Recognition of the ℓ_1-graphs with Complexity O(nm), or Football in a Hypercube

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LIENS - 95 - 11

May 1995

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(June 19, 1995)

Abstract

We fill in the details of the algorithm sketched in [Sh] and determine its complexity. As a part of this main algorithm, we also describe an algorithm which recognizes graphs which are isometric subgraphs of halved cubes. We discuss possible further applications of the same ideas and give a nice example of a non ℓ_1-graph allowing a highly isometric embedding into a halved cube.

1 Introduction

For a set Ω, let 2^Ω denote the set of all the subsets of Ω. We turn 2^Ω into an *n-dimensional cube graph* Q_k, where $k = |\Omega|$, by making two subsets A and B adjacent whenever the symmetric difference $A \triangle B$ has size 1. The graph Q_k is bipartite, the bipartite half of Q_k being known as the halved cube graph (we denote it HQ_k). Therefore, HQ_k can be defined as the graph on the even size subsets in 2^Ω, in which two such subsets A and B are adjacent whenever $|A \triangle B| = 2$.

We usually identify a graph Γ with its set of vertices, and we use the same notation Γ for both. For a connected graph Γ let d_Γ be the distance function on Γ. For a given positive integer λ, we call a mapping $\phi : \Gamma \longrightarrow \Delta$, where Δ is another connected graph, a λ-*embedding* if, for any two vertices $x, y \in \Gamma$, we have $d_\Delta(\phi(x), \phi(y)) = \lambda d_\Gamma(x, y)$. If $\lambda = 1$ then ϕ is called an *isometric embedding.* A connected graph Γ is called an ℓ_1-graph if it allows a λ-embedding ϕ into a cube Q_k for some λ and k. It can be easily seen that the distance function on Q_k is given by $d_{Q_n}(A, B) = |A \triangle B|$; hence the condition on ϕ can be rewritten as follows: $|\phi(x) \triangle \phi(y)| = \lambda d_\Gamma(x, y)$.

Notice that an isometric embedding into a halved cube HQ_k can be viewed as a 2-embedding into Q_k; hence any isometric subgraph of a halved cube is an ℓ_1-graph.

It was proved (or rather, noticed) in [Sh] that the property of Γ being an ℓ_1-graph can be recognized in a polynomial time. The main purpose of the present paper is to fill in all the details of the algorithm and determine its complexity. Let, as usual, n denote the number of vertices of Γ, and m the number of edges of Γ. We prove the following result.

Theorem 1 *There exists an algorithm with time complexity $O(n^2 + nm)$ and space complexity $O(n^2)$, which decides, for a graph Γ, whether Γ is an ℓ_1-graph.*

We present such an algorithm in Section 3. The key part of it checks that certain graphs, constructed in terms of Γ, are isometric subgraphs of halved cubes. When singled out (in Section 2), this sub-algorithm allows us to claim the following.

Theorem 2 *There exists an algorithm with time complexity $O(n^2+nm)$ and space complexity $O(n^2)$, which decides whether a graph Γ can be isometrically embedded in a halved cube. The algorithm constructs an embedding, if one exists.*

The recognition problem for the general ℓ_1-metrics is NP-complete. The relevant references are [Chv] and [Kar].

Normally, the input graph Γ is represented by its adjacency matrix. In our case, however, it seems to be more convenient to work with the 2-dimensional array containing, for each vertex $v \in \Gamma$, the list of all the neighbours of v. The size of this array is $O(m)$ and it takes $O(n^2)$ time to construct it starting from the adjacency matrix. In what follows we assume that the array is known from the very beginning.

Notice also, that, naturally, ℓ_1-graphs and, in particular, isometric subgraphs of halved cubes are all connected. If Γ is connected then $m \geq n-1$; this means that, for connected inputs, the complexity $O(n^2+nm)$ simplifies in both Theorems 1 and 2 to $O(nm)$.

In the final section of the paper we introduce the notion of an s-isometric embedding (embedding isometric to distance s). The ideas, which we used for the results above, apply to some extent also to the s-isometric embeddings. We discuss a nice example: the football graph (the skeleton graph of the truncated icosahedron; also known, in chemistry, as the fulleren graph C_{60}) has an embedding into a 20-dimensional halved cube, which is isometric to distance 7! (The diameter of the football graph is 9.) We prove the uniqueness of this embedding.

