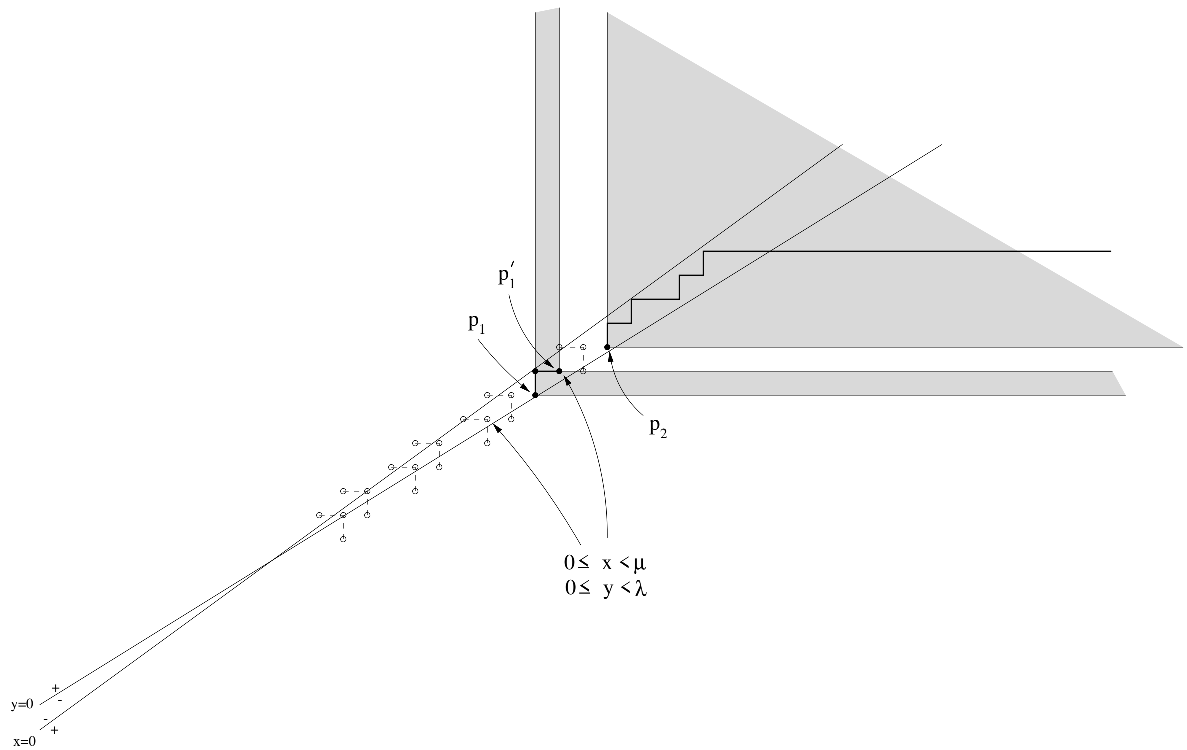


Figure 17

5 Programs with Three Recursive Rules

In the rest of the report, we focus on programs with three recursive rules:

$$\begin{array}{llcl} & p(x_0, y_0, z_0). & & \\ h: & p(x+e, y+\lambda, z+\gamma) & \leftarrow & x \geq 0, \quad p(x,y,z). \\ v: & p(x+\mu, y+f, z+\delta) & \leftarrow & y \geq 0, \quad p(x,y,z). \\ t: & p(x+\alpha, y+\beta, z+g) & \leftarrow & z \geq 0, \quad p(x,y,z). \end{array}$$

The behaviour for such programs are far more complicated than for programs with only two recursive rules.

Proposition 13:
Knowing the least fixpoint (lfp) of a program with the associated matrix Φ, one can infer the lfp of the program with associated matrix $-\Phi$. $\diamond$

Proof:
Briefly, consider a rule of a program P:

$$\begin{array}{llcl} 0: & p(\overline{x}', \overline{x}'). & & \\ & \vdots & & \\ i: & p(\overline{x}+\overline{k}_i, \overline{x}') & \leftarrow & x_i \geq 0, \quad p(\overline{x}, \overline{x}'). \\ & \vdots & & \end{array}$$

By turning the program "backwards", one gets

$$\begin{array}{ll} 0: & p'(\overline{x}', \overline{x}'). \\ & \vdots \\ i: & p'(\overline{x}, \overline{x}' - \overline{k}_i) \quad \leftarrow \quad x'_i - \overline{k}_i \geq 0, \quad p'(\overline{x}, \overline{x}'). \\ & \vdots \end{array}$$

which computes the same relation, but the coefficients has changed signs. (See [1] for a detailed investigation of such program transformations and the relationships between theire least fixpoints.) $\diamond$

Proposition 14:
The lfp of a program, the associated matrix Φ of which contains a fully negative row (a row made of 3 negative coefficients), is known. (see [4] pp. 139-152). $\diamond$

Proposition 15:
The lfp of a program, the associated matrix Φ of which contains a fully negative column (a column made of 3 negative coefficients) and a row containing 2 positive coefficients, is known. (see [4] pp. 139-152). $\diamond$

Proposition 16:
The lfp of a program with associated matrix Φ of one of the following forms:

$$\begin{pmatrix} + & + & - \\ - & \bullet & + \\ - & - & - \end{pmatrix} \quad \begin{pmatrix} + & - & + \\ + & - & - \\ - & + & \bullet \end{pmatrix} \quad \begin{pmatrix} \bullet & + & - \\ - & - & + \\ + & - & + \end{pmatrix}$$

$$\begin{pmatrix} - & - & + \\ + & \bullet & - \\ - & + & + \end{pmatrix} \quad \begin{pmatrix} - & + & - \\ - & + & + \\ + & - & \bullet \end{pmatrix} \quad \begin{pmatrix} \bullet & - & + \\ + & + & - \\ - & + & - \end{pmatrix}$$

is known (see [4] pp. 139-152). $\diamond$

Proposition 17:
Any matrix (or its negative form) falls into one of the following classes:

(a) matrix with a fully negative row

(b) matrix with a fully negative column and a row containing two positive coefficients

(c) matrix of the form

$$\begin{pmatrix} + & + & - \\ - & \bullet & + \\ + & - & - \end{pmatrix} \quad \begin{pmatrix} + & - & + \\ + & - & - \\ - & + & \bullet \end{pmatrix} \quad \begin{pmatrix} \bullet & + & - \\ - & - & + \\ + & - & + \end{pmatrix}$$

(d) matrix of the form

$$\begin{pmatrix} - & - & + \\ + & - & - \\ - & - & + \end{pmatrix} \quad \begin{pmatrix} + & - & - \\ - & - & + \\ + & - & - \end{pmatrix} \quad \begin{pmatrix} - & + & - \\ - & - & + \\ - & - & + \end{pmatrix}$$

$$\begin{pmatrix} - & - & + \\ - & + & - \\ - & + & - \end{pmatrix} \quad \begin{pmatrix} - & + & - \\ - & + & - \\ + & - & - \end{pmatrix} \quad \begin{pmatrix} + & - & - \\ + & - & - \\ - & + & - \end{pmatrix}$$

(e) matrix with (at least) one minor associated with a diagonal element of the form

$$\begin{pmatrix} - & + \\ + & - \end{pmatrix}$$

That is, a matrix of the form

$$\begin{pmatrix} - & + & \bullet \\ + & - & \bullet \\ \bullet & \bullet & \bullet \end{pmatrix} \quad \begin{pmatrix} - & \bullet & + \\ \bullet & \bullet & \bullet \\ + & \bullet & - \end{pmatrix} \quad \begin{pmatrix} \bullet & \bullet & \bullet \\ \bullet & - & + \\ \bullet & + & - \end{pmatrix}$$

(f) matrix of the form

$$\begin{pmatrix} + & + & - \\ - & + & + \\ + & - & + \end{pmatrix} \quad \begin{pmatrix} + & - & + \\ + & + & - \\ - & + & + \end{pmatrix}$$

$\diamond$

From propositions 13 to 17 it follows that the matrices for which the associated lfp are (a priori) unknown, are matrices of type (e) and (f) of proposition 17.

Programs, the associated matrices of which are of types (a) to (d) of proposition 17, will in this report be refered to as programs of class 1, and those described by case (f) as class 5.

In the rest of this report, we mainly focus on matrices of type (e).
The three subcases of form (e) correspond one-by-one too three elements of the diagonals (chosen for the associated minors). The corresponding least fixpoints are identical up to a permutation of variables x, y, and z. Therefore we will focus on matrices of the form

$$\begin{pmatrix} \bullet & \bullet & \bullet \\ \bullet & - & + \\ \bullet & + & - \end{pmatrix}$$

We will suppose furthermore that the determinant of the submatrix (the minor) is nonnull.

The matrices of type (e) are subdivided into three classes:

class 2:
This class contains the following 6 matrices (but two of them have already been treated elsewhere):

$$\begin{pmatrix} - & + & + \\ - & - & + \\ - & + & - \end{pmatrix}$$

[For a matrix such as the one above, the associated lfp is known by proposition 15]

$$\begin{pmatrix} - & + & - \\ - & - & + \\ + & + & - \end{pmatrix} \quad \begin{pmatrix} - & + & - \\ - & - & + \\ - & + & - \end{pmatrix}$$

$$\begin{pmatrix} - & - & + \\ + & - & + \\ - & + & - \end{pmatrix} \quad \begin{pmatrix} - & - & + \\ - & - & + \\ - & + & - \end{pmatrix}$$

$$\begin{pmatrix} - & - & - \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{pmatrix}$$

[By proposition 14, the lfp associated with a matrix of the last of the above forms, is known.]

class 3:
This class contains the following 6 matrices:

$$\begin{pmatrix} - & + & + \\ + & - & + \\ + & + & - \end{pmatrix} \quad \begin{pmatrix} - & + & + \\ + & - & + \\ - & + & - \end{pmatrix} \quad \begin{pmatrix} - & + & + \\ - & - & + \\ + & + & - \end{pmatrix}$$

$$\begin{pmatrix} - & + & - \\ + & - & + \\ - & + & - \end{pmatrix} \quad \begin{pmatrix} - & + & - \\ + & - & + \\ + & + & - \end{pmatrix} \quad \begin{pmatrix} - & - & + \\ - & - & + \\ + & + & - \end{pmatrix}$$

class 4:
This class contains the following 13 matrices (but two of them have already been treated elsewhere):

$$\begin{pmatrix} + & + & + \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{pmatrix}$$

[By taking the negative of the matrix above, one obtaines a matrix with a fully negative row. By proposition 14, the associated lfp is known.]

$$\begin{pmatrix} + & + & - \\ + & - & + \\ + & + & - \end{pmatrix} \quad \begin{pmatrix} + & + & - \\ + & - & + \\ - & + & - \end{pmatrix} \quad \begin{pmatrix} + & + & - \\ - & - & + \\ + & + & - \end{pmatrix} \quad \begin{pmatrix} + & + & - \\ - & - & + \\ - & + & - \end{pmatrix}$$

$$\begin{pmatrix} + & - & + \\ + & - & + \\ + & + & - \end{pmatrix} \quad \begin{pmatrix} + & - & + \\ + & - & + \\ - & + & - \end{pmatrix} \quad \begin{pmatrix} + & - & + \\ - & - & + \\ + & + & - \end{pmatrix} \quad \begin{pmatrix} + & - & + \\ - & - & + \\ - & + & - \end{pmatrix}$$

$$\begin{pmatrix} + & - & - \\ + & - & + \\ + & + & - \end{pmatrix}$$

[By taking the negative of a matrix of this last form, one obtains a matrix with a fully negativ column and a row containing two possitive coefficients. By proposition 15, the associated lfp is known.]

$$\begin{pmatrix} + & - & - \\ + & - & + \\ - & + & - \end{pmatrix} \quad \begin{pmatrix} + & - & - \\ - & - & + \\ + & + & - \end{pmatrix} \quad \begin{pmatrix} + & - & - \\ - & - & + \\ - & + & - \end{pmatrix}$$

6 The Pigeon-Hole Principle

The analysis of programs with three recursive rules is not as straightforward as for 2-rule programs. In this section we derive a tool for computing certain patterns that exists within the lfp of some classes of programs. The idea is to show that there is a finitie family of (infinite) sets of points, and that a sufficiently long path of some particular form must visit a set of the family at least twice. This yields a solution to a set of equations from which a corresponding motif may be computed.

We denote by $\Gamma_{xy}(\tau,\tau')$, $\Gamma_{xz}(\tau,\tau')$ or $\Gamma_{yz}(\tau,\tau')$ the set of pairs $\langle x,y\rangle \geq \langle \tau,\tau'\rangle$, $\langle x,z\rangle \geq \langle \tau,\tau'\rangle$ or $\langle y,z\rangle \geq \langle \tau,\tau'\rangle$, respectively. By $[\Gamma_{xy}(\tau,\tau')]$ we denote the set of points $\overline{p}$ such that $\langle x_{\overline{p}}, y_{\overline{p}}\rangle \in \Gamma_{xy}(\tau,\tau')$ (and similarly for the other sets). We write $\overline{p}_{\Gamma_{xy}(\tau,\tau')}$ to indicate that $\overline{p} \in [\Gamma_{xy}(\tau,\tau')]$.

By $\Gamma_{xyz}(\tau,\tau',\tau'')$ we denote the set of triples $\langle x,y,z\rangle \geq \langle \tau,\tau',\tau''\rangle$. We use the same conventions as above for $\overline{p}_{\Gamma_{xyz}(\tau,\tau',\tau'')}$ and $[\Gamma_{xyz}(\tau,\tau',\tau'')]$.

Lemma 21:
Consider a matrix of the form

$$\Phi = \begin{pmatrix} \bullet & \lambda & \gamma \\ \bullet & f & \bullet \\ \bullet & \bullet & g \end{pmatrix}$$

where no assumptions are made on f, g, λ or γ. Let $\tau = \min\{0,f\}$ and $\tau' = \min\{0,g\}$. Suppose $\overline{p} \xrightarrow{(h+v+t)^*} \overline{p}'$ for some $\overline{p}$ and $\overline{p}'$.

1. If $\lambda \geq 0 \ \vee \ \gamma < 0$, then

$$\begin{array}{c} \exists \overline{q} \in \Gamma_{yz}(\tau,\tau') : \overline{p} \xrightarrow{(h+t)^*(h+v)^*} \overline{q} \xrightarrow{h^* t(h+v+t)^*} \overline{p}' \\ \vee \\ \overline{p} \xrightarrow{(h+t)^*(h+v)^*} \overline{p}' \end{array}$$

   where $h^*t(h+v+t)^*$ can be replaced by $t(h+v+t)^*$ in the case where $\lambda \geq 0$.

2. If $\lambda < 0 \ \vee \ \gamma \geq 0$, then

$$\begin{array}{c} \exists \overline{q} \in \Gamma_{yz}(\tau,\tau') : \overline{p} \xrightarrow{(h+v)^*(h+t)^*} \overline{q}' \xrightarrow{h^* v(h+v+t)^*} \overline{p}' \\ \vee \\ \overline{p} \xrightarrow{(h+v)^*(h+t)^*} \overline{p}' \end{array}$$

   where $h^*v(h+v+t)^*$ can be replaced by $v(h+v+t)^*$ in the case where $\gamma \geq 0$.

$\diamond$

Proof:
We give the proof for the case when $\lambda \geq 0 \ \vee \ \gamma < 0$. The other case is identical.

Since

$$\overline{p} \xrightarrow{(h+v+t)^*} \overline{p}' \ \Rightarrow \ \exists \overline{q} : \overline{p} \xrightarrow{(h+t)^*(h+v)^*} \overline{q} \xrightarrow{(h+v+t)^*} \overline{p}'$$

it is enough to prove

$$\overline{p} \xrightarrow{(h+t)^*(h+v)^*} \overline{q} \xrightarrow{(h+v+t)^*} \overline{p}' \ \Rightarrow \left(\begin{array}{c} \exists \overline{q}' \in \Gamma_{yz}(\tau,\tau') : \overline{p} \xrightarrow{(h+t)^*(h+v)^*} \overline{q} \xrightarrow{h^* t(h+v+t)^*} \overline{p}' \\ \vee \\ \overline{p} \xrightarrow{(h+t)^*(h+v)^*} \overline{p}' \end{array} \right)$$

This follows easily by induction on n from the following implication

$$\overline{p} \xrightarrow{(h+t)^*(h+v)^*} \overline{q} \xrightarrow{(h+v+t)^n} \overline{p}' \ \Rightarrow \left(\begin{array}{c} n = 0 \ \wedge \ \overline{q} = \overline{p}' \\ \vee \\ \exists \overline{q}' \in \Gamma_{yz}(\tau,\tau') : \overline{p} \xrightarrow{(h+t)^*(h+v)^*} \overline{q} \xrightarrow{h^* t(h+v+t)^*} \overline{p}' \\ \vee \\ \exists \overline{q}' : \overline{p} \xrightarrow{(h+t)^*(h+v)^*} \overline{q}' \xrightarrow{(h+v+t)^{n-1}} \overline{p}' \end{array} \right)$$

Consider points $\overline{p}$, $\overline{q}$ and $\overline{p}'$ such that $\overline{p} \xrightarrow{(h+t)^*(h+v)^*} \overline{q} \xrightarrow{(h+v+t)^n} \overline{p}'$ for some n. The case when $n = 0$ is trivial, so assume $n > 0$. Then $\overline{q} \xrightarrow{w} \overline{p}'$ for some $w \in (h+v+t)^*$ with $|w| = n > 0$. There are three cases: $w = hw'$, $w = vw'$ and $w = tw'$ for some $w' \in (h+v+t)^*$ with $|w'| = n-1$. If $w = hw'$ or $w = vw'$, then clearly $\overline{p} \xrightarrow{(h+t)^*(h+v)^*} \overline{q} \xrightarrow{h+v} \overline{q}' \xrightarrow{w'} \overline{p}'$ which implies $\overline{p} \xrightarrow{(h+t)^*(h+v)^*} \overline{q}' \xrightarrow{(h+v+t)^{n-1}} \overline{p}'$ and we're done. Suppose therefore that $w = tw'$. Now, $\overline{p} \xrightarrow{u} \overline{q}$ for some $u \in (h+t)^*(h+v)^*$. If u contains no v-move, as above we then have $\overline{p} \xrightarrow{(h+t)^*(h+v)^*} \overline{q}' \xrightarrow{(h+v+t)^{n-1}} \overline{p}'$ for some $\overline{q}'$. Thus assume $u = u'vh^*$ for some $u' \in (h+t)^*(h+v)^*$. We have the situation illustrated in figure 18. Consider the point $\overline{q}''$

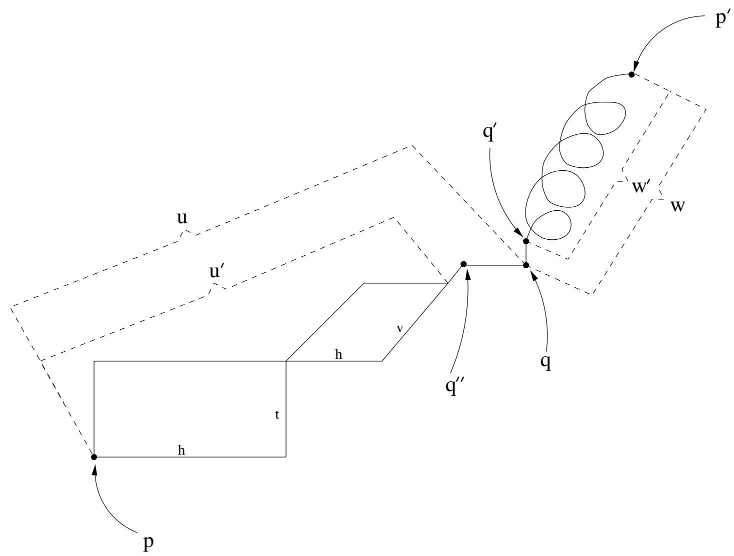


Figure 18

such that $\overline{p} \xrightarrow{u'v} \overline{q}''$. Clearly $\min\{f, 0\} \leq y_{\overline{q}''}$ and $0 \leq z_{\overline{q}}$. If $\lambda \geq 0$, then $\min\{f, 0\} \leq y_{\overline{q}}$ since an h-move increases y. Thus $\overline{q} \in [\Gamma_{yz}(\tau, \tau')]$. If $\gamma < 0$, then $0 \leq z_{\overline{q}''}$ since an h-move decreases z. Thus $\overline{q}' \in [\Gamma_{yz}(\tau, \tau')]$.

We have $\overline{q} \xrightarrow{t(h+v+t)^*} \overline{p}'$ and $\overline{q}' \xrightarrow{h^*t(h+v+t)^*} \overline{p}'$. This concludes the proof. $\diamond$

Corollary:
Consider a matrix of the form

$$\Phi = \begin{pmatrix} e & \bullet & \bullet \\ \mu & \bullet & \delta \\ \bullet & \bullet & g \end{pmatrix}$$

where no assumptions are made on e, g, μ or δ. Let $\tau = \min\{0, e\}$ and $\tau' = \min\{0, g\}$. Suppose $\overline{p} \xrightarrow{(h+v+t)^*} \overline{p}'$ for some $\overline{p}$ and $\overline{p}'$.

1. If $\mu \geq 0 \;\vee\; \delta < 0$, then

$$\exists \overline{q} \in \Gamma_{xz}(\tau, \tau') : \overline{p} \xrightarrow{(v+t)^*(h+v)^*} \overline{q} \xrightarrow{v^*t(h+v+t)^*} \overline{p}' \\ \vee \\ \overline{p} \xrightarrow{(v+t)^*(h+v)^*} \overline{p}'$$

where $v^*t(h+v+t)^*$ can be replaced by $t(h+v+t)^*$ in the case where $\mu \geq 0$.

2. If $\mu < 0 \;\vee\; \delta \geq 0$, then

$$\exists \overline{q} \in \Gamma_{xz}(\tau, \tau') : \overline{p} \xrightarrow{(h+v)^*(v+t)^*} \overline{q} \xrightarrow{v^*h(h+v+t)^*} \overline{p}'$$
$$\vee$$
$$\overline{p} \xrightarrow{(h+v)^*(v+t)^*} \overline{p}'$$

where $v^*h(h+v+t)^*$ can be replaced by $h(h+v+t)^*$ in the case where $\delta \geq 0$.

$\diamond$

Corollary:
Consider a matrix of the form

$$\Phi = \begin{pmatrix} e & \lambda & \bullet \\ \bullet & f & \bullet \\ \alpha & \beta & \bullet \end{pmatrix}$$

where no assumptions are made on e, f, α or β. Let $\tau = \min\{0, e\}$ and $\tau' = \min\{0, f\}$. Suppose $\overline{p} \xrightarrow{(h+v+t)^*} \overline{p}'$ for some $\overline{p}$ and $\overline{p}'$.

1. If $\beta \geq 0 \;\vee\; \alpha < 0$, then

$$\exists \overline{q} \in \Gamma_{xy}(\tau, \tau') : \overline{p} \xrightarrow{(h+t)^*(v+t)^*} \overline{q} \xrightarrow{t^*h(h+v+t)^*} \overline{p}'$$
$$\vee$$
$$\overline{p} \xrightarrow{(h+t)^*(v+t)^*} \overline{p}'$$

where $t^*h(h+v+t)^*$ can be replaced by $h(h+v+t)^*$ in the case where $\beta \geq 0$.

2. If $\beta < 0 \;\vee\; \alpha \geq 0$, then

$$\exists \overline{q} \in \Gamma_{xy}(\tau, \tau') : \overline{p} \xrightarrow{(v+t)^*(h+t)^*} \overline{q} \xrightarrow{t^*v(h+v+t)^*} \overline{p}'$$
$$\vee$$
$$\overline{p} \xrightarrow{(v+t)^*(h+t)^*} \overline{p}'$$

where $t^*v(h+v+t)^*$ can be replaced by $v(h+v+t)^*$ in the case where $\alpha \geq 0$.

$\diamond$

Proof:
Follows from lemma 21 by permutation of h, v and t. $\diamond$

Lemma 21 and its corollaries says that after at most two changes of planes, either xy, xz or yz cannot be very negative.

The next lemma is a slight generalization of proposition 11.

Lemma 22:
Consider a matrix of the form

$$\Phi = \begin{pmatrix} \bullet & \bullet & \bullet \\ \bullet & f & \delta \\ \bullet & \beta & g \end{pmatrix}$$

where $\delta, \beta \geq 0$ and no assumptions are made on f or g. Let $\tau = \min\{0, f\}$ and $\tau' = \min\{0, g\}$. Then

$$\overline{p} \in [\Gamma_{yz}(\tau, \tau')] \;\wedge\; \overline{p} \xrightarrow{(v+t)^*} \overline{p}' \;\Rightarrow\; \overline{p}' \in [\Gamma_{yz}(\tau, \tau')]$$

must hold. $\diamond$

Corollary:
Consider a matrix of the form

$$\Phi = \begin{pmatrix} e & \bullet & \gamma \\ \bullet & \bullet & \bullet \\ \alpha & \bullet & g \end{pmatrix}$$

where $\gamma, \alpha \geq 0$ and no assumptions are made on e or g. Let $\tau = \min\{0, e\}$ and $\tau' = \min\{0, g\}$. Then

$$\overline{p} \in [\Gamma_{xz}(\tau, \tau')] \ \wedge \ \overline{p} \xrightarrow{(h+t)^*} \overline{p}' \ \Rightarrow \ \overline{p}' \in [\Gamma_{xz}(\tau, \tau')]$$

must hold. $\diamond$

Corollary:
Consider a matrix of the form

$$\Phi = \begin{pmatrix} e & \lambda & \bullet \\ \mu & f & \bullet \\ \bullet & \bullet & \bullet \end{pmatrix}$$

where $\lambda, \mu \geq 0$ and no assumptions are made on e or f. Let $\tau = \min\{0, e\}$ and $\tau' = \min\{0, f\}$. Then

$$\overline{p} \in [\Gamma_{xy}(\tau, \tau')] \ \wedge \ \overline{p} \xrightarrow{(h+v)^*} \overline{p}' \ \Rightarrow \ \overline{p}' \in [\Gamma_{xy}(\tau, \tau')]$$

must hold. $\diamond$

Proof:
Follows from lemma 22 by permutation of h, v and t. $\diamond$

Lemma 22 and its corollaries says that (under the stated conditions) $\Gamma_{xy}(\tau, \tau')$ is an invariant for all paths in the hv-plane (and the corresponding statements for the ht- and vt-planes).

Lemma 23:
Consider a matrix of the form

$$\Phi = \begin{pmatrix} \bullet & \lambda & \gamma \\ \bullet & f & \delta \\ \bullet & \beta & g \end{pmatrix}$$

where $\beta, \delta \geq 0$ and no assumptions are made on f, g, λ or γ. Let $\tau = \min\{0, f\}$, $\tau' = \min\{0, g\}$. Assume $\overline{p} \xrightarrow{(h+v+t)^*} \overline{p}'$ for some points $\overline{p}$ and $\overline{p}'$ where $\overline{p} \in [\Gamma_{yz}(\tau, \tau')]$. We have:

1. If $\lambda \geq 0 \ \wedge \ \gamma \geq 0$, then

$$\exists \overline{q} \in \Gamma_{yz}(\tau, \tau') : \overline{p} \xrightarrow{h+v+t} \overline{q} \xrightarrow{(h+v+t)^*} \overline{p}' \\ \vee \\ \overline{p} = \overline{p}'$$

2. If $\lambda \geq 0 \ \wedge \ \gamma < 0$, then

$$\exists \overline{q} \in \Gamma_{yz}(\tau, \tau') : \begin{pmatrix} \overline{p} \xrightarrow{v+t} \overline{q} \xrightarrow{(h+v+t)^*} \overline{p}' \\ \vee \\ \overline{p} \xrightarrow{h(h+v)^*} \overline{q} \xrightarrow{t(h+v+t)^*} \overline{p}' \end{pmatrix} \\ \vee \\ \overline{p} \xrightarrow{(h+v)^*} \overline{p}'$$

3. If $\lambda < 0 \ \wedge \ \gamma \geq 0$, then

$$\exists \overline{q} \in \Gamma_{yz}(\tau,\tau') : \left(\begin{array}{c} \overline{p} \xrightarrow{v+t} \overline{q} \xrightarrow{(h+v+t)^*} \overline{p}' \\ \vee \\ \overline{p} \xrightarrow{h(h+t)^*} \overline{q} \xrightarrow{v(h+v+t)^*} \overline{p}' \end{array} \right)$$
$$\vee$$
$$\overline{p} \xrightarrow{(h+t)^*} \overline{p}'$$

4. If $\lambda < 0 \ \wedge \ \gamma < 0$, then

$$\exists \overline{q} \in \Gamma_{yz}(\tau,\tau') : \left(\begin{array}{c} \overline{p} \xrightarrow{v+t} \overline{q} \xrightarrow{(h+v+t)^*} \overline{p}' \\ \vee \\ \overline{p} \xrightarrow{h(h+v)^*v} \overline{q} \xrightarrow{h^*t(h+v+t)^*} \overline{p}' \\ \vee \\ \overline{p} \xrightarrow{h(h+t)^*t} \overline{q} \xrightarrow{h^*v(h+v+t)^*} \overline{p}' \end{array} \right)$$
$$\vee$$
$$\overline{p} \xrightarrow{(h+v)^*+(h+t)^*} \overline{p}'$$

$\diamond$

Proof:
Follows by an easy adaptation of the proof of lemma 21 and using lemma 22 by which $\Gamma_{yz}(\tau,\tau')$ is invariant under $(v+t)^*$-paths. $\diamond$

The motivation for this lemma is that we will construct an algorithm that searches for certain points in $\Gamma_{yz}(\tau,\tau')$ starting from a point in $\Gamma_{yz}(\tau,\tau')$. The lemma above allows us to restrict the form of the paths that needs be considered. Furthermore, we will use the fact that very restricted paths are sufficient to derive bounds on the search to guarantee termination.

For completeness, we state the obvious corollaries to lemma 23.

Corollary:
Consider a matrix of the form

$$\Phi = \begin{pmatrix} e & \bullet & \gamma \\ \mu & \bullet & \delta \\ \alpha & \bullet & g \end{pmatrix}$$

where $\alpha, \gamma \geq 0$ and no assumptions are made on e, g, μ or δ. Let $\tau = \min\{0, e\}$, $\tau' = \min\{0, g\}$. Assume $\overline{p} \xrightarrow{(h+v+t)^*} \overline{p}'$ for some points $\overline{p}$ and $\overline{p}'$ where $\overline{p} \in [\Gamma_{xz}(\tau,\tau')]$. We have:

1. If $\mu \geq 0 \ \wedge \ \delta \geq 0$, then

$$\exists \overline{q} \in \Gamma_{xz}(\tau,\tau') : \overline{p} \xrightarrow{h+v+t} \overline{q} \xrightarrow{(h+v+t)^*} \overline{p}'$$
$$\vee$$
$$\overline{p} = \overline{p}'$$

2. If $\mu \geq 0 \ \wedge \ \delta < 0$, then

$$\exists \overline{q} \in \Gamma_{xz}(\tau,\tau') : \left(\begin{array}{c} \overline{p} \xrightarrow{h+t} \overline{q} \xrightarrow{(h+v+t)^*} \overline{p}' \\ \vee \\ \overline{p} \xrightarrow{v(h+v)^*} \overline{q} \xrightarrow{t(h+v+t)^*} \overline{p}' \end{array} \right)$$
$$\vee$$
$$\overline{p} \xrightarrow{(h+v)^*} \overline{p}'$$

3. If $\mu < 0 \ \wedge \ \delta \geq 0$, then

$$\begin{array}{c} \exists \overline{q} \in \Gamma_{xz}(\tau, \tau') : \left(\begin{array}{c} \overline{p} \xrightarrow{h+t} \overline{q} \xrightarrow{(h+v+t)^*} \overline{p}' \\ \vee \\ \overline{p} \xrightarrow{v(v+t)^*} \overline{q} \xrightarrow{h(h+v+t)^*} \overline{p}' \end{array} \right) \\ \vee \\ \overline{p} \xrightarrow{(v+t)^*} \overline{p}' \end{array}$$

4. If $\mu < 0 \ \wedge \ \delta < 0$, then

$$\begin{array}{c} \exists \overline{q} \in \Gamma_{xz}(\tau, \tau') : \left(\begin{array}{c} \overline{p} \xrightarrow{h+t} \overline{q} \xrightarrow{(h+v+t)^*} \overline{p}' \\ \vee \\ \overline{p} \xrightarrow{v(h+v)^*h} \overline{q} \xrightarrow{v^*t(h+v+t)^*} \overline{p}' \\ \vee \\ \overline{p} \xrightarrow{v(v+t)^*t} \overline{q} \xrightarrow{v^*h(h+v+t)^*} \overline{p}' \end{array} \right) \\ \vee \\ \overline{p} \xrightarrow{(h+v)^*+(v+t)^*} \overline{p}' \end{array}$$

$\diamond$

Corollary:
Consider a matrix of the form

$$\Phi = \begin{pmatrix} e & \lambda & \bullet \\ \mu & f & \bullet \\ \alpha & \beta & \bullet \end{pmatrix}$$

where $\mu, \lambda \geq 0$ and no assumptions are made on e, f, α or β. Let $\tau = \min\{0, e\}$, $\tau' = \min\{0, f\}$. Assume $\overline{p} \xrightarrow{(h+v+t)^*} \overline{p}'$ for some points $\overline{p}$ and $\overline{p}'$ where $\overline{p} \in [\Gamma_{xy}(\tau, \tau')]$. We have:

1. If $\alpha \geq 0 \ \wedge \ \beta \geq 0$, then

$$\begin{array}{c} \exists \overline{q} \in \Gamma_{xz}(\tau, \tau') : \overline{p} \xrightarrow{h+v+t} \overline{q} \xrightarrow{(h+v+t)^*} \overline{p}' \\ \vee \\ \overline{p} = \overline{p}' \end{array}$$

2. If $\alpha \geq 0 \ \wedge \ \beta < 0$, then

$$\begin{array}{c} \exists \overline{q} \in \Gamma_{xz}(\tau, \tau') : \left(\begin{array}{c} \overline{p} \xrightarrow{h+v} \overline{q} \xrightarrow{(h+v+t)^*} \overline{p}' \\ \vee \\ \overline{p} \xrightarrow{t(h+t)^*} \overline{q} \xrightarrow{v(h+v+t)^*} \overline{p}' \end{array} \right) \\ \vee \\ \overline{p} \xrightarrow{(h+t)^*} \overline{p}' \end{array}$$

3. If $\alpha < 0 \ \wedge \ \beta \geq 0$, then

$$\begin{array}{c} \exists \overline{q} \in \Gamma_{xy}(\tau, \tau') : \left(\begin{array}{c} \overline{p} \xrightarrow{h+v} \overline{q} \xrightarrow{(h+v+t)^*} \overline{p}' \\ \vee \\ \overline{p} \xrightarrow{t(v+t)^*} \overline{q} \xrightarrow{h(h+v+t)^*} \overline{p}' \end{array} \right) \\ \vee \\ \overline{p} \xrightarrow{(v+t)^*} \overline{p}' \end{array}$$

4. If $\alpha < 0 \ \wedge \ \beta < 0$, then

$$\exists \overline{q} \in \Gamma_{xy}(\tau,\tau') : \left(\begin{array}{c} \overline{p} \xrightarrow{h+v} \overline{q} \xrightarrow{(h+v+t)^*} \overline{p}' \\ \vee \\ \overline{p} \xrightarrow{t(h+t)^* h} \overline{q} \xrightarrow{t^* v(h+v+t)^*} \overline{p}' \\ \vee \\ \overline{p} \xrightarrow{t(v+t)^* v} \overline{q} \xrightarrow{t^* h(h+v+t)^*} \overline{p}' \end{array} \right)$$
$$\vee$$
$$\overline{p} \xrightarrow{(h+t)^*+(v+t)^*} \overline{p}'$$

$\diamond$

Proof:
Follows from lemma 23 by permutation of h, v and t. $\diamond$

Lemma 24:
Consider a matrix of the form

$$\Phi = \begin{pmatrix} e & \bullet & \bullet \\ \bullet & f & \delta \\ \bullet & \beta & g \end{pmatrix}$$

where $\delta, \beta \geq 0$ and no assumptions are made on e, f or g. Let $\tau = \min\{0, e\}$, $\tau' = \min\{0, f\}$ and $\tau'' = \min\{0, g\}$. Then

$$\overline{p} \in [\Gamma_{yz}(\tau',\tau'')] \quad \wedge \ \overline{p} \xrightarrow{(h+v+t)^*} \overline{p}' \ \Rightarrow \left(\begin{array}{c} \exists \overline{q} \in \Gamma_{xyz}(\tau,\tau',\tau'') : \overline{p} \xrightarrow{(v+t)^*} \overline{q} \xrightarrow{h(h+v+t)^*} \overline{p}' \\ \vee \\ \overline{p} \xrightarrow{(v+t)^*} \overline{p}' \end{array} \right)$$

must hold. $\diamond$

Corollary:
Consider a matrix of the form

$$\Phi = \begin{pmatrix} e & \bullet & \gamma \\ \bullet & f & \bullet \\ \alpha & \bullet & g \end{pmatrix}$$

where $\gamma, \alpha \geq 0$ and no assumptions are made on e, f or g. Let $\tau = \min\{0, e\}$, $\tau' = \min\{0, f\}$ and $\tau'' = \min\{0, g\}$. Then

$$\overline{p} \in [\Gamma_{xz}(\tau,\tau'')] \quad \wedge \ \overline{p} \xrightarrow{(h+v+t)^*} \overline{p}' \ \Rightarrow \left(\begin{array}{c} \exists \overline{q} \in \Gamma_{xyz}(\tau,\tau',\tau'') : \overline{p} \xrightarrow{(h+t)^*} \overline{q} \xrightarrow{v(h+v+t)^*} \overline{p}' \\ \vee \\ \overline{p} \xrightarrow{(h+t)^*} \overline{p}' \end{array} \right)$$

must hold. $\diamond$

Corollary:
Consider a matrix of the form

$$\Phi = \begin{pmatrix} e & \lambda & \bullet \\ \mu & f & \bullet \\ \bullet & \bullet & g \end{pmatrix}$$

where $\lambda, \mu \geq 0$ and no assumptions are made on e, f or g. Let $\tau = \min\{0, e\}$, $\tau' = \min\{0, f\}$ and $\tau'' = \min\{0, g\}$. Then

$$\overline{p} \in [\Gamma_{xy}(\tau, \tau')] \quad \wedge \quad \overline{p} \xrightarrow{(h+v+t)^*} \overline{p}' \quad \Rightarrow \\ \left(\begin{array}{c} \exists \overline{q} \in \Gamma_{xyz}(\tau, \tau', \tau'') : \overline{p} \xrightarrow{(h+v)^*} \overline{q} \xrightarrow{t(h+v+t)^*} \overline{p}' \\ \vee \\ \overline{p} \xrightarrow{(h+v)^*} \overline{p}' \end{array} \right)$$

must hold. $\diamond$

Proof:
Follows from lemma 24 by permutation of h, v and t. $\diamond$

Lemma 24 and its corollaries says (under the stated conditions) that starting from a point in $[\Gamma_{xy}(\tau, \tau')]$, when a t-move is applied, one must be in $[\Gamma_{xyz}(\tau, \tau', \tau'')]$ (and the corresponding statements for $[\Gamma_{xz}(\tau, \tau'')]$ and $[\Gamma_{yz}(\tau', \tau'')]$).

