The Combinatorial Structure of Small Cut and Metric Polytopes

Antoine DEZA
Michel DEZA

Laboratoire d'Informatique, URA 1327 du CNRS
Département de Mathématiques et d'Informatique
Ecole Normale Supérieure

LIENS - 95 - 1

January 1995

The Combinatorial Structure of Small Cut and Metric Polytopes

Antoine DEZA Michel DEZA

January 1995

Abstract. We study the combinatorial structure of the cut and metric polytopes on n nodes for $n \leq 5$. Those two polytopes have a complicated geometrical structure, but using their large symmetry group, we can completely describe their face lattices. We present, for any n, some orbits of faces and give new result on the tightness of the wrapping of the cut polytope by the metric polytope, disproving a conjecture of [14] on their lattices.

Key words: metric polytope, complete bipartite subgraphs polytope, face lattice.

1 Introduction

We first recall the definition of the *metric polytope* m_n, the *cut polytope* c_n, and their relatives, the *metric cone* and the *cut cone*. Then we present some applications to well known optimization problems and some combinatorial and geometric properties of those polyhedra. The general references are [4, 24] for polytopes and [5] for graphs. For a complete study of the applications and the combinatorial optimization aspects of those polyhedra, we refer, respectively, to the surveys [13] and [21].

For all 3-sets $\{i, j, k\} \subset \{1, \ldots, n\}$, we consider the following inequalities:

$$x_{ij} - x_{ik} - x_{jk} \leq 0 \tag{1}$$

$$x_{ij} + x_{ik} + x_{jk} \leq 2. \tag{2}$$

The inequalities (1) define the metric cone and the metric polytope m_n is obtained by bounding the latter by the inequalities (2). The $3\binom{n}{3}$ facets defined by the inequalities (1), which can be seen as triangle inequalities for distance x_{ij} on $\{1, 2, \ldots, n\}$, are called *homogeneous triangle facets* and are denoted by $Tr_{ij,k}$. The $\binom{n}{3}$ facets defined by the inequalities (2) are called *non-homogeneous triangle facets* and are denoted by Tr_{ijk}.

Given a subset S of $V_n = \{1, 2, \ldots, n\}$, the *cut* determined by S consists of the pairs (i, j) of elements of V_n such that exactly one of i, j is in S. $\delta(S)$ denotes both the cut and its incidence vector in $\mathbb{R}^{\binom{n}{2}}$, that is, $\delta(S)_{ij} = 1$ if exactly one of i, j is in S and 0 otherwise for $1 \leq i < j \leq n$. By abuse of language, we use the term cut for both the cut itself and its incidence vector, so $\delta(S)_{ij}$ are considered as coordinates of a point in $\mathbb{R}^{\binom{n}{2}}$. The cut polytope of the complete graph c_n, which is also called the *complete bipartite subgraphs polytope*, is the convex hull of all 2^{n-1} cuts, and the cut cone is the conic hull of all $2^{n-1} - 1$ nonzero cuts. Those polyhedra were considered by many authors, see for instance [2, 3, 9, 11, 12, 13, 14, 15, 17, 18] and references therein. One of the motivations for the study of these polyhedra comes from their applications in combinatorial optimization, the most important being the max-cut and multicommodity flow problems.

Given a graph $G = (V_n, E)$ and nonnegative weights w_e, $e \in E$, assigned to its edges, the *max-cut* problem consists in finding a cut $\delta(S)$ whose weight $\sum_{e \in \delta(S)} w_e$ is as large as possible. It is a well-known NP-complete problem. By setting $w_e = 0$ if e is not an edge of G, we can consider without loss of generality the complete graph on V_n. Then the max-cut problem can be stated as a linear programming problem over the cut polytope c_n as follows:

$$\begin{cases} \max \quad w^T \cdot x \\ x \in c_n. \end{cases}$$

Since the metric polytope is a relaxation of the cut polytope, optimizing $w^T \cdot x$ over c_n instead of m_n provides an upper bound for the max-cut problem [3].

