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Event Structures Representing Domains with Coherence

Antonio BUCCIARELLI

LIENS - 95 - 16

Département de Mathématiques et Informatique

CNRS URA 1327

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Antonio BUCCIARELLI

LIENS - 95 - 16

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Laboratoire d'Informatique de l'Ecole Normale Supérieure
45 rue d'Ulm 75230 PARIS Cedex 05

Tel : (33)(1) 44 32 00 00

Adresse électronique : .. @dmi.ens.fr

Event Structures representing domains with coherence

Antonio Bucciarelli*

Abstract

We present two cartesian closed subcategories of dIC, the category of dI domains with coherence and strongly stable maps introduced in [2, 3] to provide an extensional and "sequential" model of PCF ([7]). The interest of these subcategories lies in the fact that the order structure of their objects is decomposed in two logically independent parts: the order on *prime* elements and the *coherence* predicate on them. This leads to a simplified description of these domains, in terms of suitable classes of event structures.

keywords: denotational semantics, sequentiality, stability, strong stability, hypercoherences, event strucures.

1 Introduction

The category of *dI-domains with coherence* and *strongly stable* functions (**dIC**) introduced and studied in [2, 3] provides a model of PCF ([7]) in which first-order morphisms are sequential in the sense of Kahn-Plotkin ([6]) and in which morphisms are functions "in extenso".

The new feature of these structures is that domains are endowed with a *coherence*, which is a predicate on finite subsets of the domain.

In any domain there exists a natural notion of *compatibility*: a set of elements is compatible if the whole information carried by these elements is non-contradictory, i.e. if the set is upper bounded.

The expressive power of this basic notion is very strong: actually Scott theory of continuous function and Berry's theory of stable function on partially ordered sets may be completely described in terms of the compatibility relation.

The theory of strongly stable functions requires a kind of information about the structure of domains which cannot in general be deduced from the compatibility relation. This information is provided by the coherence. Very roughly, a finite subset of a given domain is coherent if whenever all its elements verify a "sequential" property, then its greatest lower bound verify it too. Here "sequential" means "which can be checked sequentially". Typically the property "at least one component is non-$\perp$" defined on a cartesian product of domains is not sequential. An example of coherent and non-compatible set in $Bool^3$ is the following:

$$B = \{(\perp, true, false),\ (false, \perp, true),\ (true, false, \perp)\}$$

*LIENS-DMI, Ecole Normale Supérieure, 45 rue d'Ulm 75014 Paris, France, email: buccia@dmi.ens.fr

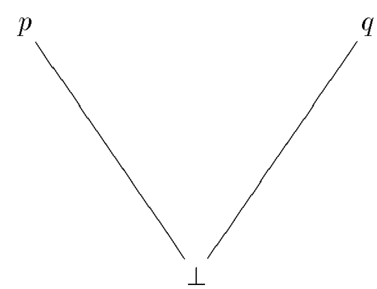
the elements of B are pairwise unbounded, but B is coherent if we choose as sequential properties on $Bool^3$ P_1, P_2, P_3, where $P_i((b_1, b_2, b_3)) \equiv (b_i \neq \perp)$ (this choice is somehow canonical; it gives the *linear* coherence associated to $Bool^3$, see [2]).

Having realized that we need coherence, it is natural to ask if we need compatibility anymore. In other terms, can the order theoretic structure of a domain with coherence be completely described by the coherence? T. Ehrhard has showed in [5] that the answer is "yes" in the particular case in which the sub-domain of *prime* elements of the given domain is flat. An *hypercoherence* is a set endowed with a coherence relation. Compatibility becomes a derived notion. This approach leads to the construction of a strongly stable model of classical linear logic.

In general, if prime elements may be comparable, the coherence relation is not enough: we still need to know the order of prime elements.

Let us go back to general dI-domains with coherence. A drawback of these structures consists in the *extrinsic* character of coherences: the coherence $\mathcal{C}(X)$ which endows a dI-domain X is very weakly related to the order of X. Typically the coherence is not fully described by its restriction to prime elements of X, as it was the case for linear coherences ([2]). This leads from one side to the logical complexity of the definition of $\mathcal{C}(X)$, which is a predicate on finite and non-empty subset of X composed of *arbitrary* elements (not prime in general), from the other to the possibility of defining dIC's where the computational intuition which is behind the "domains-with-coherence" construction is lost: consider for instance the following $(X, \mathcal{C}(X))$:

$X =$



$\mathcal{C}(X) = \{\{p\}, \{q\}, \{p, q\}\}$. The function $f : X \to O$ (O denotes the Sierpinsky domain $\perp < \top$) defined by $f(x) = \top$ if and only if $x = p$ or $x = q$ is not strongly stable, even if it is intuitively sequential (f is actually strongly stable if we redefine $\mathcal{C}(X) = \{\{p\}, \{q\}\}$). The mismatch is due in this case to the fact that $\{p, q\}$ is not upper-bounded, even if any of its non-empty subset is coherent. These observations leads to consider subcategories of **dIC** in which domains and coherences be more strictly related, following two directions:

- coherences fully described by their restriction to prime elements (we call them *prime generated*).

- coherences which are related to the order structure of the associated domain X, in that $\{x_1, \ldots, x_n, \ldots\} \subseteq X$ is upper bounded if and only if any of its non-empty and finite subsets is coherent (we call them *strong* coherences)

It turns out that the category of dI-domain with prime-generated coherence and strongly stable maps is a sub-ccc of **dIC**, while in general strong coherences are not preserved by strongly stable exponentiation. Nevertheless if we require that coherences be prime-generated *and* strong we get cartesian closedness. The objects of this latter subcategory of **dIC** are called dI-domains with *hypercoherence* (the terminology is borrowed from [5]). One of the main interests of these subcategories of **dIC** lies in the possibility of representing concretely their objects by means of suitable classes of event structures [8, 9] in which objects and computations on objects are built on starting from elementary tokens of informations (events). This decomposition of the domains we deal with highlights the causal dependencies hidden in the order-theoretic structure, namely for function spaces, and provides a simpler description of them.

The next section provide a quick introduction to strong stability. The categories of prime generated and strong coherences are introduced in sections 3 and 4 respectively, and their associated categories of event structures in section 5.

2 Preliminaries on strong stability

Definition 1 *A dI-domain with coherence is a couple* $(X, \mathcal{C}(X))$*, where* X *is a dI-domain and* $\mathcal{C}(X)$ *is a set of finite subsets of* X *such that:*

- *For all* $x \in X$ $\{x\} \in \mathcal{C}(X)$*.*
- *For all* $A \in \mathcal{C}(X)$*, for all* B *finite subset of* X *such that* $B \sqsubseteq A$*,* $B \in \mathcal{C}(X)$*.*
- *If* $D_1, \ldots, D_n$ *are directed subsets of* X *such that for any family* $x_1 \in D_1, \ldots, x_n \in D_n$ *we have* $\{x_1, \ldots, x_n\} \in \mathcal{C}(X)$ *then* $\{\vee D_1, \ldots, \vee D_n\} \in \mathcal{C}(X)$*.*

In this definition $\sqsubseteq$ stands for the Egli-Milner preordering: $A \sqsubseteq B$ if $\forall x \in A\ \exists y \in B$ such that $x \leq y$ and $\forall y \in B\ \exists x \in A$ such that $x \leq y$. From now on, if X is a dI-domain , $|X|$ denotes the set of prime elements of X.