Lemma 24 and its first corollary has the following consequence: Consider a matrix of the form

$$\Phi = \begin{pmatrix} e & \bullet & \gamma \\ \bullet & f & \delta \\ \alpha & \beta & g \end{pmatrix}$$

where $\alpha, \gamma, \beta, \delta \geq 0$ and no assumptions are made on e, f or g. Let $\tau = \min\{0, e\}$, $\tau' = \min\{0, f\}$ and $\tau'' = \min\{0, g\}$. If $\overline{p} \in [\Gamma_{yz}(\tau', \tau'')]$ and $\overline{p} \xrightarrow{(h+v+t)^*} \overline{p}'$ for some $\overline{p}$ and $\overline{p}'$, then there exists a sequence of points $\overline{p}_0$, $\overline{p}_1$, ..., $\overline{p}_n$ such that $\overline{p}_i \in \Gamma_{xyz}(\tau, \tau', \tau'')$ and

$$\overline{p} \xrightarrow{(v+t)^*} \overline{p}_0 \xrightarrow{h(h+t)^*} \overline{p}_1 \xrightarrow{v(v+t)^*} \overline{p}_2 \xrightarrow{h(h+t)^*} \overline{p}_3 \cdots \overline{p}_n \xrightarrow{(v+t)^*} \overline{p}'$$

This is illustrated in figure 19. Consider more specifically the matrix

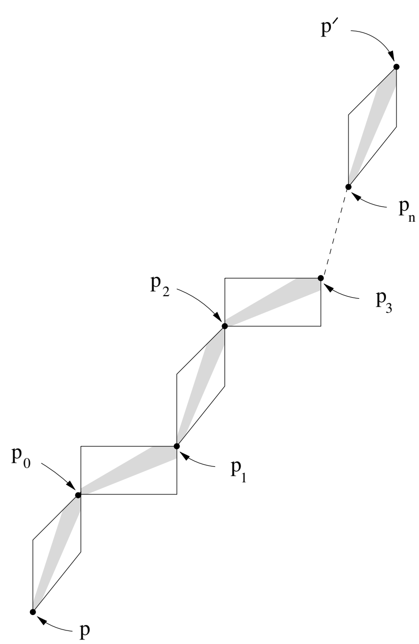


Figure 19

$$\Phi = \begin{pmatrix} -e & -\lambda & \gamma \\ -\mu & -f & \delta \\ \alpha & \beta & g \end{pmatrix}$$

where $e, \mu, \alpha, \lambda, f, \beta, \gamma, \delta \geq 0$ and no assumptions are made on g. If $\overline{p} \xrightarrow{(h+v+t)^*} \overline{p}'$ and $\overline{p}$ and $\overline{p}'$ are not in the same hv-plane, then there exists a sequence of points in $\Gamma_{xyz}(\tau, \tau', \tau'')$ such that

$$\overline{p} \xrightarrow{(v+t)^*} \overline{p}_0 \xrightarrow{h^*t(h+t)^*} \overline{p}_1 \xrightarrow{v^*t(v+t)^*} \overline{p}_2 \xrightarrow{h^*t(h+t)^*} \overline{p}_3 \cdots \overline{p}_n \xrightarrow{(v+t)^*} \overline{p}'$$

or

$$\overline{p} \xrightarrow{(h+t)^*} \overline{p}_0 \xrightarrow{v^*t(v+t)^*} \overline{p}_1 \xrightarrow{h^*t(h+t)^*} \overline{p}_2 \xrightarrow{v^*t(v+t)^*} \overline{p}_3 \cdots \overline{p}_n \xrightarrow{(h+t)^*} \overline{p}'$$

That is, the sequence of planes does not degenerate to $\overline{p} \xrightarrow{(h+v)^*} \overline{p}'$. This can be seen since $\overline{p} \xrightarrow{(h+v)^*} \overline{p}' \Leftrightarrow \overline{p} \xrightarrow{h^*v^*+v^*h^*} \overline{p}'$, so any sequence of ht- and vt-moves that do not contain any t-move, collapses.

We will be interested in dividing the set $[\Gamma_{yz}]$ into subsets in some way. Given a family of sets, say $\{M_i\}$ such that $[\Gamma_{yz}] = \bigcup M_i$, it is obvious that the lemmas and their corollaries above hold when $[\Gamma_{yz}]$ is replaced by $\bigcup M_i$ (the same is true for $[\Gamma_{xy}]$, $[\Gamma_{xz}]$ et.c.). In particular we introduce the two sets, $[\Omega_{yz}]$ and $[\Pi_{yz}]$, such that $[\Gamma_{yz}] = [\Omega_{yz}] \cup [\Pi_{yz}]$ and which are defined as follows:

We define

$$\Pi_{xy}(d, d') = \{\langle x, y\rangle : \langle x, y\rangle \in \Gamma_{xy}(\tau, \tau') \ \wedge\ x < d \ \wedge\ y < d'\}$$
$$\Pi_{xz}(d, d') = \{\langle x, z\rangle : \langle x, z\rangle \in \Gamma_{xz}(\tau, \tau') \ \wedge\ x < d \ \wedge\ z < d'\}$$
$$\Pi_{yz}(d, d') = \{\langle y, z\rangle : \langle y, z\rangle \in \Gamma_{yz}(\tau, \tau') \ \wedge\ y < d \ \wedge\ z < d'\}$$

It is clear that the sets $\Pi_{xy}(d, d')$, $\Pi_{xz}(d, d')$ and $\Pi_{yz}(d, d')$ are finite. We will refer to pairs in a set $\Pi_{yz}(d, d')$ as *pigeon-holes*. By $[\Pi_{yz}(d, d')]$ we denote the set of points $\overline{p}$ such that $\langle y_{\overline{p}}, z_{\overline{p}}\rangle \in \Pi_{yz}(d, d')$ (and analogously for the other sets).

The motivation for introducing these sets is that we will construct a sequence of points $\overline{p}_0 \xrightarrow{(h+v+t)^*} \overline{p}_1 \cdots \overline{p}_{r-1} \xrightarrow{(h+v+t)^*} \overline{p}_r$ such that $\overline{p}_i \in [\Pi_{yz}(d, d')]$ for all $0 \leq i \leq r$. By the pigeon-hole principle, for sufficiently large r, we must have $\langle y_{\overline{p}_i}, z_{\overline{p}_i}\rangle = \langle y_{\overline{p}_j}, z_{\overline{p}_j}\rangle$ for some $0 \leq i < j \leq r$. This yields a solution to a set of equations which can be exploited for giving an expression for the least fixpoint.

We also introduce sets $\Omega_{xy}(d, d')$, $\Omega_{xz}(d, d')$ and $\Omega_{yz}(d, d')$ defined by

$$\Omega_{xy}(d, d') = \{\langle x, y\rangle : \langle x, y\rangle \in \Gamma_{xy}(\tau, \tau') \ \wedge\ (x \geq d \ \vee\ y \geq d')\}$$
$$\Omega_{xz}(d, d') = \{\langle x, z\rangle : \langle x, z\rangle \in \Gamma_{xz}(\tau, \tau') \ \wedge\ (x \geq d \ \wedge\ z \geq d')\}$$
$$\Omega_{yz}(d, d') = \{\langle y, z\rangle : \langle y, z\rangle \in \Gamma_{yz}(\tau, \tau') \ \wedge\ (y \geq d \ \wedge\ z \geq d')\}$$

For a point $\overline{p} \in [\Omega_{yz}(d, d')]$, some simple reasoning often applies to characterize the points reachable from $\overline{p}$. The required property or transformation usually fails for points in $[\Pi_{yz}(d, d')]$. But since $\Pi_{yz}(d, d')$ is finite and the elements are known, special arguments may be applied and repeated a bounded number of times before some pattern must arize.

It is obvious that the definitions of the Ω- and Π-sets implies that $[\Gamma_{yz}(\tau, \tau')] = [\Omega_{yz}(d, d')] \cup [\Pi_{yz}(d, d')]$ (and the analogous for the other sets).

Strictly, the sets $\Pi_{yz}(d, d')$ and $\Omega_{yz}(d, d')$ should be parameterized by τ and τ'. We will supress the parameters τ, τ', d and d' when they are understood from the context.

6.1 Derivation of Bounds

In this section we will answer the question (for some matrices) of how long a path must be to be guaranteed to be able to reach some point in $[\Pi_{yz}]$, starting from $[\Pi_{yz}]$.

Let us restrict our attention to matrices of the form

$$\Phi = \begin{pmatrix} \bullet & \lambda & \gamma \\ \bullet & -f & \delta \\ \bullet & \beta & -g \end{pmatrix}$$

where $f, g, \beta, \delta \geq 0$ and no assumptions are made on λ or γ. We will throughout this section consider $[\Gamma_{yz}(-f, -g)]$, so we dropp the parameters and write $[\Gamma_{yz}]$. Thus if a point $\overline{p}$ is in $[\Gamma_{yz}]$, then $-f \leq y_{\overline{p}}$ and $-g \leq z_{\overline{p}}$.

We state a refinement of lemma 23 and its corollaries.

Lemma 25:
Consider the matrix

$$\Phi = \begin{pmatrix} \bullet & \lambda & \gamma \\ \bullet & -f & \delta \\ \bullet & \beta & -g \end{pmatrix}$$

where $f, g, \beta, \delta \geq 0$ and no assumptions are made on λ or γ. Assume $\overline{p} \xrightarrow{(h+v+t)^*} \overline{p}'$ for some points $\overline{p}$ and $\overline{p}'$ where $\overline{p} \in [\Pi_{yz}(d, d')]$ (remember that $[\Gamma_{yz}] = [\Pi_{yz}] \cup [\Omega_{yz}]$ and $[\Pi_{yz}] \cap [\Omega_{yz}] = \emptyset$).

1. If $\lambda \geq 0 \ \wedge \ \gamma \geq 0$, then

$$\begin{gathered} \exists \overline{q} \in \Gamma_{yz} : \overline{p} \xrightarrow{h+v+t} \overline{q} \xrightarrow{(h+v+t)^*} \overline{p}' \\ \vee \\ \overline{p} = \overline{p}' \end{gathered}$$

2. If $\lambda \geq 0 \ \wedge \ \gamma < 0$, then

$$\begin{gathered} \exists \overline{q} \in \Gamma_{yz} : \overline{p} \xrightarrow{v+t} \overline{q} \xrightarrow{(h+v+t)^*} \overline{p}' \\ \vee \\ \exists \overline{q} \in \Pi_{yz} : \overline{q} \leq \overline{p} + \overline{s}_0 \ \wedge \ \overline{p} \xrightarrow{h(h+v)^*} \overline{q} \xrightarrow{t(h+v+t)^*} \overline{p}' \\ \vee \\ \exists \overline{q} \in \Omega_{yz} : \overline{p} \xrightarrow{h(h+v)^*} \overline{q} \xrightarrow{t(h+v+t)^*} \overline{p}' \\ \vee \\ \overline{p} \xrightarrow{(h+v)^*} \overline{p}' \end{gathered}$$

   where $\overline{s}_0 = \langle h_0, v_0, 0 \rangle^T$ and $h_0 = \lceil \frac{(-\delta f - fg) - (\delta d + f d')}{\lambda\delta + f\gamma} \rceil + 1$ if

$$\begin{vmatrix} \lambda & \gamma \\ -f & \delta \end{vmatrix} < 0$$

   and $h_0 = \lceil \frac{(\delta d - f d') - (-\delta f - fg)}{\lambda\delta + f\gamma} \rceil + 1$ if

$$\begin{vmatrix} \lambda & \gamma \\ -f & \delta \end{vmatrix} > 0$$

   and where $v_0 = \lceil \frac{d + \lambda h_0}{f} \rceil + 1$.

3. If $\lambda < 0 \;\wedge\; \gamma \geq 0$, then

$$\begin{array}{c}
\exists \overline{q} \in \Gamma_{yz} : \overline{p} \xrightarrow{v+t} \overline{q} \xrightarrow{(h+v+t)^*} \overline{p}' \\
\vee \\
\exists \overline{q} \in \Pi_{yz} : \overline{q} \leq \overline{p} + \overline{s}_0 \;\wedge\; \overline{p} \xrightarrow{h(h+t)^*} \overline{q} \xrightarrow{v(h+v+t)^*} \overline{p}' \\
\vee \\
\exists \overline{q} \in \Omega_{yz} : \overline{p} \xrightarrow{h(h+t)^*} \overline{q} \xrightarrow{v(h+v+t)^*} \overline{p}' \\
\vee \\
\overline{p} \xrightarrow{(h+t)^*} \overline{p}'
\end{array}$$

where $\overline{s}_0 = \langle h_0, 0, t_0 \rangle^T$ and $h_0 = \lceil \frac{(-gf-\beta g)-(gd+\beta d')}{-(-\lambda g-\beta\gamma)} \rceil + 1$ if

$$\begin{vmatrix} \lambda & \gamma \\ \beta & -g \end{vmatrix} > 0$$

and $h_0 = \lceil \frac{(gd+\beta d')-(-gf-\beta g)}{-(-\lambda g-\beta\gamma)} \rceil + 1$ if

$$\begin{vmatrix} \lambda & \gamma \\ \beta & -g \end{vmatrix} < 0$$

and where $t_0 = \lceil \frac{d'+\lambda h_0}{g} \rceil + 1$.

4. If $\lambda < 0 \;\wedge\; \gamma < 0$, then

$$\begin{array}{c}
\exists \overline{q} \in \Gamma_{yz} : \overline{p} \xrightarrow{v+t} \overline{q} \xrightarrow{(h+v+t)^*} \overline{p}' \\
\vee \\
\exists \overline{q} \in \Pi_{yz} : \overline{q} \leq \overline{p} + \overline{s}_0 \;\wedge\; \overline{p} \xrightarrow{h(h+v)^* v} \overline{q} \xrightarrow{h^* t(h+v+t)^*} \overline{p}' \\
\vee \\
\exists \overline{q} \in \Omega_{yz} : \overline{p} \xrightarrow{h(h+v)^* v} \overline{q} \xrightarrow{h^* t(h+v+t)^*} \overline{p}' \\
\vee \\
\exists \overline{q} \in \Pi_{yz} : \overline{q} \leq \overline{p} + \overline{s}'_0 \;\wedge\; \overline{p} \xrightarrow{h(h+t)^* t} \overline{q} \xrightarrow{h^* v(h+v+t)^*} \overline{p}' \\
\vee \\
\exists \overline{q} \in \Omega_{yz} : \overline{p} \xrightarrow{h(h+t)^* t} \overline{q} \xrightarrow{h^* v(h+v+t)^*} \overline{p}' \\
\vee \\
\overline{p} \xrightarrow{(h+v)^*+(h+t)^*} \overline{p}'
\end{array}$$

where $\overline{s}_0 = \langle h_0, v_0, 0 \rangle^T$, $h_0 = \lceil \frac{-f-d}{\lambda} \rceil + 1$ and $v_0 = \lceil \frac{d}{f} \rceil + 1$, and where $\overline{s}'_0 = \langle h'_0, 0, t'_0 \rangle^T$, $h'_0 = \lceil \frac{-g-d'}{\gamma} \rceil + 1$ and $t'_0 = \lceil \frac{d'}{g} \rceil + 1$.

$\diamond$

Proof:
If $\lambda, \gamma \geq 0$, by lemma 23, $[\Gamma_{yz}]$ is invariant for all paths, so in this case the search for a point in $[\Gamma_{yz}]$ will succeed after one step in any direction. Actually Ω_{yz} is also an invariant, so it follows that starting from $[\Pi_{yz}]$, going one step in any direction will either yield a new point in $[\Pi_{yz}]$ or no such point will ever be reached.

Assume $\lambda \geq 0 \;\wedge\; \gamma < 0$ and consider a point $\overline{p} \in [\Pi_{yz}(d, d')]$. Then $-f \leq y_{\overline{p}} < d$ and $-g \leq z_{\overline{p}} < d'$. By lemma 23, we need only consider paths of the form $h(h+v)^*$ since a v- or a t-move applied to $\overline{p}$ leads directly to a point in $[\Gamma_{yz}]$. Consider $\delta y_{\overline{p}} + f z_{\overline{p}}$. Since $\overline{p} \in [\Pi_{yz}]$, we have $-\delta f - fg \leq \delta y_{\overline{p}} + fz_{\overline{p}} < \delta d + fd'$. Suppose $\overline{p} \xrightarrow{(h+v)^*} \overline{p}'$ for some point $\overline{p}'$. Then for some h and v,

$$\begin{array}{rclll} y_{\overline{p}'} & = & y_{\overline{p}} & +\lambda h & -fv \\ z_{\overline{p}'} & = & z_{\overline{p}} & +\gamma h & +\delta v \end{array}$$

and thus

$$\delta y_{\overline{p}'} + fz_{\overline{p}'} \quad = \quad \delta y_{\overline{p}} + fz_{\overline{p}} \quad +(\lambda\delta + f\gamma)h \tag{1}$$

From equation 1 we see that the sum $\delta y_{\overline{p}'} + fz_{\overline{p}'}$ only depends on the number of h-moves in an $(h+v)^*$-path (that is, the sum is invariant under v^*-paths). Furthermore, since $\lambda \geq 0$, $-f \leq y$ is an invariant under $(h+v)^*$-paths. So if $\delta y_{\overline{p}'} + fz_{\overline{p}'} < -\delta f - fg$, then $z_{\overline{p}'} < -g$ and thus $\overline{p}' \notin [\Gamma_{yz}]$ and no sequence of v-moves will lead to a point in $[\Gamma_{yz}]$ (and consequently in $[\Pi_{yz}]$). And if $\delta y_{\overline{p}'} + fz_{\overline{p}'} \geq \delta fd + fd'$, then $\overline{p}'$ cannot be in Π_{yz} and no sequence of v-moves will lead to a point in $[\Pi_{yz}]$.

There are two cases to consider: $(\lambda\delta + f\gamma) < 0$ and $(\lambda\delta + f\gamma) > 0$.

Assume $(\lambda\delta + f\gamma) < 0$. Since $\delta y_{\overline{p}} + fz_{\overline{p}} < \delta d + fd'$, after at most $h_0 = \lceil \frac{(-\delta f - fg) - (\delta d + fd')}{\lambda\delta + f\gamma} \rceil + 1$ (not necessarily consecutive) h-moves $[\Gamma_{yz}]$ and thus $[\Pi_{yz}]$ cannot be reached by any $(h+v)^*$-path. Since $y_{\overline{p}} < d$ and a v-move decreases y by $-f$, h_0 horizontal moves will make at most $v_0 = \lceil \frac{d + \lambda h_0}{f} \rceil + 1$ v-moves possible. Thus, if $\overline{p} \xrightarrow{w} \overline{p}'$ for some $w \in (h+v)^*$, and $\overline{p}' \in [\Gamma_{yz}]$, then $\overline{w} \leq \langle h_0, v_0, 0 \rangle^T$.

Assume $(\lambda\delta + f\gamma) > 0$. Since $\delta y_{\overline{p}} + fz_{\overline{p}} \geq -\delta f - fg$, after at most $h_0 = \lceil \frac{(\delta d - fd') - (-\delta f - fg)}{\lambda\delta + f\gamma} \rceil + 1$ (not necessarily consequtive) h-moves, $\delta y_{\overline{p}} + fz_{\overline{p}} \geq \delta d + fd'$ must hold and thus, any $(h+v)^*$ that leads to $[\Gamma_{yz}]$, must lead to $[\Omega_{yz}]$. Thus, if $\overline{p} \xrightarrow{w} \overline{p}'$ for some $w \in (h+v)^*$, and $\overline{p}' \in [\Pi_{yz}]$, then $\overline{w} \leq \langle h_0, v_0, 0 \rangle^T$, where v_0 is determined as above.

For the case when $\lambda < 0 \ \wedge \ \gamma \geq 0$ one consider paths of the form $(h+t)^*$ and reasons with the sum $gy_{\overline{p}} + \beta z_{\overline{p}}$. If $\overline{p} \xrightarrow{(h+t)^*} \overline{p}'$ for some points $\overline{p} \in [\Pi_{yz}(d,d')]$ and $\overline{p}'$, then for some h and t,

$$\begin{array}{rclll} y_{\overline{p}'} & = & y_{\overline{p}} & +\lambda h & +\beta t \\ z_{\overline{p}'} & = & z_{\overline{p}} & +\gamma h & -gt \end{array}$$

and thus

$$gy_{\overline{p}'} + \beta z_{\overline{p}'} \quad = \quad gy_{\overline{p}} + \beta z_{\overline{p}} \quad -(-\lambda g - \beta\gamma)h \tag{2}$$

By a similar reasoning as above, if $\overline{p} \xrightarrow{w} \overline{p}'$ for some $w \in (h+t)^*$, and $\overline{p}' \in [\Pi_{yz}]$, then $\overline{w} \leq \langle h_0, v_0, 0 \rangle^T$ where $h_0 = \lceil \frac{(-gf - \beta g) - (gd + \beta d')}{-(-\lambda g - \beta\gamma)} \rceil + 1$ if $(-\lambda g - \beta\gamma) > 0$ and $h_0 = \lceil \frac{(gd + \beta d') - (-gf - \beta g)}{-(-\lambda g - \beta\gamma)} \rceil + 1$ if $(-\lambda g - \beta\gamma) < 0$, and where $t_0 = \lceil \frac{d' + \lambda h_0}{g} \rceil + 1$.

Consider now the case when $\lambda < 0 \ \wedge \ \gamma < 0$. By lemma 23, $(h+v)^*$- or $(h+t)^*$-paths must be considered. But since now all $(h+v)^*$-paths decrease y and all $(h+t)^*$-paths decrease z, if $\overline{p} \in [\Pi_{yz}]$ (which means that $y_{\overline{p}} < d$ and $z_{\overline{p}} < d'$), then after at most $h_0 = \lceil \frac{-f-d}{\lambda} \rceil + 1$ horizontal moves (not necessarily consequtive), $y_{\overline{p}'} < -f$ so $[\Gamma_{yz}]$ will not reachable by any $(h+v)^*$-path, and similarly after at most $h_0' = \lceil \frac{-g-d'}{\gamma} \rceil + 1$ horizontal moves (not necessarily consequtive), $z_{\overline{p}'} < -g$ $[\Gamma_{yz}]$ will not reachable by any $(h+t)^*$-path. is guaranteed to hold (so $\overline{p}' \notin [\Gamma_{yz}]$). It is also clear that no more than $v_0 = \lceil \frac{d}{f} \rceil + 1$ vertical moves can be made in a $(h+v)^*$-path, and no more than $t_0' = \lceil \frac{d'}{g} \rceil + 1$ transversal moves can be made in a $(h+t)^*$-path. Thus, if $\overline{p} \xrightarrow{w} \overline{p}'$ for some points $\overline{p}, \overline{p}' \in [\Pi_{yz}]$, then $\overline{w} \leq \langle h_0, v_0, 0 \rangle^*$ if $w \in (h+v)^*$ and $\overline{w} \leq \langle h_0', 0, t_0' \rangle^*$ if $w \in (h+t)^*$.

This concluses the proof. $\diamond$

Note that lemma 25 does not cover the situation when the relevant subdeterminant for the different cases is zero. At present, we have no means of treating such programs.

We also stress an important point. The derivation of the bounds does not depend on the fact that all pairs $\langle y, z\rangle$ that satisfy $-f \le y < d \ \wedge \ -g \le z < d'$ actually belongs to Π_{yz}. What *is* important is that all pairs in Π_{yz} satisfy $-f \le y < d \ \wedge \ -g \le z < d'$ for some d and d'. But this is true for *any finite subset of* Γ_{yz}. This is important, since choosing the elements in Π_{yz} cleverly, the number of elements may be dramatically reduced.

For completeness we state the corollaries to lemma 25 obtained by permutation of h, v and t.

Corollary:
Consider the matrix

$$\Phi = \begin{pmatrix} -e & \bullet & \gamma \\ \mu & \bullet & \delta \\ \alpha & \bullet & -g \end{pmatrix}$$

where $e, g, \alpha, \gamma \ge 0$ and no assumptions are made on μ or δ. Assume $\overline{p} \xrightarrow{(h+v+t)^*} \overline{p}'$ for some points $\overline{p}$ and $\overline{p}'$ where $\overline{p} \in [\Pi_{xz}(d, d')]$.

1. If $\mu \ge 0 \ \wedge \ \delta \ge 0$, then

$$\begin{gathered} \exists \overline{q} \in \Gamma_{xz} : \overline{p} \xrightarrow{h+v+t} \overline{q} \xrightarrow{(h+v+t)^*} \overline{p}' \\ \vee \\ \overline{p} = \overline{p}' \end{gathered}$$

2. If $\mu \ge 0 \ \wedge \ \delta < 0$, then

$$\begin{gathered} \exists \overline{q} \in \Pi_{xz} : \overline{p} \xrightarrow{h+t} \overline{q} \xrightarrow{(h+v+t)^*} \overline{p}' \\ \vee \\ \exists \overline{q} \in \Pi_{xz} : \overline{q} \le \overline{p} + \overline{s}_0 \ \wedge \ \overline{p} \xrightarrow{v(h+v)^*} \overline{q} \xrightarrow{t(h+v+t)^*} \overline{p}' \\ \vee \\ \exists \overline{q} \in \Omega_{xz} : \overline{p} \xrightarrow{v(h+v)^*} \overline{q} \xrightarrow{t(h+v+t)^*} \overline{p}' \\ \vee \\ \overline{p} \xrightarrow{(h+v)^*} \overline{p}' \end{gathered}$$

   where $\overline{s}_0 = \langle h_0, v_0, 0\rangle^T$ and $v_0 = \lceil \frac{(-\gamma e - eg) - (\gamma d + ed')}{\mu\gamma + e\delta} \rceil + 1$ if

$$\begin{vmatrix} -e & \gamma \\ \mu & \delta \end{vmatrix} > 0$$

   and $v_0 = \lceil \frac{(\gamma d - ed') - (-\gamma e - fg)}{\mu\gamma + e\delta} \rceil + 1$ if

$$\begin{vmatrix} -e & \gamma \\ \mu & \delta \end{vmatrix} < 0$$

   and where $h_0 = \lceil \frac{d + \mu v_0}{e} \rceil + 1$.

3. If $\mu < 0 \ \wedge \ \delta \geq 0$, then

$$\begin{array}{c}
\exists \overline{q} \in \Gamma_{xz} : \overline{p} \xrightarrow{h+t} \overline{q} \xrightarrow{(h+v+t)^*} \overline{p}' \\
\vee \\
\exists \overline{q} \in \Pi_{xz} : \overline{q} \leq \overline{p} + \overline{s}_0 \ \wedge \ \overline{p} \xrightarrow{v(v+t)^*} \overline{q} \xrightarrow{h(h+v+t)^*} \overline{p}' \\
\vee \\
\exists \overline{q} \in \Omega_{xz} : \overline{p} \xrightarrow{v(v+t)^*} \overline{q} \xrightarrow{h(h+v+t)^*} \overline{p}' \\
\vee \\
\overline{p} \xrightarrow{(v+t)^*} \overline{p}'
\end{array}$$

where $\overline{s}_0 = \langle 0, v_0, t_0 \rangle^T$ and $v_0 = \lceil \frac{(-ge-\alpha g)-(gd+\alpha d')}{-(-\mu g-\alpha\delta)} \rceil + 1$ if

$$\begin{vmatrix} \mu & \delta \\ \alpha & -g \end{vmatrix} > 0$$

and $v_0 = \lceil \frac{(gd+\alpha d')-(-ge-\alpha g)}{-(-\mu g-\alpha\delta)} \rceil + 1$ if

$$\begin{vmatrix} \mu & \delta \\ \alpha & -g \end{vmatrix} < 0$$

and where $t_0 = \lceil \frac{d'+\delta v_0}{g} \rceil + 1$.

4. If $\mu < 0 \ \wedge \ \delta < 0$, then

$$\begin{array}{c}
\exists \overline{q} \in \Gamma_{xz} : \overline{p} \xrightarrow{h+t} \overline{q} \xrightarrow{(h+v+t)^*} \overline{p}' \\
\vee \\
\exists \overline{q} \in \Pi_{xz} : \overline{q} \leq \overline{p} + \overline{s}_0 \ \wedge \ \overline{p} \xrightarrow{v(h+v)^*h} \overline{q} \xrightarrow{v^*t(h+v+t)^*} \overline{p}' \\
\vee \\
\exists \overline{q} \in \Omega_{xz} : \overline{p} \xrightarrow{v(h+v)^*h} \overline{q} \xrightarrow{v^*t(h+v+t)^*} \overline{p}' \\
\vee \\
\exists \overline{q} \in \Pi_{xz} : \overline{q} \leq \overline{p} + \overline{s}'_0 \ \wedge \ \overline{p} \xrightarrow{v(v+t)^*t} \overline{q} \xrightarrow{v^*h(h+v+t)^*} \overline{p}' \\
\vee \\
\exists \overline{q} \in \Omega_{xz} : \overline{p} \xrightarrow{v(v+t)^*t} \overline{q} \xrightarrow{v^*h(h+v+t)^*} \overline{p}' \\
\vee \\
\overline{p} \xrightarrow{(h+v)^*+(v+t)^*} \overline{p}'
\end{array}$$

where $\overline{s}_0 = \langle h_0, v_0, 0 \rangle^T$, $v_0 = \lceil \frac{-e-d}{\mu} \rceil + 1$ and $h_0 = \lceil \frac{d}{e} \rceil + 1$, and where $\overline{s}'_0 = \langle 0, v'_0, t'_0 \rangle^T$, $v'_0 = \lceil \frac{-g-d'}{\delta} \rceil + 1$ and $t'_0 = \lceil \frac{d'}{g} \rceil + 1$.

$\diamond$

Corollary:
Consider the matrix

$$\Phi = \begin{pmatrix} -e & \lambda & \bullet \\ \mu & -f & \bullet \\ \alpha & \beta & \bullet \end{pmatrix}$$

where $e, f, \mu, \lambda \geq 0$ and no assumptions are made on α or β. Assume $\overline{p} \xrightarrow{(h+v+t)^*} \overline{p}'$ for some points $\overline{p}$ and $\overline{p}'$ where $\overline{p} \in [\Pi_{xy}(d, d')]$.

1. If $\alpha \geq 0 \ \wedge \ \beta \geq 0$, then

$$\begin{array}{c}
\exists \overline{q} \in \Gamma_{xy} : \overline{p} \xrightarrow{h+v+t} \overline{q} \xrightarrow{(h+v+t)^*} \overline{p}' \\
\vee \\
\overline{p} = \overline{p}'
\end{array}$$

2. If $\alpha \geq 0 \;\wedge\; \beta < 0$, then

$$\begin{array}{c}
\exists \overline{q} \in \Pi_{xy} : \overline{p} \xrightarrow{h+v} \overline{q} \xrightarrow{(h+v+t)^*} \overline{p}' \\
\vee \\
\exists \overline{q} \in \Pi_{xy} : \overline{q} \leq \overline{p} + \overline{s}_0 \;\wedge\; \overline{p} \xrightarrow{t(h+t)^*} \overline{q} \xrightarrow{v(h+v+t)^*} \overline{p}' \\
\vee \\
\exists \overline{q} \in \Omega_{xy} : \overline{p} \xrightarrow{t(h+t)^*} \overline{q} \xrightarrow{v(h+v+t)^*} \overline{p}' \\
\vee \\
\overline{p} \xrightarrow{(h+t)^*} \overline{p}'
\end{array}$$

where $\overline{s}_0 = \langle h_0, 0, t_0 \rangle^T$ and $t_0 = \lceil \frac{(-\lambda e - eg) - (\lambda d + ed')}{\alpha\lambda + e\beta} \rceil + 1$ if

$$\begin{vmatrix} -e & \lambda \\ \alpha & \beta \end{vmatrix} < 0$$

and $t_0 = \lceil \frac{(\lambda d - ed') - (-\lambda e - eg)}{\alpha\lambda + e\beta} \rceil + 1$ if

$$\begin{vmatrix} -e & \lambda \\ \alpha & \beta \end{vmatrix} > 0$$

and where $h_0 = \lceil \frac{d + \alpha t_0}{e} \rceil + 1$.