Theorem 3 *The football graph has a unique 3-isometric embedding into a halved cube.*

Throughout the paper the input graph (of whatever algorithm) is denoted Γ. To simplify the notation, we let d denote the distance function d_Γ.

2 Isometric embeddings into halved cubes

If ϕ is an isometric embedding of a graph Γ into HQ_k then, for an edge $e = \{x, y\}$ of Γ, the subset $p(e) = \phi(x) \triangle \phi(y) \subseteq \Omega$ has size 2. We call $p(e)$ the *label* of e. The algorithm below attempts to construct an isometric embedding of an arbitrary Γ by defining the labels of specially chosen edges. The method is based on the following observation.

Lemma 2.1 *Suppose $e = \{x, y\}$ and $e' = \{z, t\}$ are two edges of Γ and suppose, for some vertex u, we have $d(u, y) = d(u, x) + 1$ and $d(u, t) = d(u, z) + 1$. Then $|p(e) \cap p(e')| = -d(y, t) + d(x, t) + d(y, z) - d(x, z)$.*

Proof. The mapping $\phi'(x) = \phi(x) \triangle \Omega_0$, where Ω_0 is a fixed subset of Ω, is an isometric embedding, moreover, ϕ and ϕ' define exactly the same labels on the edges of Γ. Therefore we may assume without loss that $\phi(u) = \emptyset$. Lemma 2.2 from [Sh] implies that $|\phi(a) \cap \phi(b)| = d(u, a) + d(u, b) - d(a, b)$ (in our case $\lambda = 2!$). By assumption, $\phi(x) \subset \phi(y)$ and $\phi(z) \subset \phi(t)$. Therefore,

$|p(e) \cap p(e')| = |\phi(y) \cap \phi(t)| - |\phi(x) \cap \phi(t)| - |\phi(y) \cap \phi(z)| + |\phi(x) \cap \phi(z)|$. Substituting each of these intersection sizes with its expression in terms of the distances, and cancelling, we end up with the formula $|p(e) \cap p(e')| = -d(y,t) + d(x,t) + d(y,z) - d(x,z)$. □

The lemma gives us a practical method to find the relation between the labels of two edges. With this in mind, we can now define the algorithm. We assume that for each vertex $x \in \Gamma$ a list is given, containing all the neighbours of x.

STEP 1. *Find the distance matrix of the graph.*

It is well-known that this computation requires $O(mn)$ time and $O(n^2)$ space. If Γ is disconnected, it cannot be isometrically embedded into HQ_k, hence it should be rejected. From now on we assume that Γ is connected.

For the rest of the algorithm we fix a vertex $u \in \Gamma$.

STEP 2. *Construct a spanning tree Λ for Γ, such that $d_\Lambda(u,x) = d(u,x)$ for all $x \in \Gamma$.*

This step requires $O(m)$ time. Indeed, we initiate two lists as follows: $V = \{u\}$ and $E = \emptyset$, and mark u to indicate that it is contained in V. For each vertex $y \in V$ and for each neighbour y' of y we check whether y' is contained in V (*i.e.*, whether it has been marked previously). If it is contained in V, take the next neighbour of y, or the next y. Otherwise, add y' to the end of V, add the edge $\{e, e'\}$ (in this order) to the end of E and mark e' as taken care of. Since Γ is connected, when the algorithm stops, V contains all vertices of Γ. The graph $\Lambda = (V, E)$ is the desired spanning tree. During the computation every edge of Γ is touched twice, hence the complexity is $O(m)$, as claimed.

Clearly, $|E| = n - 1$. Whenever $\{x, y\}$ is an edge from E, we have that $d(u,x) < d(u,y)$. According to Lemma 2.1, the labels (with respect to an arbitrary embedding ϕ, if any such exists) of two edges $e = \{x, y\}$ and $e' = \{z, t\}$ from E intersect each other in exactly $i(e,e') = -d(y,t) + d(x,t) + d(y,z) - d(x,z)$ elements. In particular, $i(e,e')$ must be nonnegative for all $e, e' \in E$. (It is easy to see that, in general, $i(e,e') \in \{-2,-1,0,1,2\}$.)

