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Abstract

A degree of parallelism is an equivalence class of Scott-continuous functions which are relatively definable each other with respect to the language PCF (a paradigmatic sequential language). We introduce an infinite ("bi-dimensional") hierarchy of degrees. This hierarchy is inspired by a representation of first order continuous functions by means of a class of hypergraphs. We assume some familiarity with the language PCF and with its continuous model.

Keywords: sequentiality, stability, strong stability, logical relations, sequentiality relations.

1 Introduction

A natural notion of relative definability in the continuous type hierarchy is given by the following definition:

Definition 1 *Given two continuous functions f and g, we say that f is* less parallel than g *($f \leq_{par} g$) if there exists a PCF-term M such that $[\![M]\!]g = f$.*

A *degree of parallelism* is a class of the equivalence relation associated to the preorder $\leq_{par}$.

In this paper we deal with degrees of parallelism of first order boolean functions, i.e. of functions which take tuples of booleans as arguments and give booleans as results. PCF-definability for first order functions is fully characterized by the notion of sequentiality (in any of its formulations), and Sieber's *sequentiality relations* ([6]) provide a characterization of first order degrees of parallelism. Moreover this characterization is effective: given f and g one can decide if $f \leq_{par} g$, and recently A. Stoughton ([7]) has implemented an algorithm which solves this decision problem.

Nevertheless, as far as I know, there is little knowledge of the structure of the partial order $\leq_{par}$ on first order boolean functions.

A well known fact is that any continuous function(al) is less parallel than the "parallel or" function (the non-strict binary disjunction) ([4]), and we also know a that any first order stable function is less parallel than the Berry-Plotkin function ([3], p. 334), but there is a lack of general results about the poset of degrees, whose structure turns out to be quite complicated, already at first order. Sazonov's paper [5] may be considered as a first step toward a systematic study of the poset of degrees of parallelism.

In this paper we give a geometric account of first order degrees of parallelism, by representing first order functions as *hypergraphs* which higlight the structure of *linearly coherent*[1] subsets in the trace of the function. Then we introduce a hierarchy of functions $\{f_{(n,m)}\}_{n\leq m\in\omega}$ which has the property that $f_{(n,m)} \leq_{par} f_{(n',m')}$ if and only if there exists a morphism from the hypergraph associated to $f_{(n,m)}$ to the hypergraph asociated to $f_{(n',m')}$.

Throughout the paper PCF terms will be written in uncurryed form (as n-ary functions), and some "macros" like a syntactic $\perp$ and a sequential conjunction $\wedge$ wil be used.

2 Preliminaries

We denote by B the flat domain of boolean values $\{\perp, true, false\}$. Tuples of boolean values are ordered componentwise. Given a continuous function $f : B^n \to B$, the *trace* of f is defined by

$$tr(f) = \{(v, b) \mid v \in B^n,\ b \in B,\ b \neq \perp,\ f(v) = b \text{ and } \forall v' < v\ f(v') = \perp\}$$

A continuous function $f : B^n \to B$ is *stable* if for all $v_1, v_2 \in \pi_1(tr(f))$, $v_1 \not\uparrow v_2$. A subset $A = \{v_1, \ldots, v_k\}$ of B^n is *linearly coherent* (or simply *coherent*) if for any linear function $\alpha : B^n \to O^2$, $\alpha(\bigwedge A) = \bigwedge \alpha(A)$, or equivalently if

$$\forall j\ 1 \leq j \leq n\ (\forall l\ 1 \leq l \leq k\ v_l^j \neq \perp\ \Rightarrow\ \forall l_1, l_2\ 1 \leq l_1 \leq l_2 \leq k\ v_{l_1}^j = v_{l_2}^j)$$

The set of coherent subsets of B^n is noted $\mathcal{C}(B^n)$.

fact 1: If $A \in \mathcal{C}(B^n)$ and B is an Egli-Milner lower bound of A^3, then $B \in \mathcal{C}(B^n)$. ∎

Definition 2 *A continuous function $f : B^n \to B^m$ is* linearly strongly stable *(or simply* strongly stable*) if for any $A \in \mathcal{C}(B^n)$*

[1] in the sense of [2]

[2] O denotes the Sierpinsky domain $\{\perp, \top\}$

[3] that is

$$\forall x \in A \exists y \in B\ y \leq x \text{ and } \forall y \in B \exists x \in A\ y \leq x$$

- $f(A) \in \mathcal{C}(B^m)$.
- $f(\bigwedge A) = \bigwedge(f(A))$.