3. If $\alpha < 0 \;\wedge\; \beta \geq 0$, then

$$\begin{array}{c}
\exists \overline{q} \in \Gamma_{xy} : \overline{p} \xrightarrow{h+v} \overline{q} \xrightarrow{(h+v+t)^*} \overline{p}' \\
\vee \\
\exists \overline{q} \in \Pi_{xy} : \overline{q} \leq \overline{p} + \overline{s}_0 \;\wedge\; \overline{p} \xrightarrow{t(v+t)^*} \overline{q} \xrightarrow{h(h+v+t)^*} \overline{p}' \\
\vee \\
\exists \overline{q} \in \Omega_{xy} : \overline{p} \xrightarrow{t(v+t)^*} \overline{q} \xrightarrow{h(h+v+t)^*} \overline{p}' \\
\vee \\
\overline{p} \xrightarrow{(v+t)^*} \overline{p}'
\end{array}$$

where $\overline{s}_0 = \langle 0, v_0, t_0 \rangle^T$ and $t_0 = \lceil \frac{(-fe - \mu f) - (fd + \mu d')}{-(-\alpha f - \mu\beta)} \rceil + 1$ if

$$\begin{vmatrix} \mu & -f \\ \alpha & \beta \end{vmatrix} < 0$$

and $t_0 = \lceil \frac{(fd + \mu d') - (-fe - \mu f)}{-(-\alpha f - \mu\beta)} \rceil + 1$ if

$$\begin{vmatrix} \mu & -f \\ \alpha & \beta \end{vmatrix} > 0$$

and where $v_0 = \lceil \frac{d' + \beta t_0}{f} \rceil + 1$.

4. If $\alpha < 0 \;\wedge\; \beta < 0$, then

$$\begin{array}{c}
\exists \overline{q} \in \Gamma_{xy} : \overline{p} \xrightarrow{h+v} \overline{q} \xrightarrow{(h+v+t)^*} \overline{p}' \\
\vee \\
\exists \overline{q} \in \Pi_{xy} : \overline{q} \leq \overline{p} + \overline{s}_0 \;\wedge\; \overline{p} \xrightarrow{t(h+t)^*h} \overline{q} \xrightarrow{t^*v(h+v+t)^*} \overline{p}' \\
\vee \\
\exists \overline{q} \in \Omega_{xy} : \overline{p} \xrightarrow{t(h+t)^*h} \overline{q} \xrightarrow{t^*v(h+v+t)^*} \overline{p}' \\
\vee \\
\exists \overline{q} \in \Pi_{xy} : \overline{q} \leq \overline{p} + \overline{s}'_0 \;\wedge\; \overline{p} \xrightarrow{t(v+t)^*v} \overline{q} \xrightarrow{t^*h(h+v+t)^*} \overline{p}' \\
\vee \\
\exists \overline{q} \in \Omega_{xy} : \overline{p} \xrightarrow{t(v+t)^*v} \overline{q} \xrightarrow{t^*h(h+v+t)^*} \overline{p}' \\
\vee \\
\overline{p} \xrightarrow{(h+t)^*+(v+t)^*} \overline{p}'
\end{array}$$

where $\overline{s}_0 = \langle h_0, 0, t_0\rangle^T$, $t_0 = \lceil\frac{-e-d}{\alpha}\rceil + 1$ and $h_0 = \lceil\frac{d}{e}\rceil + 1$, and where $\overline{s}'_0 = \langle 0, v'_0, t'_0\rangle^T$, $t'_0 = \lceil\frac{-f-d'}{\beta}\rceil + 1$ and $v'_0 = \lceil\frac{d'}{f}\rceil + 1$.

$\diamond$

6.2 Pigeon-Hole Graph

In this section we show how to construct a finite graph describing the reachability relation between points in $[\Pi_{yz}]$. Each node in the graph represents a (possibly infinite) set of points $\overline{p}$, and the links are labeled with relative positions in *hvt*-coordinates of points of the sets represented by the nodes in the graph.

We define the graph and the algorithm for $[\Pi_{yz}]$. It is obvious how it is done for $[\Pi_{xz}]$ and $[\Pi_{xy}]$.

Definition 1:
A p-graph $\mathcal{G}$ is a pair $\langle\Pi_{yz}, \Psi\rangle$ where Π_{yz} is the set of vertices and Ψ is a set of labeled edges (triples). The interpretation of the graph is the following (remember that $\overline{p} \in [\Pi_{yz}]$ iff $\langle y_{\overline{p}}, z_{\overline{p}}\rangle \in \Pi_{yz}$): If $\langle\langle y, z\rangle, \langle y', z'\rangle, \langle h, v, t\rangle^T\rangle \in \Psi$ then

$$\langle y_{\overline{p}}, z_{\overline{p}}\rangle = \langle y, z\rangle \;\wedge\; \overline{p}' = \overline{p} + \begin{pmatrix} h \\ v \\ t \end{pmatrix} \;\Rightarrow\; \langle y_{\overline{p}'}, z_{\overline{p}'}\rangle = \langle y', z'\rangle$$

$\diamond$

It is clear from the definition that if there is a cycle in the graph, then the sum of the labels along the cycle is a solution to the equations

$$\begin{array}{llllll}
y: & \lambda h & -fv & +\beta t & = & 0 \\
z: & \gamma h & +\delta v & -gt & = & 0
\end{array}$$

We will later in this section discuss the relation between the graph and solutions to the equations above.

We will now present an algorithm for computing the graph above. The idea is simply that for each pair $\langle y, z\rangle \in \Pi_{yz}$, we search exhaustively for pairs $\langle y', z'\rangle \in \Pi_{yz}$, reachable by paths of the restricted form $h + v + t$, $v + t + (h + v)^*$ or $v + t + (h + t)^*$ depending on the signs of λ and γ, and such that the paths are bounded by $\overline{s}_0$ (and $\overline{s}'_0$ when $\lambda < 0 \wedge \gamma < 0$). See lemma 25.

We present the algorithm for the case when $\lambda \geq 0 \ \wedge \ \gamma < 0$. It is obvious how it should be changed for the other cases.

Let $\Psi(y,z) = \{\langle\langle y,z\rangle, \psi, \xi\langle : \ \langle\langle y,z\rangle, \psi, \xi\langle \in \Psi\}$. Then

$$\Psi = \bigcup_{\langle y,z\rangle \in \Pi} \Psi(y,z)$$

The set $\Psi(y,z)$ are all links in the graph that leaves the node $\langle y,z\rangle$ and it is computed as follows:

$\Psi(y,z) := \emptyset;$ % No pigeon-hole has yet % been reached

If $y \geq 0$ **and** $\langle y-f, z+\delta\rangle \in \Pi_{yz}$ **then**
  $\Psi(y,z) := \Psi(y,z) \cup \{\langle\langle y,z\rangle, \langle y-f, z+\delta\rangle, \langle 0,1,0\rangle^T\rangle\}$
  % When moving one v-step, % $-f$ is added to y and % δ is added to z.

If $z \geq 0$ **and** $\langle y+\beta, z-g\rangle \in \Pi_{yz}$ **then**
  $\Psi(y,z) := \Psi(y,z) \cup \{\langle\langle y,z\rangle, \langle y+\beta, z-g\rangle, \langle 0,0,1\rangle^T\rangle\}$
  % When moving one t-step, % β is added to y and % $-g$ is added to z.

If $\langle y+\lambda, z+\gamma\rangle \in \Pi_{yz}$ **then**
  $\Psi(y,z) := \Psi(y,z) \cup \{\langle\langle y,z\rangle, \langle y+\lambda, z+\gamma\rangle, \langle 1,0,0\rangle^T\rangle\}$
  % When moving one h-step, % λ is added to y and % γ is added to z.

elseif $\langle y+\lambda, z+\gamma\rangle \notin \Omega_{yz}$ **then** % Do not investigate % points in $[\Omega_{yz}]$.

  $\mathcal{E} := \{\langle\langle y+\lambda, z+\gamma\rangle, \langle 1,0,0\rangle^T\rangle;$ % The set of nodes to be % investigated.

  while $\mathcal{E} \neq \emptyset$ **do**
    choose any $\langle\langle y', z'\rangle, \langle h', v', t'\rangle^T\rangle \in \mathcal{E};$
    $\mathcal{E} := \mathcal{E} - \{\langle\langle y', z'\rangle, \langle h', v', t'\rangle^T\rangle\};$
    if $\langle h', v', t'\rangle^T \leq \overline{s}_0$ **then** % The search bound has not % been reached.

      If $\langle y'+\lambda, z'+\gamma\rangle \in \Pi_{yz}$ **then**
        $\Psi(y,z) := \Psi(y,z) \cup \{\langle\langle y,z\rangle, \langle y'+\lambda, z'+\gamma\rangle, \langle h'+1, v', t'\rangle^T\rangle\}$
        % When moving one h-step, % λ is added to y and % γ is added to z.

      elseif $\langle y'+\lambda, z'+\gamma\rangle \notin \Omega_{yz}$ **then** % Do not investigate % points in $[\Omega_{yz}]$.
        $\mathcal{E} := \mathcal{E} \cup \{\langle\langle y'+\lambda, z'+\gamma\rangle, \langle h'+1, v', t'\rangle^T\rangle;$
        % New point to investigate.

      If $y \geq 0$ **and** $\langle y'-f, z'+\delta\rangle \in \Pi_{yz}$ **then**
        $\Psi(y,z) := \Psi(y,z) \cup \{\langle\langle y,z\rangle, \langle y'-f, z'+\delta\rangle, \langle h', v'+1, t'\rangle^T\rangle\}$
        % When moving one v-step, % $-f$ is added to y and % β is added to z.

elseif $\langle y' - f, z' + \delta \rangle \notin \Omega_{yz}$ **then** % Do not investigate
% points in $[\Omega_{yz}]$.
$\mathcal{E} := \mathcal{E} \cup \{\langle\langle y' - f, z' + \delta\rangle, \langle h', v' + 1, t'\rangle^T\rangle$;
% New point to investigate.

od;

By lemma 25, we must first search one step verticaly and transversally. Then the search continue with $h(h+v)^*$-paths until the bound has been reached.

Note that points in Π_{yz} or Ω_{yz} are not investigated. Note also that there is no test to wether the paths investigated are actually admissible. The tests for $y \geq 0$ and $z \geq 0$ are used merely to reduce the search space since, by definition, all points in a set represented by a node in the graph has the same associated y and z values, so if in the algorithm above $y < 0$, for example, then a v-move can never be applied, so no points are lost. But the x-value may differ for different points with the same y- and z-values. Therefore no test on the x-value in the algorithm. The admissibility of a path will be taken care of in the construction that follows. However, such a test could be incorporated in the algorithm above as an optimisation to avoid computing things twice.

Example 1:
Consider the matrix

$$\Phi = \begin{pmatrix} -1 & 2 & -3 \\ -1 & -5 & 7 \\ 13 & 4 & -4 \end{pmatrix}$$

Let us choose Π_{yz} to be the following set of pairs:

$$\begin{aligned} \Pi_{yz} \quad = \quad & \{\langle y, z\rangle; -5 \leq y < 0 \ \wedge \ -4 \leq z < 0\} \\ & \cup \\ & \{\langle -4, 0\rangle, \langle -5, 0\rangle, \langle -5, 1\rangle, \langle -5, 2\rangle, \langle -5, 3\rangle\} \end{aligned}$$

We see that if $\langle y, z\rangle \in \Pi_{yz}$, then $-5 \leq y < 1$ and $-4 \leq z < 4$ (with a good marginal) so $d = 1$ and $d' = 4$. There are 25 pigeon holes. The matrix above falls into case three of lemma 25, and the upper right determinant is

$$\begin{vmatrix} \lambda & \gamma \\ -f & \delta \end{vmatrix} = \begin{vmatrix} 2 & -3 \\ -5 & 7 \end{vmatrix} = -1 < 0$$

so the limit h_0 is given by

$$h_0 = \lceil \frac{-35 - 20 - 7 - 20}{-1} \rceil + 1 = 83$$

$$v_0 = \lceil \frac{1 + 2 \cdot 83}{7} \rceil + 1 = 25$$

Thus

$$\overline{s}_0 = \langle 83, 25, 0\rangle^T$$

Figures 20 and 21 shows parts of the search space of the algorithm for some pigeon holes. Numbers inside ellipses correspond to pairs in Π_{yz} and numbers inside rectangles correspond to Ωyz. Figure 22 shows a "compressed" reachability graph (It is the transitive closure of the original graph where some points have been removed. We have done so in this example for illustration to reduce the number of nodes). Note that there are three simple cycles. Note also that the lengths of the simple cycles measured as the sum of the labels along the path, all equals $\langle 8, 4, 1\rangle^T$. This is because the equations

$$\begin{array}{rcrrrcl} y & = & \lambda h & -fv & +\beta t & = & 0 \\ z & = & -\gamma h & +\delta v & -gt & = & 0 \end{array}$$

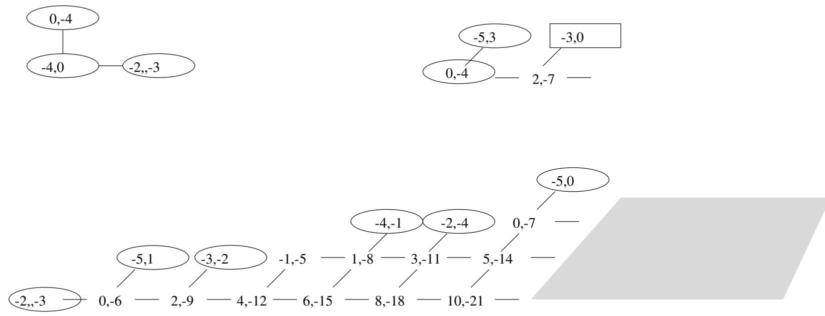


Figure 20

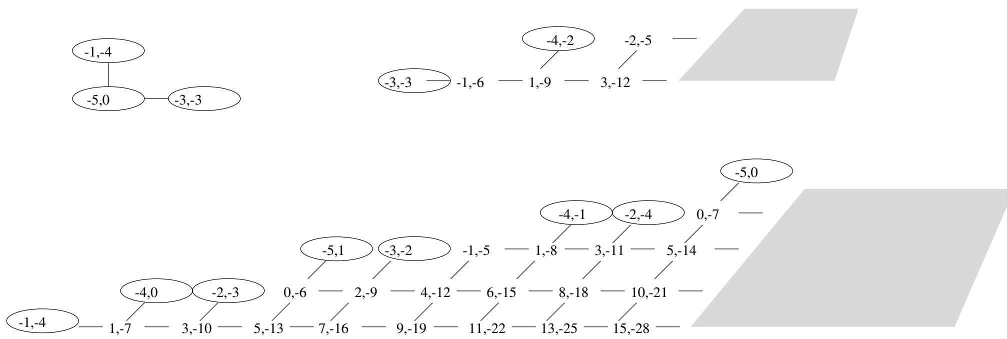


Figure 21

has a unique minimal solution when

$$\begin{vmatrix} \lambda & -\gamma \\ -f & \delta \end{vmatrix} < 0$$

In case when

$$\begin{vmatrix} \lambda & -\gamma \\ -f & \delta \end{vmatrix} > 0$$

the equations lack solutions, which means that the graph would have no cycles. The fixpoint

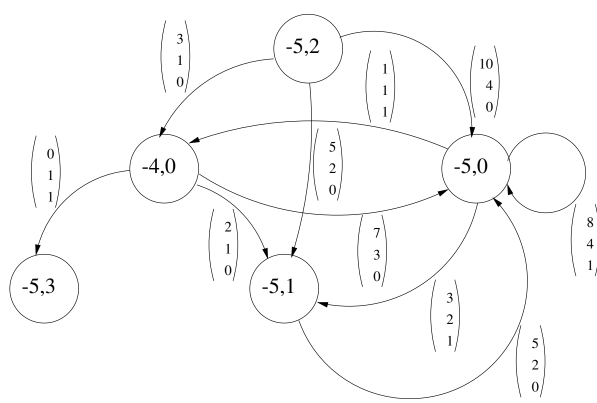


Figure 22

computed with base value $\langle -13, -5, 0\rangle^T$ is shown in figure 23 $\diamond$

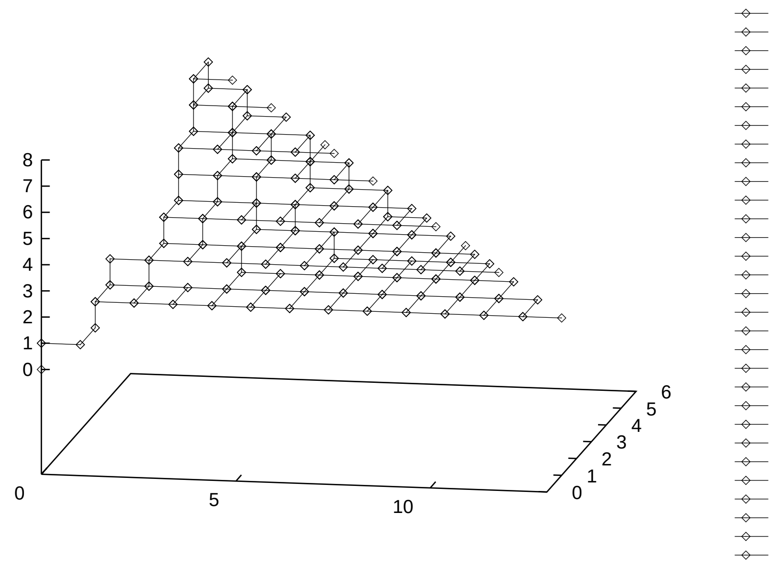


Figure 23

6.3 Linear arithmetic expressions for reachability between pigeon-hole points

Let us make the following observation:

$$\begin{vmatrix} -f & \delta \\ \beta & -g \end{vmatrix}, \quad \begin{vmatrix} \lambda & \gamma \\ -f & \delta \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} \lambda & \gamma \\ \beta & -g \end{vmatrix}$$

are all nonnull iff all solutions $\langle h, v, t \rangle$ to

$$\begin{array}{lrrrcl} y: & \lambda h & -fv & +\beta t & = & 0 \\ z: & \gamma h & +\delta v & -gt & = & 0 \end{array} \tag{3}$$

satisfies $h \neq 0$, $v \neq 0$ and $t \neq 0$.

From now on we assume

$$\begin{vmatrix} -f & \delta \\ \beta & -g \end{vmatrix} < 0$$

and that the determinants above are nonzero.

Lemma 26:
If

$$\begin{vmatrix} \lambda & \gamma \\ -f & \delta \end{vmatrix} \neq 0 \quad \text{and} \quad \begin{vmatrix} \lambda & \gamma \\ \beta & -g \end{vmatrix} \neq 0$$

and the equations (3) have solutions, then

$$\begin{vmatrix} \lambda & \gamma \\ -f & \delta \end{vmatrix} < 0 \quad \text{and} \quad \begin{vmatrix} \lambda & \gamma \\ \beta & -g \end{vmatrix} > 0$$

◇

Proof:
From (3) we get

$$\begin{array}{rcl}(\delta\lambda + f\gamma)h & = & (fg - \beta\delta)t \\ (\delta\lambda + f\gamma)v & = & -(-\lambda g - \beta\gamma)t\end{array}$$

It is immediatelly seen that the signs of the determinants are necessary for h, v and t to have the same signs. $\diamond$

Corollary:
If $\lambda \geq 0 \ \wedge \gamma > 0$ or $\lambda > 0 \ \wedge \gamma \geq 0$ then (3) has no solutions. $\diamond$

If (3) has solutions, it has infinitely many. Since we assume that all subdeterminants are nonzero, all solutions to (3) are comparable. That is, if $\overline{\xi}$ and $\overline{\xi}'$ are solutions, either $\overline{\xi} \leq \overline{\xi}'$ or $\overline{\xi}' \leq \overline{\xi}$.

A cycle in the graph yields a nontrivial solution to (3). By the pigeon-hole principle, a (simple) cycle cannot be longer than $|\Pi_{yz}|$. Furthermore, the derivation above of the bounds h_0 on the number of horizontal steps from a pigeon hole to the next, yields a bound $|\Pi_{yz}|h_0$ on the total length of an hvt-path corresponding to the cycle. Thus there exists a unique maximal solution corresponding to the maximum of the sums along all simple cycles. Note that several simple cycles may have the same sum. Also note that (3) may have solutions even though there are no cycles in the graph. This may happen when $\lambda < 0$ and $\gamma < 0$ for instance. Of course, when (3) has no solutions, there cannot be any cycles.

Lemma 27:
Consider the matrix

$$\Phi = \begin{pmatrix} \bullet & \lambda & \gamma \\ \bullet & -f & \delta \\ \bullet & \beta & -g \end{pmatrix}$$

where $f, g, \beta, \delta \geq 0$ and no assumptions are made on λ or γ. Let $\overline{\xi}$ be the sum of the labels along a simple cycle in the graph with maximum sum of labels, and let ξ be the motif of $\overline{\xi}$. If no cycle exists let $\overline{\xi} = \overline{0}$ and ξ the empty language. Consider a point $\overline{p} \in [\Pi_{yz}]$. If $\overline{p} \xrightarrow{(h+v+t)^*} \overline{p}'$ for some point $\overline{p}' \in [\Pi_{yz}]$, then one of the following holds:

$$\overline{p} \xrightarrow{(h+v+t)^{\leq k}\xi^*(h+v+t)^{\leq k}} \overline{p}'$$

where $k = |\Pi_{yz}| \cdot (|\overline{s}_0| + 1)$ and $\overline{s}_0$ is given by lemma 25 under the proper assumptions on the signs of submatrices. $\diamond$

Proof:
From any $\overline{q} \in [\Pi_{yz}]$, it is sufficient to walk a path of length at most $|\overline{s}_0| + 1$ to reach another point $\overline{q}' \in [\Pi_{yz}]$ since either a single h, v or t move suffices, or a $(h+v)^*$- or $(h+t)^*$-path of length $|\overline{s}_0|$ suffices (depending on the signs of the coefficients by lemma 25).

Since there are no more then $|\Pi_{yz}|$ elements in Π_{yz}, a path of length at most $|\Pi_{yz}| \cdot (|\overline{s}_0| + 1)$ can be taken before entering a cycle. Since $\overline{\xi}$ is a multiple of every cycle, any number of applications of any cycle is a multiple of $\overline{\xi}$ plus some residue path. After any number of applications of cycles, some extra pigeon-holes may be visited before $\overline{p}'$ is reached. The length of this tail together with the residue, cannot exceed $|\Pi_{yz}| \cdot (|\overline{s}_0| + 1)$, otherwise a new cycle must have been entered.

Thus, from $\overline{p}$ a path of the form $(h+v+t)^{\leq k}$ suffices to reach a cycle. Then some path of the form ξ^* may be applied, followed again by a path of the form $(h+v+t)^{\leq k}$ to leave the cycle suffices to reach any reachable point $\overline{p}' \in [\Pi_{yz}]$. $\diamond$

The proof of lemma 27 does not actually depend on an explicit construction of the pigeon-hole graph. It only depends on the bounds given in lemma 25 and the size of Π_{yz}, since ξ can be taken as the motif associated with the largest solution $\overline{\xi}$ to (3), such that $|\overline{\xi}| \leq k$. However, the formula obtained this way is unnecessarily large, as suggested by example 1 where $k = |\Pi_{yz}| \cdot (|\overline{s}_0| + 1) = 25 \cdot (108 + 1) = 2725$ but all simple cycles in the graph satisfy $|\overline{\xi}| = 8 + 4 + 1 = 13$. For simplicity $(h + v + t)^{\leq k}$ is used in the expression to guarantee that all reachable pigeon-hole points are covered, but in practice $(h + v + t)^{\leq k}$ should be replaced by $(\xi_1 + \cdots + \xi_n)^{\leq k'}$ where $k' = |\Pi_{yz}|$ and ξ_i are the motifs associated with the labels of the edges in the graph, just as for the cycles. As noted above, the construction of the motifs can be incorporated in the graph construction algorithm. The explicit construction of the graph can thus be considered as a way of searching for relevant solutions to the equations

$$\begin{array}{ccccccc} y & +\lambda h & -fv & +\beta t & = & y' \\ z & +\gamma h & +\delta v & -gt & = & z' \end{array}$$

where $\langle y, z\rangle, \langle y', z'\rangle \in \Pi_{yz}$.

Theorem 7:
Consider the matrix

$$\Phi = \begin{pmatrix} \bullet & \lambda & \gamma \\ \bullet & -f & \delta \\ \bullet & \beta & -g \end{pmatrix}$$

where $f, g, \beta, \delta \geq 0$ and no assumptions are made on λ or γ. Let $\overline{\xi}$ be the sum of the labels along a simple cycle in the graph with maximum sum of labels, and let ξ be the motif of $\overline{\xi}$. If no cycle exists let $\overline{\xi} = \overline{0}$ and ξ the empty language. Assume $\overline{p} \xrightarrow{(h+v+t)^*} \overline{p}'$ for some points $\overline{p}$ and $\overline{p}'$. Then

1. If $\lambda \geq 0 \;\wedge\; \gamma \geq 0$, then

$$\begin{array}{c} \overline{p} \xrightarrow{(h+t)^*(h+v)^*(h+v+t)^{\leq k}\xi^*(h+v+t)^{\leq k}} \overline{p}' \\ \vee \\ \exists \overline{q} \in [\Omega_{yz}] : \overline{p} \xrightarrow{(h+t)^*(h+v)^*(h+v+t)^{\leq k}\xi^*(h+v+t)^{\leq k}} \overline{q} \xrightarrow{(h+v+t)^*} \overline{p}' \end{array}$$

2. If $\lambda \geq 0 \;\wedge\; \gamma < 0$, then

$$\begin{array}{c} \overline{p} \xrightarrow{(h+t)^*(h+v)^*(h+v+t)^{\leq k}\xi^*(h+v+t)^{\leq k}(h+v)^*} \overline{p}' \\ \vee \\ \exists \overline{q} \in [\Omega_{yz}] : \overline{p} \xrightarrow{(h+t)^*(h+v)^*(h+v+t)^{\leq k}\xi^*(h+v+t)^{\leq k}(h+v)^*} \overline{q} \xrightarrow{(h+v+t)^*} \overline{p}' \end{array}$$

3. If $\lambda < 0 \;\wedge\; \gamma \geq 0$, then

$$\begin{array}{c} \overline{p} \xrightarrow{(h+v)^*(h+t)^*(h+v+t)^{\leq k}\xi^*(h+v+t)^{\leq k}(h+t)^*} \overline{p}' \\ \vee \\ \exists \overline{q} \in [\Omega_{yz}] : \overline{p} \xrightarrow{(h+v)^*(h+t)^*(h+v+t)^{\leq k}\xi^*(h+v+t)^{\leq k}(h+t)^*} \overline{q} \xrightarrow{(h+v+t)^*} \overline{p}' \end{array}$$

4. If $\lambda < 0 \;\wedge\; \gamma < 0$, then

$$\begin{array}{c} \overline{p} \xrightarrow{(h+t)^*(h+v)^*(h+v+t)^{\leq k}\xi^*(h+v+t)^{\leq k}(h+v)^*} \overline{p}' \\ \vee \\ \overline{p} \xrightarrow{(h+t)^*(h+v)^*(h+v+t)^{\leq k}\xi^*(h+v+t)^{\leq k}(h+t)^*} \overline{p}' \\ \vee \\ \exists \overline{q} \in [\Omega_{yz}] : \overline{p} \xrightarrow{(h+t)^*(h+v)^*(h+v+t)^{\leq k}\xi^*(h+v+t)^{\leq k}(h+v)^*} \overline{q} \xrightarrow{(h+v+t)^*} \overline{p}' \\ \vee \\ \exists \overline{q} \in [\Omega_{yz}] : \overline{p} \xrightarrow{(h+t)^*(h+v)^*(h+v+t)^{\leq k}\xi^*(h+v+t)^{\leq k}(h+t)^*} \overline{q} \xrightarrow{(h+v+t)^*} \overline{p}' \end{array}$$

where $k = |\Pi_{yz}| \cdot (|\overline{s}_0| + 1)$ and $\overline{s}_0$ is given by lemma 25 under the proper assumptions on the signs of submatrices. $\diamond$

Proof:
Follows by combining lemmas 21, 25 and 27. $\diamond$

6.4 An observation

Consider figure 24. Let ξ be the motif associated with a solution $\overline{\xi} = \langle h_\xi, v_\xi, t_\xi \rangle^T$ to the

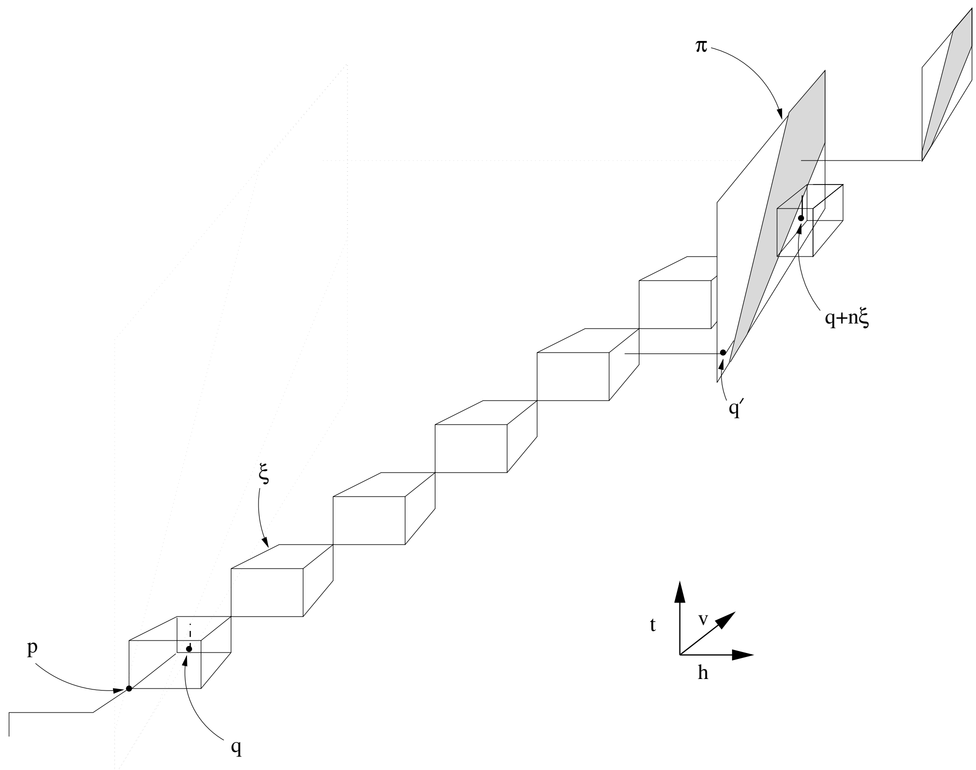


Figure 24

equations

$$\begin{array}{rcrrrcl} y & = & \lambda h & -fv & +\beta t & = & 0 \\ z & = & -\gamma h & +\delta v & -gt & = & 0 \end{array}$$

and assume that the motif is applicable at a point $\overline{p}$. Also assume that $\overline{\xi}$ *increases* x. Suppose that after a number of applications of ξ it is possible to reach a point $\overline{q}'$ such that there exists an infinite path in the vt-plane, π, leaving $\overline{q}'$. Since the motif keeps $y = 0$ and $z = 0$, and increases x, it is possible to continue to apply the motif. After some number of applications, the plane π will be crossed at a point $\overline{q}'' = \overline{q} + n \cdot \overline{\xi}$, say. But since $\overline{q}'' \geq \overline{q}'$, there must be an infinite path in the plane π leaving $\overline{q}''$ as well. Now, $\overline{q}''$ is a point of the path in ξ that is repeated, and ξ keeps $y = 0$ and $z = 0$, thus there must be an infinite path in the vt-plane leaving the point $\overline{q}$. So either one can walk infinitely in the vt-plane starting from some point $\overline{p}'$ such that $\overline{p} \leq \overline{p}' \leq \overline{p} + \overline{\xi}$, or no such point can be reached by iterating the motif ξ.

Assume that the solution $\overline{\xi}$ *decreases* x. This situation is shown in figure 25. Suppose

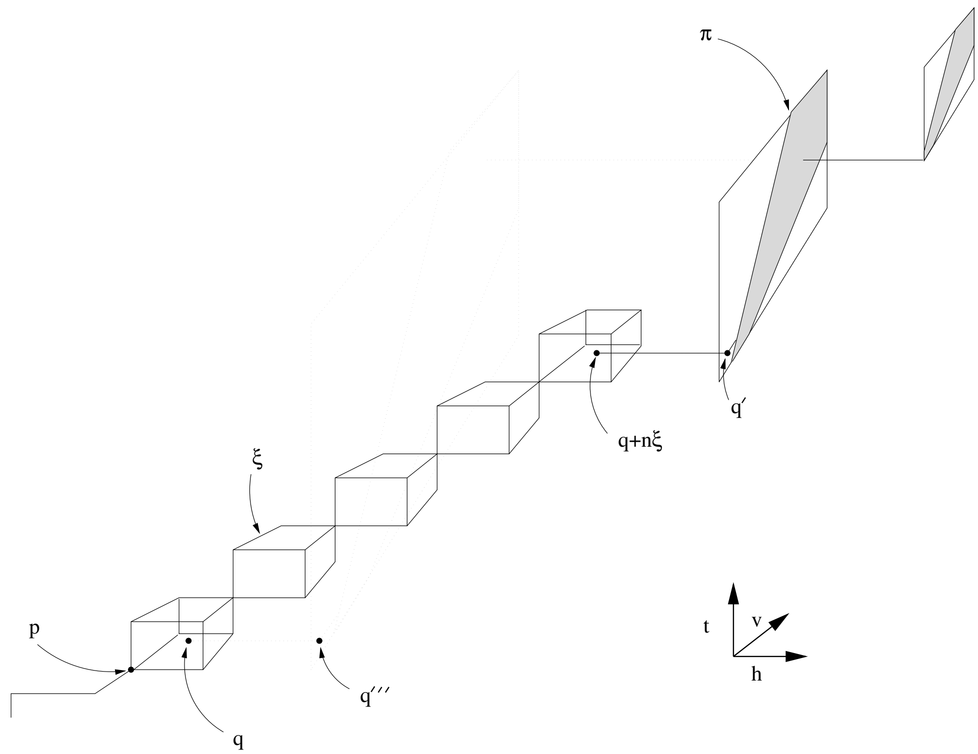


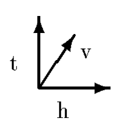
Figure 25

that after a number of applications of ξ one reaches a point $\overline{q}'' = \overline{q} + n \cdot \overline{\xi}$ such that by a number of h-moves, $h_{\overline{q}''}$ say, a point $\overline{q}'$ can be reached for which there is an infinite path in the vt-plane leaving $\overline{q}'$. Since ξ decreases x, $x_{\overline{q}} \geq x_{\overline{q}''}$ so $h_{\overline{q}''}$ h-moves can be applied at $\overline{q}$ to reach $\overline{q}'''$. Since ξ lets $y = 0$ and $z = 0$, we have that $y_{\overline{q}} = y_{\overline{q}''}$ and $z_{\overline{q}} = z_{\overline{q}''}$ and thus $y_{\overline{q}'''} = y_{\overline{q}'}$ and $z_{\overline{q}'''} = z_{\overline{q}'}$, so there must exist an infinite path leaving $\overline{q}'''$. So either one can walk infinitely in the vt-plane starting from some point $\overline{p}' + m \cdot 1_h$ for some m such that $\overline{p} \leq \overline{p}' \leq \overline{p} + \overline{\xi}$, or no such point can be reached by iterating the motif ξ.

7 Class 1

We include in this class the programs corresponding to cases a to d of proposition 17. That is, the class of 3-rule programs treated in [4]. We call this class *hierarchic* because there is one rule of the program that has priority over the other ones. Every admissible path is contained in at most four planes. In this section we give examples of programs from each of these types.