STEP 3. *Compute the function i. Check that it is nonnegative.*

Since $|E| = n - 1$, this step has time and space complexity $O(n^2)$. Next, we utilize the fact that i does not depend on ϕ, hence it can be used to recover the equivalence relation on E defined by (equal) labels.

STEP 4. *Construct the relation on E defined as follows:*

$$e \sim e' \text{ if and only if } i(e,e') = 2,$$

and verify that this is an equivalence relation; pick a representative in each equivalence class.

Clearly, if $\sim$ is not an equivalence then Γ does not have an isometric embedding into a halved cube.

We do everything in parallel. With each edge $e \in E$ we associate a reference to the first edge $e' \in E$ with $e \sim e'$ (e' might be equal to e). This can be arranged as two nested cycles. For each $e \in E$ and for each $e' \in E$, which precedes e, we do the following. (For each of the two cycles we assume the natural order of E.) If $e \not\sim e'$ we proceed with the next pair (e,e'). Otherwise, if e is assigned a reference to an edge e'', we check that e' also refers to $e''' = e''$ (if $e''' \neq e''$ then $e \not\sim e'''$ or $e' \not\sim e'''$; in either case $\sim$ can not be an equivalence). Finally, if e hasn't been assigned a reference, we have two possibilities. If e' refers to itself, add a reference from e to e'. If e' refers to a previous edge e'' then again $\sim$ is not an equivalence, since $e \not\sim e''$.

If this procedure is completed without $\sim$ being rejected then, clearly, $\sim$ *is* an equivalence relation and for each $e \in E$ we know the first edge in the equivalence class of e. In what follows we denote the equivalence class of e by $[e]$; $f_{[e]}$ (or simply, f_e) is the first edge in $[e]$.

STEP 5. *Define a graph Σ on the set of equivalence classes by letting $[e]$ and $[e']$ be adjacent whenever $i(e,e') = 1$. Check that this is well-defined, that is, $i(e,e')$ does not depend on the choice of edges in $[e]$ and $[e']$.*

(If the check fails, Γ cannot be embedded into a halved cube.)

It suffices to verify that $i(e,e') = i(f_e, f_{e'})$ for all $e, e' \in E$. Therefore, we can accomplish Step 5 with complexity $O(n^2)$.

Now we are ready to assign real label to our equivalence classes. Each label should be a 2-element subset of a certain set Ω. Different classes should be given different labels. The labels of two adjacent classes should have an element in common. Clearly, such a labeling establishes an isomorphism between Σ and the line graph of a graph on Ω in which edges are all the labels.

Therefore, if Σ is not a line graph then Γ is not an isometric subgraph of a halved cube. This observation leads to our next step.

STEP 6. *Check that Σ is a line graph and construct a root graph Ω for Σ. If Σ is not a line graph, reject Γ.*

According to Lehot [Le], this computation requires $O(m')$ time, where m' is the number of edges in Σ. Let n' denote the number of vertices of Σ. Clearly, $n' \le |E| = n-1$. Therefore, $m' \le (n')^2 < n^2$. We conclude that the complexity of this step is $O(n^2)$.

With Step 6 done, every equivalence class $[e]$ (and hence every edge e of Λ) is given a label $p(e)$, a 2-element subset in a set Ω. It remains to construct the embedding of Γ into the halved cube defined by Ω.

STEP 7. *For each vertex $x \in \Gamma$ construct its image $\phi(x) \subseteq \Omega$.*

We set $\phi(u) = \emptyset$ and then repeat the following in a cycle: read the next vertex $y \in V$ and the next edge $\{y', y\} \in E$. As y' precedes y in V, the set $\phi(y')$ has been determined previously. Set $\phi(y) = \phi(y') \cup p(\{y', y\})$.

Step 7 requires $O(n^2)$ time and space. Indeed, V contains all the n vertices of Γ, while the size of Ω is no greater than $2|\Sigma| \le 2(n-1)$.

The total complexity of this algorithm is as indicated in Theorem 1. We claim that the input graph Γ that successfully passed the tests of Steps 1 (connectivity), 3 (i nonnegative), 4 (equivalence $\sim$ well-defined), 5 (graph Σ well-defined) and 6 (Σ a line graph), this Γ is necessarily an isometric subgraph of a halved cube.