Definition 2 *Let* $(X, \mathcal{C}(X)), (Y, \mathcal{C}(Y))$ *be dI-domains with coherence. A Scott continuous function* $f : X \to Y$ *is strongly stable if*

- $\forall A \in \mathcal{C}(X)\ f(A) \in \mathcal{C}(Y)$
- $\forall A \in \mathcal{C}(X)\ f(\wedge A) = \wedge f(A)$

A first remark is that any strongly stable function is stable, because if $f : X \to Y$ is strongly stable and $\{x, x'\}$ is bounded by $\{z\}$ in X, then $\{x, x'\} \sqsubseteq \{z\}$ and, as $\{z\} \in \mathcal{C}(X)$, we get $\{x, x'\} \in \mathcal{C}(X)$; hence by definition $f(x \wedge x') = f(x) \wedge f(x')$.

It is easy to see that composition preserves strong stability and that the identity function is strongly stable. Hence we have a category of dI-domains with coherence and strongly stable maps. Products in this category are defined as follows:

$$(X, \mathcal{C}(X)) \times (Y, \mathcal{C}(Y)) = (X \times Y, \mathcal{C}(X \times Y))$$

where $(X \times Y)$ is the set of couples (x, y) $x \in X, y \in Y$ ordered componentwise, and $B \subseteq X \times Y$ is coherent iff its projections on X and Y are. We define the function space $(X \to Y, \mathcal{C}(X \to Y))$ by taking $X \to Y$ as the domain of strongly stable functions with stable ordering, and $\mathcal{C}(X \to Y)$ as the largest coherence which makes evaluation strongly stable, that is

Definition 3 *$\mathcal{F} \in (X \to Y)$ is coherent if it is finite and for all $A \in \mathcal{C}(X)$ and for all pairing $\mathcal{E}$ of $\mathcal{F}$ and A the set $Ev(\mathcal{E}) = \{f(x) \mid (f,x) \in \mathcal{E}\}$ is in $\mathcal{C}(Y)$ and furthermore:*

$$(\bigwedge \mathcal{F})(\bigwedge A) = \bigwedge Ev(\mathcal{E})$$

In this definition, a pairing of two sets A and B is a subset of $A \times B$ the projections of which are A and B.

Actually this category is cartesian closed. The domain of strongly stable functions from X to Y is noted $X \to_{ss} Y$. Our main purpose in the definition of strong stability is to capture the basic ideas of Kahn-Plotkin sequentiality in a new framework; this goal is achieved as soon as we start with a notion of linear coherence at ground types:

Definition 4 *Let X be a dI-domain; $A \subseteq X$ is linearly coherent if*

$$\forall \alpha : X \multimap O \quad (\forall x \in A \ \alpha(x) = \top \ \rightarrow \alpha(\bigwedge A) = \top)$$

where O denotes the two value domain $\bot < \top$ and $\multimap$ the linear maps, that is the stable maps which commute with arbitrary lubs. (We use linear maps because cells in CDSs [4] may be viewed as linear maps from the domain to Sierpinsky space.)

A first remark about linear coherence is that it satisfies the condition expressed in definition 2, and hence we can endow ground domains with linear coherence. The basic result is:

Proposition 1 *A function is Kahn-Plotkin sequential iff it is strongly stable with respect to linear coherence.*

See [2] for details.

3 dI domains with prime coherence

In this section we study the class of dI-domains with coherence whose coherence is *prime-generated* in the sense expressed by the following definition (Throughout the paper, if A is a set, we write $B \in \mathcal{P}^*_{\text{fin}}(A)$ to denote a subset B of A which is finite and non-empty).

Definition 5 *The coherence $\mathcal{C}(X)$ of the dIC $(X, \mathcal{C}(X))$ is* prime generated *if, for any $B \in \mathcal{P}^*_{fin}(X)$,*

$$B \in \mathcal{C}(X) \textit{ if and only if for all } A \in \mathcal{P}^*_{fin}(|X|) \ A \sqsubseteq B \Rightarrow A \in \mathcal{C}(X)$$

The dIC's whose coherence is prime-generated will be called dI-domains with prime-generated coherence *(dIPC for short).*

The following is immediate:

Proposition 2 *Any linear coherence is prime-generated.*

An interesting property of prime generated coherences is that the corresponding category of dI-domains is cartesian closed:

Proposition 3 *Let $(X, \mathcal{C}(X))$, $(Y, \mathcal{C}(Y))$ be dIC's, and $\mathcal{C}(X)$, $\mathcal{C}(Y)$ be prime-generated; then $\mathcal{C}(X \times Y)$ is prime-generated.*

Proof: Let $B \in \mathcal{P}^*_{\text{fin}}\ (X \times Y)$ be such that for any $A \in \mathcal{P}^*_{\text{fin}}\ |X \times Y|$, if $A \sqsubseteq B$ then $A \in \mathcal{C}(X \times Y)$. We have to show that $B \in \mathcal{C}(X \times Y)$, i.e. that $\pi_1(B) \in \mathcal{C}(X)$ and $\pi_2(B) \in \mathcal{C}(Y)$. Since we know that $\mathcal{C}(X)$ and $\mathcal{C}(Y)$ are prime-generated it is enough to show that for all $A \in \mathcal{P}^*_{\text{fin}}\ |X|$, if $A \sqsubseteq \pi_1(B)$, then $A \in \mathcal{C}(X)$ (and simmetrically for $\pi_2(B)$). For a given $A \in \mathcal{P}^*_{\text{fin}}\ |X|$ such that $A \sqsubseteq \pi_1(B)$, consider $(A \times \perp) = \{(a, \perp_Y) \mid a \in A\}$. Clearly $A \times \perp \in \mathcal{P}^*_{\text{fin}}\ |X \times Y|$ (any pair $(a, \perp_Y)$ is a prime element of $X \times Y$), and $(A \times \perp) \sqsubseteq B$, hence by hypothesis $(A \times \perp) \in \mathcal{C}(X \times Y)$ and hence by definition $A \in \mathcal{C}(X)$. ∎

In order to show that strongly stable exponentiation preserves prime-generation of coherences, we need some preliminary notions:

Definition 6 *Given $f \in (X \rightarrow_{ss} Y)$, $x \in X$ and $p \in |Y|$ such that $f(x) \geq p$, let $e_{(f,x,p)}$ be the function defined by the following trace:*

$$tr(e_{(f,x,p)}) = \{(a, q) \in tr(f) \mid a \leq x,\ q \leq p\}$$

Proposition 4 *Given f, x and p as above, $e_{(f,x,p)} \in |(X \rightarrow_{ss} Y)|$. Moreover $e_{(f,x,p)} \leq f$ and $e_{(f,x,p)}(x) = p$.*