With E the set of edges of the complete graph on V_n, an instance of the *multicommodity flow problem* is given by two nonnegative vectors indexed by E: a capacity $c(e)$ and a requirement $r(e)$ for each $e \in E$. Let $U = \{e \in E : r(e) > 0\}$. If T denotes the subset of V_n spanned by the edges in U, then we say that the graph $G = (T, U)$ denotes the *support* of r. For each edge $e = (s, t)$ in the support of r, we seek a flow of $r(e)$ units between s and t in the complete graph. The sum of all flows along any edge $e' \in E$ must not exceed $c(e')$. If such a flow exists, we call c, r *feasible*. A necessary and sufficient condition for feasibility is given by the Japanese theorem [16]: a pair c, r is feasible if and only if $(c - r)^T x \geq 0$ is valid over the metric cone. For example, $Tr_{ij,k}$ can be seen as an elementary solvable flow problem with $c(ij) = r(ik) = r(jk) = 1$ and $c(e) = r(e) = 0$ otherwise, so the inequalities (1) correspond to $(c - r)^T x \geq 0$ for x in the metric cone. Therefore, the metric cone is the dual cone to the cone of feasible multicommodity flow problems.

2 Combinatorial and geometric properties of the cut and metric polytopes

The polytope c_n is a $\binom{n}{2}$ dimensional $0-1$ polyhedron with 2^{n-1} vertices and m_n is a polytope of same dimension with $4\binom{n}{3}$ facets inscribed in the cube $[0,1]^{\binom{n}{2}}$. We have $c_n \subseteq m_n$ with equality only for $n \leq 4$. It is easy to see that the point $\omega_n = (\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$ is the center of gravity of both c_n and m_n and is also the center of the sphere of radius $r = \frac{1}{2}\sqrt{n(n-1)}$ where all the cuts lie. Another two geometric characteristics of the cut polytope c_n are its *width* and *geometric diameter*. We recall that while the width of a polytope P is equal to the minimum distance between a pair of parallel hyperplanes containing P in the slice between them, the geometric diameter of P is the maximum distance between a pair of supporting hyperplanes. The width of c_n is 1 ([22]) and its geometric diameter is $\frac{n}{2}$ for n even and $\frac{1}{2}\sqrt{n^2-1)}$ for n odd, see [21]. Any facet, respectively ridge (that is, a face of codimension 2), of the metric polytope contains a facet, respectively a ridge, of the cut polytope and the vertices of the cut polytope are vertices of the metric polytope, in fact the cuts are precisely the integral vertices of the metric polytope. Actually the metric polytope m_n wraps the cut polytope c_n very tightly since, in addition to the vertices, all edges and 2-faces of c_n are also faces of m_n ([14]). In other words, c_n is a *segment of order* 2 of m_n and its dual, m_n^*, is a segment of order 1 of c_n^* in terms of [19]: a polytope P is a segment of order s of a polytope Q if they have the same dimension and if every i-face of P is a face of Q for $0 \leq i \leq s$. The polytope c_n is 3-neighbourly ([14]). Any two cuts are adjacent both on c_n ([3]) and on m_n ([20]); in other words m_n is *quasi-integral* in terms of [23], that is, the skeleton of the convex hull of its integral vertices, i.e. the skeleton of the c_n, is an induced subgraph of the edge graph of the metric polytope itself. While the diameter of m_n^* is 2 ([6]), the diameters of c_n^* and m_n are respectively conjectured to be 4 and 3 ([18, 6]). For a detailed study of the combinatorial and geometric properties of c_n and m_n, we refer to [8].