Lemma 2.2 *The mapping ϕ constructed by the above algorithm is an isometric embedding of Γ into the halved cube defined by Ω.*

Proof. The labels $p(e)$, $e \in E$, constructed at Step 6, have the following property: if $e = \{x, y\}$ and $e' = \{z, t\}$ are two edges from E then $|p(e) \cap p(e')| = -d(y,t) + d(x,t) + d(y,z) - d(x,z)$.

Suppose that the claim of the lemma is false, that is, there exist pairs $y, t \in \Gamma$ with $|\phi(y) \triangle \phi(t)| \neq 2d(y,t)$. Choose such a pair with $s = d(u,y) + d(u,t)$ minimal. First consider the case $y = u$ or $t = u$. Without loss of generality we may assume that $y = u$. The path in Λ from u to t is geodetic in Γ. If e and e' are two edges on this path then, as we can see from the distances, $p(e) \cap p(e') = \emptyset$. Therefore, the labels along the path are disjoint, that is, $|\phi(t)| = 2d(u,t)$. This proves that $y \neq u$ (and, of course, also $t \neq u$).

Let $e = \{x, y\}$ (respectively, $e' = \{z, t\}$) be the edge from E corresponding to $y \in V$ (respectively, $t \in V$). Let $X = \phi(x)$, $Y = \phi(y)$, $Z = \phi(z)$ and $T = \phi(t)$. Clearly, $|Y \triangle T| = |Y| + |T| - 2|Y \cap T| = 2[d(u,y) + d(u,t) - |Y \cap T|]$. On the other hand, as $Y \setminus X = p(e)$ and $T \setminus Z = p(e')$, we have $|Y \cap T| = |p(e) \cap p(e')| + |X \cap T| + |Y \cap Z| - |X \cap Z|$. By our choice of y and t (minimality of $d(y,t)$), we have $|X \triangle T| = 2d(x,t)$, which implies $|X \cap T| = d(u,x) + d(u,t) - \frac{1}{2}|X \triangle T| = d(u,x) + d(u,t) - d(x,t)$. Similarly, $|Y \cap Z| = d(u,y) + d(u,z) - d(y,z)$ and $|X \cap Z| = d(u,x) + d(u,z) - d(x,z)$. Also, $p(e) \cap p(e') = -d(y,t) + d(x,t) + d(y,z) - d(x,z)$. Substituting all this into our formula for $|Y \cap T|$ and cancelling, we end up with $|Y \cap T| = -d(y,t) + d(u,y) + d(u,t)$. Therefore, $|Y \triangle T| = 2(d(u,y) + d(u,t) - |Y \cap T|) = 2d(y,t)$. This conclusion contradicts our choice of y and t. $\square$

Clearly, Lemma 2.2 implies Theorem 2.

3 The main algorithm

Our algorithm detecting ℓ_1-graphs consists of two steps.

STEP 1. *Construct the canonical direct product graph* $\hat{\Gamma} = \Gamma_1 \times \Gamma_2 \times \ldots \times \Gamma_s$ *related to* Γ*, as defined by Graham and Winkler* [GW].

Aurenhammer and Hagauer [AH] demonstrated that this step can be performed in $O(nm)$ time and $O(n^2)$ space.

It was proved in [Sh] that Γ is an ℓ_1-graph if and only if each factor Γ_i is. Moreover, an idecomposible graph ("idecomposible" means that $\Gamma = \hat{\Gamma}$; each factor Γ_i has this property) Γ is an ℓ_1-graph if and only if Γ is a subgraph of a cocktail party graph $K_{r \times 2}$, or Γ allows an isometric embedding into a halved cube. This justifies

STEP 2. *For each factor* Γ_i *determine whether it is a subgraph of a cocktail party graph. If not, check that* Γ_i *allows an isometric embedding into a halved cube. If this check fails for some* i*, report that* Γ *is not an* ℓ_1*-graph and quit. Otherwise, report that it is an* ℓ_1*-graph.*