Proof: $f(x) \geq p \Rightarrow \bigvee\{q \mid (a, q) \in tr(f),\ a \leq x\} \geq p \Rightarrow \exists p' \geq p,\ a' \leq x$ such that $(a', p') \in tr(f) \Rightarrow \exists a \leq a'$ such that $(a, p) \in tr(f)$. Hence $tr(e_{(f,x,p)}) = \{(a, p)\} \bigcup \{(a'', p'') \in tr(f) \mid a'' \leq a,\ p'' \leq p\}$. $e_{(f,x,p)} \in |X \rightarrow Y|$ (see [3]), moreover it is clear that $e_{(f,x,p)} \leq f$ and $e_{(f,x,p)}(x) = p$. ∎

We show now that prime-generation is preserved by exponentiation:

Proposition 5 *Let $(X, \mathcal{C}(X))$, $(Y, \mathcal{C}(Y))$ be dIC's, and $\mathcal{C}(X)$, $\mathcal{C}(Y)$ be prime-generated; then $\mathcal{C}(X \rightarrow_{ss} Y)$ is prime-generated.*

Proof: Let $\mathcal{F} \in \mathcal{P}^*_{\text{fin}}\ (X \rightarrow_{ss} Y)$ be such that for any $\mathcal{A} \in \mathcal{P}^*_{\text{fin}}\ |(X \rightarrow_{ss} Y)|$, $\mathcal{A} \sqsubseteq \mathcal{F} \Rightarrow \mathcal{A} \in \mathcal{C}(X \rightarrow_{ss} Y)$. We have to show that $\mathcal{F} \in \mathcal{C}(X \rightarrow_{ss} Y)$. Let $A \in \mathcal{C}(X)$ and let $\mathcal{E}$ be a coupling of $\mathcal{F}$ and A:

- $\mathcal{E}v(\mathcal{E}) = \{f(x) \mid (f, x) \in \mathcal{E}\} \in \mathcal{C}(Y)$

  As $\mathcal{C}(Y)$ is prime generated, it is enough to show that, for any given $B \in \mathcal{P}^*_{\text{fin}}\ |Y|$, $B \sqsubseteq \mathcal{E}V(\mathcal{E}) \Rightarrow B \in \mathcal{C}(Y)$. Let B be such that $B \in \mathcal{P}^*_{\text{fin}}\ |Y|$, $B \sqsubseteq \mathcal{E}v(\mathcal{E})$, and, for $(f, x) \in \mathcal{E}$, let $B_{(f,x)} = \{q \in B \mid q \leq f(x)\}$ (remark that $B_{(f,x)} \neq \perp$, since $B \sqsubseteq \mathcal{E}v(\mathcal{E})$). Consider now the set

  $$\mathcal{A} = \bigcup_{(f,x) \in \mathcal{E}} \bigcup_{q \in B(f,x)} e_{(f,x,q)} \in \mathcal{P}^*_{\text{fin}}\ |(X \rightarrow_{ss} Y)|$$

  By proposition 4 we get $\mathcal{A} \sqsubseteq \mathcal{F}$, hence by hypothesis $\mathcal{A} \in \mathcal{C}(X \rightarrow_{ss} Y)$. Consider now the coupling of $\mathcal{A}$ and A

  $$\mathcal{E}' = \{(e_{(f,x,q)}, x) \mid e_{(f,x,q)} \in \mathcal{A}\}$$

  Again by proposition 4 we get $\mathcal{E}v(\mathcal{E}') = B$ and we have done.

- $\bigwedge \mathcal{E}v(\mathcal{E}) = \bigwedge \mathcal{F}(\bigwedge A)$

  As usual we only have to show $\bigwedge \mathcal{E}v(\mathcal{E}) \leq \bigwedge \mathcal{F}(\bigwedge A)$, the converse being assured by monotonicity. Let $q \in |Y|$ be such that $q \leq \bigwedge \mathcal{E}v(\mathcal{E})$ and let $\mathcal{A} \in \mathcal{P}^*{}_{\mathrm{fin}} \; |(X \rightarrow_{ss} Y)|$ be defined by

$$\mathcal{A} = \bigcup_{(f,x) \in \mathcal{E}} e_{(f,x,q)}$$

  It is easy to see that $\mathcal{A} \sqsubseteq \mathcal{F}$, and hence $\mathcal{A} \in \mathcal{C}(X \rightarrow_{ss} Y)$. Consider now the coupling of $\mathcal{A}$ and A

$$\mathcal{E}' = \{(e_{(f,x,q)}, x) \mid e_{(f,x,q)} \in \mathcal{A}\}$$

  We get

$$q = \mathcal{E}v(\mathcal{E}') = \bigwedge \mathcal{A}(\bigwedge A) \leq \bigwedge \mathcal{F}(\bigwedge A)$$

  and we have done by prime algebraicity of Y.

■

We can now summarize:

Proposition 6 *The category **dIPC** of dIPC's and strongly stable maps is a cartesian closed subcategory of **dIC**.*

Proof: **dIPC** is a full subcategory of **dIC**, which contains the terminal object $(\{\perp\}, \{\{\perp\}\})$ and is closed by product and exponentiation (of **dIC**), hence it is cartesian closed. ■

A remarkable fact about dIPC's is that their coherence is fully described by its restriction to prime elements:

Definition 7 *For a given dIPC $(X, \mathcal{C}(X))$, let $\Gamma(X)$ be the following set*

$$\Gamma(X) = \{A \in \mathcal{P}^*{}_{\mathit{fin}} \; |X| \mid A \in \mathcal{C}(X)\}$$

From now on we use the following notation: if A is a set of sets and B is a set, we say that B is a *section* of A and we write $B \triangleleft A$ if $B \subseteq \bigcup A$ and for all $C \in A$ there exists $x \in B$ such that $x \in C$. It turns out that it is possible to characterize of $\Gamma(X \times Y)$ and $\Gamma(X \rightarrow_{ss} Y)$, for two given dIPC $(X, \mathcal{C}(X)), (Y, \mathcal{C}(Y))$:

Proposition 7 *If $(X, \mathcal{C}(X))$, $(Y, \mathcal{C}(Y))$ are dIPC's, then*

$$\Gamma(X \times Y) = \{U \in \mathcal{P}^*{}_{\mathit{fin}} \; |X| + |Y| \mid U_1 = \emptyset \Rightarrow U_2 \in \Gamma(Y) \text{ and } U_2 = \emptyset \Rightarrow U_1 \in \Gamma(X)\}$$

and

$$\Gamma(X \rightarrow_{ss} Y) = \{U \in \mathcal{P}^*{}_{\mathit{fin}} \; |X \rightarrow_{ss} Y| \mid \forall W \triangleleft U \;\; \pi_1(W) \in \mathcal{C}(X) \Rightarrow$$
$$(\pi_2(W) \in \Gamma(Y) \text{ and } (\#\pi_2(W) = 1 \Rightarrow \#\pi_1(W) = 1))\}$$

Proof: We start by proving the statement concerning cartesian products: it is sufficient to remark that $|(X \times Y)| = \{(p, \perp) \mid p \in |X|\} \bigcup \{(\perp, q) \mid q \in |Y|\}$, is isomorphic to $|X| + |Y|$, and that if $U \in \mathcal{P}^*{}_{\mathrm{fin}} \; (|X \times Y|)$ is such that $\pi_1(U) \neq \{\perp\}$ and $\pi_2(U) \neq \{\perp\}$, then $\perp \in \pi_1(U)$ and $\perp \in \pi_2(U)$, hence $U \in \mathcal{C}(X \times Y)$, since $\mathcal{C}(X)$ and $\mathcal{C}(Y)$ are prime-generated[1].