The following proposition states that strong stability captures the notion of sequential definability, at least at first order.

Proposition 1 *Any strongly stable, first order function $f : B^n \to B$ is PCF-definable.*

2.1 Sequential Logical Relations

Definition 3 (Sieber) *For each $n \geq 0$ and each pair of sets $A \subseteq B \subseteq \{1, \ldots, n\}$ let $S_n^{A,B} \subseteq B^n$ be defined by*

$$S_n^{A,B}(b_1, \ldots, b_n) \Leftrightarrow (\exists i \in A \ b_i = \perp) \ \vee \ (\forall i, j \in B \ b_i = b_j)$$

An n-ary logical relation R is called a sequentiality relation *if it is an intersection of relations of the form $S_n^{A,B}$.*

A function $f : B^n \to B$ is invariant with respect to the m-ary logical relation R if for any

$$(x_1^1, \ldots, x_1^m) \in R, (x_2^1, \ldots, x_2^m) \in R, \ldots, (x_n^1, \ldots, x_n^m) \in R$$

one has that

$$(f(x_1^1, x_2^1, \ldots, x_n^1), f(x_1^2, x_2^2, \ldots, x_n^2), \ldots, f(x_1^m, x_2^m, \ldots, x_n^m)) \in R$$

Proposition 2 *For any $f : B^n \to B$ and $g : B^m \to B$ continuous functions, $f \leq_{par} g$ if and only if for any sequentiality relation R, if g is invariant with respect to R then f is invariant too.*

fact 2: A set $A = \{(x_1^1, \ldots, x_1^n), \ldots, (x_k^1, \ldots, x_k^n)\} \subseteq B^n$ is linearly coherent if and only if

$$\forall 1 \leq i \leq n \ (x_1^i, x_2^i, \ldots, x_k^i) \in S_k^{\{1,\ldots,k\},\{1,\ldots,k\}}$$

■

3 Hypergraphs for boolean functions

We consider a category whose objects are (colored) hypergraphs and whose morphisms are arcs-preserving and coloring-preserving maps:

Definition 4 *A* colored hypergraph $h = (V_h, A_h, C_h)$ *is given by a set* V_h *of* vertexes*, a set* $A_h \subseteq \{A \subseteq V_h | \# A \geq 2\}$ *of* (hyper)arcs *and a* coloring function $C_h : V_h \to \{black, white\}$*. A morphism from an hypergraph* h *to an hypergraph* h' *is a function* $m : V_h \to V_{h'}$ *such that:*

- *for all* $A \subseteq V_h$*, if* $A \in A_h$ *then* $m(A) \in A_{h'}$*.*
- *for all* $x, x' \in V_h$*,* $C_h(x) = C_h(x')$ *if and only if* $C_{h'}(m(x)) = C_{h'}(m(x'))$*.*

Definition 5 *Let* $f : B^n \to B$ *be the* n*-ary function defined by* $tr(f) = \{(v_1, b_1), \ldots, (v_k, b_k)\}$*. The hypergraph* $H(f)$ *is defined by*

- $V_{H(f)} = \{1, 2, \ldots, k\}$.
- $A_{H(f)} = \{\{i_1, i_2, \ldots, i_l\} \subseteq V_{H(f)} \mid l \geq 2$ *and* $\{v_{i_1}, v_{i_2}, \ldots, v_{i_l}\} \in \mathcal{C}(B^n)\}$.
- $C_{H(f)}(i) =$ *if* b_i *then white else black.*

example 1: Consider the functions $G : B^3 \to B$ and $Por : Bool^2 \to B$ defined by

$$tr(G) = \{((\perp, true, false), true), ((false, \perp, true), true), ((true, false, \perp), false)\}$$

and

$$tr(Por) = \{((\perp, true), true), ((true, \perp), true), ((false, false), false)\}$$