Let n_i and m_i be the number of vertices and the number of edges of Γ_i. Being a subgraph of a cocktail party graph means simply that for each vertex there is at most one other vertex non-adjacent to it. This property can clearly be checked for Γ_i in $O(n_i^2)$ time and space. According to Section 2, $O(n_i m_i)$ time and $O(n_i^2)$ space suffices in order to determine whether Γ_i does or does not allow an isometric embedding into a halved cube. We claim that the total time complexity of Step 2 is $O(mn)$. Indeed, according to [GW], for each i, $n_i \leq n$ and $m_1 + m_2 + \ldots + m_s = m$. Therefore, $n_1 m_1 + n_2 m_2 + \ldots + n_s m_s \leq nm$. Similarly, the total space complexity $O(n_1^2 + n_2^2 + \ldots + n_s^2)$ is $O(n^2)$. Therefore, the overall complexity of the algorithm is as claimed in Theorem 1.

4 Embeddings isometric to distance s

For connected graphs Γ and Δ, we call a mapping $\phi : \Gamma \longrightarrow \Delta$ an *s-isometric embedding* if $d_\Delta(\phi(x), \phi(y)) = d_\Gamma(x, y)$ for all $x, y \in \Gamma$ with $d_\Gamma(x, y) \leq s$. As an example, a 1-isometric embedding is any mapping sending edges to edges.

The methods from Section 2 can also be used for investigation of s-isometric embeddings of graphs into halved cubes. Indeed, let ϕ be an s-embedding of a graph Γ into a halved cube Δ, with $s \geq 1$. For an edge $e = \{x, y\}$ of Γ, the set $p(e) = \phi(x) \triangle \phi(y)$ is a 2-element subset in the underlying set Ω of Δ. As in Section 2, we call $p(e)$ the *label* of e.

The main tool from Section 2, Lemma 2.1 can also be generalized for our new setting.

Lemma 4.1 *Suppose $e = \{x, y\}$ and $e' = \{z, t\}$ are two edges of Γ and suppose, for some vertex u, we have $d(u, y) = d(u, x) + 1$ and $d(u, t) = d(u, z) + 1$. If the pairwise distances between u, x, y, z and t do not exceed s then $|p(e) \cap p(e')| = -d(y, t) + d(x, t) + d(y, z) - d(x, z)$.* □

The proof repeats the one given for Lemma 2.1. Notice that although u does not participate in the formula for $|p(e) \cap p(e')|$ it plays an important role in the proof and we cannot skip it in the assumption part of the lemma. On the other hand, we only have to garantee that such an element u exists. Say, if $d(x, z) \neq d(x, t)$, we can take $u = x$ (if $d(x, z) > d(x, t)$, we also have to switch the roles of z and t).

For large s, Lemma 4.1 allows to find the intersection of labels for many pairs of edges and those intersection sizes do not depend on a particular embedding. We single out the following special cases.

Lemma 4.2 (1) *If $x_1, x_2, \ldots, x_t$ is a geodetic path in Γ of length $\leq s$, then the labels along the path are pairwise disjoint.*

(2) *If C is an isometric cycle in Γ of length $n = 2t$, $t \leq s$, then the opposite edges on C have equal labels. The labels of non-opposite edges are disjoint.*

(3) *If C is an isometric cycle in Γ of length $2t + 1$, $t \leq s$, then the labels of opposite edges (that is, the edges which are at the maximal possible distance from each other) on C have exactly one element in common. The labels of non-opposite edges are disjoint.*

Proof. A geodetic path is an isometric subgraph of Γ, so in each case we have a subgraph (a path or a cycle) which is isometric in Γ. This means that the distances between the vertices of the subgraph can be determined within the subgraph itself. We apply Lemma 4.1 with u being a suitable vertex of the subgraph. □

Let Γ be the football graph, *i.e.*, the skeleton graph of the truncated icosahedron. The icosahedron has 12 vertices and 20 triangular faces, each vertex adjacent to 5 faces. Therefore, the truncated icosahedron has 12 pentagonal and 20 hexagonal faces. The football graph is shown in Fig. 1, the 12 pentagonal faces being shadowed. Notice that the faces are the only 5- and 6-cycles in Γ, and that they are isometric subgraphs of Γ.