We pass now to exponentials: given $U \in \mathcal{P}^*_{\text{fin}}\,(|X \rightarrow_{ss} Y|)$ satisfiyng the condition above, we have to show that $U \in \mathcal{C}(X \rightarrow_{ss} Y)$. Let $A \in \mathcal{C}(X)$ and $\mathcal{E}$ be a coupling of U and A.

- $\mathcal{E}v(\mathcal{E}) \in \mathcal{C}(Y)$

  Let $B \in \mathcal{P}^*_{\text{fin}}\,|Y|$ such that $B \sqsubseteq \mathcal{E}v(\mathcal{E})$. Consider now the set

  $$W = \{(a', q') \in \bigcup_{u \in U} tr(u) \mid \exists a \in A\ a' \leq a \text{ and } \exists q \in B\ q \leq q'\}$$

  We have $W \triangleleft U$, since

  - for any $u \in U$ $\exists a \in A, q \in B$ such that $q \leq u(a)$, hence $\bigvee\{q' \mid \exists a' \leq a,\ (a', q') \in tr(u)\} \geq q$, hence, since q is prime, there exist $a' \leq a$, $q' \geq q$ such that $(a', q') \in tr(u)$.
  - for any $(a', q') \in W$, we have by definition $\exists u \in U$ such that $(a', q') \in tr(u)$.

  Moreover it is easy to see that $\pi_1(W) \sqsubseteq A$, hence $\pi_1(W) \in \mathcal{C}(X)$, and this entails $\pi_2(W) \in \Gamma(Y)$. Since $B \sqsubseteq \pi_2(W)$ we get $B \in \Gamma(Y)$ and we have done.

- $\bigwedge \mathcal{E}v(\mathcal{E}) = \bigwedge U(\bigwedge A)$

  We begin by the following observation: if, for a given $q \in |Y|$, there exists $a \in X_0$ such that $(a, q) \in \bigcap_{u \in U} tr(u)$, then $(a, q) \in tr(\bigwedge U)$, (see for instance [1], lemma 4).

  Let $q \in |Y|$ be such that $q \leq \mathcal{E}v(\mathcal{E})$. We know that for any $q' \leq q$, for any $u \in U$, there exists $a \in A$, $a' \leq a$ such that $(a', q') \in tr(u)$ (by the same argument as in proposition 4). Consider

  $$W_q = \{(a', q) \mid \exists a \in A,\ a' \leq a,\ u \in U \ \text{ such that } (a', q) \in tr(u)\}$$

  We get $W_q \triangleleft U$, $\pi_1(W_q) \sqsubseteq A$ and $\#\pi_2(W_q) = 1$. Hence $\pi_1(W_q)$ is a singleton, say $\{a_0\}$, and moreover $a_0 \leq \bigwedge A$, since $\{a_0\} \sqsubseteq A$. By the observation above, $(a_0, q) \in tr(\bigwedge U)$ and hence $\bigwedge U(\bigwedge A) \geq q$, and this concludes the proof by prime-algebraicity of Y.

we have showed that any element of $\mathcal{P}^*_{\text{fin}}\,(|X \rightarrow_{ss} Y|)$ satisfying the condition above is coherent; the converse is easy. ∎

4 dI-domains with hypercoherence

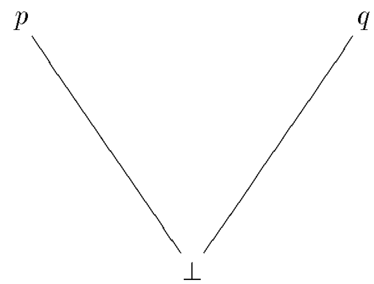
In a dIPC $(X, \mathcal{C}(X))$ the coherence $\mathcal{C}(X)$ is fully described by its restriction to prime elements. In view of the representation theorems, by which we will give a *concrete* description of some subcategories of **dIC**, we need domains in which also the compatibility (remind that a subset of a domain X is compatible if it is upper bounded) is fully described by $\Gamma(X)$.

[1] A finite set containing $\perp$ is always coherent in a domain with prime-generated coherence, since it has no Egli-Milner lower bound of prime elements.

Definition 8 *Given a dIC* $(X, \mathcal{C}(X))$ *and a subset* A *of* X*, we say that* A *is* hereditarily coherent *(hc for short) if for any* $B \in {\mathcal{P}^*}_{fin}(A)$ $\;B \in \mathcal{C}(X)$

Any bounded set A of a dIC $(X, \mathcal{C}(X))$ is hereditarily coherent, since if $B \subseteq A$ and $P \in {\mathcal{P}^*}_{\text{fin}}(|X|)$ is finite and non-empty, then $B \sqsubseteq \bigvee A$, and hence $B \in \Gamma(X)$. The converse does not hold in general, as showed by the following example:

example 1: Let X be the following dI-domain



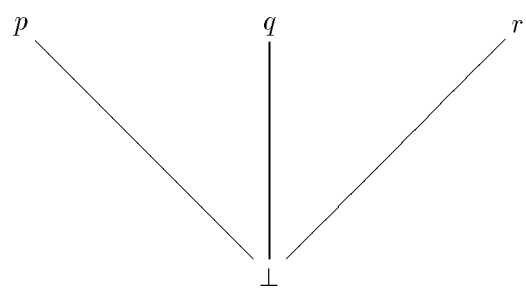
and $\Gamma(X) = \{\{p\}, \{q\}, \{p, q\}\}$. The set {p,q} is hereditarily coherent but non upper bounded. ∎

In the following we will study the class of dIC's such that any hereditarily coherent subset be upper bounded.

Definition 9 *A* dI-domain with strong coherence *(dISC for short) is a dIC* $(X, \mathcal{C}(X))$ *such that for any* $A \subseteq X$*,* A *is upper bounded if and only if* A *is hereditarily coherent.*

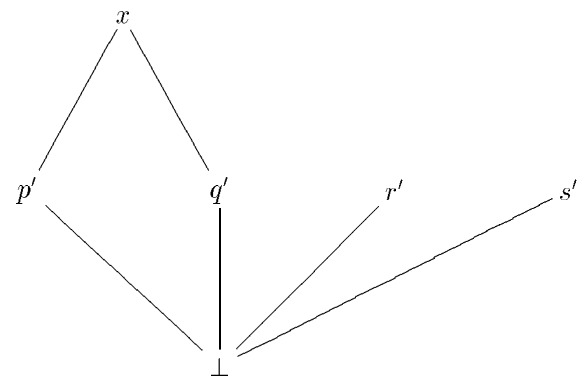
Example 1 shows that a linear coherence is not strong in general. It is easy to see that cartesian products of dISC's are dISC's, but in general strong coherence is not preserved by strongly stable exponentiation, as showed in the following example:

example 2: Let X be the following dI-domain:



and $\mathcal{C}(X) = \{\{p, q, r\}\}$ (this means that $\{p, q, r\}$ is the only non-singleton and Egli-Milner maximal coherent subset of X; clearly X has other coherent subsets, that we do not list since they can be derived from the axioms of coherences). The structure $(X, \mathcal{C}(X))$ is clearly a dISC, since the only hereditarily coherent subsets of X are the singletons.