We have:

$$H(G) = (\{1,2,3\}, \{\{1,2,3\}\}, C_{H(G)}(1) = C_{H(G)}(2) = white, C_{H(G)}(3) = black)$$

$$H(Por) = (\{1,2,3\}, \{\{1,2\}, \{1,2,3\}\}, C_{H(Por)}(1) = C_{H(Por)}(2) = white, C_{H(Por)}(3) = black)$$

The map $m : H(G) \to H(Por)$ defined by $m(i) = i$, for $i = 1, 2, 3$, is a morphism. A term M such that $[\![M]\!]Por = G$ is

$$M = \lambda f \ \lambda x_1 x_2 x_3 \ f(t_1, t_2)$$

where

$$t_1 = if \ x_1 \ then \ (if \ x_2 \ then \ \perp \ else \ false) \ else \ if \ x_3 \ then \ true \ else \ \perp$$

$$t_2 = if \ x_2 \ then \ (if \ x_3 \ then \ \perp \ else \ true) \ else \ if \ x_1 \ then \ false \ else \ \perp$$

■

example 2: Let $3-Por : B^3 \to B$ be defined by

$$tr(3-Por) = \{((true, \perp, \perp), true), ((\perp, true, \perp), true), ((\perp, \perp, true), true)\}$$

The associated hypergraph is:

$$H(3{-}Por) = (\{1,2,3\}, \{\{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}, C(1) = C(2) = C(3) = white)$$

It is easy to see that there exists no morphism $m : H(3-Por) \to H(Por)$. Nevertheless $3-Por \leq_{par} Por$, since for instance

$$3-Por = [\![M]\!]Por$$

where

$$M = \lambda f\ \lambda x_1 x_2 x_3\ f(f(x_1, x_2), x_3)$$

■

In the rest of this section we list some simple properties relating hypergraphs and degrees of parallelism:

fact 3: Let $f : B^n \to B$ be a continuous function:

- f is stable if and only if $H(f)$ has no 2-arc.
- f is strongly stable if and only $H(f)$ has no arc.

■

Proposition 3 *Let $f : B^n \to B$ and $g : B^m \to B$ be such that*

$$min\{\#A \mid A \in A_{H(f)}\} < min\{\#A \mid A \in A_{H(g)}\}.$$

Then $f \not\leq_{par} g$.

Proof: Let $k = min\{\#A \mid A \in A_{H(f)}\}$. We show that the logical relation $S_{k+1}^{\{1,2,\ldots,k\},\{1,2,\ldots,k+1\}}$ is such that g is invariant with respect to it, and f is not.

- f is not invariant:

  let

$$v_1 = (x_1^1, x_1^2, \ldots, x_1^n)$$
$$\vdots$$
$$v_k = (x_k^1, x_k^2, \ldots, x_k^n)$$

be a coherent subset of $\pi_1(tr(f))$. For $1 \leq \jmath \leq n$, let w_j be the $k+1$-tuple defined by:

$$w_j = (x_1^j, x_2^j, \ldots, x_k^j, \bigwedge_{1 \leq l \leq k} x_l^j)$$

The coherence of $\{v_1, \ldots, v_k\}$ entails that, for all $1 \leq j \leq n$, $w_j \in S_{k+1}^{\{1,2,\ldots,k\},\{1,2,\ldots,k+1\}}$. Appliyng $(\underbrace{f, \ldots, f}_{k+1})$ to

$$(v_1, \ldots, v_k, (\bigwedge_{1 \leq l \leq k} x_l^1, \ldots, \bigwedge_{1 \leq l \leq k} x_l^n))$$

we get $(b_1, \ldots, b_k, \perp) \notin S_{k+1}^{\{1,2,\ldots,k\},\{1,2,\ldots,k+1\}}$, where $b_i = f(v_i)$, and we have done.