The figures depict all the possible paths associated with the nondeterministic application of all the rules. The orientation of the figures in terms of h, v and t is:



7.1 case a

$$\Phi = \begin{pmatrix} -4 & -2 & -1 \\ 2 & -1 & -3 \\ -1 & 3 & 1 \end{pmatrix}$$

Starting with the base value $\langle -1, -4, 3\rangle^T$ one gets the fixpoint ploted in figure 26. Every point is reachable by a path of the form $(t+v)^* h^* (t+v)^*$.

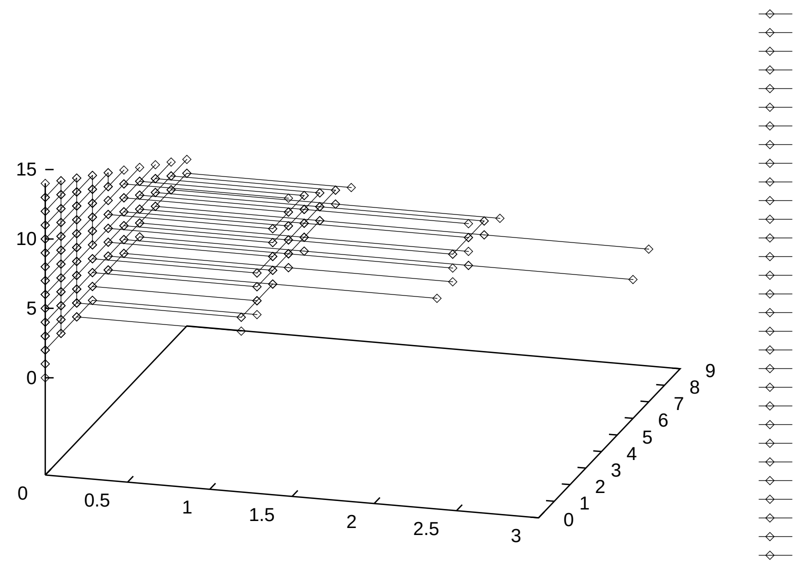


Figure 26

7.2 case b

$$\Phi = \begin{pmatrix} -4 & 2 & -1 \\ 2 & -1 & -3 \\ 1 & 3 & -1 \end{pmatrix}$$

Starting with the base value $\langle -1, -4, 3\rangle^T$ one gets the fixpoint ploted in figure 27. Every point is reachable by a path of the form $t^*(h+v)^*$.

7.3 case c

$$\Phi = \begin{pmatrix} 4 & -2 & 1 \\ 2 & -1 & -3 \\ -1 & 3 & -1 \end{pmatrix}$$

Starting with the base value $\langle -1, -4, 3\rangle^T$ one gets the fixpoint ploted in figure 28. Every point is reachable by a path of the form $(t+v)^* h^* t^* v^* (h+t)^*$.

7.4 case d

$$\Phi = \begin{pmatrix} -4 & -2 & 1 \\ 2 & -1 & -3 \\ -1 & -3 & 1 \end{pmatrix}$$

Starting with the base value $\langle 4, 4, 4\rangle^T$ one gets the fixpoint ploted in figure 29. Every point is reachable by a path of the form $v^* h^* t^* v^* h^* t^* v^*$.

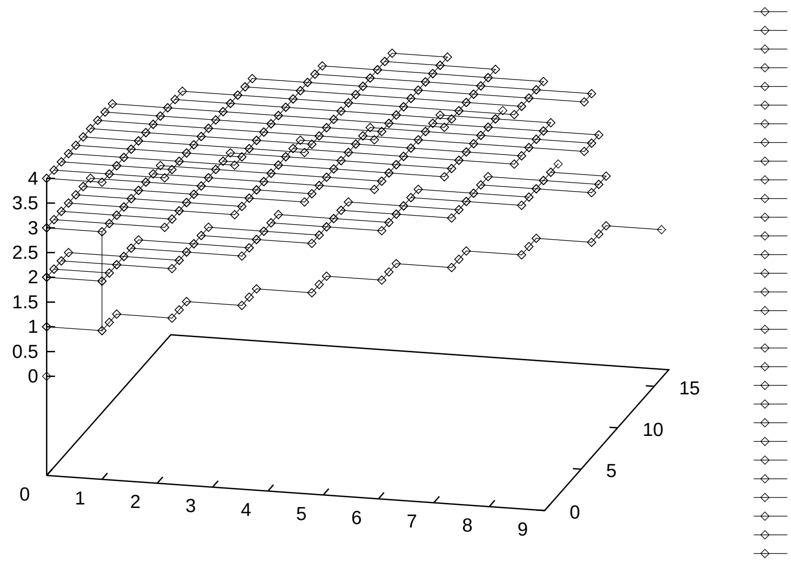


Figure 27

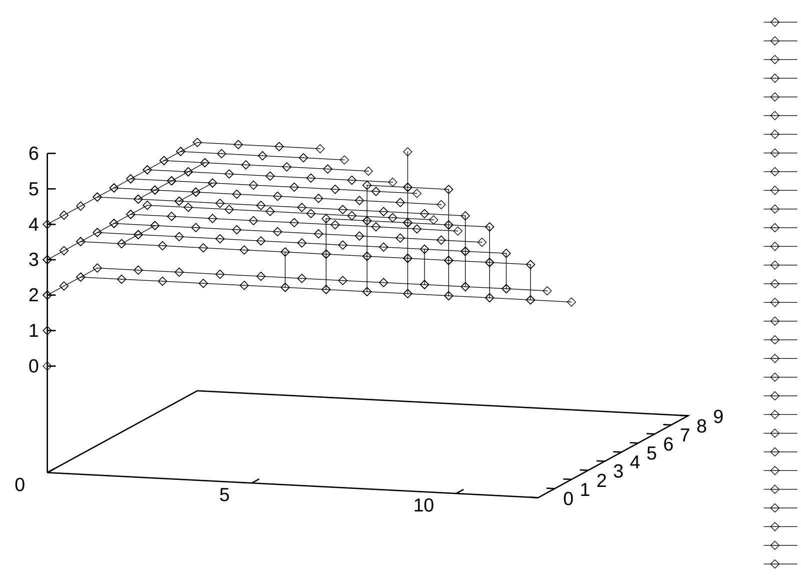


Figure 28

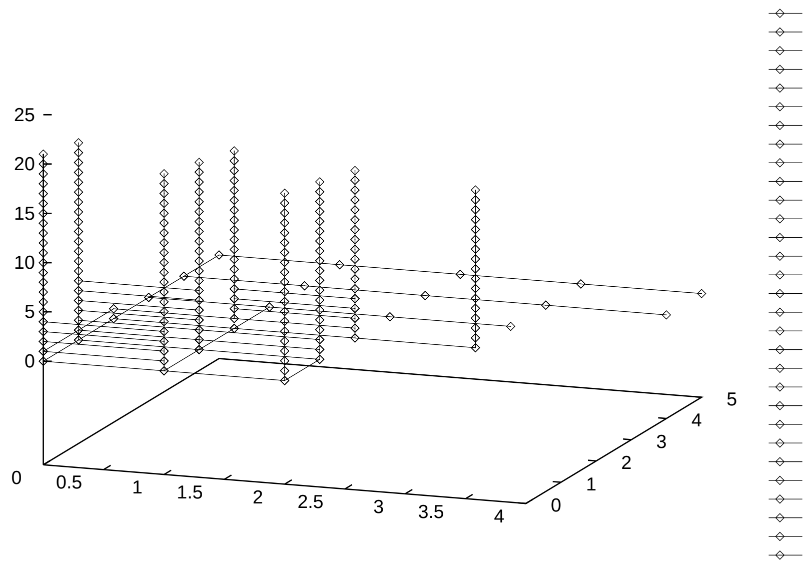


Figure 29

8 Class 2

First let us make the following simple observation: Consider a point $\overline{p}_{\Omega_{yz}} \in [\Omega_{yz}]$ and some point $\overline{p}'$ such that $\overline{p}_{\Omega_{yz}} \leq \overline{p}'$. Then there exists a point $\overline{q}$ such that $\overline{p}_{\Omega_{yz}} \xrightarrow{(v+t)^*} \overline{q}$, $\overline{p}' = \overline{q} + \langle h, v, t\rangle^T$ and either $v = 0$ or $t = 0$ (which is illustrated in figures 30 and 31 respectively). This is

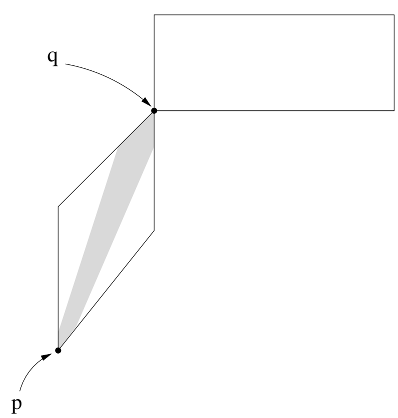


Figure 30

true since there exists an infinite path leaving $\overline{p}_{\Omega_{yz}}$ in the vt-plane, so at some point along this path either $v_{\overline{q}} = v_{\overline{p}'}$ or $t_{\overline{q}} = t_{\overline{p}'}$ must hold. Thus by moving in the vt-plane starting from $\overline{p}_{\Omega_{yz}}$, either the ht- or the hv-plane of $\overline{p}'$ will be crossed.

In this section we always assume

$$\begin{vmatrix} -f & \delta \\ \beta & -g \end{vmatrix} \neq 0$$

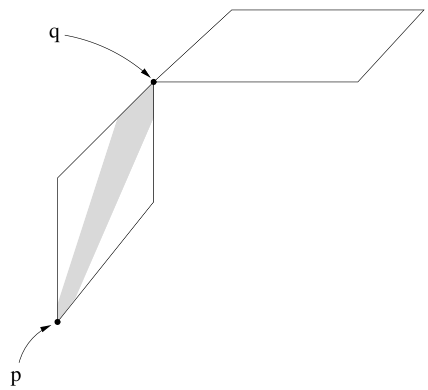


Figure 31

where $e, f, g > 0$ and $\delta, \beta \geq 0$. In order to fix the ideas, we will assume as a matter of fact

$$\begin{vmatrix} -f & \delta \\ \beta & -g \end{vmatrix} < 0$$

(The case where the subdeterminant is strictly positive can be treated similarly).

Remember that, by proposition 12, if $\delta y_{\overline{p}} + g z_{\overline{p}} \geq 0 \ \vee \ g y_{\overline{p}} + \beta z_{\overline{p}} \geq 0$ then there is an infinite path in the vt-plane starting at $\overline{p}$. Then if d and d' are chosen sufficiently large, $\overline{p} \in [\Omega_{yz}(d, d')]$ will guarantee that $\delta y_{\overline{p}} + g z_{\overline{p}} \geq 0 \ \vee \ g y_{\overline{p}} + \beta z_{\overline{p}} \geq 0$ holds.

Lemma 28:
Consider a program with the matrix

$$\Phi = \begin{pmatrix} -e & \bullet & -\gamma \\ \bullet & -f & \delta \\ \bullet & \beta & -g \end{pmatrix}$$

where $\gamma \geq 0$. Let $\overline{p}_{\Omega_{yz}} \in [\Omega_{yz}]$ and $\overline{p}'$ be points such that $\overline{p}_{\Omega_{yz}} \xrightarrow{(h+v+t)^*} \overline{p}'$ and $x_{\overline{p}'} \geq 0$. Assume also that there is a point $\overline{q}$ such that $\overline{p}_{\Omega_{yz}} \xrightarrow{(v+t)^*} \overline{q}$ and $\overline{p}' = \overline{q} + \langle h, 0, t\rangle^T$ for some h and t. Then $\overline{q} \xrightarrow{t^* h^*} \overline{p}'$ (and therefore $\overline{p}_{\Omega_{yz}} \xrightarrow{(v+t)^* h^*} \overline{p}'$). $\diamond$

Proof:
Since $\overline{p}_{\Omega_{yz}} \xrightarrow{(h+v+t)^*} \overline{p}'$, either $\overline{p}_{\Omega_{yz}} \xrightarrow{(h+v)^*} \overline{p}'$ or $\overline{p}_{\Omega_{yz}} \xrightarrow{(h+v+t)^* t(h+v)^*} \overline{p}'$ must hold. The first case is trivial, so assume the second case holds. Then there exists a point $\overline{q}' = \langle h_{\overline{q}'}, v_{\overline{q}'}, t_{\overline{q}'}\rangle^T$, such that $\overline{p}_{\Omega_{yz}} \xrightarrow{(h+v+t)^*} \overline{q}' \xrightarrow{t(h+v)^*} \overline{p}'$. Then $z_{\overline{q}'} \geq 0$ must hold. But then $z_{\overline{q}''} \geq 0$ must hold, where $\overline{q}'' = \langle h_{\overline{q}'}, v_{\overline{p}'}, t_{\overline{q}'}\rangle^T$, since a v-move increases z. But then $z_{\overline{q}'''} \geq 0$ must hold, where $\overline{q}''' = \langle h_{\overline{p}_{\Omega_{yz}}}, v_{\overline{p}'}, t_{\overline{q}'}\rangle^T$, since an h-move decreases z. Finaly then $z_{\overline{q}} \geq 0$ must hold, where $\overline{q} = \langle h_{\overline{p}_{\Omega_{yz}}}, v_{\overline{p}'}, t_{\overline{q}}\rangle^T$, since a t-move decreases z (note that by assumption $h_{\overline{p}} = h_{\overline{q}}$ and $v_{\overline{p}'} = v_{\overline{q}}$). Therefore $\overline{q} \xrightarrow{t^*} \overline{q}'''$. Since $x_{\overline{p}'} \geq 0$ and an h-move decreases x, $x_{\overline{q}'''} \geq 0$ and hence $\overline{q}''' \xrightarrow{h^*} \overline{p}'$. We get $\overline{q} \xrightarrow{t^*} \overline{q}''' \xrightarrow{h^*} \overline{p}'$. That is, $\overline{q} \xrightarrow{t^* h^*} \overline{p}'$. The situation is illustrated in figure 32. $\diamond$

Lemma 29:
Consider a program with the matrix

$$\Phi = \begin{pmatrix} -e & \bullet & \bullet \\ \bullet & -f & \delta \\ -\alpha & \beta & -g \end{pmatrix}$$

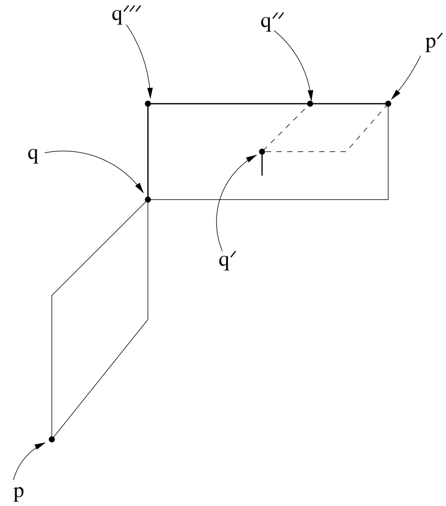


Figure 32

where $\alpha \geq 0$. Let $\overline{p}_{\Omega_{yz}} \in [\Omega_{yz}]$ and $\overline{p}'$ be points such that $\overline{p}_{\Omega_{yz}} \xrightarrow{(h+v+t)^*} \overline{p}'$ and $x_{\overline{p}'} \geq 0$. Assume also that there is a point $\overline{q}$ such that $\overline{p}_{\Omega_{yz}} \xrightarrow{(v+t)^*} \overline{q}$ and $\overline{p}' = \overline{q} + \langle h, 0, t\rangle^T$ for some h and t. Then $\overline{q} \xrightarrow{h^*t^*h^*} \overline{p}'$. $\diamond$

Proof:
The proof is similar to that of lemma 28. Consider figure 33.

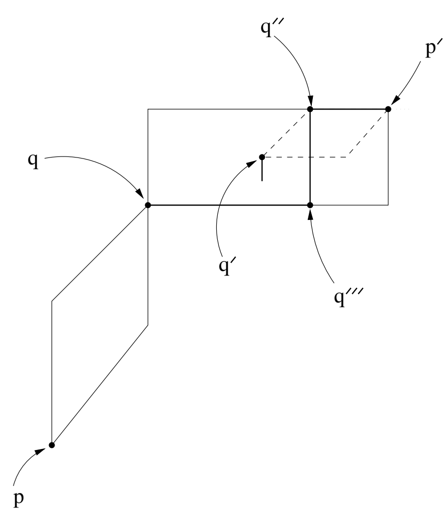


Figure 33

$x_{\overline{q}} \geq x_{\overline{q}'''} \geq x_{\overline{q''}} \geq x_{\overline{p}'} \geq 0$.
$z_{\overline{q}'''} \geq z_{\overline{q}''} \geq z_{\overline{q}'} \geq 0$. $\diamond$

Lemma 30:

Consider a program with the matrix

$$\Phi = \begin{pmatrix} -e & -\lambda & \bullet \\ \bullet & -f & \delta \\ \bullet & \beta & -g \end{pmatrix}$$

where $\lambda \geq 0$. Let $\overline{p}_{\Omega_{yz}} \in [\Omega_{vt}]$ and $\overline{p}'$ be points such that $\overline{p}_{\Omega_{yz}} \xrightarrow{(h+v+t)^*} \overline{p}'$ and $x_{\overline{p}'} \geq 0$. Assume also that there is a point $\overline{q}$ such that $\overline{p}_{\Omega_{yz}} \xrightarrow{(v+t)^*} \overline{q}$ and $\overline{p}' = \overline{q} + \langle h, v, 0\rangle^T$ for some h and v. Then $\overline{q} \xrightarrow{v^*h^*} \overline{p}'$. $\diamond$

Proof:
The proof is similar to that of lemma 28. Consider figure 34.

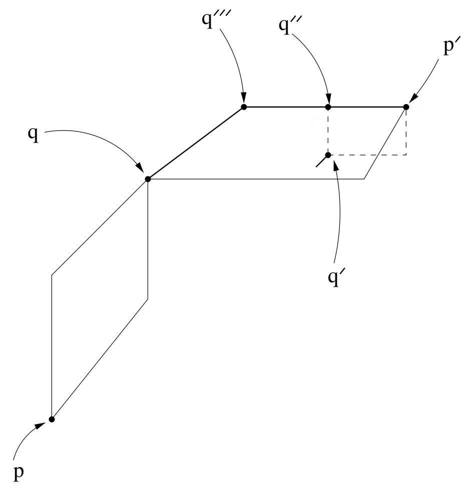


Figure 34

$x_{\overline{q}'''} \geq x_{\overline{p}'} \geq 0$.
$y_{\overline{q}} \geq y_{\overline{q}'''} \geq y_{\overline{q}''} \geq y_{\overline{q}'} \geq 0$. $\diamond$

Lemma 31:
Consider a program with the matrix

$$\Phi = \begin{pmatrix} -e & \bullet & \bullet \\ -\mu & -f & \delta \\ \bullet & \beta & -g \end{pmatrix}$$

where $\mu \geq 0$. Let $\overline{p}_{\Omega_{yz}} \in [\Omega_{yz}]$ and $\overline{p}'$ be points such that $\overline{p}_{\Omega_{yz}} \xrightarrow{(h+v+t)^*} \overline{p}'$ and $x_{\overline{p}'} \geq 0$. Assume also that there is a point $\overline{q}$ such that $\overline{p}_{\Omega_{yz}} \xrightarrow{(v+t)^*} \overline{q}$ and $\overline{p}' = \overline{q} + \langle h, v, 0\rangle^T$ for some h and v. Then $\overline{q} \xrightarrow{h^*v^*h^*} \overline{p}'$. $\diamond$

Proof:
The proof is similar to that of lemma 28. Consider figure 35.

$x_{\overline{q}} \geq x_{\overline{q}'''} \geq x_{\overline{q}''} \geq x_{\overline{p}'} \geq 0$.
$y_{\overline{q}'''} \geq y_{\overline{q}''} \geq y_{\overline{q}'} \geq 0$. $\diamond$

Corollary:

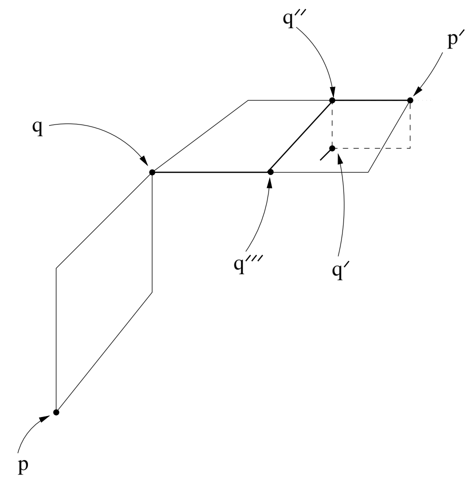


Figure 35

Let $\overline{p}_{\Omega_{yz}} \in [\Omega_{yz}]$ and $\overline{p}'$ be points such that $\overline{p}_{\Omega_{yz}} \xrightarrow{(h+v+t)^*} \overline{p}'$ and $x_{\overline{p}'} \geq 0$. Then either $\overline{p}_{\Omega_{yz}} \xrightarrow{(v+t)^*h^*t^*h^*} \overline{p}'$ or $\overline{p}_{\Omega_{yz}} \xrightarrow{(v+t)^*h^*v^*h^*} \overline{p}'$ holds in the following cases (where $\lambda, \gamma, \alpha, \mu \geq 0$):

$$\Phi = \begin{pmatrix} -e & -\lambda & -\gamma \\ \bullet & -f & \delta \\ \bullet & \beta & -g \end{pmatrix}$$

(by lemmas 28 and 30),

$$\Phi = \begin{pmatrix} -e & \lambda & -\gamma \\ -\mu & -f & \delta \\ \bullet & \beta & -g \end{pmatrix}$$

(by lemmas 28 and 31),

$$\Phi = \begin{pmatrix} -e & -\lambda & \bullet \\ \mu & -f & \delta \\ -\alpha & \beta & -g \end{pmatrix}$$

(by lemmas 29 and 30),

$$\Phi = \begin{pmatrix} -e & \bullet & \bullet \\ -\mu & -f & \delta \\ -\alpha & \beta & -g \end{pmatrix}$$

(by lemmas 29 and 31). $\diamond$

Finally note, that if $\overline{p}_{\Omega_{yz}} \xrightarrow{w} \overline{p}''$ for some path $w \in (h+v+t)^*$ and some point $\overline{p}''$, then $w \in (v+t)^*$ or $w \in (h+v+t)^*h(v+t)^*$ hold (since if h is used, it must somewhere in w be used for the last time) and thus $\overline{p}_{\Omega_{yz}} \xrightarrow{(v+t)^*} \overline{p}''$ or $\overline{p}_{\Omega_{yz}} \xrightarrow{(h+v+t)^*} \overline{p}' \xrightarrow{h(v+t)^*} \overline{p}''$ where clearly $x_{\overline{p}'} \geq 0$ hold. But then, by the above corollary, $\overline{p}_{\Omega_{yz}} \xrightarrow{(v+t)^*h^*t^*h^*} \overline{p}' \xrightarrow{h(v+t)^*} \overline{p}''$ or $\overline{p}_{\Omega_{yz}} \xrightarrow{(v+t)^*h^*v^*h^*} \overline{p}' \xrightarrow{h(v+t)^*} \overline{p}''$ holds depending on the signs of the coefficients. In either of the cases when $w \in (v+t)^*$ or $w \in (h+v+t)^*h(v+t)^*$, we must then have $\overline{p}_{\Omega_{yz}} \xrightarrow{(v+t)^*h^*t^*h^*(v+t)^*} \overline{p}''$ or $\overline{p}_{\Omega_{yz}} \xrightarrow{(v+t)^*h^*v^*h^*(v+t)^*} \overline{p}''$ for any point $\overline{p}''$ such that $\overline{p}_{\Omega_{yz}} \xrightarrow{w} \overline{p}''$.

We are now in the position to give expressions for the full reachability relations for the matrices discussed so far.

Theorem 8:
In the matrices below $\lambda, \gamma, \alpha, \mu \geq 0$ holds, and proper assumptions are made on signs of subdeterminants. Consider two points $\overline{p}$ and $\overline{p}'$ such that $\overline{p} \xrightarrow{(h+v+t)^*} \overline{p}'$. Then

1. for
$$\Phi = \begin{pmatrix} -e & -\lambda & -\gamma \\ \bullet & -f & \delta \\ \bullet & \beta & -g \end{pmatrix}$$
we have
$$\overline{p} \xrightarrow{(h+t)^*(h+v)^*(h+v+t)^{\leq k}\xi^*(h+v+t)^{\leq k}(h+v)^*(v+t)^*h^*v^*h^*(v+t)^*} \overline{p}'$$
or
$$\overline{p} \xrightarrow{(h+t)^*(h+v)^*(h+v+t)^{\leq k}\xi^*(h+v+t)^{\leq k}(h+t)^*(v+t)^*h^*v^*h^*(v+t)^*} \overline{p}'$$

2. for
$$\Phi = \begin{pmatrix} -e & \lambda & -\gamma \\ -\mu & -f & \delta \\ \bullet & \beta & -g \end{pmatrix}$$
we have
$$\overline{p} \xrightarrow{(h+t)^*(h+v)^*(h+v+t)^{\leq k}\xi^*(h+v+t)^{\leq k}(h+v)^*(v+t)^*h^*v^*h^*(v+t)^*} \overline{p}'$$

3. for
$$\Phi = \begin{pmatrix} -e & -\lambda & \bullet \\ \mu & -f & \delta \\ -\alpha & \beta & -g \end{pmatrix}$$
we have
$$\overline{p} \xrightarrow{(h+v)^*(h+t)^*(h+v+t)^{\leq k}\xi^*(h+v+t)^{\leq k}(h+t)^*(v+t)^*h^*t^*h^*(v+t)^*} \overline{p}'$$

4. for
$$\Phi = \begin{pmatrix} -e & \bullet & \bullet \\ -\mu & -f & \delta \\ -\alpha & \beta & -g \end{pmatrix}$$
we have
$$\overline{p} \xrightarrow{(h+v)^*(h+t)^*(h+v+t)^{\leq k}\xi^*(h+v+t)^{\leq k}(h+t)^*(v+t)^*h^*t^*h^*(v+t)^*} \overline{p}'$$

where ξ and k are given by theorem 7 $\diamond$

Proof:
Follows from the corollary above and theorem 7. $\diamond$

Note that the first case where all elements of the top row of the matrix are negative, actually belongs to class 1 (case (a) of proposition 17).

9 Class 3

In this section we consider matrices (not belonging to class 2) of the form

$$\Phi = \begin{pmatrix} -e & \bullet & \bullet \\ \bullet & -f & \delta \\ \bullet & \beta & -g \end{pmatrix}$$

where $e, f, g, \beta, \delta > 0$,

$$\begin{vmatrix} -f & \delta \\ \beta & -g \end{vmatrix} \neq 0$$

As a matter of fact we assume

$$\begin{vmatrix} -f & \delta \\ \beta & -g \end{vmatrix} < 0$$

but the case where the subdeterminant is strictly positive can be treated similarly.
We investigate reachability $\overline{p}_{\Omega_{yz}} \xrightarrow{(h+v+t)^*} \overline{p}'$ under the assumption that $\overline{p}_{\Omega_{yz}} \in [\Omega_{yz}(d, d')]$ for sufficiently large d and d' that guarantees that a planar vt-pattern is applicable, and $x_{\overline{p}'} \geq 0$.

We distinguish two subclasses in class 3: the subclass where there is at least one planar vt-pattern that makes x strictly increase, and the subclass where both planar vt-patterns make x decrease (or let invariant).

9.1 increasing vt-pattern

Throughout this subsection we assume that at least one planar vt-pattern makes x *strictly* increase. That is, for a matrix

$$\Phi = \begin{pmatrix} -e & \lambda & \gamma \\ \mu & -f & \delta \\ \alpha & \beta & -g \end{pmatrix}$$

we assume

$$\mu g + \alpha\delta > 0$$

or

$$\mu\beta + \alpha f > 0$$

Lemma 32:
Consider the matrix

$$\Phi = \begin{pmatrix} -e & \lambda & \gamma \\ \mu & -f & \delta \\ \alpha & \beta & -g \end{pmatrix}$$

where $\lambda, \gamma \geq 0$ (we make no assumptions on α and μ),

$$\begin{vmatrix} -e & \gamma \\ \alpha & -g \end{vmatrix} \neq 0 \quad \text{and} \quad \begin{vmatrix} -e & \lambda \\ \mu & -f \end{vmatrix} \neq 0$$

Consider two points $\overline{p}_{\Omega_{yz}}$ and $\overline{p}'$ such that $\overline{p}_{\Omega_{yz}} \notin [\Pi_{yz}]$, $\overline{p}' \notin [\Pi_{xz}]$, $\overline{p}' \notin [\Pi_{xy}]$ and $x_{\overline{p}'} \geq 0$.
Then $\overline{p}_{\Omega_{yz}} \xrightarrow{(h+v+t)^*} \overline{p}'$ iff one of the following holds.

1. $\overline{p}_{\Omega_{yz}} \xrightarrow{(v+t)^* h^* (vh^* + th^*)^{\leq k}} \overline{p}'$ for some k (depending only on the coefficients of Φ).

2. $\overline{p}_{\Omega_{yz}} \xrightarrow{(v+t)^*(h+t)^*} \overline{p}'$

3. $\overline{p}_{\Omega_{yz}} \xrightarrow{(v+t)^*(h+v)^*} \overline{p}'$

$\diamond$

Proof:
The proof is a case analysis with respect to the signs of α and μ.

Consider the case when $\alpha, \mu \geq 0$. Without loss of generality one may assume that

$$\overline{p}_{\Omega_{yz}} \xrightarrow{(v+t)^*} \overline{p}''_{\Omega_{yz}} \xrightarrow{h(h+v+t)^*} \overline{p}'$$

for some $\overline{p}''$ (that is, the path from $\overline{p}_{\Omega_{yz}}$ to $\overline{p}'$ contains at least one h-move, the other case is trivial). This is illustrated in figures 36 and 37 respectively. Since $\overline{p}_{\Omega_{yz}} \in [\Omega_{yz}]$, it must be

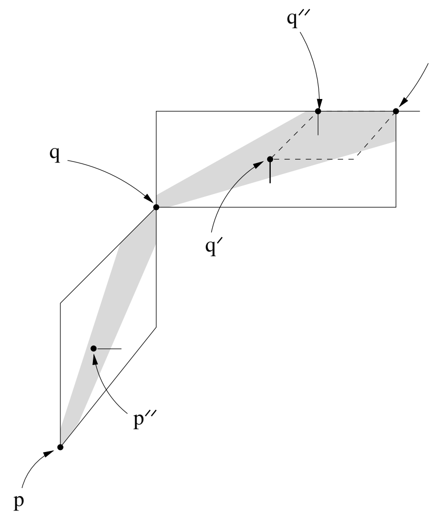


Figure 36

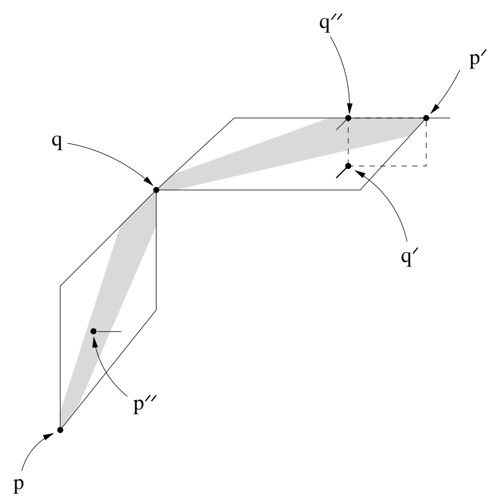


Figure 37

the case that $\overline{p}''_{\Omega_{yz}} \in [\Omega_{yz}]$, so both patterns in the vt-plane can be applied to $\overline{p}''_{\Omega_{yz}}$, and in particular the one that makes x increase. Call this pattern ξ_{vt}. Suppose there exists a point $\overline{q}_{\Omega_{yz}}$

such that $\overline{p}''_{\Omega_{yz}} \xrightarrow{\xi_{vt}^{\geq k}(v+t)^{\leq k'}} \overline{q}_{\Omega_{yz}}$ for some k and some $k' \leq |\xi_{vt}|$, and $\overline{p}' = \overline{q}_{\Omega_{yz}} + \langle h', v', t' \rangle^T$ where either $v' = 0$ (figure 36) or $t' = 0$ (figure 37).

Assume $v' = 0$. Since $\overline{q}_{\Omega_{yz}} \in [\Omega_{yz}]$, $-g \leq z_{\overline{q}_{\Omega_{yz}}}$ holds, and since $0 \leq x_{\overline{p}''_{\Omega_{yz}}}$ and ξ_{vt} makes x strictly increase, after at least k applications of ξ_{vt} followed by a path $(v+t)^{\leq k'}$, $d \leq x_{\overline{q}_{\Omega_{yz}}}$ holds for some d such that $\gamma x_{\overline{q}_{\Omega_{yz}}} + e z_{\overline{q}_{\Omega_{yz}}} \geq \gamma d - eg \geq 0$ or $g x_{\overline{q}_{\Omega_{yz}}} + \alpha z_{\overline{q}_{\Omega_{yz}}} \geq gd - \alpha g \geq 0$ holds. Thus $\overline{q}_{\Omega_{yz}} \in [\Omega_{yz}]$. Either $\overline{p}_{\Omega_{yz}} \xrightarrow{(v+t)^*} \overline{p}'' \xrightarrow{h(h+v)^*} \overline{p}'$, in which case reachability is trivial, or $\overline{p}_{\Omega_{yz}} \xrightarrow{(v+t)^*} \overline{p}'' \xrightarrow{h(h+v+t)^*t} \overline{q}' \xrightarrow{(h+v)^*} \overline{p}'$ for some $\overline{q}'$. Consider the second case. Then $z_{\overline{q}'} \geq -g$ must hold. Since a v-move increases z, $z_{\overline{q}''} \geq -g$ must hold for some $\overline{q}''$ such that $\overline{q}'' \leq_h \overline{p}'$ (see 36). But since an h-move increases z, $z_{\overline{p}'} \geq -g$ must hold. By assumption, $x_{\overline{p}'} \geq 0$ and $\overline{p}' \notin [\Pi_{xz}]$, so $\overline{p}' \in [\Omega_{xz}]$. Thus $\overline{q} \xrightarrow{(h+t)^*} \overline{p}'$, and thus $\overline{p}_{\Omega_{yz}} \xrightarrow{(v+t)^*(h+t)^*} \overline{p}'$.

If $v' = 0$ but $\overline{p}''_{\Omega_{yz}} \xrightarrow{\xi_{vt}^{\geq k}(v+t)^{\leq k'}} \overline{q}_{\Omega_{yz}}$ does not hold, the difference between $\overline{p}''_{\Omega_{yz}}$ and $\overline{p}'$ in terms of vt-moves is bounded by ξ_{vt}^k. Therefore $\overline{p}''_{\Omega_{yz}} \xrightarrow{h^*(vh^*+th^*)^{\leq k}} \overline{p}'$. k is chosen so that

$$\left(\begin{array}{c} x_{\overline{p}''_{\Omega_{yz}}} \geq 0 \\ \wedge \\ z_{\overline{p}''_{\Omega_{yz}}} \geq -g \\ \wedge \\ \overline{p}''_{\Omega_{yz}} \xrightarrow{\xi_{vt}^{\geq k}(v+t)^{\leq k'}} \overline{q}_{\Omega_{yz}} \end{array} \right) \Rightarrow \left(\begin{array}{c} \gamma x_{\overline{q}_{\Omega_{yz}}} + e z_{\overline{q}_{\Omega_{yz}}} \geq 0 \\ \vee \\ g x_{\overline{q}_{\Omega_{yz}}} + \alpha z_{\overline{q}_{\Omega_{yz}}} \geq 0 \end{array} \right)$$

This is possible since ξ_{vt} strictly increases x. Thus $\overline{p}_{\Omega_{yz}} \xrightarrow{(v+t)^*h^*(vh^*+th^*)^{\leq k}} \overline{p}'$.