Consider now the dISC $(Y, \mathcal{C}(Y))$ where Y is the following domain:



and $\mathcal{C}(Y)$ is the following (we only list the non-singleton and Egli-Milner maximal elements)

$$\mathcal{C}(Y) = \{\{p', q'\}, \{p', r', s'\}, \{q', r', s'\}, \{p', q', r', s'\}\}$$

Again $(Y, \mathcal{C}(Y))$ is clearly a dISC, since the only non-singleton hereditarily coherent subset of Y $(\{p', q'\})$ is upper bounded (by x). Consider now the strongly stable functions $f_1, f_2 : X \to Y$ defined by:

$$tr(f_1) = \{(p, p'), (q, r'), (r, s')\} \quad tr(f_2) = \{(p, q'), (q, r'), (r, s')\}$$

It is easy to see that $\{f_1, f_2\}$ is a hereditarily coherent subset of $X \to_{ss} Y$ but its least upper bound is not strongly stable, since it maps the coherent subsetset of X $\{p, q, r\}$ onto the non-coherent subset of Y $\{x, r', s'\}$. ■

Remark that the coherence of the dIC $(Y, \mathcal{C}(Y))$ of the example above is not prime-generated, since $B = \{x, r', s'\}$ is not coherent even if for any $A \in \mathcal{P}^*_{\mathrm{fin}}\ |Y|$, $A \sqsubseteq B \Rightarrow A \in \Gamma(X)$ (the Egli-Milner lower bound of A in $\mathcal{P}^*_{\mathrm{fin}}\ |Y|$ being $\{p', r', s'\}$ and $\{q', r', s'\}$, which are both coherent). It turns out that if we require that coherences be strong *and* prime generated we get categorical exponentials. We call a dIC with strong and prime-generated coherence *dI domain with hypercoherence* (the terminology is borrowed from [5]).

Definition 10 *A* dI-domain with hypercoherence *(dIHC for short) is a dIC $(X, \mathcal{C}(X))$ such that $C(X)$ is strong and prime-generated.*

Proposition 8 *The category **dIHC** of dI-domains with hypercoherence and strongly stable maps is cartesian closed.*

Proof: We have alredy showed that **dIPC** is cartesian closed. Moreover cartesian products of dISC's are trivially dISC's, hence all we have to prove is that, if $(X, \mathcal{C}(X))$ and $(Y, \mathcal{C}(Y))$ are dIHC's, then $\mathcal{C}(X \to_{ss} Y))$ is a strong coherence. Let $\mathcal{F}$ be a hereditarly coherent set of strongly stable functions from X to Y. We have to show that $\mathcal{F}$ is upper bounded in $X \to_{ss} Y$. Remark that we can suppose that $\mathcal{F}$ is finite, since the corresponding property for an infinite set of function will easily follow from directed-completeness of $X \to_{ss} Y$. Let hence $\mathcal{F} = \{f_1, \ldots, f_n\}$, and define, for $x \in X$, $f(x) = \bigvee_{i \leq n} f_i(x)$. We have to show that f is strongly stable. First of all we need that, for $x \in X$, $\{f_i(x)\}_{i \leq n}$ be upper bounded. Since $\mathcal{C}(Y)$ is strong, it is enough to show that $\{f_i(x)\}_{i \leq n}$ is hereditarely coherent. Let $B = \{f_{i_1}(x), \ldots, f_{i_k}(x)\} \subseteq \{f_i(x)\}_{i \le n}$; since $\{f_{i_j}\}_{j \le k}$ is coherent (F being h.c.), we get trivially that $B \in \mathcal{C}(Y)$.

We have now to prove that f is strongly stable: let $A \in \mathcal{C}(X)$ and $B = f(A)$.

To show that $B \in \mathcal{C}(Y)$ we use prime-generation of $\mathcal{C}(Y)$: let $Q \in \mathcal{P}^*{}_{\text{fin}}\ |Y|$ be such that $Q \sqsubseteq B$, if we show that $Q \in \mathcal{C}(Y)$ we have done. Given $q \in Q$, $a \in A$, let $\mathcal{F}_q^a$ be the set

$$\mathcal{F}_q^a = \{g \in \mathcal{F} \mid q \le g(a)\}$$

and for $g \in \mathcal{F}_q^a$, let $c_{g,a,q} \le a$ be such that $(c_{g,a,q}, q) \in tr(g)$. Consider the coupling

$$\mathcal{E} = \bigcup_{q \in Q} \bigcup_{a \in A} \{(g, c_{g,a,q}) \mid g \in \mathcal{F}_q^a\}$$

- $\pi_1(\mathcal{E}) \subseteq \mathcal{F}$, hence $\pi_1(\mathcal{E}) \in \mathcal{C}(X \rightarrow_{ss} Y)$
- $\pi_2(\mathcal{E}) \sqsubseteq A$, hence $\pi_2(\mathcal{E}) \in \mathcal{C}(Y)$
- $\mathcal{E}v(\mathcal{E}) = Q$

From the three points above we get $Q \in \mathcal{C}(Y)$, and we have done.

To show that $f(\bigwedge A) = \bigwedge_{a \in A} f(a)$, let $q \in Q$ be such that $q \le \bigwedge_{a \in A} f(a)$. Define, for $a \in A$

$$\mathcal{F}_a = \{g \in \mathcal{F} \mid q \le g(a)\}$$

and, for $g \in \mathcal{F}_a$, let $c_{g,a} \le a$ be such that $(c_{g,a}, q) \in tr(g)$. Consider now the coupling

$$\mathcal{E} = \bigcup_{a \in A} \{(g, c_{g,a}) \mid g \in \mathcal{F}_a\}$$

- $\pi_1(\mathcal{E}) \subseteq \mathcal{F}$, hence $\pi_1(\mathcal{E}) \subseteq\in \mathcal{C}(X \rightarrow_{ss} Y)$
- $\pi_2(\mathcal{E}) \sqsubseteq A$, hence $\pi_2(\mathcal{E}) \in \mathcal{C}(Y)$
- $\bigwedge \mathcal{E}v(\mathcal{E}) = q$

From the above facts we get $q = \bigwedge \pi_1(\mathcal{E})(\bigwedge \pi_2(\mathcal{E})) \le f(\bigwedge A)$ and we have done. ∎

It is easy to see that in a dIHC $(X, \mathcal{C}(X))$ the compatibility of X is fully described by $\Gamma(X)$, as stated below:

Fact 1 *Given a dIHC $(X\mathcal{C}(X))$ and a subset A of X, A is upper bounded if and only if*

$$\forall B \in \mathcal{P}^*{}_{\mathit{fin}}\ A, P \in \mathcal{P}^*{}_{\mathit{fin}}\ |X|, P \sqsubseteq B \Rightarrow P \in \Gamma(X)$$

5 Representation theorems

In this section we show that **dIPC** and **dIHC** can be represented by suitable classes of event structures ([8, 9]).