- g is invariant:

  let us suppose, by *reductio ad absurdum*, that there exist $k+1$ m-tuple

$$u_1 = (y_1^1, \ldots, y_1^m)$$

$$\vdots$$

$$u_{k+1} = (y_{k+1}^1, \ldots, y_{k+1}^m)$$

  such that for all $1 \leq j \leq m$,

$$(y_1^j, y_2^j, \ldots, y_{k+1}^j) \in S_{k+1}^{\{1,2,\ldots,k\},\{1,2,\ldots,k+1\}}$$

  and

$$(g(u_1), \ldots, g(u_{k+1})) \notin S_{k+1}^{\{1,2,\ldots,k\},\{1,2,\ldots,k+1\}}$$

  This entails that $U = \{u_1, \ldots, u_k\}$ is coherent, and that any element of U has a lower bound in $\pi_1(tr(g))$. Hence there exists an Egli-Milner lower bound A of U in $\pi_1(tr(g))$, and $1 < \#A \leq k^4$, which is absurd.

∎

[4] $\#A > 1$ since, if all the elements of U are upper bounds of an element $v \in \pi_1(tr(g))$, then $u_{k+1} \geq v$, and hence

$$(g(u_1), \ldots, g(u_{k+1})) \in S_{k+1}^{\{1,2,\ldots,k\},\{1,2,\ldots,k+1\}}$$

4 A hierarchy of degrees

Definition 6 *Given two natural numbers $m \geq n \geq 3$, let $h_{(n,m)}$ be the hypergraph defined by:*

$$h_{(n,m)} = (\{1,2,\ldots,m\}, \{A \subseteq \{1,2,\ldots,m\} \mid \#A \geq n\}, \ \textit{for all} \ \ i \ \ C(i) = white)$$

Given $h_{(n,m)}$ and $h_{(n',m')}$, we are interested in determining the conditions under which there exists a morphism $f : h_{(n,m)} \rightarrow h_{(n',m')}$. Since the $h_{(i,j)}$'s are mono-colored, the only condition to be satified for a function $f : \{1,\ldots,m\} \rightarrow \{1,\ldots,m'\}$ to be a morphism is the preservation of arcs. It is easy to see that f is a morphism if and only if

$$max\{\#f^{-1}(B) \mid B \subseteq \{1,\ldots,m'\} \text{ and } \#B = n'-1\} < n$$

since only in that case any arc of $h_{(n,m)}$ is mapped by f on an arc of $h_{(n',m')}$. Hence there exists a morphism from $h_{(m,n)}$ to $h_{(m',n')}$ if and only if

$$n > min_{f:\{1,\ldots,m\}\rightarrow\{1,\ldots,m'\}} max\{\#f^{-1}(B) \mid B \subseteq \{1,\ldots,m'\} \text{ and } \#B = n'-1\}$$

It is quite easy to see that one of the functions $f : \{1,\ldots,m\} \rightarrow \{1,\ldots,m'\}$ which realize the minimum above is $f_0(i) = ((i-1) \ \text{ MOD } \ m') + 1$, and that

$$max\{\#f_0^{-1}(B) \mid B \subseteq \{1,\ldots,m'\} \text{ and } \#B = n'-1\} =$$
$$= (min\{n'-1, m \ \text{ MOD } \ m'\} * \uparrow (\frac{m}{m'})) + (max\{0, (n'-1) - (m \ \text{ MOD } \ m')\} * \downarrow (\frac{m}{m'})$$

where $\uparrow x$ and $\downarrow x$ denote the integer parts of $x+1$ and x respectively. If we denote this natural number by $C^{m,n',m'}$, we have that there exists a morphism from $h_{(n,m)}$ to $h_{(n',m')}$ if and only if $n > C^{m,n',m'}$.