The metric polytope and the cut polytope share the same symmetry group, that is, the group of isometries preserving a polytope. This group is isomorphic to the automorphism group of the *folded n-cube*, that is, $Is(m_p) = Is(c_p) \approx Aut(\Box_n)$, see [11, 17]. We recall that the folded n-cube is the graph whose vertices are the partitions of $V_n = \{1, \ldots, n\}$ into two subsets, two partitions being adjacent when their common refinement contains a set of size one, see [5]. More precisely, for $n \geq 5$, $Is(m_n) = Is(c_n)$ is induced by permutations on $V_n = \{1, \ldots, n\}$ and *switching reflections by a cut*. Given a cut $\delta(S)$, the switching reflection $r_{\delta(S)}$ is defined by $y = r_{\delta(S)}(x)$ where $y_{ij} = 1 - x_{ij}$ if $(i,j) \in \delta(S)$ and $y_{ij} = x_{ij}$ otherwise. These symmetries preserve the adjacency relations and the linear independency. For the study of their face lattices, we frequently use the fact that the faces of m_n and c_n are partitioned into orbits of their symmetry group.

We finally mention the following link with metrics: there is an evident $1-1$ correspondence between the elements of the metric cone and all the semi-metrics on n points, and the elements of the cut cone correspond precisely to the semi-metrics on n points that are isometrically embeddable into some l_1^m, see [1], it is easy to check that $m \leq \binom{n}{2}$.

3 Face lattices of small cut and metric polytopes

3.1 Face lattice of the $c_n = m_n$ for $n \leq 4$

For $n \leq 4$, we have $c_n = m_n$, moreover c_3 and c_4 are both well-known polytopes. While c_3 is the regular tetrahedron of edge length $\sqrt{2}$ and volume $v_3 = \frac{1}{3}$, c_4 is combinatorially equivalent to the 6-dimensional cyclic polytope with 8 vertices and its volume is $v_4 = \frac{2}{45}$. The f-vector of c_4 is obviously $f(c_4) = (8, 28, 56, 68, 48, 16)$; more precisely all proper faces of c_4 are partitioned into the following orbits of the symmetry group $Is(c_4) \approx Aut(\Box_4)$:

- the 8 vertices of c_4 form the orbit O_0^1,
- the 2 orbits O_1^1 and O_1^2 of edges $\{\delta(S), \delta(S')\}$ are respectively formed by the 16 edges with $|S \triangle S'|$ odd and the 12 ones with $|S \triangle S'|$ even, (that is respectively represented by $\{\delta(\emptyset), \delta(1)\}$ and $\{\delta(\emptyset), \delta(1,2)\}$),
- the 2 orbits of 2-faces are: O_2^1 of size 48 which is represented by $\{\delta(\emptyset), \delta(2), \delta(1,2)\}$, and O_2^2 of size 8 which is represented by $\{\delta(1), \delta(2), \delta(3)\}$,
- the 3 orbits of 3-faces are: O_3^1 of size 12 which is represented by $\{\delta(\emptyset), \delta(1), \delta(2), \delta(1,2)\}$, O_3^2 of size 24 which is represented by $\{\delta(\emptyset), \delta(1), \delta(2), \delta(1,3)\}$, and O_3^3 of size 32 and represented by $\{\delta(\emptyset), \delta(1), \delta(2), \delta(3)\}$,
- the 48 ridges form the orbit O_4^1; they are the cofaces (that is the convex hull of the vertices not belonging to a face) corresponding to the 2-faces from the orbit O_2^1.
- the 16 facets form the orbit O_5^1; they are the cofaces corresponding to the edges from the orbit O_1^1.

Remark 3.1
(i) The skeleton of c_4^* is the (4×4)-grid, which is also the line graph of $K_{4,4} = \Box_4$.
(ii) A set of vertices is not a face of c_4^* if and only if it contains one of the following 2 sets of 4 vertices: $\{\delta(1), \delta(2), \delta(3), \delta(4)\}$ and $\{\delta(\emptyset), \delta(1,2), \delta(1,3), \delta(1,4)\}$.