The automorphism group of the footbal graph is the Coxeter group $H_3 \cong Alt_5 \times 2$. The group is transitive on vertices, on pentagonal and hexagonal faces. It has two orbits on edges: (1) the edges adjacent to two hexagons, and (2) edges adjacent to a hexagon and a pentagon.

Let us apply labels to construct and investigate the only 3-isometric embedding of Γ into a halved cube. We start with some properties of any such embedding ϕ.

Since $s \geq 3$, Lemma 4.2 (2) implies that the opposite edges of a hexagonal face bear equal labels. This can be extended by transitivity to an equivalence relation on the edge set of Γ; there are 30 equivalence classes, each consisting of three edges—one (the middle one) of type (1) and two of type (2) (an example of an equivalence class can be seen in Fig. 1; say, S is one of them). Let us call the equivalence classes *triplets*. Clearly, the three edges in a triplet all have the same label, so that the labels in fact correspond to triplets. We call two triplets *dependent* if they are represented by opposite edges of a pentagonal face. According to Lemma 4.2, the labels of dependent triplets have exactly one element in common.

The three triplets in Fig. 1 (denoted S, T and U) are pairwise dependent.

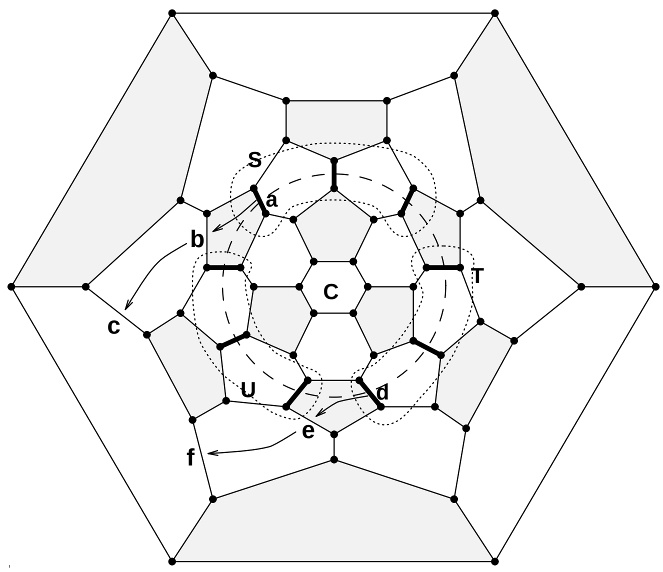

Figure 1: The footbal graph, triplets, a circle and the proof of Lemma 4.3

These nine edges separate the vertices, which are at distance at most 2 from the central hexagon C, from the rest of the vertices. Because of this, we call the nine edges (three triplets) a *circle* (of radius $2\frac{1}{2}$!) around C.

Lemma 4.3 *The labels of the three triplets in a circle have a common element.*

Proof. Suppose the labels of the triplets S and T (as in Fig. 1 share an element i, and suppose i is not contained in the label of U. Since every label consists of two elements, it follows from Lemma 4.2 (3) that the label of an edge e of a pentagonal face is contained in the union of the labels of the two edges opposite to e. Let us apply this observation to the edge a in Fig. 1. This edge is contained in S, therefore, $i \in p(a)$. Since i is not contained in the label of U, by the above comment, $i \in p(b)$. The edges b and c are in the same triplet. Hence $i \in p(c)$. Similarly, starting from the edge $d \in T$, we conclude that $i \in p(f)$. However, c and f are non-opposite edges of a hexagonal face. By Lemma 4.2, their labels must be disjoint; a contradiction. □

This result suggests that the elements of Ω may be in a one-to-one correspondence with the circles.