Definition 11 *An* event structure with coherence *(ESC for short) is a tuple*

$$E = (Ev_E, Coh_E, Con_E, \vdash_E)$$

such that

- Ev_E *is a countable set of* event.
- $Coh_E \subseteq \mathcal{P}^*{}_{fin}\,(Ev_E)$ *is such that:*

$$\forall e \in Ev_E \ \{e\} \in Coh_E$$

- $Con_E \subseteq \mathcal{P}_{\ fin}\,(Ev_E)$ *is such that*

$$\forall P \in Con_E, Q \subseteq P \Rightarrow Q \in Con_E \ \textit{and}\ (Q \in \mathcal{P}^*{}_{fin}\,(P) \Rightarrow Q \in Coh_E)$$

- $\vdash \subseteq Con_E \times Ev_E$ *is such that the binary relation* $\ll$ *on* Ev_E *defined by* $e \ll e'$ *if and only if there exists* $P \in Con_E$ *such that* $e \in P$ *and* $P \vdash e'$, *has no infinite chain* $\ldots \ll e_n \ll \ldots \ll e_2 \ll e_1$.

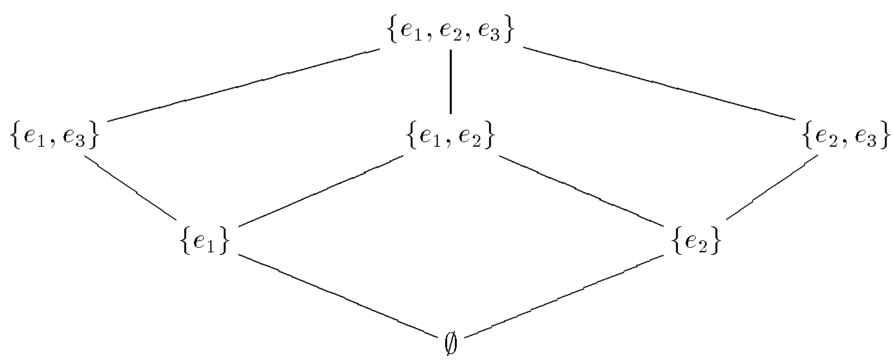
When $P \vdash e$ we say that e is enabled by P. The set of *states* of an ESC is defined exactly as for ordinary event structures:

Definition 12 *A* state *of an ESC E is a subset A of Ev_E which is*

- *consistent: any finite subset of A is in Con_E*
- *secured: for all $e \in A$ there exists $P \subseteq A$ such that $P \vdash e$.*

The set of states of E ordered by inclusion is noted $D(E)$.

$D(E)$ is not a distributive domain in general, as showed by the following example:
example 3: Let $E = (\{e_1, e_2, e_3\}, \mathcal{P}^*{}_{\text{fin}}\,(E), \mathcal{P}(E), \{\emptyset \vdash e_1, \emptyset \vdash e_2, \{e_1\} \vdash e_3, \{e_2\} \vdash e_3\}$. $D(E)$ is the following domain:



which is not distributive since

$$\{e_1, e_3\} \wedge (\{e_1, e_2\} \vee \{e_2, e_3\}) \neq (\{e_1, e_3\} \wedge \{e_1, e_2\}) \vee (\{e_1, e_3\} \wedge \{e_2, e_3\})$$

■

For $D(E)$ to be distributive, the following stability condition on E is needed:

Definition 13 *An ESC E is* stable *if for any state $x \in D(E)$, for any $P, Q \subseteq x$, $e \in E$, $P \vdash e, Q \vdash e \Rightarrow P = Q$*

The following is a classical result in the theory of event structure, which holds for ESC's since the notion of state is unchanged:

Proposition 9 *The set of states of a stable ESC ordered by inclusion is a dI-domain.*

The extra-information (with respect to event structures) provided by Coh_E allows to endow $D(E)$ with a coherence:

Definition 14 *Given a stable ESC E, let $\mathcal{C}(E) \subseteq \mathcal{P}^*_{fin}(D(E))$ be defined by*

$$A = \{a_1, \ldots, a_n\} \in \mathcal{C}(E) \textit{ if and only if } \textit{ for any } P \in \mathcal{P}^*_{fin}(E),\ P \lhd A \Rightarrow P \in Coh_E$$

Proposition 10 *For a given stable ESC E, $(D(E), \mathcal{C}(E))$ is a dI-domain with prime-generated coherence.*

Proof: We have to show that $\mathcal{C}(E)$ is a prime-generated coherence on $D(E)$:

- $x \in D(E) \Rightarrow \{x\} \in \mathcal{C}(E)$

  Let $P \in \mathcal{P}^*_{\text{fin}}(E)$ be such that $P \lhd \{x\}$. Clearly $P \subseteq x$, hence $P \in Con_E$ and finally $P \in Coh_E$.

- $A \in \mathcal{C}(E), B \sqsubseteq A \Rightarrow B \in \mathcal{C}(E)$

  Let $P \in \mathcal{P}^*_{\text{fin}}(E)$ be such that $P \lhd B$. It is easy to see that $P \lhd A$, hence $P \in Coh_E$ and we have done.

- Let $D_1, \ldots, D_n$ be directed in $D(E)$ and such that $\forall x_1 \in D_1, \ldots, x_n \in D_n$ $\{x_i\}_{i \leq n} \in \mathcal{C}(E)$. If $P \in \mathcal{P}^*_{\text{fin}}(E)$ is such that $P \lhd \{\bigvee D_i\}_{i \leq n}$, let $x_i \in D_i$ be such that for any $e \in P$, if $e \in \bigvee D_i$ than $e \in x_i$ (such x_i exist by finiteness of P). It is easy to see that $P \lhd \{x_i\}_{i \leq n}$, hence $P \in Coh_E$ and we have done.

To show that $\mathcal{C}(E)$ is prime generated, let $A = \{x_1, \ldots, x_n\} \subseteq D(E)$ be such that for any $B \in \mathcal{P}^*_{\text{fin}} |D(E)|$, $B \sqsubseteq A \Rightarrow B \in \mathcal{C}(E)$, and suppose that $P \in \mathcal{P}^*_{\text{fin}} E$ is such that $P \lhd A$. Consider for $x \in A$ the set $P_x = \{e \in P \mid e \leq x\}$. For any $e \in P_x$ there exists a prime element of $D(E)$ p^e_x such that $p^e_x \leq x$ and $e \in p^e_x$, since $D(E)$ is prime algebraic and least upper bounds in $D(E)$ are set unions. It is clear that

$$B = \bigcup_{x \in A} \bigcup_{e \in P_x} \{p^e_x\} \sqsubseteq A$$

and that $P \lhd B$. Hence $P \in Coh_E$ and we have done. ∎

Hence the dI-domain with coherence associated to a given stable ESC is a dIPC. Moreover for any given dIPC $(X, \mathcal{C}(X)$ we can define a stable ESC E_X such that $(D(E_X), \mathcal{C}(E_X))$ is isomorphic to $(X, \mathcal{C}(X))$ in **dIPC**, in the following way:

- $Ev_{E_X} = |X|$
- $Coh_{E_X} = \{P \in \mathcal{P}^*_{\text{fin}} |X| \mid P \in \mathcal{C}(X)\}$
- $Con_{E_X} = \{P \in \mathcal{P}_{\text{fin}} |X| \mid P \text{ is upper bounded in } X\}$

- $P \vdash e$ if $P = \{e' \in |X| \mid e' < e\}$

It is easy to see that E_X is actually an ESC, and it is stable since any element of Ev_{E_X} has exactly one enabling.