We define now a set of boolean functions $\{f_{(n,m)}\}$ such that for all n, m (with $3 \leq n \leq m$), $H(f_{(n,m)}) = h_{(n,m)}$, and we show that for all n, m, n', m' $f_{(n,m)} \leq_{par} f_{(n',m')}$ if and only if $n > C^{m,n',m'}$. We start by showing how to construct, for any given $h_{(n,m)}$, a boolean function f such that $H(f) = h_{(n,m)}$. The trace of f has to contain m elements, its second projection has to be the singleton $\{true\}$ and for any subset A of the first projection of the trace, A has to be coherent if and only if $\#A \geq n$. Before describing the general method for constructing such a function f, let us consider an example:

example 3: The function f described by the following trace (that we represent as a matrix), is such that $H(f) = h_{(3,4)}$

true	true	true	$\perp$	$\perp$	$\perp$	true
false	$\perp$	$\perp$	true	true	$\perp$	true
$\perp$	false	$\perp$	false	$\perp$	true	true
$\perp$	$\perp$	false	$\perp$	false	false	true

Actually a subset of the first projection of this trace is coherent if and only if its cardinality is at least 3, since for any binary subset $\{i, j\}$ of rows there exists a column l such that the elements (i, l) and (j, l) are defined and different. ∎

For constructing a function $f_{(n,m)}$ whose associated hypergraph be $h_{(n,m)}$ we have just to generalize the idea above: for any subset of less then n rows (and of at least two rows), it must exist a column which makes that subset uncoherent. The arity of the function is $\sum_{i=2}^{n-1} C_m^i$ [5], and in the jth column, only elements corresponding to rows in the jth subset (with respect to an enumeration whatsoever) will be defined, say by $true$ for the first row in that subset and by $false$ for the other rows.

example 4:

The following matrix represents $\pi_1(tr(f_{(4,4)}))$:

$$
\begin{array}{lcccccccccc}
v_1= & \text{true} & \text{true} & \text{true} & \perp & \perp & \perp & \text{true} & \text{true} & \text{true} & \perp \\
v_2= & \text{false} & \perp & \perp & \text{true} & \text{true} & \perp & \text{false} & \perp & \text{false} & \text{true} \\
v_3= & \perp & \text{false} & \perp & \text{false} & \perp & \text{true} & \text{false} & \text{false} & \perp & \text{false} \\
v_4= & \perp & \perp & \text{false} & \perp & \text{false} & \text{false} & \perp & \text{false} & \text{false} & \text{false}
\end{array}
$$

and the following one represents $\pi_1(tr(f_{(3,3)}))$:

$$
\begin{array}{lccc}
w_1= & \text{true} & \text{true} & \perp \\
w_2= & \text{false} & \perp & \text{true} \\
w_3= & \perp & \text{false} & \text{false}
\end{array}
$$

∎

Proposition 4 *If $n, m, n'm' \in \omega$ are such that $3 \leq n \leq m$, $3 \leq n' \leq m'$ and $n > C^{m,n',m'}$, then $f_{(n,m)} \leq_{par} f_{(n',m')}$.*

Proof: Let $k = \sum_{i=2}^{n-1} C_m^i$ and $k' = \sum_{i=2}^{n'-1} C_{m'}^i$, and let $A = \pi_1(tr(f_{(n,m)})) = \{v_1, \ldots, v_m\}$ and $B = \pi_1(tr(f_{(n',m')})) = \{w_1, \ldots, w_{m'}\}$. By hypothesis there exists a function $f : \{1, \ldots, m\} \to \{1, \ldots, m'\}$ which maps any non-singleton coherent subset of A on a non-singleton coherent subset of B. Consider the function $F : B^k \to B^{k'}$ defined by $tr(F) = \{(v_i, w_{f(i)}) \mid 1 \leq i \leq m\}$. We show that F is strongly stable: given $C \in \mathcal{C}(B^k)$ we have to prove that $F(C) \in \mathcal{C}(B^{k'})$ and that $F(\bigwedge C) = \bigwedge F(C)$:

- If $(\underbrace{\perp, \perp, \ldots, \perp}_{k'}) \in F(C)$ then $F(C)$ is linearly coherent, else there exists an Egli-Milner lower bound C' of C such that $C' \subseteq \pi_1(tr(F))$, and

[5] C_m^i denotes the binomial coefficient of m and i

$F(C) = F(C') = \{w_{f(i)} \mid v_i \in C'\}$. By fact 1 C' is coherent, and hence $\{w_{f(i)} \mid v_i \in C'\}$ is coherent. Hence F preserves linear coherence.