Lemma 4.4 *The elements of Ω, which appear in labels, bijectively correspond to the hexagonal faces. For a hexagon C and an edge e, the element corresponding to C appears in $p(e)$ if and only if e is contained in the circle around C.*

Proof. It suffices to show that the labels of the edges from two different circles cannot all have an element in common. As indicated above, circles correspond to the hexagonal faces. The group $\mathrm{Aut}\,\Gamma$ has 5 orbits on the pairs (C, C'), $C \neq C'$, of hexagons, depending on whether the distance between C and C' is 0, 1, 3, 5, or 7. Therefore, for a fixed C, we have to check 5 hexagons $C' = C_0, C_1, C_3, C_5$ and C_7. This is shown in Fig. 2. The edges e and f belong to the circle around the hexagon C. Similarly, for $i \in \{0, 1, 3, 5, 7\}$, e_i is contained in the circle around C_i. It follows from Lemma 4.2 (1) that the labels of e_0, e_1, e_3 and e_7 are disjoint from $p(e)$. Also, by the same lemma, $p(e_5)$ is disjoint from $p(f) = p(e)$. □

Corollary 4.5 *Up to isomorphism, there is only one way to assign labels to edges.*

Proof. Every triplet is contained in exactly two circles. Say, in Fig. 2 the triplet T is contained in the circles around C and C_5. The rest follows from Lemma 4.4. □

So far we have recovered the labels on the edges of Γ. Let us now construct the embedding itself. Notice that if ϕ is an s-isometric embedding into the

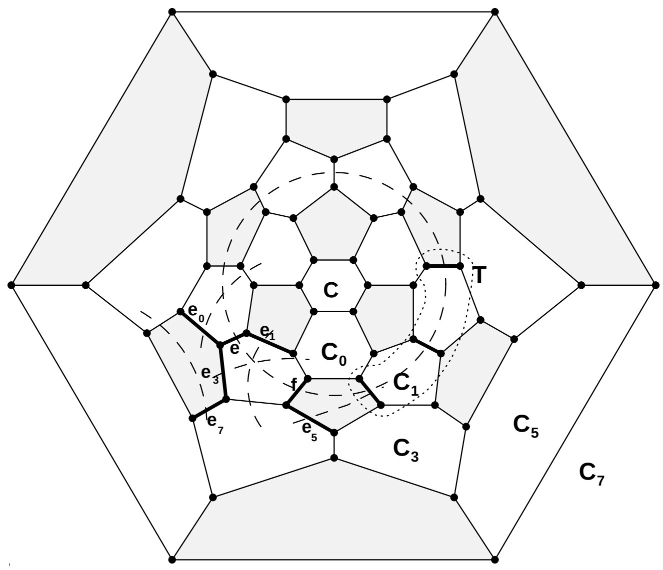

Figure 2: Proof of Lemma 4.4

halved cube Δ defined by Ω and A is any (even size) subset of Ω then $x \mapsto \phi(x) \triangle A$ is also an s-isometric embedding of Γ into the same halved cube Δ. We say that the embeddings ϕ and $\phi \triangle A$ are *equivalent up to a shift.* Let us fix a vertex $v \in \Gamma$. Up to a shift we may assume that $\phi(v) = \emptyset$. Since $s \geq 1$, we then have that, for $x \in \Gamma$,

$$\phi(x) = p(\{x_0, x_1\}) \triangle p(\{x_1, x_2\}) \triangle \ldots \triangle p(\{x_{t-1}, x_t\}),$$

where $v = x_0, x_1, \ldots, x_t = x$ is a geodetic (shortest) path joining v and x. Indeed, this follows from the definition: $p(\{x_{i-1}, x_i\}) = \phi(x_{i-1}) \triangle \phi(x_i)$.

Since $\phi(x)$ has been expressed in terms of the labels only, Corollary 4.5 implies the following

Proposition 4.6 *Up to isomorphism there is at most one* 3*-isometric embedding of* Γ *into a halved cube.* $\square$

Let us now construct this unique embedding. The preceeding discussion hints us how to do this. We formally define Ω to be the set of all the hexagonal faces in Γ. For an edge $e \in \Gamma$, we define $p(e)$ to be the set of all the hexagons $C \in \Omega$ with the property that e belongs to the circle around C. As we noticed above, e is contained in two circles, that is, $|p(e)| = 2$. This gives us all the labels. To define the embedding ϕ, pick an arbitrary vertex $v \in \Gamma$. We set $\phi(v) = \emptyset$. For $x \in \Gamma$, $x \neq v$, and for a shortest path $v = x_0, x_1, \ldots, x_t = x$, we define $\phi(x)$ as above, as the symmetric difference of all the labels along the path.