Assume $t' = 0$ (see figure 37). The treatment is analogous to the case when $v' = 0$:
$x_{\overline{q}_{\Omega_{yz}}} \geq x_{\overline{p}''_{\Omega_{yz}}} + d \geq d$,
$v_{\overline{q}_{\Omega_{yz}}} \geq -f$, so $\overline{q}_{\Omega_{yz}} \in [\Omega_{xy}]$,
$x_{\overline{p}'} \geq 0$,
$y_{\overline{p}'} \geq y_{\overline{q}''} \geq y_{\overline{q}'} \geq -f$, so $\overline{p}' \in [\Omega_{xy}]$ since $\overline{p}' \notin [\Pi_{xy}]$ and thus $\overline{q}_{\Omega_{yz}} \xrightarrow{(h+v)^*} \overline{p}'$ so $\overline{p}_{\Omega_{yz}} \xrightarrow{(v+t)^*(h+v)^*} \overline{p}'$.

Consider the case when $\mu \leq 0$ and $\alpha \geq 0$. This is illustrated in figures 38 and 39

Consider figure 38 which illustrates the situation when $v' = 0$. We get:
$x_{\overline{q}_{\Omega_{yz}}} \geq x_{\overline{p}''_{\Omega_{yz}}} + d \geq d$,
$z_{\overline{q}_{\Omega_{yz}}} \geq -g$, so $\overline{q}_{\Omega_{yz}} \in [\Omega_{xz}]$,
$x_{\overline{p}'} \geq 0$,
$z_{\overline{p}'} \geq z_{\overline{q}''} \geq z_{\overline{q}'} \geq -g$, so $\overline{p}' \in [\Omega_{xz}]$ since $\overline{p}' \notin [\Pi_{xz}]$ and thus $\overline{q}_{\Omega_{yz}} \xrightarrow{(h+t)^*} \overline{p}'$.

Consider figure 39 which illustrates the situation when $t' = 0$. We get:
$y_{\overline{q}'''} \geq y_{\overline{q}''} \geq y_{\overline{q}'} \geq -f$,
$x_{\overline{q}_{\Omega_{yz}}} \geq x_{\overline{q}'''} \geq x_{\overline{q}''} \geq x_{\overline{p}'} \geq 0$.
If $\overline{q}''' \leq_h \overline{p}'$, then $\overline{q}_{\Omega_{yz}} \xrightarrow{h^*} \overline{p}'$. If not, then $y_{\overline{q}'''} \geq 0$ and thus $\overline{q}_{\Omega_{yz}} \xrightarrow{h^*v^*h^*} \overline{p}'$. Note that there is no assumption $\overline{p}' \notin [\Pi_{xy}]$ needed here.

Consider the case when $\mu \geq 0$ and $\alpha \leq 0$. This is illustrated in figures 40 and 41

Consider figure 40 which illustrates the situation when $v' = 0$. We get:
$z_{\overline{q}'''} \geq z_{\overline{p}'} \geq z_{\overline{q}''} \geq z_{\overline{q}'} \geq -g$,
$x_{\overline{q}_{\Omega_{yz}}} \geq x_{\overline{q}'''} \geq x_{\overline{p}'} \geq 0$,
and thus $\overline{q}_{\Omega_{yz}} \xrightarrow{h^*t^*} \overline{p}'$ (since if $0 > z_{\overline{q}'''}$ then $\overline{q}_{\Omega_{yz}} \leq_h \overline{p}'$ so $\overline{q}_{\Omega_{yz}} \xrightarrow{h^*} \overline{p}'$). No assumption

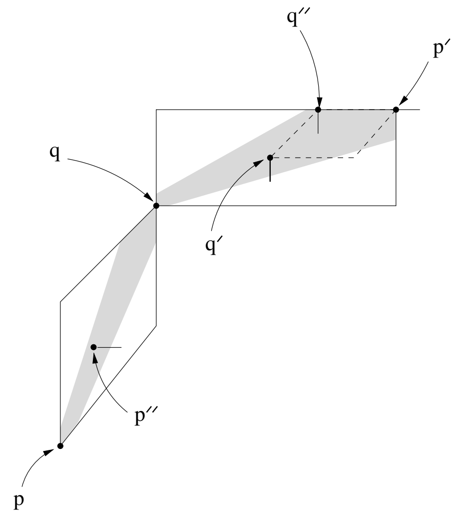


Figure 38

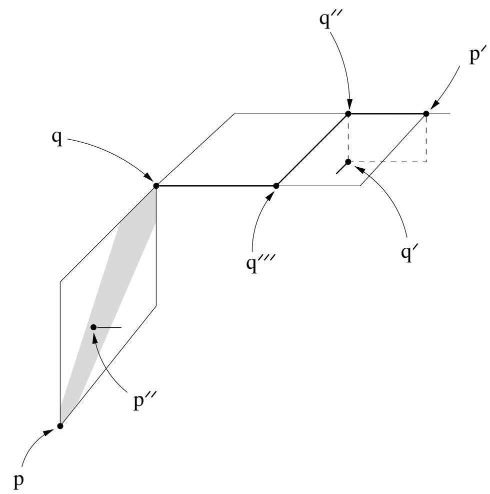


Figure 39

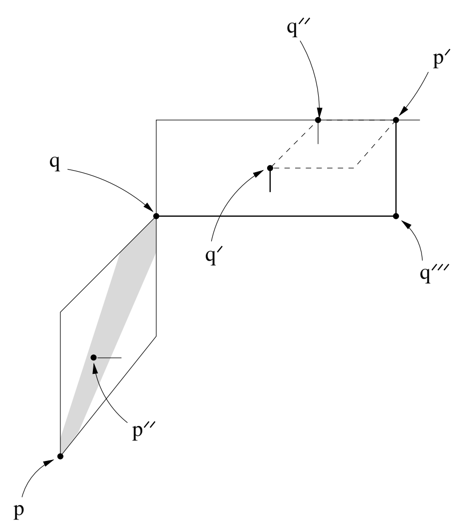


Figure 40

$\overline{p}' \notin [\Pi_{ht}]$ is needed.

Consider figure 41 which illustrates the situation when $t' = 0$. We get:

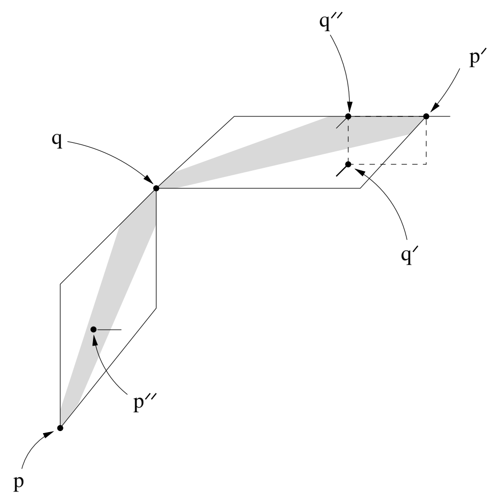


Figure 41

$x_{\overline{q}_{\Omega_{yz}}} \geq x_{\overline{p}''_{\Omega_{yz}}} + d \geq d$,
$y_{\overline{q}_{\Omega_{yz}}} \geq -f$, so $\overline{q}_{\Omega_{yz}} \in [\Omega_{xy}]$,
$x_{\overline{p}'} \geq 0$,
$y_{\overline{p}'} \geq y_{\overline{q}''} \geq y_{\overline{q}'} \geq -f$, so $\overline{p}' \in [\Omega_{xy}]$ since $\overline{p}' \notin [\Pi_{xy}]$ and thus $\overline{q}_{\Omega_{yz}} \xrightarrow{(h+t)^*} \overline{p}'$. ◇

Lemma 33:
Consider the matrix

$$\Phi = \begin{pmatrix} -e & -\lambda & \gamma \\ \mu & -f & \delta \\ \alpha & \beta & -g \end{pmatrix}$$

where $\lambda, \gamma, \alpha \geq 0$ (no constraints on μ) and

$$\begin{vmatrix} -e & \gamma \\ \alpha & -g \end{vmatrix} \neq 0$$

Consider two points $\overline{p}_{\Omega_{yz}}$ and $\overline{p}'$ such that $\overline{p}_{\Omega_{yz}} \notin [\Pi_{yz}]$, $\overline{p}' \notin [\Pi_{xz}]$ and $x_{\overline{p}'} \geq 0$.
Then $\overline{p}_{\Omega_{yz}} \xrightarrow{(h+v+t)^*} \overline{p}'$ iff one of the following holds.

1. $\overline{p}_{\Omega_{yz}} \xrightarrow{(v+t)^* h^* (vh^* + th^*)^{\leq k}} \overline{p}'$ for some k (depending only on the coefficients in Φ).
2. $\overline{p}_{\Omega_{yz}} \xrightarrow{(v+t)^* (h+t)^*} \overline{p}'$
3. $\overline{p}_{\Omega_{yz}} \xrightarrow{(v+t)^* (h+v)^*} \overline{p}'$

◇

Proof:
Analogous to the proof of lemma 32. Consider figure 42 which illustrates the situation when $v' = 0$. We get:
$x_{\overline{q}_{\Omega_{yz}}} \geq x_{\overline{p}''} + d \geq d$,

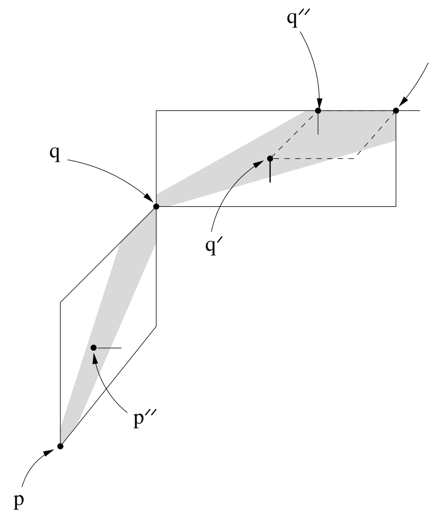


Figure 42

$z_{\overline{q}} \geq -g$, so $\overline{q}_{\Omega_{yz}} \in [\Omega_{xz}]$.
$x_{\overline{p}'} \geq 0$,
$z_{\overline{p}'} \geq z_{\overline{q}''} \geq z_{\overline{q}'} \geq -g$, so $\overline{p}' \in [\Omega_{xz}]$ since $\overline{p}' \notin [\Pi_{xz}]$, and thus $\overline{q}_{\Omega_{yz}} \xrightarrow{(h+t)^*} \overline{p}'$.

Consider figure 43 which illustrates the situation when $t' = 0$. We get:

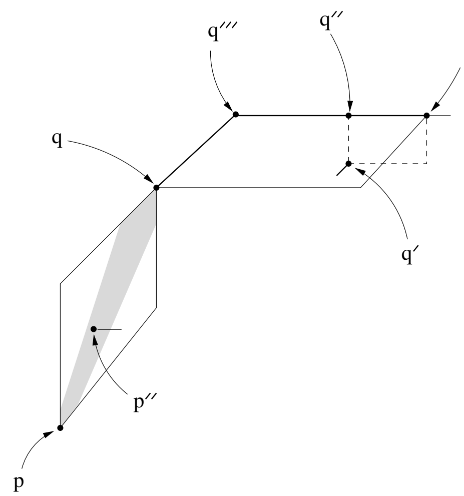


Figure 43

$x_{\overline{q}'''_{\Omega_{yz}}} \geq x_{\overline{p}'} \geq 0$,
$y_{\overline{q}} \geq y_{\overline{q}'''} \geq y_{\overline{q}''} \geq y_{\overline{q}'} \geq -f$, and thus $\overline{q}_{\Omega_{yz}} \xrightarrow{v^*h^*} \overline{p}'$ (since if $0 > y_{\overline{q}'''}$ then $\overline{q}_{\Omega_{yz}} \leq_h \overline{p}'$ so $\overline{q}_{\Omega_{yz}} \xrightarrow{h^*} \overline{p}'$). No assumption $\overline{p}' \notin [\Pi_{xz}]$ is needed. $\diamond$

Lemma 34:
Consider the matrix

$$\Phi = \begin{pmatrix} -e & \lambda & -\gamma \\ \mu & -f & \delta \\ \alpha & \beta & -g \end{pmatrix}$$

where $\lambda, \gamma, \mu \geq 0$ (no constraints on α) and

$$\begin{vmatrix} -e & \lambda \\ \mu & -f \end{vmatrix} \neq 0$$

Consider two points $\overline{p}_{\Omega_{yz}}$ and $\overline{p}'$ such that $\overline{p}_{\Omega_{yz}} \notin [\Pi_{yz}]$, $\overline{p}' \notin [\Pi_{xy}]$ and $x_{\overline{p}'} \geq 0$.
Then $\overline{p}_{\Omega_{yz}} \xrightarrow{(h+v+t)^*} \overline{p}'$ iff one of the following holds.

1. $\overline{p}_{\Omega_{yz}} \xrightarrow{(v+t)^* h^* (vh^* + th^*)^{\leq k}} \overline{p}'$ for some k (depending only on the coefficients in Φ).
2. $\overline{p}_{\Omega_{yz}} \xrightarrow{(v+t)^*(h+t)^*} \overline{p}'$
3. $\overline{p}_{\Omega_{yz}} \xrightarrow{(v+t)^*(h+v)^*} \overline{p}'$

$\diamond$

Proof:
Analogous to the proof of lemma 32. Consider figure 44 which illustrates the situation when

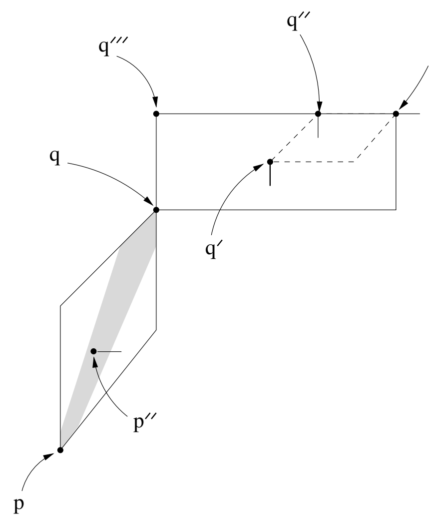


Figure 44

$v' = 0$. We get:
$x_{\overline{q}'''_{\Omega_{yz}}} \geq x_{\overline{p}'} \geq 0$,
$z_{\overline{q}} \geq z_{\overline{q}'''} \geq z_{\overline{q}''} \geq z_{\overline{q'}} \geq -g$, and thus $\overline{q}_{\Omega_{yz}} \xrightarrow{t^*+h^*} \overline{p}'$ (since if $0 > z_{\overline{q}'''}$ then $\overline{q}_{\Omega_{yz}} \leq_h \overline{p}'$ so $\overline{q}_{\Omega_{yz}} \xrightarrow{h^*} \overline{p}'$). No assumption $\overline{p}' \notin [\Pi_{xy}]$ is needed.

Consider figure 45 which illustrates the situation when $t' = 0$. We get:

$x_{\overline{q}_{\Omega_{yz}}} \geq x_{\overline{p}''} + d \geq d$,
$y_{\overline{q}} \geq -f$, so $\overline{q}_{\Omega_{yz}} \in [\Omega_{xz}]$.
$x_{\overline{p}'} \geq 0$,
$y_{\overline{p}'} \geq y_{\overline{q}''} \geq y_{\overline{q}'} \geq -f$, so $\overline{p}' \in [\Omega_{xy}]$ since $\overline{p}' \notin [\Pi_{xy}]$ and thus $\overline{q}_{\Omega_{yz}} \xrightarrow{(h+v)^*} \overline{p}'$. $\diamond$

As for class 2, if $\overline{p}_{\Omega_{yz}} \xrightarrow{w} \overline{p}''$ for some point $\overline{p}''$ where $w \in (h+v+t)^*$, and if an h-move occurs in w, then there is a point $\overline{p}'$ such that h is applied for the last time at $\overline{p}'$. Thus $x_{\overline{p}'} \geq 0$ and $\overline{p}' \xrightarrow{h(v+t)^*} \overline{p}''$.

We summarize the results of this subsection as

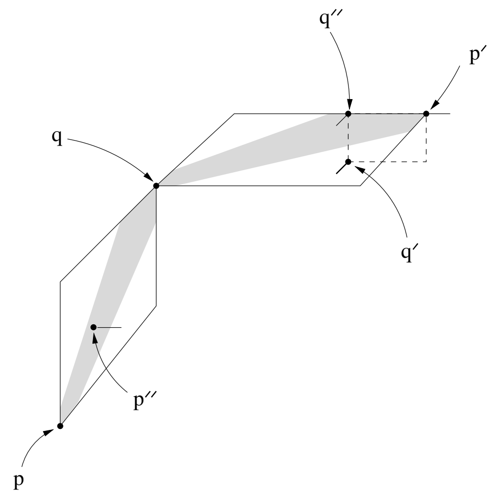


Figure 45

Theorem 9:
For the matrix of lemma 32 the reachability relation is given by

$$\overline{p} \xrightarrow{(h+t)^*(h+v)^*(h+v+t)^{\leq k}\xi^*(h+v+t)^{\leq k}(v+t)^*(h+t)^*(v+t)^*} \overline{p}'$$
$$\vee$$
$$\overline{p} \xrightarrow{(h+t)^*(h+v)^*(h+v+t)^{\leq k}\xi^*(h+v+t)^{\leq k}(v+t)^*(h+v)^*(v+t)^*} \overline{p}'$$
$$\vee$$
$$\overline{p} \xrightarrow{(h+t)^*(h+v)^*(h+v+t)^{\leq k}\xi^*(h+v+t)^{\leq k}(v+t)^*h^*(vh^*+th^*)^{\leq k'}(v+t)^*} \overline{p}'$$

for the matrix of lemma 33 the reachability relation is given by

$$\overline{p} \xrightarrow{(h+v)^*(h+t)^*(h+v+t)^{\leq k}\xi^*(h+v+t)^{\leq k}(h+t)^*(v+t)^*(h+t)^*(v+t)^*} \overline{p}'$$
$$\vee$$
$$\overline{p} \xrightarrow{(h+v)^*(h+t)^*(h+v+t)^{\leq k}\xi^*(h+v+t)^{\leq k}(h+t)^*(v+t)^*(h+v)^*(v+t)^*} \overline{p}'$$
$$\vee$$
$$\overline{p} \xrightarrow{(h+t)^*(h+v)^*(h+v+t)^{\leq k}\xi^*(h+v+t)^{\leq k}(h+t)^*(v+t)^*h^*(vh^*+th^*)^{\leq k'}(v+t)^*} \overline{p}'$$

and for the matrix of lemma 34 the reachability relation is given by

$$\overline{p} \xrightarrow{(h+t)^*(h+v)^*(h+v+t)^{\leq k}\xi^*(h+v+t)^{\leq k}(h+v)^*(v+t)^*(h+t)^*(v+t)^*} \overline{p}'$$
$$\vee$$
$$\overline{p} \xrightarrow{(h+t)^*(h+v)^*(h+v+t)^{\leq k}\xi^*(h+v+t)^{\leq k}(h+v)^*(v+t)^*(h+v)^*(v+t)^*} \overline{p}'$$
$$\vee$$
$$\overline{p} \xrightarrow{(h+t)^*(h+v)^*(h+v+t)^{\leq k}\xi^*(h+v+t)^{\leq k}(h+v)^*(v+t)^*h^*(vh^*+th^*)^{\leq k'}(v+t)^*} \overline{p}'$$

for some ξ, k, and k' (depending only on the matrix coefficients). $\diamond$

Proof:
Follows by combining lemmas 32, 33 and 34 with theorem 7. $\diamond$

9.2 decreasing vt-pattern

Consider the matrix

$$\Phi = \begin{pmatrix} -e & -\lambda & \gamma \\ -\mu & -f & \delta \\ \alpha & \beta & -g \end{pmatrix}$$

where $\lambda, \gamma, \mu, \alpha \geq 0$ and where both planar vt-patterns make x decrease. That is

$$-\mu g + \alpha\delta \leq 0$$

and

$$-\mu\beta + \alpha f \leq 0$$

Consider a path w ending at a point $\overline{p}'$ and starting from some point $\overline{p}_{\Omega_{yz}}$. The path can be represented (by considering it backwards from the end to the beginning) as a sequence:

$$\overline{p}'_0,\ \overline{p}_0,\ \overline{p}'_1,\ \overline{p}_1,\ \overline{p}'_2,\ \cdots\ \overline{p}'_r,\ \overline{p}_r$$

where $\overline{p}_i \xrightarrow{(v+t)^*} \overline{p}'_i$ for all $0 \leq i \leq r$ and $\overline{p}'_{i+1} \xrightarrow{(h+t)^*} \overline{p}_i$ for all $0 \leq i \leq r-1$ and such that for no $\overline{p}_{i+1}, \overline{p}'_{i+1}, \overline{p}_i$ and $\overline{p}'_i, \overline{p}_{i+1} \xrightarrow{(h+v)^*} \overline{p}_i$ or $\overline{p}'_{i+1} \xrightarrow{(h+v)^*} \overline{p}'_i$ holds. We say that the sequence is not degenerate (see lemma 24 and its corollaries, section 6). Furthermore we have

$$\begin{array}{llllll} x_{\overline{p}_i} \geq -e, & y_{\overline{p}_i} \geq 0, & z_{\overline{p}_i} \geq -g \\ x_{\overline{p}'_i} \geq 0, & y_{\overline{p}'_i} \geq -f, & z_{\overline{p}'_i} \geq -g \end{array}$$

This is illustrated in figure 46.

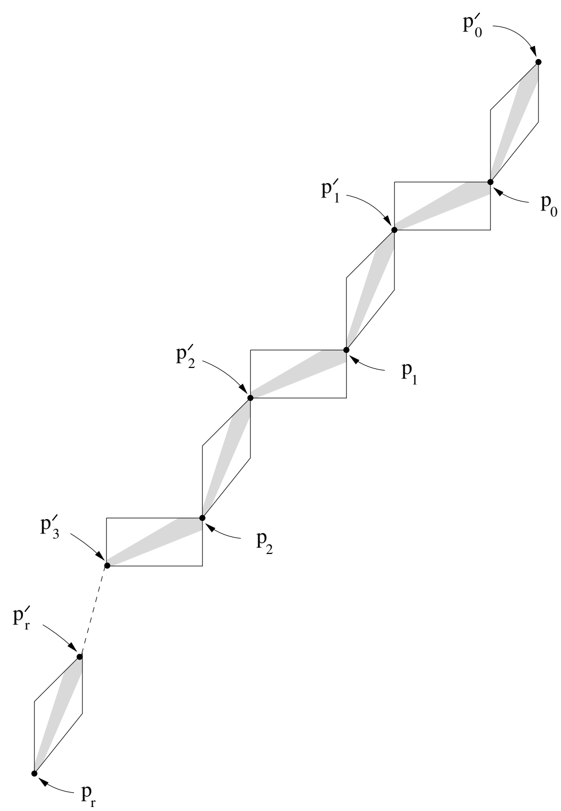


Figure 46

Lemma 35:
Let us consider a path w represented, as above, under the form

$$\overline{p}'_0,\ \overline{p}_0,\ \overline{p}'_1,\ \overline{p}_1,\ \overline{p}'_2,\ \cdots\ \overline{p}'_r,\ \overline{p}_r$$

Then the path w can be transformed into a path w' represented as the sequence

$$\overline{p}'_0,\ \overline{q}_0,\ \overline{q}'_1,\ \overline{q}_1,\ \overline{q}'_2,\ \cdots\ \overline{q}_{s-1},\ \overline{q}'_s$$

such that

1. $\overline{q}_0 \xrightarrow{(v+t)^*} \overline{p}'_0$
2. $\overline{q}_i \xrightarrow{(v+t)^*} \overline{q}'_i$ for all $1 \leq i \leq s-1$
3. $\overline{q}'_{i+1} \xrightarrow{(h+t)^*} \overline{q}_i$ for all $0 \leq i \leq s-1$
4. $(y_{\overline{q}_i} \leq d_1 \ \vee\ x_{\overline{q}_i} \leq d_0)\ \wedge\ z_{\overline{q}_i} \leq d_2$ or $x_{\overline{q}'_{i+1}} \leq m\ \wedge\ z_{\overline{q}'_{i+1}} \leq n$ for all $0 \leq i \leq s-1$
5. $\overline{q}'_s = \overline{p}_r$

For some constants d_0, d_1, d_2, m and n. $\diamond$

Proof:

part 1

In the first part of the proof we show the existence of a point $\overline{q}_0$ which is vt-connected to $\overline{p}'_0$ ($\overline{q}_0 \xrightarrow{(v+t)^*} \overline{p}'_0$) and such that $y_{\overline{q}_0} \leq d_1\ \vee\ x_{\overline{q}_0} \leq d_0$ and $z_{\overline{q}_0} \leq d_2$.

Let us consider the beginning of the sequence (that is, the end of the path)

$$\overline{p}'_0,\ \overline{p}_0,\ \overline{p}'_1,\ \overline{p}_1,\ \overline{p}'_2,\ \cdots$$

Let us recall that

$$x_{\overline{p}_0} \geq -e, \quad y_{\overline{p}_0} \geq 0, \quad z_{\overline{p}_0} \geq -g$$

One assumes furthermore that $y_{\overline{p}_0} > d_1(k_1,k_2)\ \wedge\ x_{\overline{p}_0} > d_0(k_1,k_2)$ or $z_{\overline{p}_0} > d_2(k_1,k_2)$, where $d_0(k_1,k_2)$, $d_1(k_1,k_2)$ and $d_2(k_1,k_2)$ are values that depend on some constants k_1 and k_2 that will be explained later on. We sometimes write d_0, d_1 and d_2 instead of $d_0(k_1,k_2)$, $d_1(k_1,k_2)$ and $d_2(k_1,k_2)$. Intuitively, k_1 is a lower bound which guarantees that a point $\overline{p}$ belongs to Ω_{yz} whenever $x_{\overline{p}} \geq k_1$ and $z_{\overline{p}} \geq -g$. The constant k_2 is a lower bound that guarantees that the point $\overline{q}_0$ is in Ω_{yz}.

Consider a point $\overline{q}_0 \in [\Omega_{vt}]$ such that $\overline{q}_0 \xrightarrow{(v+t)^*} \overline{p}_0$ and that satisfies

1. $k_1 \leq x_{\overline{q}_0}$
2. $k_2 \leq y_{\overline{q}_0} < k'_2\ (\leq d_1)$
3. $-g \leq z_{\overline{q}_0} < \delta\ (\leq d_2)$

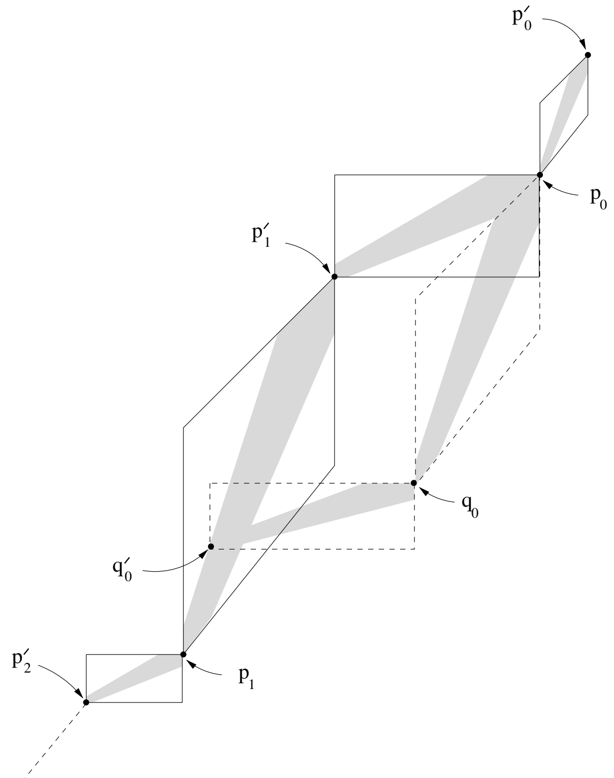


Figure 47

(see figure 47)

In this part we show that such a point $\overline{q}_0$ exists under the assumption that $x_{\overline{p}_0} > d_0(k_1, k_2) \wedge y_{\overline{p}_0} > d_1(k_1, k_2)$ or $z_{\overline{p}_0} > d_2(k_1, k_2)$.

Assume $x_{\overline{p}_0} > d_0 \wedge y_{\overline{p}_0} > d_1$. There exists a point $\overline{p}''_0$ such that $\overline{p}_0 \xrightarrow{t^*} \overline{p}''_0$ and

$$x_{\overline{p}''_0} \geq x_{\overline{p}_0} \geq d_0, \quad y_{\overline{p}''_0} \geq y_{\overline{p}_0} \geq d_1, \quad 0 > z_{\overline{p}_0} \geq -g$$

Starting from $\overline{p}''_0$, one moves backwards along the vt-pattern that keeps z invariant, thus reaching a point $\overline{q}_0$. Moving along the pattern, $-g \leq z < \delta$ holds in every point and thus $-g \leq z_{\overline{q}_0} < \delta$ ($\leq d_2$). Also, moving backwards from $\overline{p}''_0$ along the pattern makes x (globally) increase or let invariant, so that the value of x is always kept greater than $x_{\overline{p}''_0} - \epsilon$ (for some constant ϵ). Since $x_{\overline{p}''_0} \geq d_0$, the point $\overline{q}_0$ satisfies $x_{\overline{q}_0} \geq x_{\overline{p}''_0} - \epsilon \geq d_0 - \epsilon \geq k_1$. (The latter inequality holds by taking a sufficiently large value as for d_0). (see figure 48). It is clear that if $\overline{p}_0$ is in the shaded area, then $\overline{q}_0 \xrightarrow{(v+t)^*} \overline{p}_0$. Furthermore, if $\beta\Delta t \leq y_{\overline{p}_0}$ then $\overline{p}_0$ lies in the shaded area (if $\overline{p}_0 \in [\Omega_{vt}]$). Take d_1 to be $\beta\Delta t$, and its value is given by $d_1 = k_2 + (\delta\beta - fg)j$. Then $k_2 \leq y_{\overline{q}_0}$ $(< k_2 + (\delta\beta - fg) < d_2)$ is guaranteed to hold. If k_2 is chosen sufficiently large, $\overline{q}_0$ must belong to $[\Omega_{yz}]$.

Assume $z_{\overline{p}_0} > d_2$. One moves transversally as much as possible, thus reaching a point $\overline{p}''_0$. The value d_2 has been chosen so that $y_{\overline{p}''_0}$ is guaranteed to be greater than d_1 *and* sufficiently large to ensure that $\overline{q}_0 \xrightarrow{(v+t)^*} \overline{p}_0$ (This is illustrated by the shaded area in figure 48. The figure shows a situation that violates this criterion. $\overline{p}_0$ is not reachable from $\overline{q}_0$ since it does not lie above $\overline{q}_0$.). Since $f > 0$ by assumption, the line $y = 0$ cannot be parallel to the v-axis, and since the determinant $fg - \delta\beta$ is not zero, the lines $y = 0$ and $z = 0$ are not parallel. Therefore it is possible to choose d_2 sufficiently large to ensure that $\overline{p}_0$ lies in the

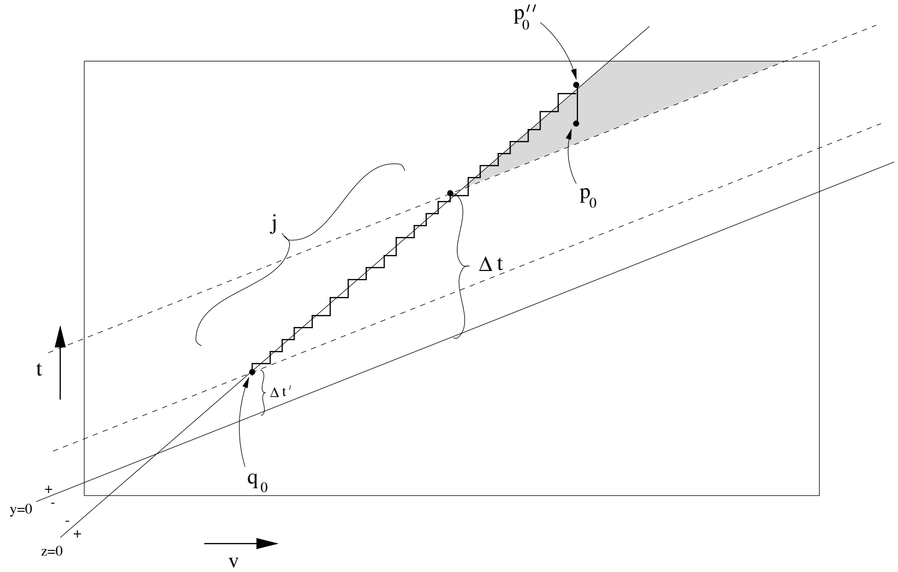


Figure 48

shaded area of figure 49. The rest proceeds as above. Thus we have proved the existence of $\overline{q}_0$.

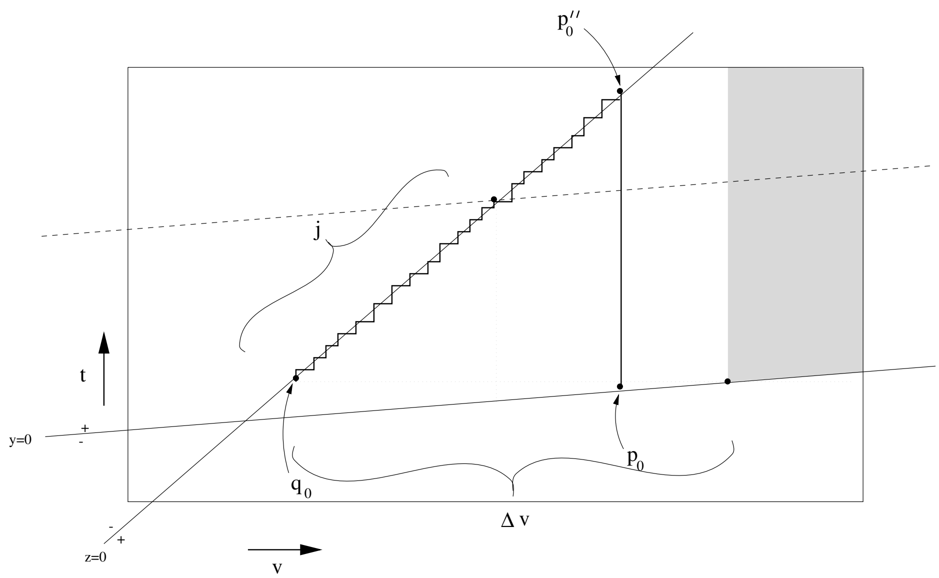


Figure 49

We now distinguish two cases depending on whether $\overline{p}_1 \leq \overline{q}_0$ or not.

part 2 $\overline{p}_1 \leq \overline{q}_0$

We suppose $\overline{p}_1 \leq \overline{q}_0$. The situation is illustrated in figure 47. We show that in such a case, one can connect $\overline{p}_1$ to $\overline{q}_0$. The interest of the constructions lies in the fact that $\overline{q}_0$ is "small" in y and z.