Proposition 11 *For a given dIPC $(X, \mathcal{C}(X))$, define $f : X \to D(E_X)$ by*

$$f(x) = \{p \in |X| \mid p \leq x\}$$

*Then f is an isomorphism in **dIPC** and its inverse is $g : D(E_X) \to X$ defined by*

$$g(a) = \bigvee \{e \in Ev_{E_X} \mid e \in a\}$$

Proof: The function f and g are clearly strongly stable, since

$$tr(f) = \{(x, p) \mid x \in X_0, p \in |X|, p \leq x\}$$

and

$$tr(g) = \{(a, p) \mid a \in D(E_X)_0, p \in a\}$$

moreover

$$g(f(x)) = \bigvee \{p \in |X| \mid p \leq x\} = x$$

and

$$f(g(a)) = \{p \in |X| \mid p \leq \bigvee \{e \in |X| \mid e \in a\}\} = a$$

∎

It is remarkable that an exponential dIPC $((X \to_{ss} Y), \mathcal{C}(X \to_{ss} Y))$ may be represented by a stable ESC which is much simpler than $E_{(X \to_{ss} Y)}$, in that its events are not prime elements of $(X \to_{ss} Y)$ but elements of $X_0 \times |Y|$

Definition 15 *Let E and F be ESC's. Define $[E \to F] = (Ev, Coh, Con, \vdash)$ by:*

- $Ev = X_0 \times |Y|$.
- $A = \{(x_1, q_1), \ldots, (x_n, q_n)\} \in Coh$ *if*

  $$\pi_1(A) \in \mathcal{C}(E) \Rightarrow (\pi_2(A) \in Coh_F \text{ and } \#\pi_2(A) = 1 \Rightarrow \#\pi_1(A) = 1)$$

  .
- $\{(x_1, q_1), \ldots, (x_n, q_n)\} \in Con$ *if forall* $B \in {\mathcal{P}^*}_{fin}(A)$ $B \in Coh$ *and* $\forall I \subseteq \{1, \ldots, n\}$ *if* $\bigcup_{j \in I} \{x_j\} \in Con_E$ *then* $\{q_i\}_{i \in I} \in Con_F$.
- $\{(x_1, q_1), \ldots, (x_n, q_n)\} \vdash (x, q)$ *if* $\{q_1, \ldots, q_n\} \vdash_F q$ *and* $\bigcup_{i \leq n} x_i \subseteq x$.

Proposition 12 *If E and F are stable ESC's, then $[E \to F]$ is a stable ESC. Moreover there exists an order isomorphisms between states of $[E \to F]$ and elements of $D(E) \to_{ss} D(S)$ defined by*

$$\varphi(t) = \lambda x \in D(E).\ \{e \in |F| \mid \exists x' \leq x\ (x', e) \in t\}$$

and

$$\psi(f) = tr(f)$$

where $t \in D([E \to F])$ and $f \in D(E) \to_{ss} D(F)$.

Proof: For proving that $[E \to F]$ is a stable ESC it is enough to show that $\vdash$ has no infinite decreasing $\ll$-chain and that the stability condition holds, the other requirements of the definition being trivially satisfied. For the first point remark that $(x,q) \ll (x',q')$ entails $q \ll q'$: since there is no infinite decreasing $\ll$-chain in F we have done. We have now to prove that $[E \to F]$ is stable. Let $t \in D([E \to F])$ and $P, Q \subseteq t$, $P = \{(x_1,q_1),\ldots,(x_n,q_n)\}$, $Q = \{(x'_1,q'_1),\ldots,(x'_m,q'_m)\}$ be such that $P, Q \vdash (x,q)$. Consider $t.x = \{q \in Ev_F \mid \exists x' \leq x \ (x',q) \in t\} (= \varphi(t)(x))$. It is easy to see that $t.x \in D(F)$, and hence $\{q_1,\ldots,q_n\} = \{q'_1,\ldots,q'_m\}$ since F is stable. Let now $(x,q) \in P$, $(x',q) \in Q$; if we show $x = x'$ we have done since $P = Q$ follows easily. Consider $W = \{(x,q),(x',q)\} \subseteq t$; since $\pi_1(W) \in \mathcal{C}(E)$ and $\#\pi_2(W) = 1$ we get $\#\pi_1(W) = 1$ and we have done.

We show now that $\varphi(t)$ is a strongly stable function from $D(E)$ to $D(F)$: actually we have already observed that, if $x \in D(E)$, $t.x = \varphi(t)(x) \in D(F)$, hence $\varphi(t)$ is a function. Continuity of $\varphi(t)$ is easy. Let us prove that $\varphi(t)$ is strongly stable: if $A = \{a_1,\ldots,a_n\} \in \mathcal{C}(E)$ and $Q \in \mathcal{P}^*_{\text{fin}}\,(Ev_F)$ is such that $Q \triangleleft \varphi(t)(A)$, then for any $x \in A$, $q \in B$, if $q \in \Phi(t)(x)$ then $\exists a^q_x \leq x$ such that $(a^q_x, q) \in t$. Consider

$$W = \bigcup_{x \in A} \ \bigcup_{q \in \{q' \in B \mid q' \in t.x\}} \{(a^q_x, q)\}$$

we get

- $\pi_1(W) \sqsubseteq A$, hence $\pi_1(W) \in \mathcal{C}(E)$.
- $\pi_2(W) = B$.
- W is a finite and non-empty subset of t.

From the three points above and the fact that any finite and non-empty subset of t is coherent we get easily $B \in Coh_F$ and we have done.

It remains to show that $\varphi(t) \bigwedge A = \bigwedge_{x \in A} \varphi(t)(x)$: let $q \in Ev_F$ be such that $q \in \bigwedge_{x \in A} \varphi(t)(x)$, and consider

$$W = \bigcup_{x \in A} \{(a_x, q) \in t \mid a_x \leq x\}$$

we have

- $\pi_1(W) \sqsubseteq A$, hence $\pi_1(W) \in \mathcal{C}(E)$.
- $\#\pi_2(W) = 1$.

hence there exists $a \in D(E)_0$ such that $\pi_1(W) = \{a\}$ and $a \leq \bigwedge A$, and hence $q \in \varphi(t) \bigwedge A$. We have showed that a state t of $[E \to F]$ is the trace of a strongly stable function $\varphi(t) : D(E) \to D(F)$, hence φ, ψ defined above actually define an isomorphism. ∎

We can define the category **ESC** of ESC's by letting $Hom(E,F) = [E \to F]$, composition being defined by usual composition of traces. The functor Ψ from **dIPC** to **ESC** such that $\Psi((X, \mathcal{C}(X))) = E_X$ and $\Psi(f) = tr(f)$, and the functor Φ from **ESC** to **dIPC** such that $\Phi(E) = (D(E), \mathcal{C}(E))$ and $\Phi(t) = \varphi(t)$ define an equivalence of categories.