Lemma 4.7 *The embedding ϕ is well-defined, i.e., $\phi(x)$ does not depend on the path chosen.*

Proof. It is easy to see that the labels are chosen in a way that garantees that the statements (2) and (3) from Lemma 4.2 hold for pentagons and hexagons. In particular, it follows that the symmetric difference of the labels along a pentagon or a hexagon is empty. Since the football graph is drown on a sphere with all faces being pentagons and hexagons, and since the sphere is simply connected, any two paths having the same end points are equivalent modulo inserting/removing subpentagons and subhexagons; the claim follows. □

The mapping ϕ, which is now known to be well-defined, maps every vertex of Γ to a even size subset of Ω, that is, to a vertex of the corresponding halved cube. It is immediate that ϕ is 1-isometric, which only reflects the fact that all the labels have size 2. Let us determine the maximum s, for which ϕ is s-isometric. Being s-isometric means in our case (when we embed Γ into a halved cube) that, for $x, y \in \Gamma$ with $d(x, y) = t \leq s$, we have $|\phi(x) \triangle \phi(y)| = 2t$. Since every label has size 2, this is equivalent to the following: labels on a geodetic path of length $\leq s$ are pairwise disjoint.

Lemma 4.8 *Labels on a geodetic path of length ≤ 7 in Γ are pairwise disjoint. In particular, ϕ is 7-isometric.*

Proof. The labels of two edges e and e' are not disjoint if and only if e and e' belong to the same circle. Therefore, we have to check that no geodetic path of length ≤ 7 crosses a circle twice. Since $\mathrm{Aut}\,\Gamma$ is transitive on the circles, it suffices, for a fixed circle, to check that no geodetic path of length ≤ 7 crosses it twice. In Fig. 3 we see a circle $\mathcal{C}$. If a path crosses $\mathcal{C}$ twice then

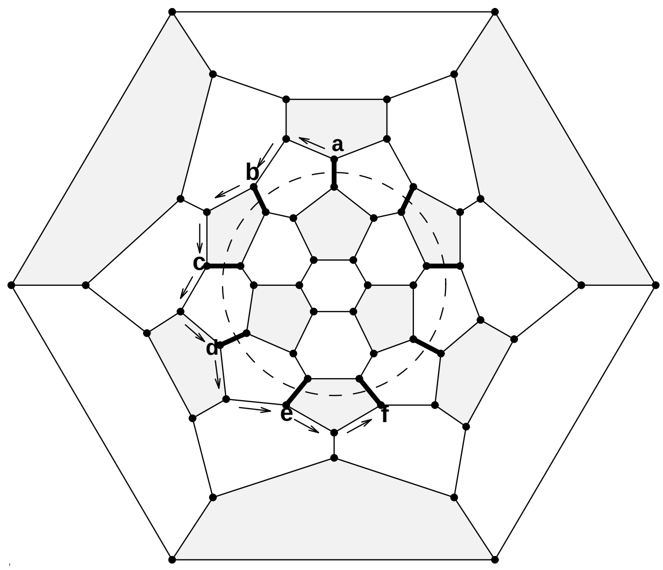

Figure 3: Proof of Lemma 4.8

either both ends are inside the circle, or both ends are outside. It is easy to see that every shortest path between two vertices inside $\mathcal{C}$ is itself fully inside $\mathcal{C}$. Therefore, we only have to check the paths with both ends outside $\mathcal{C}$. Without loss of generality, we may assume that the path starts and ends just outside the circle, in one of the vertices a, b, ... Up to automorphisms, all the variants for the end points are represented by the following pairs: (a, b), (a, c), (a, d), (a, e), (b, c), (b, e), (b, f) and (c, e). The pairs (a, e) and (b, f) represent distance $8 > 7$. For all the remaining pairs the path just outside the circle (depicted by the arrows) is shorter than any path through $\mathcal{C}$. $\square$

Notice that in case of (a, e) and (b, f) there is a geodetic path joining the two vertices and cutting across the circle. This means that ϕ is *not* 8-isometric. In fact, if $x, y \in \Gamma$ with $d(x, y) = 8$ or 9 then always $|\phi(x) \triangle \phi(y)| = 14$, which means that the distance function $d'3(x, y) = \min(d(x, y), 7)$, $x, y \in \Gamma$, is an ℓ_1-metric.

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