Let us show that if $x_{\overline{p}'_1} > m$ or $z_{\overline{p}'_1} > n$ (together with the assumption that $y_{\overline{p}_0} > d_1(k_1, k_2)$ or $z_{\overline{p}_0} > d_2(k_1, k_2)$), there exists a point $\overline{q}'_0$ such that[1]

1. $\overline{q}'_0 \xrightarrow{(h+t)^*} \overline{q}_0$

2. $\overline{p}_1 \xrightarrow{(v+t)^*} \overline{q}'_0$

Start at $\overline{q}_0$ and walk backwards horizontally until the vt-plane of $\overline{p}_1$ and $\overline{p}'_1$ is intersected in a point $\overline{q}''_0$. Since $\overline{p}_1 \leq \overline{q}_0$, it must be that $\overline{p}_1 \leq \overline{q}''_0$. By going backwards transversally (down), one will reach a point $\overline{q}'_0$ such that $\overline{p}_1 \xrightarrow{(v+t)^*} \overline{q}'_0 \xrightarrow{(v+t)^*} \overline{p}'_1$. This is shown in figure 50. Note that $\overline{p}'_1$ can always be reached from $\overline{p}_1$ by a path such as the one represented by the solid line in the figure.

Assume $x_{\overline{p}'_1} > m$. We have $x_{\overline{q}'''_0} \geq x_{\overline{p}'_1} > m$ (see figure 50) since a t-move increases x.

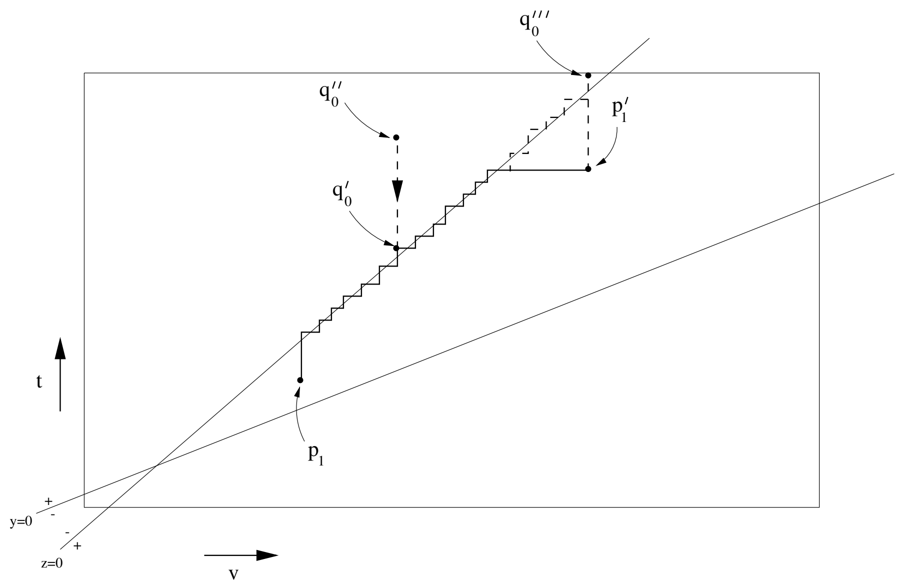


Figure 50

Since the vt-pattern that keeps z invariant decreases x, by going backwards along the path $x_{\overline{q}'_0} \geq k_1$ holds if m is chosen sufficiently large. Since also $z_{\overline{q}'_0} \geq -g$ and $\overline{q}'_0 \leq \overline{q}_0$, we have, for sufficiently large k_1, that $\overline{q}'_0 \xrightarrow{(h+t)^*} \overline{q}_0$ (recall that $x_{\overline{q}_0} \geq k_1$ and $z_{\overline{q}_0} \geq -g$).

Assume $z_{\overline{p}'_1} > n$. Since $x_{\overline{p}'_1} \geq 0$, if n is large enough, sufficiently many t-moves to make $x_{\overline{q}'''_0} > m$ can be made. The rest follows as above.

We have shown that when $(x_{\overline{p}_0} > d_0 \ \wedge \ y_{\overline{p}_0} > d_1) \ \vee \ z_{\overline{p}_0} > d_2$ and $x_{\overline{p}'_1} > m \ \vee \ z_{\overline{p}'_1} > n$, the path represented by the sequence

$$\overline{p}'_0,\ \overline{p}_0,\ \overline{p}'_1,\ \overline{p}_1,\ \overline{p}'_2,\ \overline{p}_2,\ \cdots,$$

can be replaced by a path represented by

$$\overline{p}'_0,\ \overline{q}_0,\ \overline{q}'_0, \overline{p}_1,\ \overline{p}'_2,\ \overline{p}_2,\ \cdots$$

with $y_{\overline{q}_0} \leq d_1$ and $z_{\overline{q}_0} \leq d_2$.

Let us now consider the case where $\overline{p}_1 \not\leq \overline{q}_0$. We distinguish the cases corresponding to different

[1] As for d_1 and d_2, we denote by m and n constants that depend on k_1, k_2.

signs of the determinant $eg - \alpha\gamma$, and show that, in both cases, one can reduce the number of planes that connect $\overline{p}'_2$ to $\overline{p}'_0$.

part 3 $eg - \alpha\gamma < 0$

The situation is illustrated in figure 51. Remember that we assume $x_{\overline{p}'_1} > m \ \vee \ z_{\overline{p}'_1} > n$ and $y_{\overline{p}_0} > d_1 \ \vee \ z_{\overline{p}_0} > d_2$.

By moving horizontally from $\overline{p}_1$ as much as possible ($\overline{p}_1 \xrightarrow{h^*} \overline{p}''_1$) one reaches a point $\overline{p}''_1$ such that $z_{\overline{p}''_1} \geq z_{\overline{p}_1} \geq k_1$. If k_1 is sufficiently large, $\overline{p}''_1$ is in $[\Omega_{xz}]$. It is then possible to walk in the ht-plane along the planar pattern that keeps x invariant. One thus goes from $\overline{p}''_1$ to a point $\overline{q}''_0$ at the intersection of the vt-plane of $\overline{p}_0$ ($\overline{p}''_1 \xrightarrow{(h+t)^*} \overline{q}''_0$). We have $z_{\overline{q}''_0} \geq z_{\overline{p}''_0} - \epsilon = k'_1$ (because $eg - \alpha\gamma < 0$).

On the other hand, by moving vertically backwards from $\overline{p}_0$ as much as possible, one reaches a point $\overline{p}''_0$ ($\overline{p}''_0 \xrightarrow{v^*} \overline{p}_0$) such that $0 > z_{\overline{p}''_0} \geq -g$. By walking backwards in the vt-plane along the planar pattern that preserves z, one goes from $\overline{p}''_0$ to a point $\overline{p}$ at the intersection of the ht-plane of $\overline{p}_1$ (note that this plane must be reached since by construction $\overline{p}_1 \leq \overline{p}_0$). We have $\delta > z_{\overline{p}} \geq -g$, therefore $z_{\overline{q}''_0} \geq z_{\overline{p}}$. It follows that there is a sequence of t-moves from $\overline{q}''_0$ to $\overline{p}$ ($\overline{q}''_0 \xrightarrow{t^*} \overline{p}$). Therefore we have:

$$\overline{p}_1 \xrightarrow{h^*} \overline{p}''_1 \xrightarrow{(h+t)^*} \overline{q}''_0 \xrightarrow{t^*} \overline{p} \xrightarrow{(v+t)^*} \overline{p}_0$$

Since $\overline{p}'_2 \xrightarrow{(h+t)^*} \overline{p}_1$ and $\overline{p}_0 \xrightarrow{(v+t)^*} \overline{p}'_0$, we finally get $\overline{p}'_2 \xrightarrow{(h+t)^*} \overline{q}''_0 \xrightarrow{(v+t)^*} \overline{p}'_0$.

Therefore the the sequence

$$\overline{p}'_0,\ \overline{p}_0,\ \overline{p}'_1,\ \overline{p}_1,\ \overline{p}'_2,\ \cdots$$

reduces to the sequence

$$\overline{p}'_0,\ \overline{q}''_0,\ \overline{p}'_2,\ \cdots$$

It is not necessarily the case that $(x_{\overline{q}''_0} \leq d_0 \ \vee \ y_{\overline{q}''_0} \leq d_1) \ \wedge \ z_{\overline{q}''_0} \leq d_2$. However, the number of planes always reduces.

part 4 $eg - \alpha\gamma > 0$

Suppose $z_{\overline{p}_0} > d_2$. By moving horizontally from $\overline{p}_1$ as much as possible ($\overline{p}_1 \xrightarrow{h^*} \overline{p}''_1$) one reaches a point $\overline{p}''_1$ such that $0 > x_{\overline{p}''_1} \geq -e$. Then by walking in the ht-plane along the planar pattern that keeps x invariant, one goes from $\overline{p}''_1$ to a point $\overline{q}''_0$ at the intersection of the vt-plane of $\overline{p}_0$ ($\overline{p}''_1 \xrightarrow{(h+t)^*} \overline{q}''_0$). We have $\alpha > x_{\overline{q}''_0} \geq -e$.

On the other hand, by moving transversally as much as possible from $\overline{p}_0$, one reaches a point $\overline{p}''_0$ such that $x_{\overline{p}''_0} > k_1 + \epsilon$ and $z_{\overline{p}''_0} \geq -g$. This is possible since $x_{\overline{p}_0} \geq -e$, so if d_2 is chosen sufficiently large, $x_{\overline{p}''_0} > k_1 + \epsilon$ kan be made to hold. Following the pattern backwards in the vt-plane that preserves z, one reaches a point $\overline{p}$ at the intersection of the ht-plane of $\overline{p}_1$ such

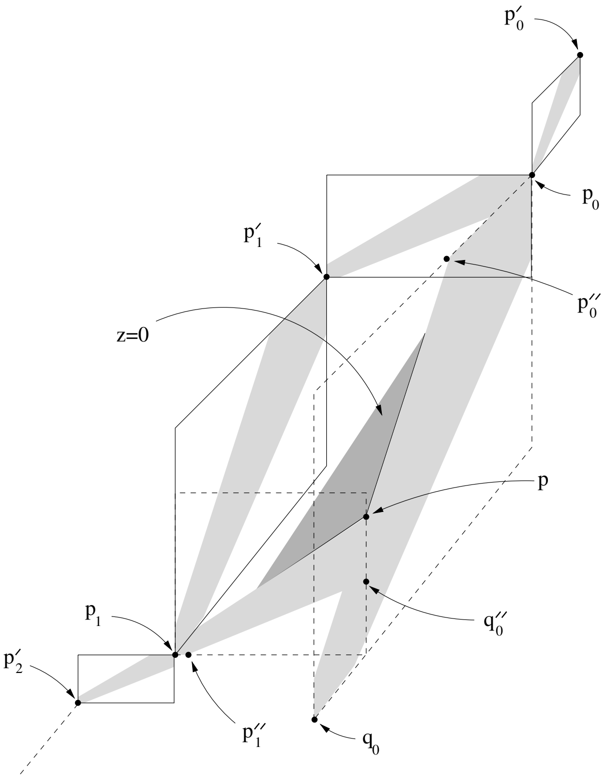


Figure 51

that $x_{\overline{p}} > k_1$. We have $x_{\overline{p}} > k_1$, therefore $x_{\overline{p}} > x_{\overline{q}''_0}$. It follows that there is a sequence of t-moves from $\overline{q}''_0$ to $\overline{p}$ ($\overline{q}''_0 \xrightarrow{t^*} \overline{p}$). Therefore we have:

$$\overline{p}_1 \xrightarrow{h^*} \overline{p}''_1 \xrightarrow{(h+t)^*} \overline{q}''_0 \xrightarrow{t^*} \overline{p} \xrightarrow{(v+t)^*} \overline{p}_0$$

Since $\overline{p}'_2 \xrightarrow{(h+t)^*} \overline{p}_1$ and $\overline{p}_0 \xrightarrow{(v+t)^*} \overline{p}'_0$, we finally get $\overline{p}'_2 \xrightarrow{(h+t)^*} \overline{q}''_0 \xrightarrow{(v+t)^*} \overline{p}'_0$.

Therefore the the sequence

$$\overline{p}'_0,\ \overline{p}_0,\ \overline{p}'_1,\ \overline{p}_1,\ \overline{p}'_2,\ \cdots$$

reduces to the sequence

$$\overline{p}'_0,\ \overline{q}''_0,\ \overline{p}'_2,\ \cdots$$

Suppose $y_{\overline{p}_0} > d_1$ (see figure 52). By moving backwards horizontally One moves backwards from $\overline{p}_0$ following the parallel to $z = 0$ until one intersects the vt-plane of $\overline{p}'_2$ in a point $\overline{q}_0$. We have $y_{\overline{q}_0} > d_1 - \epsilon$ and $z_{\overline{q}_0} \geq -g$. Therefore there is a path in the vt-plane from $\overline{p}'_2$ to $\overline{q}_0$. Thus $\overline{p}_2 \xrightarrow{(v+t)^*} \overline{q}_0 \xrightarrow{(h+t)^*} \overline{p}_0$, hence one can reduce the sequence

$$\overline{p}'_0,\ \overline{p}_0,\ \overline{p}'_1,\ \overline{p}_1,\ \overline{p}'_2,\ \overline{p}_2,\ \cdots,$$

to the sequence

$$\overline{p}'_0,\ \overline{p}_0,\ \overline{q}_0,\ \overline{p}_2,\ \cdots,$$

again the number of planes is reduced.

As a recapitulation, we have shown that one can reduce the sequence

$$\overline{p}'_0,\ \overline{p}_0,\ \overline{p}'_1,\ \overline{p}_1,\ \overline{p}'_2,\ \overline{p}_2,\ \cdots,$$

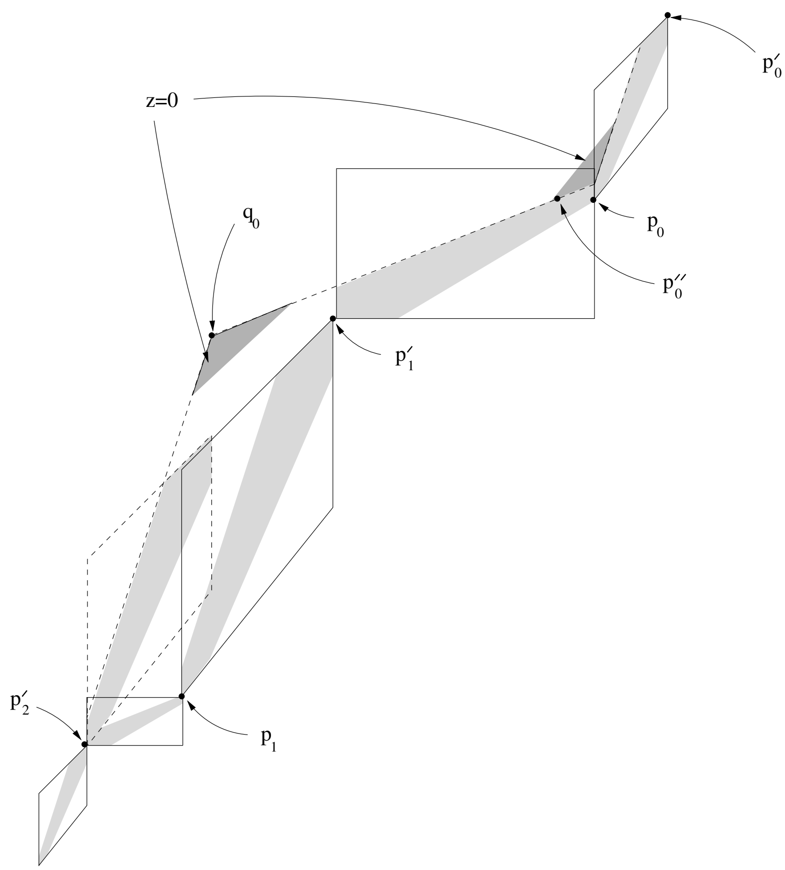


Figure 52

to a sequence

$$\overline{p}'_0,\ \overline{q}_0,\ \overline{q}'_0,\ \overline{p}_1,\ \overline{p}'_2,\ \overline{p}_2,\ \cdots,$$

where $\overline{q}_0$ is "small" in y and z, or to a shorter sequence.

By repeating the transformations above until no more transformation can be applied, the original sequence is reduced to a sequence

$$\overline{p}'_0,\ \overline{q}_0,\ \overline{q}'_1,\ \overline{q}_1,\ \overline{q}'_2,\ \cdots\ \overline{q}_{s-1},\ \overline{q}'_s$$

where $\overline{q}'_s = \overline{p}_r$, that satisfies the condition of lemma 35 such that

1. $\overline{q}_0 \xrightarrow{(v+t)^*} \overline{p}'_0$
2. $\overline{q}_i \xrightarrow{(v+t)^*} \overline{q}'_i$ for all $1 \leq i \leq r-1$
3. $\overline{q}'_{i+1} \xrightarrow{(h+t)^*} \overline{q}_i$ for all $0 \leq i \leq r-1$
4. $y_{\overline{q}_i} \leq d_1 \ \wedge\ z_{\overline{q}_i} \leq d_2$ or $x_{\overline{q}'_{i+1}} \leq d_1 \ \wedge\ z_{\overline{q}'_{i+1}} \leq d_2$ for all $0 \leq i \leq r-1$

◇

The case

$$\Phi = \begin{pmatrix} -e & \lambda & -\gamma \\ \mu & -f & \delta \\ -\alpha & \beta & -g \end{pmatrix}$$

is symmetric to

$$\Phi = \begin{pmatrix} -e & -\lambda & \gamma \\ -\mu & -f & \delta \\ \alpha & \beta & -g \end{pmatrix}$$

by permuting t and v.

Lemma 35, allows us to apply the pigeon hole principle to the points of a reduced path. Since there are finitely many points such that $(x_{\overline{p}_0} \leq d_0 \ \vee\ y_{\overline{q}_i} \leq d_1 \ \wedge\ z_{\overline{q}_i} \leq d_2$, and finitely many such that $x_{\overline{q}'_{i+1}} \leq d_1 \ \wedge\ z_{\overline{q}'_{i+1}} \leq d_2$, for a sufficiently long path, some yz- or xz-values must appear twice. This yields a three dimensional pattern that either preserves y and z, or x and z.

What remains is to show that this pattern can be computed in advance, and that it is the same for all paths. Let us assume that for some $\overline{q}_{i_0}$, $y_{\overline{q}_{i_0}} = y_{\overline{q}_{i_j}}$ and $z_{\overline{q}_{i_0}} = z_{\overline{q}_{i_j}}$, for several $\overline{q}_{i_j}$, $j = 1, 2, \ldots$ and $i_j > i_0$. That is, by the pigeon-hole principle, some yz-values occurs over and over again. Furthermore suppose $\delta y_{\overline{q}_{i_0}} + f z_{\overline{q}_{i_0}} \geq 0$ or $g y_{\overline{q}_{i_0}} + \beta z_{\overline{q}_{i_0}} \geq 0$. By proposition 12, there exists an infinite path in the vt-plane starting at $\overline{q}_{i_0}$ (and at all $\overline{q}_{i_j}$). In particular, one may choose a path that makes z invariant. Call this path w. Then $\overline{w} = k\langle 0, g, \delta\rangle^T$ where $k = g\lambda - \gamma\beta$. Thus we reach a point $\overline{q}'_{i_0} = \overline{q}_{i_0} + \overline{w}$. We have $x_{\overline{q}'_{i_0}} = x_{\overline{q}_{i_0}} - (\mu g - \alpha\delta)(g\lambda - \gamma\beta)$. Since $z_{\overline{q}'_{i_0}} \geq -g$, assuming $x_{\overline{q}_{i_0}} \geq d$ for some constant d, $\gamma x_{\overline{q}'_{i_0}} + e z_{\overline{q}'_{i_0}} \geq 0$ or $g x_{\overline{q}'_{i_0}} + \alpha z_{\overline{q}'_{i_0}} \geq 0$ must hold so that there exists an infinite path in the ht-plane leaving $\overline{q}'_{i_0}$. Once again we may choose a path that keeps z invariant. Call this path w'. Then $\overline{w}' = k'\langle g, 0, \gamma\rangle^T$ where $k' = -gf + \beta\delta$. Thus we reach a point $\overline{q}''_{i_0} = \overline{q}'_{i_0} + \overline{w}' = \overline{q}_{i_0} + \overline{w}''$, where $\overline{w}'' = \overline{w}' + \overline{w} = \langle k'g, kg, k\delta + k'\gamma\rangle^T$. It is easy to verify that $y_{\overline{q}''_{i_0}} = y_{\overline{q}_{i_0}}$ and $z_{\overline{q}''_{i_0}} = z_{\overline{q}_{i_0}}$ hold. Furthermore, $x_{\overline{q}''_{i_0}} = x_{\overline{q}_{i_0}} - gD$ hold, where

$$D = \begin{vmatrix} -e & -\lambda & \gamma \\ -\mu & -f & \delta \\ \alpha & \beta & -g \end{vmatrix}$$

Thus, $\overline{w}''$ is a vt-pattern. We will now show that $\xrightarrow{(h+v+t)^*}$ can be expressed in terms of $\overline{w}''$ under certain assumptions.

1. *Assume* that there is a unique minimal solution to

$$\begin{array}{lllllll} y(h,v,t) & = & -\lambda h & -fv & +\beta t & = & 0 \\ z(h,v,t) & = & -\gamma h & +\delta v & -gt & = & 0 \end{array}$$

   say $\overline{s}_{min}$. All other solutions are of the form $\overline{s} = l\overline{s}_{min}$ for some l.

2. *Assume* $\overline{s}_{min}$ makes x strictly increase.

3. By the path reduction and the pigeon hole principle, either for $\overline{q}'_i$ some xz-values repeat, or for $\overline{q}_i$ some yz-values repeat. *Assume* that for $\overline{q}_i$ some yz-values repeat.

4. By (3), we get a solution to $y(h,v,t) = 0$ and $z(h,v,t) = 0$. By 1, if $y_{\overline{q}_i} = y_{\overline{q}_{i'}}$ and $z_{\overline{q}_i} = z_{\overline{q}_{i'}}$ for some $\overline{q}_i$ and $\overline{q}_{i'}$ such that $\overline{q}_i \leq \overline{q}_{i'}$ it must be that $\overline{q}_{i'} = \overline{q}_i + l\overline{s}_{min}$ for some l. By (2), all solutions make x increase. Since we start with $x \geq -e$, after a bounded number of $\overline{q}_i$:s we reach $\overline{q}_{i_0}$ say, such that $x_{\overline{q}_{i_0}} \geq d$ for some constant d.

5. *Assume* $\overline{q}_{i_0}$ satisfies $\delta y_{\overline{q}_{i_0}} + fz_{\overline{q}_{i_0}} \geq 0$ or $gy_{\overline{q}_{i_0}} + \beta z_{\overline{q}_{i_0}} \geq 0$. By the reasoning above, the pattern $\overline{w}''$ is applicable at $\overline{q}_{i_0}$.

6. By (4), $\overline{q}_{i'}$ sufficiently far from $\overline{q}_{i_0}$, is of the form $\overline{q}_{i'} = \overline{q}_{i_0} + l'\overline{w}''$, and by (5), $\overline{q}_{i_0} \xrightarrow{\xi^*_{\overline{w}''}} \overline{q}_{i'}$.

7. $\overline{q}_{i'}$ can be chosen such that it is connected to $\overline{p}'_0$ by a bounded number of planes (the bound is independent of $\overline{p}_r$ and $\overline{p}'_0$).

If one assumes that

$$\left| \begin{array}{cc} -e & \gamma \\ \alpha & -g \end{array} \right| < 0$$

one may reason analogously for xz-values.

If $\overline{s}_{min}$ decreases x, one reasons as above except that one starts from the the end $\overline{p}'_0$ and go backwards towards $\overline{p}_r$. If $\overline{s}_{min}$ keeps x invariant, the determinant D is zero, which we assume it is not.

If $\overline{q}_{i_0}$ does not satisfy $\delta y_{\overline{q}_{i_0}} + fz_{\overline{q}_{i_0}} \geq 0$ or $gy_{\overline{q}_{i_0}} + \beta z_{\overline{q}_{i_0}} \geq 0$, the algorithm in sectition 6 can be adapted to compute a pattern $\overline{u}$ such that $\overline{q}_{i_0} \xrightarrow{\xi^*_{\overline{u}}} \overline{q}_{i'}$ for $\overline{q}_{i'}$ sufficiently far from $\overline{q}_{i_0}$. We summarize this section as

Theorem 10:
Consider the matrix

$$\Phi = \left(\begin{array}{ccc} -e & -\lambda & \gamma \\ -\mu & -f & \delta \\ \alpha & \beta & -g \end{array} \right)$$

where $\lambda, \gamma, \mu, \alpha \geq 0$ and where both planar vt-patterns make x decrease. Assume that there is a unique minimal solution to the equations

$$\begin{array}{lllllll} y(h,v,t) & = & -\lambda h & -fv & +\beta t & = & 0 \\ z(h,v,t) & = & -\gamma h & +\delta v & -gt & = & 0 \end{array}$$

Then the fixpoint is given by

$$\begin{array}{c} \overline{p} \xrightarrow{(h+v)^*(h+t)^*(v+t)^*(h+v+t)^{\leq k}\xi^*_{\overline{u}}(h+v+t)^{\leq k}(h+t)^*(v+t)^*} \overline{p}' \\ \vee \\ \overline{p} \xrightarrow{(h+v)^*(h+t)^*(v+t)^*((h+t)^*(v+t)^*)^{\leq k'}\xi^*_{\overline{w}''}(v+t)^*(h+t)^*(v+t)^*} \overline{p}' \end{array}$$

$\diamond$

10 Class 4

In this section we consider matrices of the form

$$\Phi = \begin{pmatrix} e & \bullet & \bullet \\ \bullet & -f & \delta \\ \bullet & \beta & -g \end{pmatrix}$$

where $e, f, g, \beta, \delta > 0$,

$$\begin{vmatrix} -f & \delta \\ \beta & -g \end{vmatrix} < 0$$

We investigate reachability $\overline{p}_{\Omega_{yz}} \xrightarrow{(h+v+t)^*} \overline{p}'$ under the assumption that $\overline{p}_{\Omega_{yz}} \in [\Omega_{yz}(d, d')]$ for sufficiently large d and d' that guarantees that a pattern is applicable, and $x_{\overline{p}'} \geq 0$.

10.1 increasing vt-pattern

Throughout this subsection we assume that at least one planar vt-pattern makes x *strictly* increase. Note that this assumption excludes matrices of the form

$$\Phi = \begin{pmatrix} e & \bullet & \bullet \\ -\mu & -f & \delta \\ -\alpha & \beta & -g \end{pmatrix}$$

where $\mu, \alpha \geq 0$ since this would imply that both vt-patterns made x decrease.

Lemma 36:
Consider the matrix

$$\Phi = \begin{pmatrix} e & \bullet & \bullet \\ \mu & -f & \delta \\ \bullet & \beta & -g \end{pmatrix}$$

where $\mu \geq 0$. Suppose $\overline{p}_{\Omega_{yz}} \xrightarrow{(h+v+t)^*} \overline{p}'$ for some points $\overline{p}_{\Omega_{yz}} \in [\Omega_{yz}]$ and $\overline{p}'$ such that $z_{\overline{p}'} \geq 0$. Then one of the following holds.

1. $\overline{p}_{\Omega_{yz}} \xrightarrow{(v+t)^* h^* (vh^* + th^*)^{\leq k}} \overline{p}'$ for some k.
2. $\overline{p}_{\Omega_{yz}} \xrightarrow{(v+t)^* h^* t^*} \overline{p}'$
3. $\overline{p}_{\Omega_{yz}} \xrightarrow{(v+t)^* h^* v^* h^*} \overline{p}'$

$\diamond$

Proof:
The proof is similar to the proof of lemma 32, so we only give the key relations. We distinguish the cases when, by moving along the pattern that increases x, one cross either the ht- or hv-plane of $\overline{p}'$. This is illustrated in figures 53 and 54 respectively.

Consider figure 53. We have
$z_{\overline{q}'} \geq z_{\overline{p}'} \geq 0$
$x_{\overline{q}'} \geq x_{\overline{q}} \geq x_{\overline{p}''} + d \geq d$

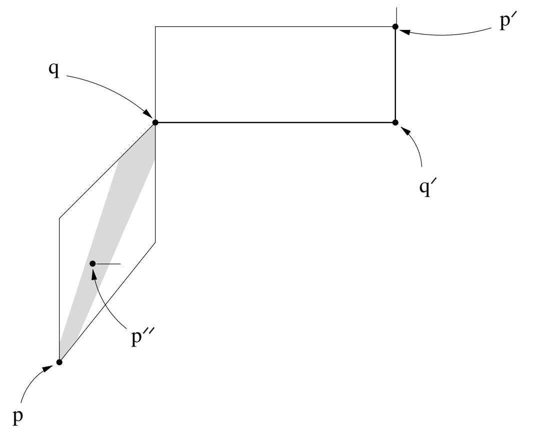


Figure 53

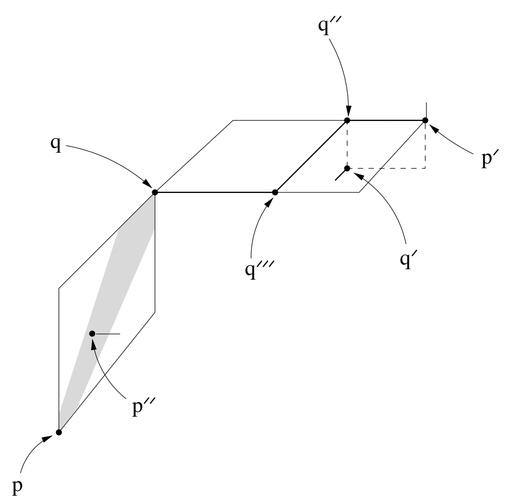


Figure 54

Thus $\overline{q} \xrightarrow{h^*} \overline{q}' \xrightarrow{t^*} \overline{p}'$

Consider figure 54. We have
$y_{\overline{q}'''} \geq y_{\overline{q}''} \geq y_{\overline{q}'} \geq 0$
$x_{\overline{p}'} \geq x_{\overline{q}''} \geq x_{\overline{q}'''} \geq x_{\overline{q}} \geq x_{\overline{p}''} + d \geq d$
Thus $\overline{q} \xrightarrow{h^*} \overline{q}''' \xrightarrow{v^*} \overline{q}'' \xrightarrow{h^*} \overline{p}'$ $\diamond$

Lemma 37:
Consider the matrix

$$\Phi = \begin{pmatrix} e & \bullet & \bullet \\ \bullet & -f & \delta \\ \alpha & \beta & -g \end{pmatrix}$$

where $\alpha \geq 0$. Suppose $\overline{p}_{\Omega_{yz}} \xrightarrow{(h+v+t)^*} \overline{p}'$ for some points $\overline{p}_{\Omega_{yz}} \in [\Omega_{yz}]$ and $\overline{p}'$ such that $y_{\overline{p}'} \geq 0$. Then one of the following holds.

1. $\overline{p}_{\Omega_{yz}} \xrightarrow{(v+t)^* h^* (vh^* + th^*)^{\leq k}} \overline{p}'$ for some k.
2. $\overline{p}_{\Omega_{yz}} \xrightarrow{(v+t)^* h^* t^* h^*} \overline{p}'$
3. $\overline{p}_{\Omega_{yz}} \xrightarrow{(v+t)^* h^* v^*} \overline{p}'$

$\diamond$

Proof:
Consider figure 55. We have

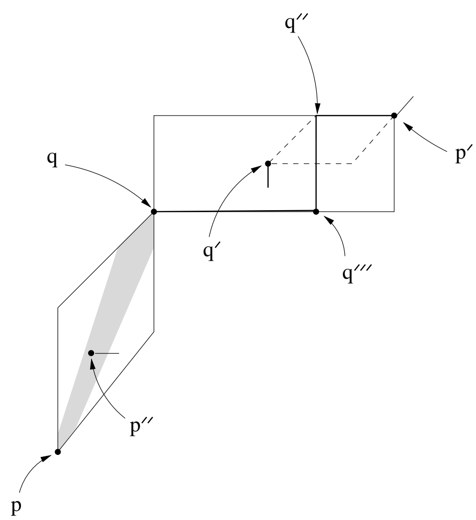


Figure 55

$z_{\overline{q}'''} \geq z_{\overline{q}''} \geq z_{\overline{q}'} \geq 0$
$x_{\overline{p}'} \geq x_{\overline{q}''} \geq x_{\overline{q}'''} \geq x_{\overline{q}} \geq x_{\overline{p}''} + d \geq d$
Thus $\overline{q} \xrightarrow{h^*} \overline{q}''' \xrightarrow{t^*} \overline{q}'' \xrightarrow{h^*} \overline{p}'$

Consider figure 56. We have
$y_{\overline{q}'} \geq y_{\overline{p}'} \geq 0$
$x_{\overline{q}'} \geq x_{\overline{q}} \geq x_{\overline{p}''} + d \geq d$

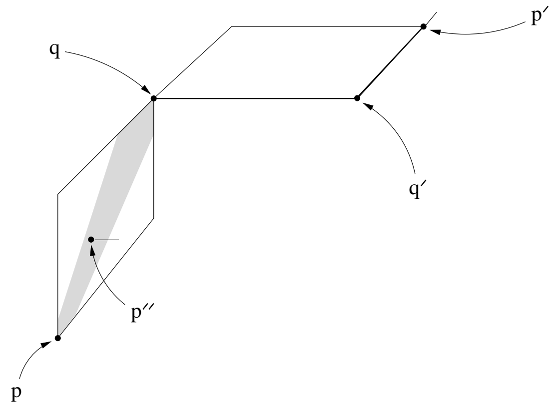


Figure 56

Thus $\overline{q} \xrightarrow{h^*} \overline{q}' \xrightarrow{v^*} \overline{p}'$ $\diamond$

As for class 2 and class 3, after $\overline{p}'$ such that $y_{\overline{p}'} \geq 0$ or $z_{\overline{p}'} \geq 0$, it is sufficient to walk a $v(h+t)^*$- or $t(h+v)^*$-path respectively, since $\overline{p}'$ can be chosen to be the point at which the last v- or t-move respectively, is applied.