- The only interesting case is when there exists an Egli-Milner lower bound of C in $\pi_1(tr(F))$ (otherwise $\bigwedge F(C) = (\underbrace{\perp, \perp, \ldots, \perp}_{k'})$, and $F(\bigwedge C) = \bigwedge F(C)$ holds trivially). Let C' be the Egli-Milner lower bound of C such that $C' \subseteq \pi_1(tr(F))$, and $F(C) = F(C') = \{w_{f(i)} \mid v_i \in C'\}$. If C' is a singleton, say $C' = \{v_i\}$, then $\bigwedge C \geq v_i$ and $F(\bigwedge C) = \bigwedge F(C)$. If C' is non-singleton then $F(C') = \{w_f(i) \mid v_i \in C'\}$ is coherent and non singleton. Hence $\#F(C') \geq n'$, and it is easy to see that by definition of $f_{(n',m')}$, $\bigwedge F(C') = (\underbrace{\perp, \perp, \ldots, \perp}_{k'}) = F(\bigwedge C)$.

Since F is strongly stable, the function $g_i : B^k \to B$ defined by $g_i = \pi_i \circ F$ is strongly stable, for any $i \leq k'$, since projections are strongly stable functions, and strong stability is preserved by composition. The g_i's are first order functions, hence by proposition 1, for all $i \leq k'$ there exists a PCF term $t_i(x_1, \ldots, x_k)$ which defines g_i. Consider the term

$$M = \lambda y\ \lambda x_1 x_2 \ldots x_k\ y(t_1(x_1 x_2 \ldots x_k), t_2(x_1 x_2 \ldots x_k), \ldots, t_{k'}(x_1 x_2 \ldots x_k))$$

In order to prove that $[\![M]\!] f_{(n',m')} = f_{(n,m)}$ we just remark that, by construction,

$$\forall v \in B^k\ (\exists j \leq m'\ (g_1(v), \ldots, g_{k'}(v)) \geq w_j \Leftrightarrow \exists i \leq m\ v \geq v_i \ \text{ and } \ f(i) = j)$$

∎

example 5: Let us apply the construction above to show that $f_{(4,4)} \leq_{par} f_{(3,3)}$ (remark that $C'^{4,3,3} = 3$) (we refers to example 4)

Any surjective function $f : \{1,2,3,4\} \to \{1,2,3\}$ satisfies the condition of being a morphism from $h_{(4,4)}$ to $h_{(3,3)}$; let us choose for instance

$$f(1) = f(4) = 1 \quad f(2) = 2 \quad f(3) = 3$$

The corresponding F is defined by

$$tr(F) = \{(v_1, w_1), (v_2, w_2), (v_3, w_3), (v_4, w_1)\}$$

and the g_i's are defined by

$$tr(g_1) = \{(v_1, true), (v_2, false), (v_4, true)\}$$

$$tr(g_2) = \{(v_1, true), (v_3, false), (v_4, true)\}$$

$$tr(g_3) = \{(v_2, true), (v_3, false)\}$$

The terms t_i's are essentially sequences of conditionals statements: for instance

$$t_3 = \lambda x_1 \ldots x_{10}\ if\ x_4\ then\ (if\ (\neg x_1 \wedge x_5 \wedge \neg x_7 \wedge \neg x_9 \wedge x_{10})\ then\ true\ else\ \ \bot)$$

$$else\ (if\ (\neg x_2 \wedge x_6 \wedge \neg x_7 \wedge \neg x_8 \wedge \neg x_{10})\ then\ true\ else\ \ \bot)$$

■

The rest of this section is devoted to prove that the condition $n > C^{m,n',m'}$. is indeed necessary for having $f_{(n,m)} \leq_{par} f(n', m')$:

Proposition 5 *If n, m, n', m' are such that $3 \leq n \leq m$, $3 \leq n' \leq m'$ and $n \leq C^{m,n',m'}$, then $f \not\leq_{par} g$.*

Proof: By proposition 2 it is sufficient to define a sequential logical relation R such that $f_{(n',m')}$ is invariant with respect to R and $f_{(n,m)}$ is not.