We now describe a class of event structure suitable to represent dI-domains with hypercoherence. These event structures will be particular cases of ESC's. Since the compatibility of a dIHC $(X, \mathcal{C}(X))$ is fully described by $\Gamma(X)$, in the corresponding event ESC

E_X the compatibility predicate Con_{E_X} can be derived from Coh_{E_X}, namely $A \in Con_{E_X}$ if and only if for any non-empty subset B of A, $B \in Coh_{E_X}$. Hence event structures corresponding to dIHC's are simpler than ESC's. We emphasize this remark by explicitly defining the *event structures with hypercoherence* (we could as well have presented them as ESC's with an additional property).

Definition 16 *An* event structure with hypercoherence *(ESH for short) is a triple*

$$E = (Ev_E, Coh_E, \vdash_E)$$

such that

- Ev_E *is a countable set of* event.
- $Coh_E \subseteq \mathcal{P}^*{}_{fin}\,(Ev_E)$ *is such that:*

$$\forall e \in Ev_E\ \{e\} \in Coh_E$$

- $\vdash_E \subseteq \{A \in \mathcal{P}^*{}_{fin}\,(Ev_E) \mid \forall B \subseteq A\ B \in Coh_E\} \times Ev_E$ *is such that the binary relation* $\ll$ *on* Ev_E *defined by* $e \ll e'$ *if and only if there exists* $P \in Con_E$ *such that* $e \in P$ *and* $P \vdash e'$*, has no infinite chain* $\ldots \ll e_n \ll \ldots \ll e_2 \ll e_1$*.*

Definition 17 *A* state *of an ESH* E *is a subset* a *of* Ev_E *which is*

- *hereditarily coherent with respect to* E*: any finite non-empty subset of* a *is in* Coh_E
- *secured: for all* $e \in a$ *there exists* $P \subseteq a$ *such that* $P \vdash e$*.*

The set of states of E *ordered by inclusion is noted* $D(E)$*.*

We are interested in stable ESH's. It is easy to see that if E is a stable ESH, then $D(E) = D(E')$, where E' is the stable ESC defined by $Ev_{E'} = Ev_E$, $Coh_{E'} = Coh_E$, $Con_{E'} = \{A \in Coh_E \mid \forall B \in \mathcal{P}_{\text{fin}}\,(A)\ B \in Coh_E\}$ and $\vdash_{E'} = \vdash_E$. Hence we know by previous results that for any ESH E, $D(E)$ is a dI-domain, and that $(D(E), \mathcal{C}(E))$ has prime-generated coherence. It is easy to see that $\mathcal{C}(E)$ is strong, since if $A \subseteq D(E)$ is hereditarily coherent and $P \in \mathcal{P}^*{}_{\text{fin}}\,(Ev_E)$ is such that $P \subseteq \bigcup A$, then there exists $B \subseteq A$ such that $P \triangleleft B$. Since $B \in \mathcal{C}(E)$ we get $P \in Coh_E$, and hence $\bigcup A$ is hereditarily coherent with respect to E. Moreover $\bigcup A$ is secured since the elements of A are states. Hence $\bigcup A$ is a state. These observations are summarized in the following proposition:

Proposition 13 *Given a stable ESH* E*,* $(D(E), \mathcal{C}(E))$ *is a dI-domain with hypercoherence.*

Moreover it is easy to see that if $(X, \mathcal{C}(X))$ is a dIHC, than E_X is actually a ESH, since $Con_{E_X} = \{P \in \mathcal{P}_{\text{fin}}\,|X| \mid P \text{ is upper bounded in X }\} = \{P \in \mathcal{P}_{\text{fin}}\,|X| \mid \forall B \in \mathcal{P}^*{}_{\text{fin}}\,(P)\ B \in Coh_{E_X}\}$. Hence by propositions 8 and 12 it follows that **dIHC** and **ESH** are equivalent subcategories of **dIPC** and **ESC** respectively (the equivalence being provided by the functors Ψ and Φ defined above). In the following exemple we show that concrete data structures and sequential functions are particular cases of ESH and strongly stable functions.

example 4: Given a CDS $M = (C_M, V_M, E_M, \vdash_M)$ we define a ESH E_M as follows:

- $Ev_{E_M} = E_M$
- Given $A \in \mathcal{P}^*_{\text{fin}}(E_M)$, $A \in Coh_{E_M}$ if and only if $\#A = 1$ or $\exists c_1, c_2 \in C_M$, $c_1 \neq c_2$, $\exists v_1, v_2 \in V_M$ such that $(c_1, v_1), (c_2, v_2) \in A$
- $\{(c_1, v_1), \ldots, (c_n, v_n) \vdash_{E_M} (c, v)$ if and only if $\{(c_1, v_1), \ldots, (c_n, v_n) \vdash_M c$

Proposition 14 *Given a CDS $M = (C_M, V_M, E_M, \vdash_M)$, $D(M) = D(E_M)$*

Proof: Let $a \in D(E_M)$; if $(c, v_1), (c, v_2) \in a$ then $v_1 = v_2$ since $\{(c, v_1), (c, v_2)\} \in C0h_{E_M}$. Moreover if $(c, v) \in a$ then c is enabled in a by definition, hence $a \in D(M)$.

If $a \in D(M)$ we get easily that $a \in D(E_M)$. ∎

Let us call $\mathcal{C}(M)$ the following (linear) coherence on $D(M)$:

$$\text{given } A \subseteq D(M)\ A \in \mathcal{C}(M)$$

$$\text{if and only if } (\exists c \in C_M,\ \forall x \in A\ \exists v_x \in V_M\ (c, v_x) \in x \Rightarrow (\forall x_1, x_2 \in A\ v_{x_1} = v_{x_2}))$$

$\mathcal{C}(M)$ is the linear coherence on $D(M)$ associated to the set of linear functions from $D(M)$ to O defined by C_M regarded as set of linear functions from $D(M)$ to O, where $c(x) = \top$ if and only if $\exists v \mid (c, v) \in x$.

Proposition 15 *If M is a sequential CDS ([4]) then $\mathcal{C}(M) = \mathcal{C}(E_M)$*

Hence $(D(M), \mathcal{C}(M))$ and $(D(E_M), \mathcal{C}(E_M))$ are the same dIHC. It follows easily that $f : D(M) \to D(M')$ is Kahn-Plotkin sequential if and only if it is strongly stable. ∎

We can now present Ehrhard's *hypercoherences* [5] as particularly simple ESH: a ESH E is a hypercoerence if for all $e \in Ev_E$, $\emptyset \vdash e$. Clearly the enabling relation becomes superfluous:

Definition 18 *An* hypercoherence *is a couple $E = (Ev_E, Coh_E)$ such that Ev_E is a countable set and $Coh_E \subseteq \mathcal{P}^*_{fin}(E)$ is such that $\forall e \in Ev_E\ \{e\} \in Coh_E$.*

Hypercoherences and strongly stable functions form a cartesian closed category, and provide a model of classical linear logic.

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