We summarize this as

Theorem 11:
Consider the matrix

$$\Phi = \begin{pmatrix} e & \lambda & \gamma \\ \mu & -f & \delta \\ \bullet & \beta & -g \end{pmatrix}$$

where $\mu \geq 0$. Suppose $\overline{p} \xrightarrow{(h+v+t)^*} \overline{p}'$ for some points $\overline{p}$ and $\overline{p}'$. Then

1. If $\lambda \geq 0 \ \wedge\ \gamma \geq 0$, then

$$\overline{p} \xrightarrow{(h+t)^*(h+v)^*(h+v+t)^{\leq k}\xi^*(h+v+t)^{\leq k}(v+t)^*h^*(vh^*+th^*)^{\leq k'}(h+v)^*} \overline{p}'$$
$$\vee$$
$$\overline{p} \xrightarrow{(h+t)^*(h+v)^*(h+v+t)^{\leq k}\xi^*(h+v+t)^{\leq k}(v+t)^*h^*t^*(h+v)^*} \overline{p}'$$

2. If $\lambda \geq 0 \ \wedge\ \gamma < 0$, then

$$\overline{p} \xrightarrow{(h+t)^*(h+v)^*(h+v+t)^{\leq k}\xi^*(h+v+t)^{\leq k}(h+v)^*(v+t)^*h^*(vh^*+th^*)^{\leq k'}(h+v)^*} \overline{p}'$$
$$\vee$$
$$\overline{p} \xrightarrow{(h+t)^*(h+v)^*(h+v+t)^{\leq k}\xi^*(h+v+t)^{\leq k}(h+v)^*(v+t)^*h^*t^*(h+v)^*} \overline{p}'$$

3. If $\lambda < 0 \ \wedge\ \gamma \geq 0$, then

$$\overline{p} \xrightarrow{(h+v)^*(h+t)^*(h+v+t)^{\leq k}\xi^*(h+v+t)^{\leq k}(h+t)^*(v+t)^*h^*(vh^*+th^*)^{\leq k'}(h+v)^*} \overline{p}'$$
$$\vee$$
$$\overline{p} \xrightarrow{(h+v)^*(h+t)^*(h+v+t)^{\leq k}\xi^*(h+v+t)^{\leq k}(h+t)^*(v+t)^*h^*t^*(h+v)^*} \overline{p}'$$

4. If $\lambda < 0 \ \wedge \ \gamma < 0$, then

$$\overline{p} \xrightarrow{(h+t)^*(h+v)^*(h+v+t)^{\leq k}\xi^*(h+v+t)^{\leq k}(h+v)^*(v+t)^*h^*(vh^*+th^*)^{\leq k'}(h+v)^*} \overline{p}'$$
$$\vee$$
$$\overline{p} \xrightarrow{(h+t)^*(h+v)^*(h+v+t)^{\leq k}\xi^*(h+v+t)^{\leq k}(h+t)^*(v+t)^*h^*(vh^*+th^*)^{\leq k'}(h+v)^*} \overline{p}'$$
$$\vee$$
$$\overline{p} \xrightarrow{(h+t)^*(h+v)^*(h+v+t)^{\leq k}\xi^*(h+v+t)^{\leq k}(h+v)^*(v+t)^*h^*t^*(h+v)^*} \overline{p}'$$
$$\vee$$
$$\overline{p} \xrightarrow{(h+t)^*(h+v)^*(h+v+t)^{\leq k}\xi^*(h+v+t)^{\leq k}(h+t)^*(v+t)^*h^*t^*(h+v)^*} \overline{p}'$$

for some ξ, k and k'. $\diamond$

Proof:
Follows from theorem 7 and lemma 36, and from the fact that $h^*v^*h^*(h+v)^* \subseteq h^*t^*(h+v)^*$.$\diamond$

Theorem 12:
Consider the matrix

$$\Phi = \begin{pmatrix} e & \lambda & \gamma \\ \bullet & -f & \delta \\ \alpha & \beta & -g \end{pmatrix}$$

where $\alpha \geq 0$. Suppose $\overline{p} \xrightarrow{(h+v+t)^*} \overline{p}'$ for some points $\overline{p}$ and $\overline{p}'$. Then

1. If $\lambda \geq 0 \ \wedge \ \gamma \geq 0$, then

$$\overline{p} \xrightarrow{(h+t)^*(h+v)^*(h+v+t)^{\leq k}\xi^*(h+v+t)^{\leq k}(v+t)^*h^*(vh^*+th^*)^{\leq k'}(h+t)^*} \overline{p}'$$
$$\vee$$
$$\overline{p} \xrightarrow{(h+t)^*(h+v)^*(h+v+t)^{\leq k}\xi^*(h+v+t)^{\leq k}(v+t)^*h^*v^*(h+t)^*} \overline{p}'$$

2. If $\lambda \geq 0 \ \wedge \ \gamma < 0$, then

$$\overline{p} \xrightarrow{(h+t)^*(h+v)^*(h+v+t)^{\leq k}\xi^*(h+v+t)^{\leq k}(h+v)^*(v+t)^*h^*(vh^*+th^*)^{\leq k'}(h+t)^*} \overline{p}'$$
$$\vee$$
$$\overline{p} \xrightarrow{(h+t)^*(h+v)^*(h+v+t)^{\leq k}\xi^*(h+v+t)^{\leq k}(h+v)^*(v+t)^*h^*v^*(h+t)^*} \overline{p}'$$

3. If $\lambda < 0 \ \wedge \ \gamma \geq 0$, then

$$\overline{p} \xrightarrow{(h+v)^*(h+t)^*(h+v+t)^{\leq k}\xi^*(h+v+t)^{\leq k}(h+t)^*(v+t)^*h^*(vh^*+th^*)^{\leq k'}(h+t)^*} \overline{p}'$$
$$\vee$$
$$\overline{p} \xrightarrow{(h+v)^*(h+t)^*(h+v+t)^{\leq k}\xi^*(h+v+t)^{\leq k}(h+t)^*(v+t)^*h^*v^*(h+t)^*} \overline{p}'$$

4. If $\lambda < 0 \ \wedge \ \gamma < 0$, then

$$\overline{p} \xrightarrow{(h+t)^*(h+v)^*(h+v+t)^{\leq k}\xi^*(h+v+t)^{\leq k}(h+v)^*(v+t)^*h^*(vh^*+th^*)^{\leq k'}(h+t)^*} \overline{p}'$$
$$\vee$$
$$\overline{p} \xrightarrow{(h+t)^*(h+v)^*(h+v+t)^{\leq k}\xi^*(h+v+t)^{\leq k}(h+t)^*(v+t)^*h^*(vh^*+th^*)^{\leq k'}(h+t)^*} \overline{p}'$$
$$\vee$$
$$\overline{p} \xrightarrow{(h+t)^*(h+v)^*(h+v+t)^{\leq k}\xi^*(h+v+t)^{\leq k}(h+v)^*(v+t)^*h^*v^*(h+t)^*} \overline{p}'$$
$$\vee$$
$$\overline{p} \xrightarrow{(h+t)^*(h+v)^*(h+v+t)^{\leq k}\xi^*(h+v+t)^{\leq k}(h+t)^*(v+t)^*h^*v^*(h+t)^*} \overline{p}'$$

for some ξ, k and k'. $\diamond$

Proof:
Follows from theorem 7 and lemma 37, and from the fact that $h^*t^*h^*(h+t)^* \subseteq h^*v^*(h+t)^*$.$\diamond$

10.2 class 4 decreasing

Throughout this subsection we assume that both planar vt-patterns makes x *strictly* decrease. This assumption excludes matrices of the form

$$\Phi = \begin{pmatrix} e & \bullet & \bullet \\ \mu & -f & \delta \\ \alpha & \beta & -g \end{pmatrix}$$

where $\mu, \alpha \geq 0$ since this would imply that both vt-patterns made x increase.

First let us note that $(h+v+t)^* \subseteq ((v+t)^*h^*t^*)^*$ which means that any path $w \in (h+v+t)^*$ between two points $\overline{p}_0 \xrightarrow{w} \overline{p}$ is included in the sequence
$\overline{p}_0 \xrightarrow{(v+t)^*} \overline{p}'_0 \xrightarrow{h^*} \overline{p}''_0 \xrightarrow{t^*} \overline{p}_1 \cdots \xrightarrow{t^*} \overline{p}_n \xrightarrow{(v+t)^*} \overline{p}'_n \xrightarrow{h^*} \overline{p}''_n \xrightarrow{t^*} \overline{p}$, (see figure 57), where the sequences $\langle \overline{p}'_i, \overline{p}''_i, \overline{p}_{i+1} \rangle$, are such that

$$\begin{array}{rclrclrcl} x_{\overline{p}'_i} & \geq & 0, & y_{\overline{p}'_i} & \geq & -f, & z_{\overline{p}'_i} & \geq & -g \\ & & & y_{\overline{p}_{i+1}} & \geq & -f, & z_{\overline{p}_{i+1}} & \geq & -g \end{array}$$

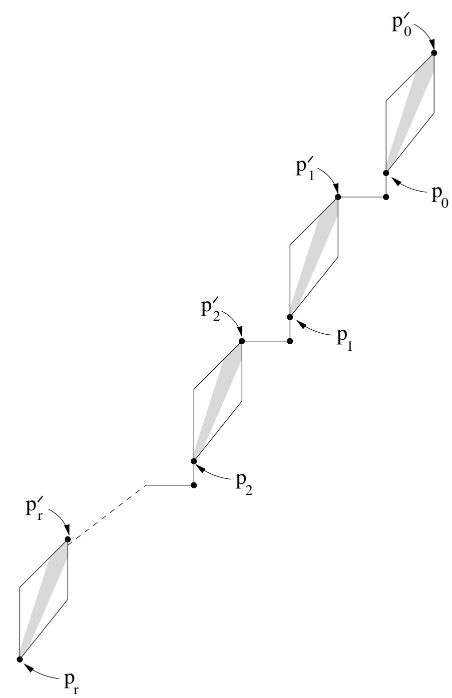


Figure 57

Lemma 38:

Consider the matrices

$$\Phi = \begin{pmatrix} e & -\lambda & \gamma \\ -\mu & -f & \delta \\ \bullet & \beta & -g \end{pmatrix} \quad \text{or} \quad \Phi = \begin{pmatrix} e & -\lambda & \bullet \\ \bullet & -f & \delta \\ -\alpha & \beta & -g \end{pmatrix}$$

where $\lambda, \gamma, \mu, \alpha \geq 0$ and both vt-patterns make x decrease. Let us consider a path w represented, as above, under the form

$$\overline{p}'_0,\ \overline{p}_0,\ \overline{p}'_1,\ \overline{p}_1,\ \overline{p}'_2,\ \overline{p}_2,\ \cdots\ \overline{p}'_r,\ \overline{p}_r$$

Then the path w can be transformed into a path w' represented as the sequence

$$\overline{p}'_0,\ \overline{q}_0,\ \overline{q}'_1,\ \overline{q}_1,\ \overline{q}'_2,\ \cdots\ \overline{q}_{s-1},\ \overline{q}'_s$$

such that

1. $\overline{q}_0 \xrightarrow{(v+t)^*} \overline{p}'_0$
2. $\overline{q}_i \xrightarrow{(v+t)^*} \overline{q}'_i$ for all $1 \leq i \leq s-1$
3. $\overline{q}'_{i+1} \xrightarrow{(h+t)^*} \overline{q}_i$ for all $0 \leq i \leq s-1$
4. $k_2 \leq y_{\overline{q}_i} \leq k'_2\ (\leq d_1)\ \wedge\ -g \leq z_{\overline{q}_i} \leq \delta\ (\leq d_2)$ or $x_{\overline{q}'_{i+1}} \leq k_1\ \wedge\ (y_{\overline{q}'_{i+1}} \leq d_1\ \vee\ z_{\overline{q}'_{i+1}} \leq d_2)$
5. $\overline{q}'_s = \overline{p}_r$

For some constants d_0, d_1, d_2, m and n. $\diamond$

Proof:

part 1

In the first part of the proof we show the existence of a point $\overline{q}_0$ associated with $\overline{p}_0$ such that $k_2 \leq y_{\overline{q}_i} \leq k'_2\ \wedge\ -g \leq z_{\overline{q}_i} \leq \delta$. The proof proceeds exactly as for class 3 (the first part of the proof of lemma 35.

part 2

Part 2 of the proof consists in showing that, if $\overline{p}_1 \leq \overline{q}_0$, there exists a path that links $\overline{p}'_2$ to $\overline{p}_0$ contained in a number of planes smaller than the number of planes crossed by the path via $\overline{p}_1$, $\overline{p}'_1$ and $\overline{p}_0$. This is illustrated in figure 58. Let $\overline{q}''_0$ be the intersection point of the horizontal line passing by $\overline{p}'_2$ and the vt-plane of $\overline{q}_0$. There exists a linear path (either transversal or vertical) that connects $\overline{q}''_0$ to a point, say $\overline{q}'_0$, of the region $y_{\overline{q}'_0} \geq -f\ \wedge\ z_{\overline{q}'_0} \geq -g$ of the vt-plane of $\overline{q}_0$. The path $\overline{p}'_2 - \overline{q}''_0 - \overline{q}'_0 - \overline{p}'_0$ is an admissible path. The subsequence $\overline{p}'_0,\ \overline{p}_0,\ \overline{p}'_1,\ \overline{p}_2,\ \overline{p}'_2$ can be replaced by the shorter sequence $\overline{p}'_0,\ \overline{q}''_0,\ \overline{p}'_2$.

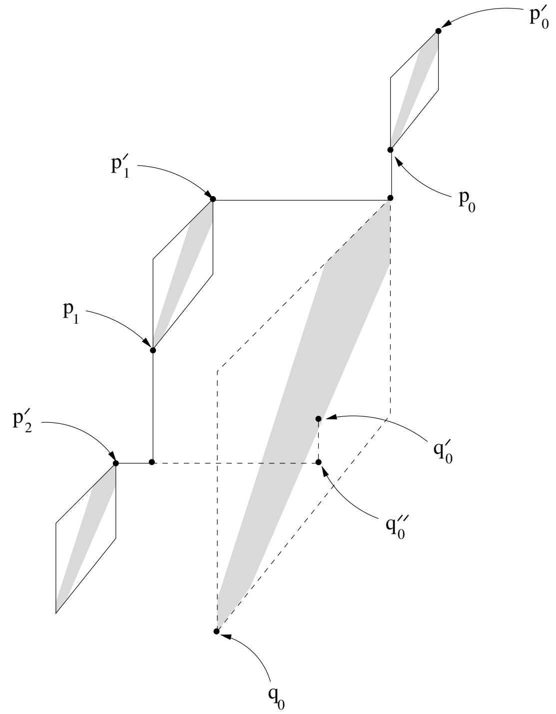


Figure 58

part 3

Part 3 of the proof consists in showing that, if $\overline{p}_1 \not\leq \overline{q}_0$, one can connect the point $\overline{p}'_2$ to the point $\overline{q}_0$.

The assumptions are:

- $x_{\overline{p}'_1} > k_1$ or $y_{\overline{p}'_1} > d_1 \;\wedge\; z_{\overline{p}'_1} > d_2$ for some sufficiently large k_1, d_1 and d_2.
- $d_1 > y_{\overline{q}_0} > k_2$ for some k_2.
- $\delta > z_{\overline{q}_0} \geq -g$

Consider the matrix

$$\Phi = \begin{pmatrix} e & -\lambda & \gamma \\ -\mu & -f & \delta \\ \bullet & \beta & -g \end{pmatrix}$$

where $\lambda, \gamma, \mu \geq 0$. There are two subcases: $x_{\overline{p}'_1} > k_1$ or $y_{\overline{p}'_1} > d_1 \;\wedge\; z_{\overline{p}'_1} > d_2$.

Consider the case when $x_{\overline{p}'_1} > k_1$ (the situation is illustrated in figure 59). By walking backwards vertically from $\overline{p}'_1$ as much as possible, one reaches a point $\overline{p}''_1$. Then by walking backward along the pattern preserving z, one reaches a point $\overline{q}'_0$ at the intersection of the vt-plane of $\overline{q}_0$.

We have: $x_{\overline{p}''_1} > x_{\overline{p}'_1} > k_1$ and $x_{\overline{q}'_0} \geq x_{\overline{p}''_1} > k_1 > 0$.

On the otherhand, let $\overline{q}''_0$ be the intersection point of the horizontal path through $\overline{q}_0$ and the vt-plane of $\overline{p}'_1$.

We have: $y_{\overline{p}''_0} \geq y_{\overline{q}_0} \geq k_2$ and $z_{\overline{q}''_0} \leq z_{\overline{q}_0} < 0$.

Therefore $\overline{q}_0''$ is located above $\overline{q}_0'$. Therefore there exists an admissible path $\overline{p}_2' - \overline{p}_1' - \overline{q}_0' - \overline{q}_0''' - \overline{q}_0$ from $\overline{p}_2'$ to $\overline{q}_0$, where $\overline{q}_0'''$ is the intersection point of the horizontal line passing through $\overline{q}_0'$ and the vertical line passing through $\overline{q}_0$.

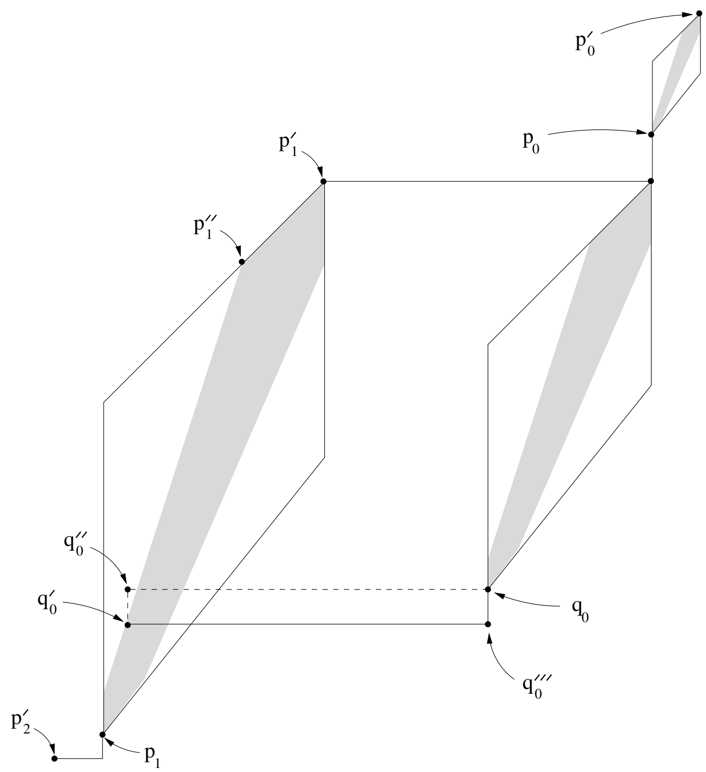


Figure 59

Consider the case when $y_{\overline{p}_1'} > d_1 \ \wedge \ z_{\overline{p}_1'} > d_2$. Since $z_{\overline{p}_1'} > d_2$ one can move backwards vertically as above and yield a point $\overline{p}_1''$ such that $x_{\overline{p}_1''} > k_1$. The rest proceeds as above.

Consider the matrix

$$\Phi = \begin{pmatrix} e & -\lambda & \bullet \\ \bullet & -f & \delta \\ -\alpha & \beta & -g \end{pmatrix}$$

where $\lambda, \alpha \geq 0$. Let us again show that there exists a path from $\overline{p}_2'$ to $\overline{q}_0$.

Consider the case when $x_{\overline{p}_1'} > k_1$ (the situation is illustrated in figure 60). By walking backwards transversally from $\overline{p}_1'$ as much as possible, one reaches a point $\overline{p}_1''$. Then by walking backwards along the pattern preserving y, one reaches a point $\overline{q}_0'$ at the intersection of the vt-plane of $\overline{q}_0$.

We have: $x_{\overline{p}_1''} > x_{\overline{p}_1'} > k_1$ and $x_{\overline{q}_0'} \geq x_{\overline{p}_1''} > k_1 > 0$

On the other hand, let $\overline{q}_0''$ be the intersection point of the horizontal path passing through $\overline{q}_0$ and the vt-plane of $\overline{p}_1'$.

We have: $y_{\overline{q}_0''} \geq y_{\overline{q}_0} \geq k_2$.

There $\overline{q}_0''$ is located above $\overline{q}_0'$. Therefore there exists a path $\overline{p}_2' - \overline{p}_1' - \overline{q}_0' - \overline{q}_0''' - \overline{q}_0$, where $\overline{q}_0'''$ is the intersection point of the horizontal line passing through $\overline{q}_0'$ and the vertical line passing through $\overline{q}_0$.

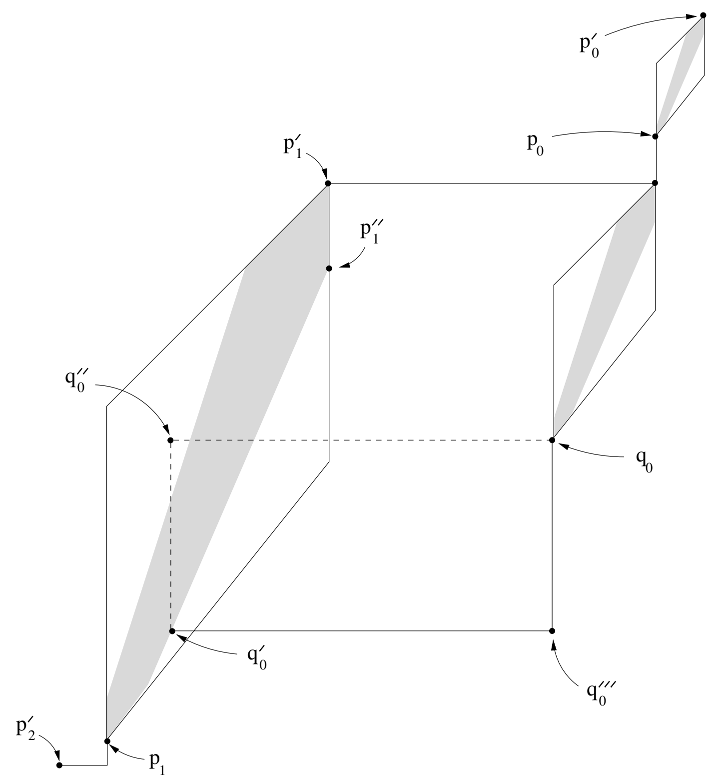


Figure 60

Consider the case where $y_{\overline{p}_1'} > d_1 \ \wedge \ z_{\overline{p}_1'} > d_2$. Since $y_{\overline{p}_1'} > d_1$ one can move backwards transversally as above, and yield a point $\overline{p}_1''$ such that $x_{\overline{p}_1''} > k_1$. The rest proceeds as above.

As a recapitulation we have shown that one always can reduce the sequence $\overline{p}_0'$, $\overline{p}_0$, $\overline{p}_1'$, $\overline{p}_1$, $\overline{p}_2'$ to a sequence $\overline{p}_0'$, $\overline{q}_0$, $\overline{q}_0'$, $\overline{p}_1'$, $\overline{p}_2'$ (where $\overline{q}_0$ is "small"), or to a shorter sequence. By iterating the process, one yields a sequence $\overline{p}_0'$, $\overline{q}_0$, $\overline{q}_1'$, $\ldots$, $\overline{q}_s$, $\overline{q}_s'$ (with $\overline{q}_s' = \overline{p}_r$) satisfying the conditions of the lemma. $\diamond$

The result applies to matrices of the form

$$\begin{pmatrix} + & - & + \\ - & -f & \delta \\ \bullet & \beta & -g \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} + & - & \bullet \\ \bullet & -f & \delta \\ - & \beta & -g \end{pmatrix}$$

Symmetric results (with hv instead of ht) can be shown for matrices of the form

$$\begin{pmatrix} + & + & - \\ \bullet & -f & \delta \\ - & \beta & -g \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} + & \bullet & - \\ - & -f & \delta \\ \bullet & \beta & -g \end{pmatrix}$$

These 4 matrices cover all the cases of class 4.

Lemma 38, allows us to apply the pigeon hole principle to the points of a reduced path. Since there are finitely many points such that $k_2 \leq y_{\overline{q}_i} \leq k_2' \ (\leq d_1) \ \wedge \ -g \leq z_{\overline{q}_i} \leq \delta \ (\leq d_2)$ or $x_{\overline{q}'_{i+1}} \leq k_1 \wedge (y_{\overline{q}'_{i+1}} \leq d_1 \vee z_{\overline{q}'_{i+1}} \leq d_2)$, for a sufficiently long path, some xy-, yz- or xz-values must appear twice. This yields a three dimensional pattern that either preserves x and y, y and z, or x and z.

What remains is to show that this pattern can be computed in advance, and that it is the same for all paths. Let us assume that for some $\overline{q}_{i_0}$, $y_{\overline{q}_{i_0}} = y_{\overline{q}_{i_j}}$ and $z_{\overline{q}_{i_0}} = z_{\overline{q}_{i_j}}$, for several $\overline{q}_{i_j}$, $j = 1, 2, \ldots$ and $i_j > i_0$. That is, by the pigeon-hole principle, some yz-values occurs over and over again. Furthermore suppose $\delta y_{\overline{q}_{i_0}} + f z_{\overline{q}_{i_0}} \geq 0$ or $g y_{\overline{q}_{i_0}} + \beta z_{\overline{q}_{i_0}} \geq 0$. By proposition 12, there exists an infinite path in the vt-plane starting at $\overline{q}_{i_0}$ (and at all $\overline{q}_{i_j}$). In particular, one may choose a path that makes z invariant. Call this path w. Then $\overline{w} = k\langle 0, g, \delta\rangle^T$ where $k = g\lambda - \gamma\beta$. Thus we reach a point $\overline{q}'_{i_0} = \overline{q}_{i_0} + \overline{w}$. We have $x_{\overline{q}'_{i_0}} = x_{\overline{q}_{i_0}} - (\mu g - \alpha\delta)(g\lambda - \gamma\beta)$. Since $z_{\overline{q}'_{i_0}} \geq -g$, assuming $x_{\overline{q}_{i_0}} \geq d$ for some constant d, $\gamma x_{\overline{q}'_{i_0}} + e z_{\overline{q}'_{i_0}} \geq 0$ or $g x_{\overline{q}'_{i_0}} + \alpha z_{\overline{q}'_{i_0}} \geq 0$ must hold so that there exists an infinite path in the ht-plane leaving $\overline{q}'_{i_0}$. Once again we may choose a path that keeps z invariant. Call this path w'. Then $\overline{w}' = k'\langle g, 0, \gamma\rangle^T$ where $k' = -gf + \beta\delta$. Thus we reach a point $\overline{q}''_{i_0} = \overline{q}'_{i_0} + \overline{w}' = \overline{q}_{i_0} + \overline{w}''$, where $\overline{w}'' = \overline{w}' + \overline{w} = \langle k'g, kg, k\delta + k'\gamma\rangle^T$. It is easy to verify that $y_{\overline{q}''_{i_0}} = y_{\overline{q}_{i_0}}$ and $z_{\overline{q}''_{i_0}} = z_{\overline{q}_{i_0}}$ hold. Furthermore, $x_{\overline{q}''_{i_0}} = x_{\overline{q}_{i_0}} - gD$ hold, where

$$D = \begin{vmatrix} -e & -\lambda & \gamma \\ -\mu & -f & \delta \\ \alpha & \beta & -g \end{vmatrix}$$

Thus, $\overline{w}''$ is a vt-pattern. We will now show that $\xrightarrow{(h+v+t)^*}$ can be expressed in terms of $\overline{w}''$ under certain assumptions.

1. *Assume* that there is a unique minimal solution to

$$\begin{array}{rcrrrcl} y(h,v,t) & = & -\lambda h & -fv & +\beta t & = & 0 \\ z(h,v,t) & = & -\gamma h & +\delta v & -gt & = & 0 \end{array}$$

   say $\overline{s}_{min}$. All other solutions are of the form $\overline{s} = l\overline{s}_{min}$ for some l.

2. *Assume* $\overline{s}_{min}$ makes x strictly increase.

3. By the path reduction and the pigeon hole principle, either for $\overline{q}'_i$ some xz-values repeat, or for $\overline{q}_i$ some yz-values repeat. *Assume* that for $\overline{q}_i$ some yz-values repeat.

4. By (3), we get a solution to $y(h,v,t) = 0$ and $z(h,v,t) = 0$. By 1, if $y_{\overline{q}_i} = y_{\overline{q}_{i'}}$ and $z_{\overline{q}_i} = z_{\overline{q}_{i'}}$ for some $\overline{q}_i$ and $\overline{q}_{i'}$ such that $\overline{q}_i \leq \overline{q}_{i'}$ it must be that $\overline{q}_{i'} = \overline{q}_i + l\overline{s}_{min}$ for some l. By (2), all solutions make x increase. Since we start with $x \geq -e$, after a bounded number of $\overline{q}_i$:s we reach $\overline{q}_{i_0}$ say, such that $x_{\overline{q}_{i_0}} \geq d$ for some constant d.

5. *Assume* $\overline{q}_{i_0}$ satisfies $\delta y_{\overline{q}_{i_0}} + f z_{\overline{q}_{i_0}} \geq 0$ or $g y_{\overline{q}_{i_0}} + \beta z_{\overline{q}_{i_0}} \geq 0$. By the reasoning above, the pattern $\overline{w}''$ is applicable at $\overline{q}_{i_0}$.

6. By (4), $\overline{q}_{i'}$ sufficiently far from $\overline{q}_{i_0}$, is of the form $\overline{q}_{i'} = \overline{q}_{i_0} + l'\overline{w}''$, and by (5), $\overline{q}_{i_0} \xrightarrow{\xi^*_{\overline{w}''}} \overline{q}_{i'}$.

7. $\overline{q}_{i'}$ can be chosen such that it is connected to $\overline{p}'_0$ by a bounded number of planes (the bound is independent of $\overline{p}_r$ and $\overline{p}'_0$).

If one assumes that

$$\begin{vmatrix} -e & \gamma \\ \alpha & -g \end{vmatrix} < 0$$

one may reason analogously for xz-values.

If $\overline{s}_{min}$ decreases x, one reasons as above except that one starts from the the end $\overline{p}'_0$ and go backwards towards $\overline{p}_r$. If $\overline{s}_{min}$ keeps x invariant, the determinant D is zero, which we assume it is not.

If $\overline{q}_{i_0}$ does not satisfy $\delta y_{\overline{q}_{i_0}} + f z_{\overline{q}_{i_0}} \geq 0$ or $g y_{\overline{q}_{i_0}} + \beta z_{\overline{q}_{i_0}} \geq 0$, the algorithm in sectition 6 can be adapted to compute a pattern $\overline{u}$ such that $\overline{q}_{i_0} \xrightarrow{\xi^*_{\overline{u}}} \overline{q}_{i'}$ for $\overline{q}_{i'}$ sufficiently far from $\overline{q}_{i_0}$. We summarize this section as

Theorem 13:
Consider the matrix

$$\Phi = \begin{pmatrix} -e & -\lambda & \gamma \\ -\mu & -f & \delta \\ \alpha & \beta & -g \end{pmatrix}$$

where $\lambda, \gamma, \mu, \alpha \geq 0$ and where both planar vt-patterns make x decrease. Assume that there is a unique minimal solution to the equations

$$\begin{array}{rcrrrcl} y(h,v,t) & = & -\lambda h & -fv & +\beta t & = & 0 \\ z(h,v,t) & = & -\gamma h & +\delta v & -gt & = & 0 \end{array}$$

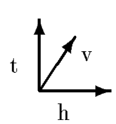
Then the fixpoint is given by

$$\begin{array}{c} \overline{p} \xrightarrow{(h+v)^*(h+t)^*(v+t)^*(h+v+t)^{\leq k}\xi^*_{\overline{u}}(h+v+t)^{\leq k}(h+t)^*(v+t)^*} \overline{p}' \\ \vee \\ \overline{p} \xrightarrow{(h+v)^*(h+t)^*(v+t)^*((h+t)^*(v+t)^*)^{\leq k'}\xi^*_{\overline{w}''}(v+t)^*(h+t)^*(v+t)^*} \overline{p}' \end{array}$$

$\diamond$

11 Fixpoint Plots

In this section we show plots of example programs with three recursive rules. We give examples for classes 2, 3 and 4. Needless to say, only an initial segment is shown since the fixpoints are infinite in general. The orientation of the figures in terms of h, v and t is:



The figures represents all the possible paths associated with the nondeterministic application of all the rules.

11.1 class 2

$$\Phi = \begin{pmatrix} -1 & 3 & -7 \\ -1 & -2 & 5 \\ 4 & 1 & -2 \end{pmatrix}$$

The patterns in the vt-plane are $\langle 0,5,2\rangle^T$ (which keeps y invariant) and $\langle 0,2,1\rangle^T$ (which keeps z invariant). There is no three dimensional pattern preserves y and z Note that $[\Omega_{vt}]$ is reached after some initial segment.

Starting with the base value $\langle -10, -6, 6\rangle^T$ one gets the fixpoint ploted in figure 61.

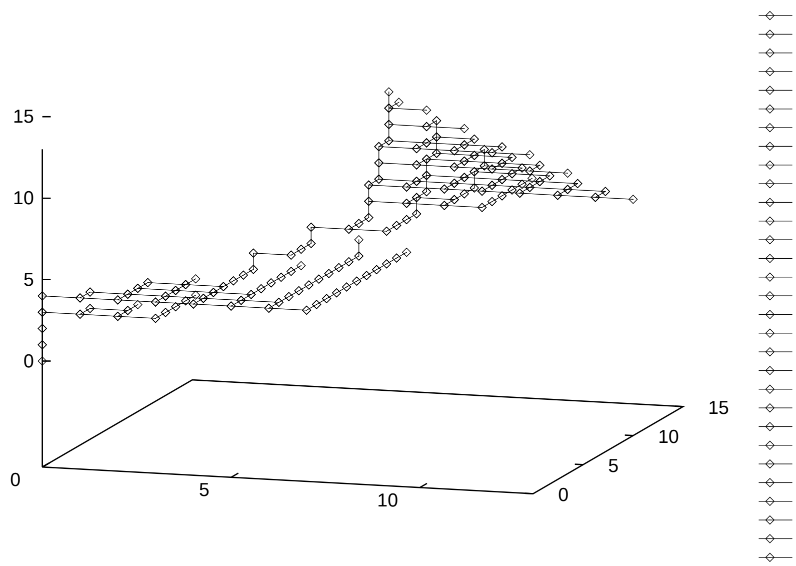


Figure 61

11.2 class 3 increasing pattern

Consider the matrix

$$\Phi = \begin{pmatrix} -1 & -3 & 1 \\ -3 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$$

The patterns in the vt-plane are $\langle 0, 1, 2\rangle^T$ (which keeps y invariant) and $\langle 0, 2, 5\rangle^T$ (which keeps z invariant). The pattern $\langle 0, 1, 2\rangle^T$ increases x.

Starting with the base value $\langle -3, -4, 1\rangle^T$ one gets the fixpoint ploted in figure 62.

Let us give another example of a program belonging to this class. Consider the matrix

$$\Phi = \begin{pmatrix} -2 & -1 & 3 \\ -1 & -1 & 2 \\ 3 & 3 & -5 \end{pmatrix}$$

The patterns in the vt-plane are $\langle 0, 3, 1\rangle^T$ (which keeps y invariant) and $\langle 0, 5, 2\rangle^T$ (which keeps z invariant). The pattern $\langle 0, 3, 1\rangle^T$ keeps x invariant.

Starting with the base value $\langle 21, 10, -36\rangle^T$ one gets the fixpoint ploted in figure 63.

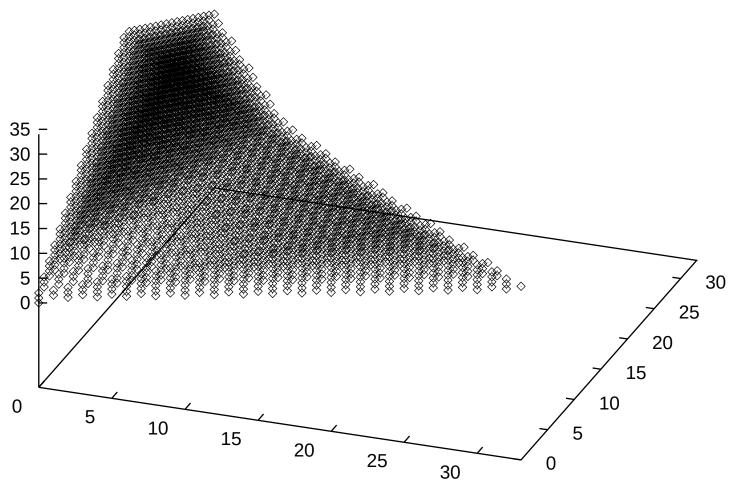


Figure 62

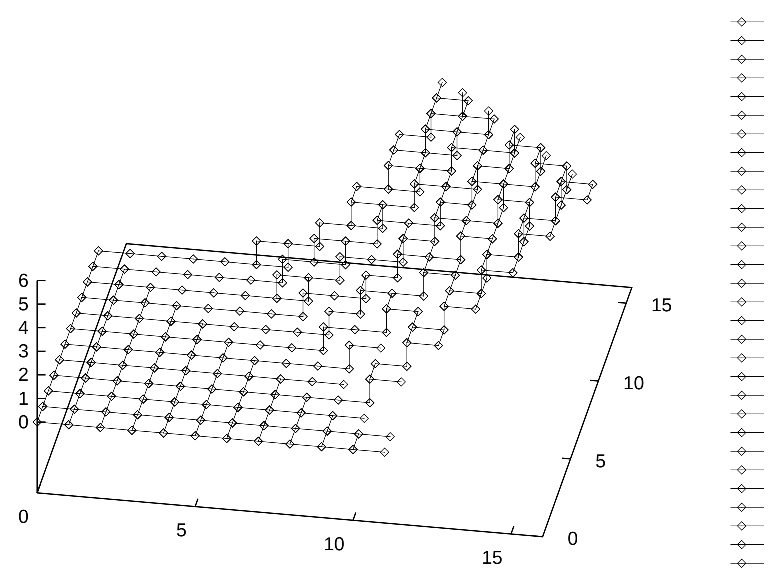


Figure 63

11.3 class 3 decreasing pattern

Consider the matrix

$$\Phi = \begin{pmatrix} -1 & -3 & 1 \\ -5 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$$

The patterns in the vt-plane are $\langle 0,1,2\rangle^T$ (which keeps y invariant) and $\langle 0,2,1\rangle^T$ (which keeps z invariant). Both patterns decrease x.