The first projection of $tr(f_{(n,m)})$ is

$$\pi_1(tr(f_{(n,m)})) = \{(x_1^1, \ldots, x_1^{\sum_{i=2}^{n-1} C_m^i}), \ldots, (x_m^1, \ldots, x_m^{\sum_{i=2}^{n-1} C_m^i})\}$$

Remark that, by definition, any "column" of the first projection of $tr(f_{(n,m)})$, i.e. any tuple

$$\{(x_1^i, x_2^i, \ldots, x_m^i)\}_{1 \leq i \leq \sum_{i=2}^{n-1} C_m^i}$$

contains at least $m - n + 1$ "$\bot$"s.

Hence it is easy to see that $f_{(n,m)}$ is not invariant with respect to the $(m+1)$-ary sequential logical relation

$$R = (\bigcap_{A \subseteq \{1,2,\ldots,m\}, \#A = n} S^{A,A}) \quad \cap \quad (S^{\{1,\ldots,m\},\{1,\ldots,m+1\}})$$

since the tuples

$$\{(x_1^i, x_2^i, \ldots, x_m^i, \bot)\}_{1 \leq i \leq \sum_{i=2}^{n-1} C_m^i}$$

are in R, and the application of $(\underbrace{f_{(n,m)}, \ldots, f_{(n,m)}}_{m+1})$ to those tuples yelds the tuple $(\underbrace{true, true, \ldots, true}_{m}, \bot)$ which is not in R

If we prove that $f_{(n',m')}$ is invariant with respect to R we have done. By *reductio ad absurdum*, let us suppose that $f_{(n',m')}$ is not invariant. Then there exist $m+1$ tuples in R

$$v_1 = (y_1^1, \ldots, y_1^{\sum_{i=2}^{n'-1} C_{m'}^i})$$

$$v_2 = (y_2^1, \ldots, y_2^{\sum_{i=2}^{n'-1} C^i_{m'}})$$

$$\vdots$$

$$v_{m+1} = (y_{m+1}^1, \ldots, y_{m+1}^{\sum_{i=2}^{n'-1} C^i_{m'}})$$

such that

$$(f_{(n',m')}(v_1), \ldots, f_{(n',m')}(v_{m+1})) \notin R$$

It is easy to see that this is the case if and only if

$$f_{(n',m')}(v_1) = f_{(n',m')}(v_2) = \ldots = f_{(n',m')}(v_m) = true \quad \text{and} \quad f_{(n',m')}(v_{m+1}) = \perp$$

Hence for any $1 \leq i \leq m$ there exists an element $w_{f(i)}$ of the first projection of $tr(f_{(n',m')})$ such that $v_i \geq w_{f(i)}$[6], for some function $f : \{1, 2, \ldots, m\} \rightarrow \{1, 2, \ldots, m'\}$. Since $n \leq C^{m,n',m'}$, there exists $B \subseteq \{1, 2, \ldots, m'\}$ such that $\#B = n' - 1$ and $\#f^{-1}(B) \geq n$. This means that there exist n elements $v_{i_1}, \ldots, v_{i_n}$, $1 \leq i_j \leq m$ and there exist $1 \leq k \leq n' - 1$, $w_1, \ldots, w_k \in \pi_1(tr(f_{(n',m')}))$, such that $\{w_1, \ldots, w_k\}$ is an Egli-Milner lower bound of $\{v_{i_1}, \ldots, v_{i_n}\}$.