Starting with the base value $\langle -3,-4,1\rangle^T$ one gets the fixpoint ploted in figure 64.

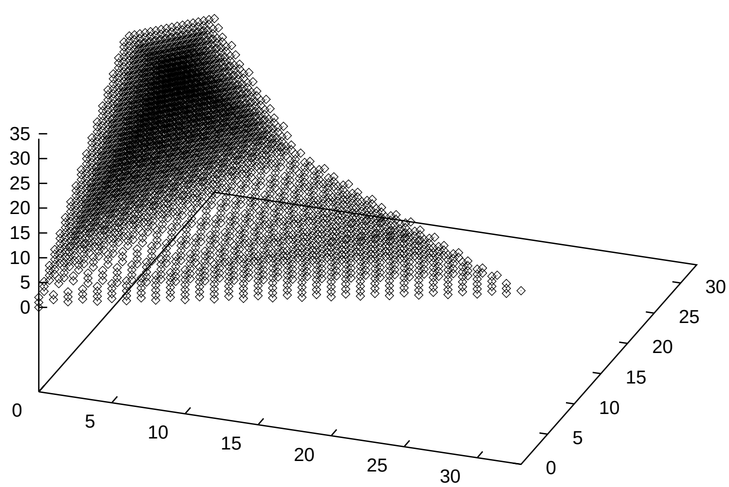


Figure 64

11.4 class 4 increasing pattern

Consider the matrix

$$\Phi = \begin{pmatrix} 1 & -3 & 1 \\ -3 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$$

The patterns in the vt-plane are $\langle 0,1,2\rangle^T$ (which keeps y invariant) and $\langle 0,2,1\rangle^T$ (which keeps z invariant). The pattern $\langle 0,1,2\rangle^T$ increase x.

Starting with the base value $\langle -3,-4,1\rangle^T$ one gets the fixpoint ploted in figure 65.

11.5 class 4 decreasing pattern

Consider the matrix

$$\Phi = \begin{pmatrix} 1 & -3 & 1 \\ -5 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$$

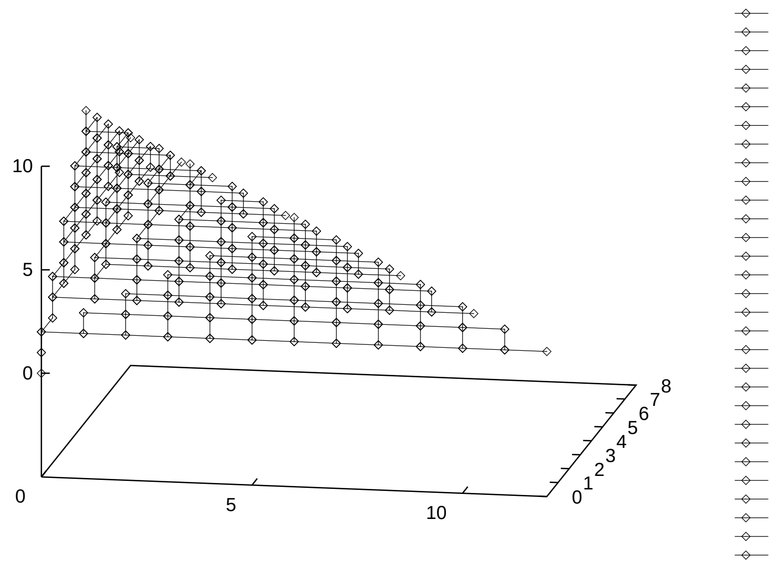


Figure 65

The patterns in the vt-plane are $\langle 0,1,2\rangle^T$ (which keeps y invariant) and $\langle 0,2,1\rangle^T$ (which keeps z invariant). Both patterns decrease x.

Starting with the base value $\langle -3,-4,1\rangle^T$ one gets the fixpoint ploted in figure 66.

12 Class 5

Let us consider class 5, which has not yet been treated, and in particular, to matrices of the form

$$\begin{pmatrix} - & + & - \\ - & - & + \\ + & - & - \end{pmatrix}$$

(see lemma 17 in section 5). We are going to see informally that, contrary to the cases studied so far, the least fixpoint cannot be expressed under a linear arithmetic form. The underlying reason for such a difference, is that for such matrices, the possitive subspace, $x \geq 0 \wedge y \geq 0 \wedge z \geq 0$ (that is, a subspace where a co-pattern of constant size is applicable), is no longer accessible from the origin through a constant number of planes.

This subspace is now only accessible from the origin through a "generating spiral" that rolls around the negative subspace, $x < 0 \ \wedge \ y < 0 \ \wedge \ z < 0$ (from this generating spiral, start also some "secondary spirals" that end into the subspace $x < 0 \ \wedge \ y < 0 \ \wedge \ z < 0$ before having a chance of reaching the positive subspace). This is illustrated in figures 67 and 68 where the matrix is

$$\Phi = \begin{pmatrix} -1 & 3 & -1 \\ -1 & -1 & 2 \\ 2 & -1 & -1 \end{pmatrix}$$

and the base values are respectively $\langle 128,-293,-1\rangle^T$ and $\langle 26,-62,-1\rangle^T$. The xyz values of

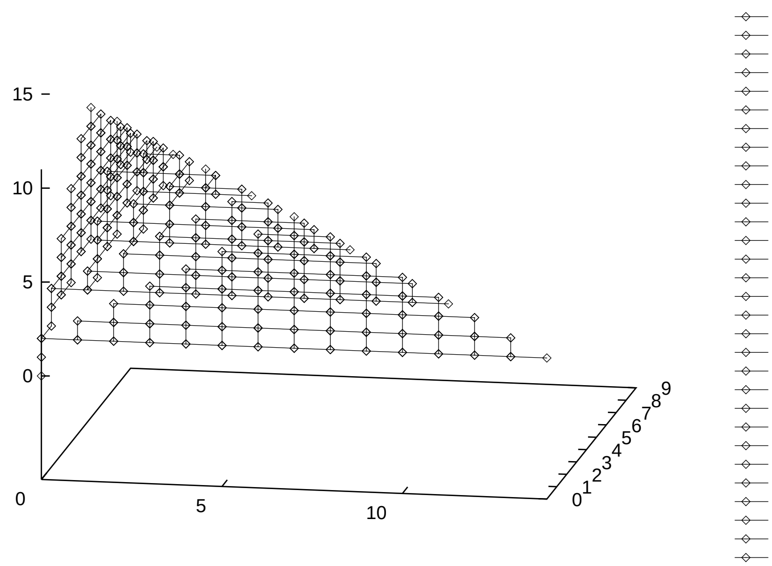


Figure 66

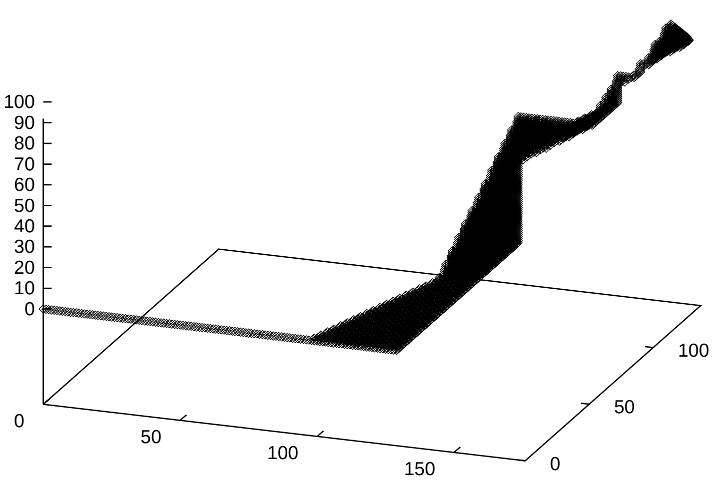


Figure 67

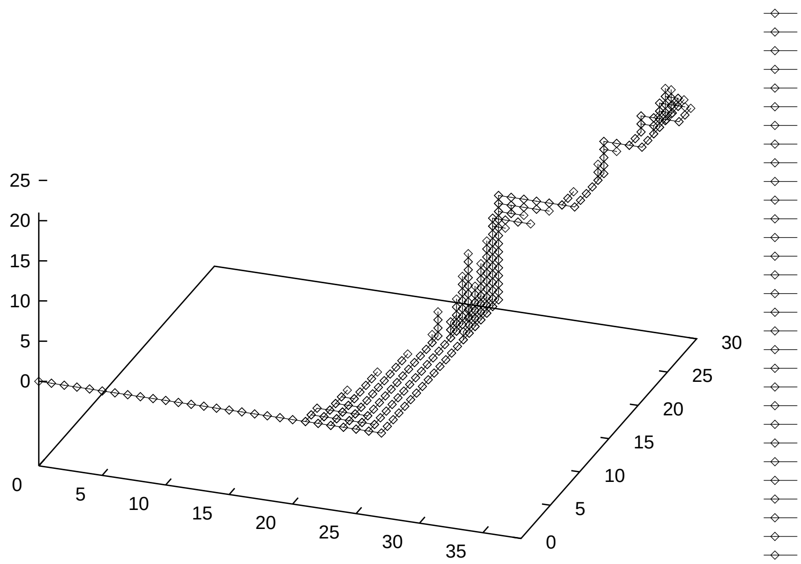


Figure 68

consecutive vertices (corners) of the generating spiral are linked together through a recurrence relation. We explain the computation of this recursive relationship for the matrix chosen above as an example, but the computation is general.

Let us suppose that we start from a point $\overline{p}_0$ such that

$$\langle x_{\overline{p}_0}, y_{\overline{p}_0}, z_{\overline{p}_0} \rangle^T = \langle x_0, -y_0, -1 \rangle^T$$

where $x_0, y_0 \geq 0$ (and satisfy some linear relationship that will be explained later on).

Since an h-move decreases x by 1, it is possible to apply $x_0 + 1$ horizontal steps thus reaching a point $\overline{p}'_0$ (that, is $\overline{p}_0 \xrightarrow{h^{x_0+1}} \overline{p}'_0$) such that

$$\langle x_{\overline{p}'_0}, y_{\overline{p}'_0}, z_{\overline{p}'_0} \rangle^T = \langle -1, 3x_0 + 3 - y_0, -x_0 - 2 \rangle^T$$

Provided that $3x_0 + 3 \geq y_0$, one can apply $3x_0 + 4 - y_0$ vertical steps to reach a point $\overline{p}''_0$ (that is, $\overline{p}'_0 \xrightarrow{v^{3x_0+4-y_0}} \overline{p}''_0$) such that

$$\langle x_{\overline{p}''_0}, y_{\overline{p}''_0}, z_{\overline{p}''_0} \rangle^T = \langle -3x_0 - 5 + y_0, -1, -x_0 + 6 - 2y_0 \rangle^T$$

Finally, one can apply $5x_0 + 7 - 2y_0$ transversal steps, provided that $5x_0 + 7 \geq 2y_0$, to reach a point $\overline{p}_1$ (that, is $\overline{p}''_0 \xrightarrow{h^{5x_0+7-2y_0}} \overline{p}_1$) such that

$$\langle x_{\overline{p}_1}, y_{\overline{p}_1}, z_{\overline{p}_1} \rangle^T = \langle 7x_0 + 9 - 3y_0, -5x_0 - 18 + 2y_0, -1 \rangle^T = \langle x_1, -y_1, -1 \rangle^T$$

Therefore, through a $h^* v^* t^*$ ("helicoidal") path, we have linked $\overline{p}_0$ to $\overline{p}_1$, and x_0, y_0, x_1, y_1 are related to each other through the linear relationship

$$\begin{pmatrix} x_1 \\ -y_1 \end{pmatrix} = \begin{pmatrix} 7 & 3 \\ -5 & -2 \end{pmatrix} \begin{pmatrix} x_0 \\ -y_0 \end{pmatrix} + \begin{pmatrix} 9 \\ -8 \end{pmatrix}$$

Therefore, a sequence of vertices $\overline{p}_0, \overline{p}_1, \overline{p}_2, \ldots, \overline{p}_n$ with corresponding xyz-values $\langle x_i, -y_i, -1\rangle^T$ such that $\langle x_{i+1}, y_{i+1}\rangle$ and $\langle x_i, y_i\rangle$ are related by the recurrence equation above, are connected as $\overline{p}_i \xrightarrow{h^*v^*t^*} \overline{p}_{i+1}$ provided that

$$
\begin{array}{rrcrlrrcr}
x_0 & & \geq & 0 & & & & & \\
3x_0 & +3 & \geq & y_0 & & & & & \\
5x_0 & +7 & \geq & 2y_0 & & & & & \\
\\
x_1 & & \geq & 0 & \text{that is:} & 7x_0 & +9 & \geq & 3y_0 \\
3x_1 & +3 & \geq & y_1 & \text{that is:} & 16x_0 & +22 & \geq & 7y_0 \\
5x_1 & +7 & \geq & 2y_1 & \text{that is:} & 25x_0 & +52 & \geq & 18y_0 \\
\\
x_2 & & \geq & 0 & \text{that is:} & 34x_0 & +82 & \geq & 29y_0 \\
3x_2 & +3 & \geq & y_2 & \text{that is:} & 77x_0 & +110 & \geq & 34y_0 \\
5x_2 & +7 & \geq & 2y_2 & \text{that is:} & 120x_0 & +123 & \geq & 53y_0 \\
\\
\vdots & & & & & & & &
\end{array}
$$

It is always possible to choose the values of x_0 and y_0 so that the spiral makes as many rolls as one desires (i.e. so that the sequence of vertices $\overline{p}_0, \overline{p}_1, \overline{p}_2, \ldots, \overline{p}_n$ is as long as desired). In the above example, starting with the base value $\langle 128, -293, -1\rangle^T$, one yields the sequence $\langle 128, -293, -1\rangle^T$, $\langle 26, -62, -1\rangle^T$, $\langle 5, -14, -1\rangle^T$, $\langle 2, -5, -1\rangle^T$ before reaching the positive subspace.

The points $\overline{p}_i$ are not expressible as a linear arithmetic formula (the x_i- and y_i-values are roughly divided by a factor of 5 when going from $\overline{p}_i$ to $\overline{p}_{i+1}$). This implies that the fixpoint itself cannot be expressed as a linear arithmetic formula because, otherwise, the generating spiral (and therefore the points $\overline{p}_i$) would be expressible as a linear arithmetic formula by filtering out points from the lfp, as explained hereafter.

If the lfp was a linear arithmetic formula G, one could indeed describe describe the generating spiral S by excluding the secondary spirals from G. It is more convenient, actually, to describe the complement of S by characterizing the points of G, which are accessible only via secondary spirals

$$
\overline{p} \in G - S \;\Leftrightarrow\; \overline{p} \in G \;\wedge\; \exists \overline{q} < \overline{p} : \left(\begin{array}{c} \overline{q} \in G \\ \wedge \\ \overline{q} + 1_h \in G \\ \wedge \\ \overline{q} + 1_v \in G \\ \wedge \\ \overline{q} + 1_v < \overline{p} \end{array}\right) \vee \left(\begin{array}{c} \overline{q} \in G \\ \wedge \\ \overline{q} + 1_v \in G \\ \wedge \\ \overline{q} + 1_t \in G \\ \wedge \\ \overline{q} + 1_t < \overline{p} \end{array}\right)
$$

One can then express the points $\overline{p}_i$ from the above formula, by taking the complement S of the above expression, then by intersecting S with the set of points having -1 as z-value. If G was a linear arithmetic formula, the points $\overline{p}_i$ would also be expressible under a linear arithmetic form, which is impossible.

Finally, let us note that, similarly, one can link a vertex $\langle -1, y, -z\rangle^*$ to a vertex $\langle -1, y', -z'\rangle^T$ through a $v^*t^*h^*$ path, with

$$
\left(\begin{array}{c} y' \\ -z' \end{array}\right) = \left(\begin{array}{cc} 7 & 5 \\ -3 & -2 \end{array}\right) \left(\begin{array}{c} y \\ -z \end{array}\right) + \left(\begin{array}{c} 11 \\ -6 \end{array}\right)
$$

Likewise, one can link a vertex $\langle -x, -1, z\rangle^T$ to a vertex $\langle -x', -1, z'\rangle^T$ through a $t^*h^*v^*$ path, with

$$\begin{pmatrix} -x' \\ z' \end{pmatrix} = \begin{pmatrix} -3 & -5 \\ 5 & 8 \end{pmatrix} \begin{pmatrix} -x \\ z \end{pmatrix} + \begin{pmatrix} -9 \\ 12 \end{pmatrix}$$

Another matrix of class 5 is analysed in appendix A.

13 Recapitulation

In this report we have studied a special form of Datalog programs made of 3 recursive rules with arithmetical cosntraints. We have decomposed such programs in three groups (hierarchic, periodic and spiralling) according to the signs of the coefficients of their "incrementation matrices".

Most of the report has been devoted to the study of the periodic group (classes 2, 3 and 4), whose incrementation matrices contain "central" submatrices of the form

$$\begin{pmatrix} - & + \\ + & - \end{pmatrix}$$

We have shown that any path of the least fixpoint is made of the repetition of a pattern (preceded and followed by prefix- and postfix-paths, which are contained within a constant number of planes).

Most of the non-periodic programs fall into the group of hierarchic programs (class 1), whose incrementation matrices are, roughly speaking, characterized by a row or a column of coefficients of the same sign. For these programs, every path of the least fixpoint is contained in at most 4 planes.

Finally we have stressed the existence of a group of programs (class 5), which correspond to 4 specific cases (over a total of 512) for which the paths are nonlinear, but have a vortical form, spiralling around the negative subspace $x < 0 \ \wedge \ y < 0 \ \wedge \ z < 0$.

References

[1] L. Fribourg and H. Olsén, "Direct, Dual and Contrapositive Proofs by Induction", Technical Report LiTH-IDA-R-94-45, Linköping University, Sweden, December 1994.

[2] L. Fribourg and M. Veloso Peixoto, "Bottom-up Evaluation of Programs with Arithmetic Constraints", Technical Report LIENS-92-13, Ecole Normale Supérieure, June 1992. (Short version in *Proc. CADE, LNCS 814*, Springer-Verlag, June 1994.)

[3] L. Fribourg and M. Veloso Peixoto, "Bottom-up Evaluation of Datalog Programs with Arithmetical Constraints: The Case of 3 rules", Technical Report, LIENS 94-9, Ecole Normale Supérieure, Paris, July 1994.

[4] M. Veloso Peixoto, "Automates à Contraintes Arithmétiques et Procédures d'Evaluation Ascendante de Programmes Logiques", PhD Thesis, Université Paris 7, December 1994.

A Another example of a program of class 5

Let us finally look at an example that illustrates the suprisingly complex behaviour of the simple class of programs we study.

Consider the matrix

$$\Phi = \begin{pmatrix} -1 & 3 & -2 \\ -2 & -1 & 3 \\ 3 & -2 & -1 \end{pmatrix}$$

whose determinant is zero. Note that every rule lets the expression $(x + y + z)$ invariant. Starting with the base value $\langle 1, -5, -1 \rangle^T$ we get the fixpoint shown in figure 69. As can be

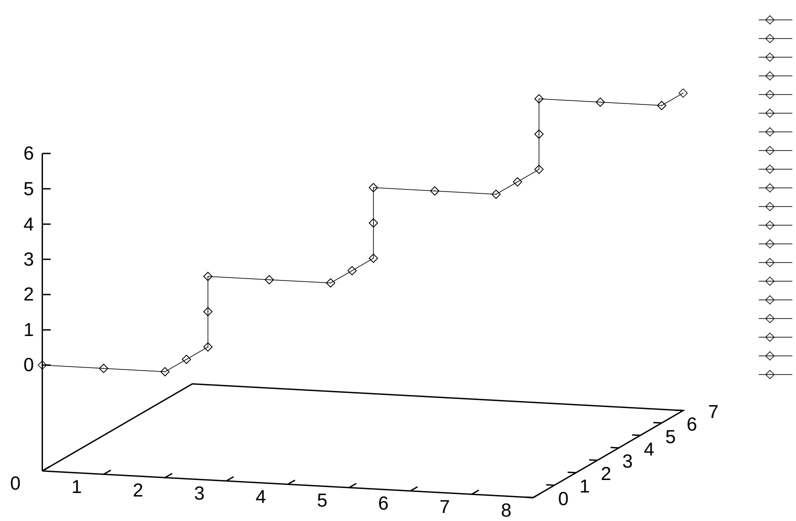


Figure 69

seen, the only admissible path is $h^2v^2t^2h^2v^2t^2 \ldots$. The pattern $\langle 1, 1, 1 \rangle^T$ preserves x, y and z, (that is, $\overline{\xi} = \langle 1, 1, 1 \rangle^T$ solves the equation $\Phi^T \overline{\xi} = \overline{0}$), but no path w such that $\overline{w} = \langle 1, 1, 1 \rangle^T$ is admissible at the origin.

Now choose $\langle 2, -7, -1 \rangle^T$ as base value. The fixpoint is shown in figure 70. In this case the only admissible path is $h^3v^3t^3h^3v^3t^3 \ldots$. We emphasize that only the base value has changed and not the matrix.

The base value $\langle 5, -13, -1 \rangle^T$ yields the fixpoint is shown in figure 71. Now the spiral is of the form $h^6v^6t^6h^6v^6t^6 \ldots$. Also some secondary spirals branch out from this spiral, but they all end up in the negative area $(x < 0 \ \wedge \ y < 0 \ \wedge \ z < 0)$.

By choosing the base value carefully, it is possible to construct a fixpoint whose only cycle is $h^nv^nt^n$ for any n. Clearly this means that a linear arithmetic expression independent of the base value cannot be given. Since the cycle must be expressed by a formula of the form

$$\overline{0} \xrightarrow{\xi^*} \overline{p}' \ \Leftrightarrow \ \exists n : \overline{p}' = n \cdot \overline{\xi}$$

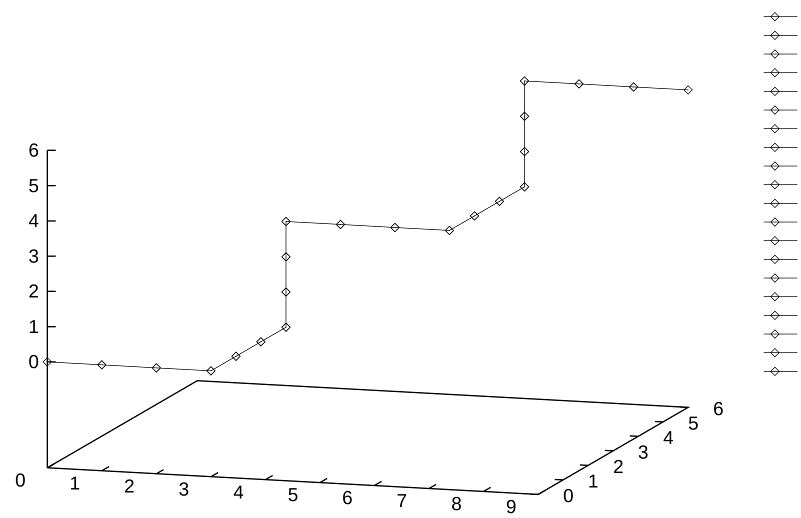


Figure 70

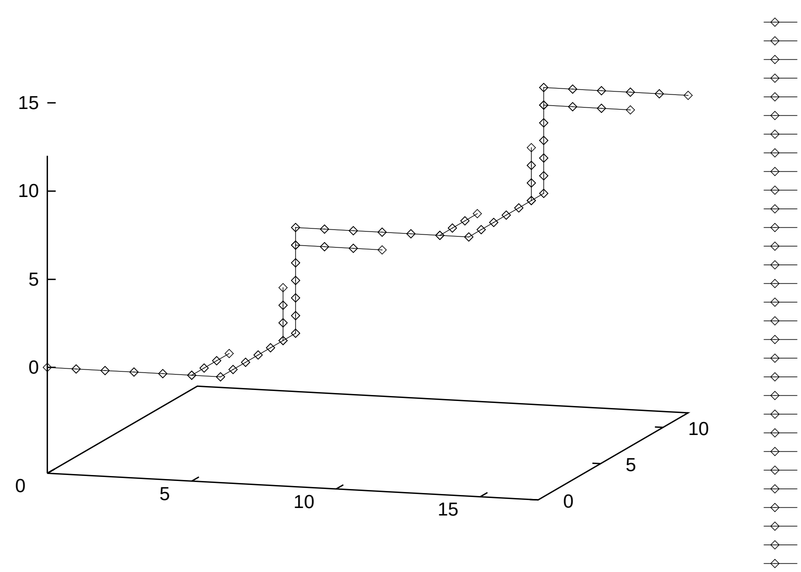


Figure 71

where ξ depends on the base value (and thus is not a constant), such an expression is not a linear arithmetic formula. This suggests that a linear arithmetic expression for the fixpoint could be given if one sacrifise the indpendense of the base value. However, choosing the base value as $\langle 2, -10, 0\rangle^T$ yields the fixpoint shown in figure 72, which has a qualitatively very different appearance. In this case all cycles $h^n v^n t^n$ such that $n \geq 6$ will appear at some point. This

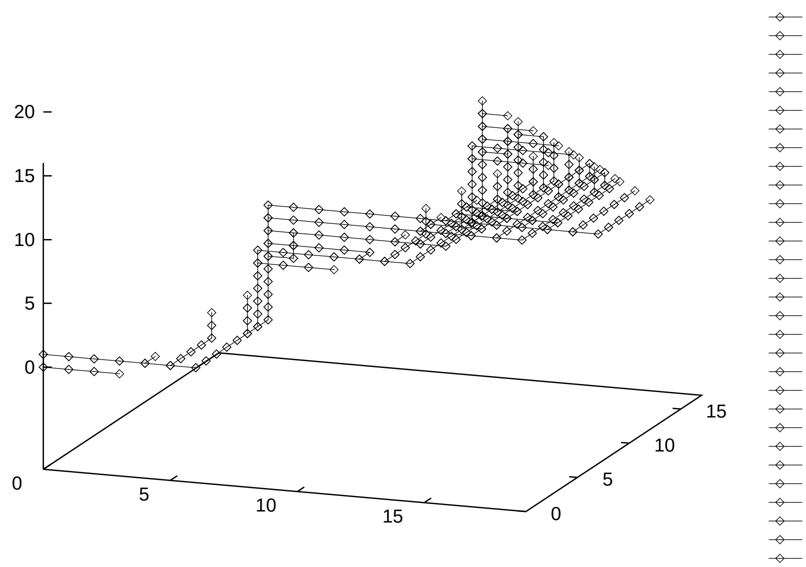


Figure 72

fixpoint is a "vortex" with a "hole" ($x < 0 \ \wedge \ y < 0 \ \wedge \ z < 0$) in the core, and the admissible paths spirals around it in ever growing cycles. The fixpoint is not a cone since the angles of the planes enclosing it grows according to a geometric progression, as discussed in section 12, that converges to some limit.

This intuitively explains why linear arithmetic expressions cannot be given in general for these programs, and it also gives a hint to the difficulties of dealing with programs with associated matrices that has zero determinants (or subterminants). The pigeon-hole approach does not work for these programs since it is not possible to derive upper bounds for the construction of the graph.

B Fixpoint Graphics

The pictures illustrating fixpoints has been generated using *gnuplot*. The program below computes (an initial segment of) the fixpoint and generates a data file that can be read by gnuplot. To generate a fixpoint plot for the program with matrix

$$\Phi = \begin{pmatrix} -1 & 2 & -3 \\ -1 & -5 & 7 \\ 13 & 4 & -4 \end{pmatrix}$$

say, with base value, $\langle -13, -5, 0\rangle^T$, the predicate

```
coefficients([-13, -5,  0,

               -1,  2, -3,
               -1, -5,  7,
               13,  4, -4]).
```

is added to the program. The goal

```
botup(15).
```

will compute the fixpoint bottom-up to a level of 15 rule applications, and write the result on the file

```
'pointpictures/pointset_3d.dat'
```

This is simply the set of points corresponding to the number of horizontal, vertical and transversal moves that are reached. The following sequence of commands to gnuplot will plot the result:

```
set parametric
set nohidden3d
set data style points
splot "pointpictures/pointset_3d.dat" using 1:2:3 notitle
```

An example of a plot of this type is seen in figure 62. The goal

```
pairs_botup(15).
```

works as

```
botup(15).
```

but now the fixpoint is represented as pairs of points and thus keeping the information about which rules are applicable at which point. The result is a directed acyclic graph (actually a finite piece of $\xrightarrow{(h+v+t)^*}$). The following sequence of commands to gnuplot will plot the result:

```
set parametric
set nohidden3d
set data style linespoints
splot "pointpictures/pointset_3d.dat" using 1:2:3 notitle
```

An example of a plot of this type is seen in figure 61.

B.1 Program Code

```
member(E,[E|_]) :- !.
member(E,[_|L]) :-
         member(E,L).

append([],L2,L2).
append([A|L1],L2,[A|L3]) :-
         append(L1,L2,L3).

%======= [Dh,Dv,Dt,Dx,Dy,Dz,Phi_x,Phi_v,Phi_t] ===============

step([H,V,T,X,Y,Z],
     [Dh,Dv,Dt,Dx,Dy,Dz,Phi_x,Phi_y,Phi_z],
     [H_new,V_new,T_new,X_new,Y_new,Z_new]) :-

         X >= Phi_x,
         Y >= Phi_y,
         Z >= Phi_z,

         X_new is X + Dx,
         Y_new is Y + Dy,
         Z_new is Z + Dz,

         H_new is H + Dh,
         V_new is V + Dv,
         T_new is T + Dt.

children([],_,Kids,Kids).
children([Operator|Ops],Point,Previous_kids,New_kids) :-
         (step(Point,Operator,New_point) ->
              (member(New_point,Previous_kids) ->
                    children(Ops,Point,Previous_kids,New_kids)
                    ; children(Ops,Point,[New_point|Previous_kids],New_kids))
              ; children(Ops,Point,Previous_kids,New_kids)).

generation([],_,New_generation,New_generation).
generation([Point|Points],Operators,Cumulative,New_generation) :-
         children(Operators,Point,Cumulative,New_cumulative),
         generation(Points,Operators,New_cumulative,New_generation).

new_generation(Previous_generation,Operators,New_generation) :-
         generation(Previous_generation,Operators,[],New_generation).

closure(N,Limit,_,Cumulative_closure,Delta,Closure) :-
         N>Limit,!,
         append(Delta,Cumulative_closure,Closure).
closure(N,Limit,Operators,Cumulative,Delta,Closure) :-
         new_generation(Delta,Operators,New_delta),
         append(Delta,Cumulative,New_cumulative),
         N_new is N + 1,
         closure(N_new,Limit,Operators,New_cumulative,New_delta,Closure).
```

```
closure(Limit,Operators,Base,Closure) :-
        closure(0,Limit,Operators,[],Base,Closure).

closure(Limit,Operators,Base,Closure) :-
        closure(0,Limit,Operators,[],Base,Closure).

%=================== Output file ==================================

init_gnu_dat_file(Stream) :-
          open('pointpictures/pointset_3d.dat',write,Stream).

output_points([],_).
output_points([[H,V,T,_,_,_]|Fixpoint],Stream) :-
          output_points(Fixpoint,Stream),
          format(Stream,"~0d ~0d ~0d~n",[H,V,T]).

end_gnu_dat_file(Stream) :-
          close(Stream).

gnuplot_path(Fixpoint) :-
          init_gnu_dat_file(Stream),
          output_points(Fixpoint,Stream),
          end_gnu_dat_file(Stream).

%============= single points botom-up ================================

botup(Limit) :-
       coefficients([A1,A2,A3,
                     K11,K12,K13,
                     K21,K22,K23,
                     K31,K32,K33]),
       closure(Limit,[[1,0,0,K11,K12,K13,0,-1000000,-1000000],
                      [0,1,0,K21,K22,K23,-1000000,0,-1000000],
                      [0,0,1,K31,K32,K33,-1000000,-1000000,0]],
                     [[0,0,0,A1,A2,A3]],Clo),
       gnuplot_path(Clo).


%==================== pairs ====================================

pairs_children([],_,Kids,Kids,Pairs,Pairs).
pairs_children([Operator|Ops],Point,Previous_kids,New_kids,
               Previous_pairs,New_pairs) :-
        (step(Point,Operator,New_point) ->
             (member(New_point,Previous_kids) ->
                   pairs_children(Ops,Point,Previous_kids,New_kids,
                                  [[Point,New_point]|Previous_pairs],
                                  New_pairs)
                   ; pairs_children(Ops,Point,[New_point|Previous_kids],
                                    New_kids,
                                    [[Point,New_point]|Previous_pairs],
                                    New_pairs))
             ; pairs_children(Ops,Point,Previous_kids,New_kids,
                              Previous_pairs,New_pairs)).
```

```
pairs_generation([],_,New_generation,New_generation,Pairs,Pairs).
pairs_generation([Point|Points],Operators,Cumulative,New_generation,
                 Cum_pairs,Pairs) :-
         pairs_children(Operators,Point,Cumulative,New_cumulative,
                        Cum_pairs,Cum_pairs2),
         pairs_generation(Points,Operators,New_cumulative,New_generation,
                          Cum_pairs2,Pairs).

new_pairs_generation(Previous_generation,Previous_pairs,Operators,
                     New_generation,New_pairs) :-
         pairs_generation(Previous_generation,Operators,[],New_generation,
                          Previous_pairs,New_pairs).

pairs_closure(N,Limit,_,_,Pairs,Pairs) :-
         N>Limit,!.
pairs_closure(N,Limit,Operators,Delta,Old_pairs,Pairs) :-
         new_pairs_generation(Delta,Old_pairs,Operators,
                              New_delta,Old_pairs2),
         N_new is N + 1,
         pairs_closure(N_new,Limit,Operators,New_delta,Old_pairs2,Pairs).

pairs_closure(Limit,Operators,Base,Pairs) :-
         pairs_closure(0,Limit,Operators,Base,[],Pairs).

%=========================================================================

output_pairs([],_).
output_pairs([[[H1,V1,T1|_],[H2,V2,T2|_]]|Fixpoint],Stream) :-
           output_pairs(Fixpoint,Stream),
           format(Stream,"~0d ~0d ~0d~n",[H1,V1,T1]),
           format(Stream,"~0d ~0d ~0d~n~n~n",[H2,V2,T2]).


gnuplot_pairs(Fixpairs) :-
           init_gnu_dat_file(Stream),
           output_pairs(Fixpairs,Stream),
           end_gnu_dat_file(Stream).

%============ pairs of points botom-up ===============

pairs_botup(Limit) :-
       coefficients([A1,A2,A3,
                     K11,K12,K13,
                     K21,K22,K23,
                     K31,K32,K33]),
       pairs_closure(Limit,[[1,0,0,K11,K12,K13,0,-1000000,-1000000],
                            [0,1,0,K21,K22,K23,-1000000,0,-1000000],
                            [0,0,1,K31,K32,K33,-1000000,-1000000,0]],
                           [[0,0,0,A1,A2,A3]],Pairs),
       gnuplot_pairs(Pairs).

%========= coefficients ==============================================
%
```

```
%       coefficients([A1,A2,A3,         base values
%
%                    K11,K12,K13,       Matrix Phi
%                    K21,K22,K23,
%                    K31,K32,K33]).
%
%======================================================================
```