If $k \geq 2$, we conclude that $\{v_{i_1}, \ldots, v_{i_n}\}$ is not linearly coherent and then that there exists $1 \leq h \leq \sum_{i=2}^{n'-1} C^i_{m'}$ such that $\{y^h_{i_1}, y^h_{i_2}, \ldots, y^h_{i_n}\} = \{true, false\}$. It follows that

$$(y_1^h, y_2^h, \ldots, y_{m+1}^h) \notin S^{\{i_1, \ldots, i_n\}, \{i_1, \ldots, i_n\}}$$

which is absurd.

If $k = 1$, there are two cases:

- there exists $1 \leq h \leq m$ such that $v_h \not\geq w_1$. In this case $v_h \geq w$ for some $w \in \pi_1(tr(f_{(n',m')}))$, $w \neq w_1$, and we conclude as before, since $\{w, w_1\}$ is not linearly coherent.

- for all $1 \leq i \leq m$, $v_i \leq w_1$. In this case for all $1 \leq j \leq \sum_{i=2}^{n'-1} C^i_{m'}$, we have $y_1^j, y_2^j, \ldots, y_m^j \geq w_1^j$ and hence, since

$$(y_1^j, y_2^j, \ldots, y_m^j, y_{m+1}^j) \in S^{\{1, \ldots, m\}, \{1, \ldots, m+1\}}$$

we get $y^j_{m+1} \geq w^j_1$. Hence we conclude that $v_{m+1} \geq w_1$ and that $f_{(n',m')}(v_{m+1}) = true$, which is absurd.

$\blacksquare$

[6] This element is unique since $f_{(n',m')}$ is stable.

Hence $f_{(n,m)} \leq_{par} f_{(n',m')}$ if and only if $n > C^{m,n',m'}$. In order to draw a picture of (a part of) this hierarchy of degrees, let us compute some typical value of $C^{i,j,l}$:

$$C^{n+1,n,n} = 2 + (n-2) = n \ \Rightarrow \forall n \geq 3 \ f_{(n+1,n+1)} \leq_{par} f_{(n,n)}$$

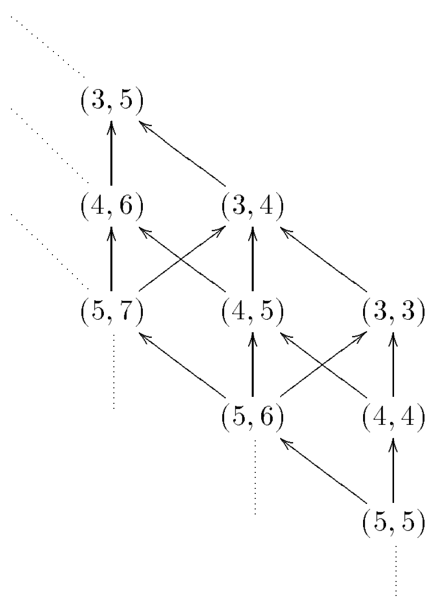
$$C^{n,n-1,n+1} = n - 2 \ \Rightarrow \forall n \geq 4 \ f_{(n-1,n)} \leq_{par} f_{(n-1,n+1)}$$

$$C^{n+1,n-1,n} = 2 + (n-3) = n - 1 \ \Rightarrow \forall n \geq 4 \ f_{(n,n+1)} \leq_{par} f_{(n-1,n)}$$

We can prove that the inequalities above are strict by using the same method: for the first one we have for instance

$$C^{n,n+1,n+1} = n \ \Rightarrow \forall n \ f_{(n,n)} \not\leq_{par} f_{(n+1,n+1)}$$

The following picture shows some degrees in the hierarchy:



5 Conclusion

The hypergraph that we associate to a function f brings some information about the degree of parallelism of f.

Actually, as shown by exemple 2, the existence of a morphism from $H(f)$ to $H(g)$ is not a necessary condition for $f \leq_{par} g$, but some of the result we got (like proposition 3, or the existence of the hierarchy $f_{(n,m)}$), comfort our feeling that the study of the combinatory of hypergraphs can result in a better understanding of the poset of degrees of parallelism.

A complete characterization of first order degrees of parallelism can be considered as preliminary to the study of the decidability problem for $\leq_{par}$ at higher order, which is open.

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