3.2 Face lattices of c_5 and m_5

The face lattices of c_5 and m_5 were obtained in the following way. We first got all the non-simplex faces by systematically checking all possible pairwise, 3-wise etc. intersections of non-simplex facets. Then, considering all $2, 3$ and 4-sets of vertices and the remaining possible pairwise, 3-wise etc. intersections of facets, we obtained all i-faces for $i = 0, 1, 2, 3, 7, 8, 9$. Finally, noticing that few i-faces contains the complete lower part of the lattice (for example any 7-face of m_5 is a facet of a face belonging to a single orbit of 8-face), we found by a case by case analysis all the remaining simplex $4, 5$ and 6-faces of m_5 and c_5. The dimensions of the faces were computed using the list of all affine dependencies of m_5 and c_5 given below.

Using [10] one can that check all affine dependencies on the vertices of m_5 and c_5, that is equations $\sum \lambda_i x_i = 0$ with $\sum \lambda_i = 0$, are, up to permutations, switchings and the bijection $\delta(S) \leftrightarrow \hat{\delta}(S)$ (which clearly preserves affine dependencies):

- $\displaystyle\sum_{S\subset\{1,2,3,4\}} (-1)^{|S|}\delta(S) = 0,$

- $\displaystyle\sum_{S\subset\{1,2,3,4,5\},|\{4,5\}\cap S|=1} (-1)^{|S|}\delta(S) = 0,$

- $\displaystyle 2\delta(\emptyset) - 2\delta(1) + \sum_{i=2}^{5}(\delta(1,i) - \delta(i)) = 0,$

- $\displaystyle 3\hat{\delta}(\emptyset) + \delta(1) - \sum_{i=2}^{5}\delta(1,i) = 0.$

The restriction of the face lattices of c_5 and m_5 to their non-simplex faces are respectively given in Figures 3.1 and 3.8. All simplicial and all maximal (under inclusion) simplex faces are given in Proposition 3.2 and their complete face lattices are presented in detail in Section 3.2.1 and Section 3.2.2. The f-vectors of c_5 and m_5 are respectively:

- $f(c_5) = (16, 120, 560, 1780, \dots, 3080, 640, 56)$,

- $f(m_5) = (32, 280, 1280, 3620, \dots, 2840, 480, 40)$.

By f^i_j and g^i_j we respectively denote a representative j-face of the i^{th} orbit C^i_j, respectively M^i_j, of c_5 and m_5.

3.2.1 Face lattice of c_5

In Figure 3.1 all the 4 orbits of proper non-simplex faces of m_5 are given. Each orbit is represented by the set of vertices belonging to a representative face of the orbit. A cut $\delta(i)$, respectively $\delta(ij)$, is denoted by a circled point i and respectively by an edge $\{i, j\}$.

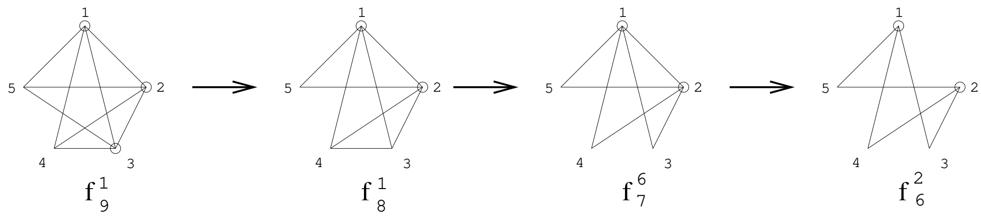

Figure 3.1: Non-simplices of the face lattice of c_5

The face lattice of c_5 is partitioned into the following orbits of $Is(c_5)$:

- the 56 facets are partitioned into the 2 orbits C_9^1 and C_9^2 respectively formed by the 40 triangle facets represented by $f_9^1 = Tr_{123}$ and the 16 switchings of the *equicut facet* f_9^2 which is defined by the inequality: $\sum_{1\le i<j\le 5} x_{ij} \le 6$,
- the 640 ridges are partitioned into the 3 orbits C_8^1, C_8^2 and C_8^3 of size 240, 240 and 160 and respectively represented by $f_8^1 = Tr_{123} \cap Tr_{124}$, $f_8^2 = Tr_{123} \cap Tr_{145}$ and $f_8^3 = Tr_{123} \cap f_9^2$,
- the 3080 7-faces are partitioned into the 7 orbits C_7^1, $C_7^2 \dots C_7^7$ of size 120, 960, 480, 240, 160, 160 and 960 respectively represented by the graphs given in Figure 3.2. We have: $f_7^1 = f_8^1 \cap Tr_{24,3} \cap Tr_{13,4}$, $f_7^2 = f_8^2 \cap Tr_{23,4}$, $f_7^3 = f_8^3 \cap Tr_{125}$, $f_7^4 = f_8^2 \cap f_9^2$ $(= f_8^3 \cap Tr_{145})$, $f_7^5 = f_8^1 \cap Tr_{134}$, $f_7^6 = f_8^1 \cap Tr_{125}$ and $f_7^7 = f_8^1 \cap Tr_{135}$,

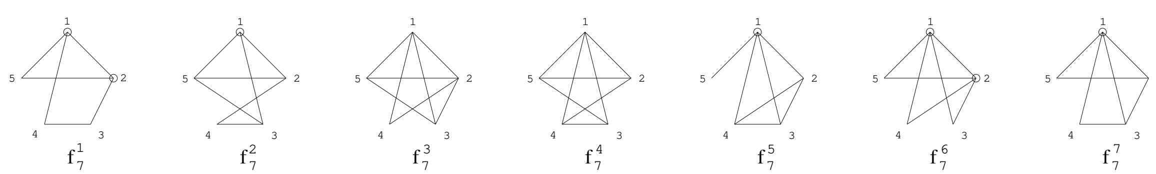

Figure 3.2: 7-faces of c_5

- the 6-faces are partitioned into the 10 orbits $C_6^1, \ldots, C_6^{10}$ respectively represented by the graphs given in Figure 3.3,

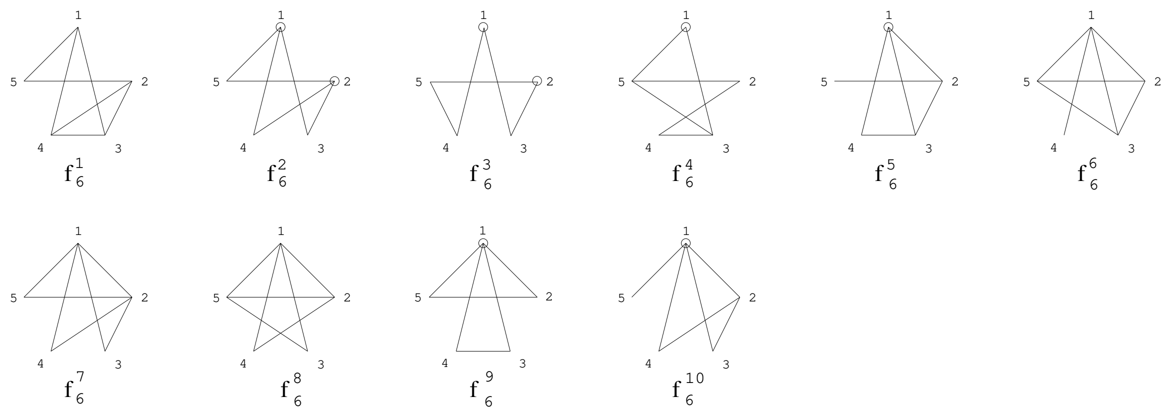

Figure 3.3: 6-faces of c_5

- the 5-faces are partitioned into the 11 orbits $C_5^1, \ldots, C_5^{11}$ respectively represented by the graphs given in Figure 3.4,

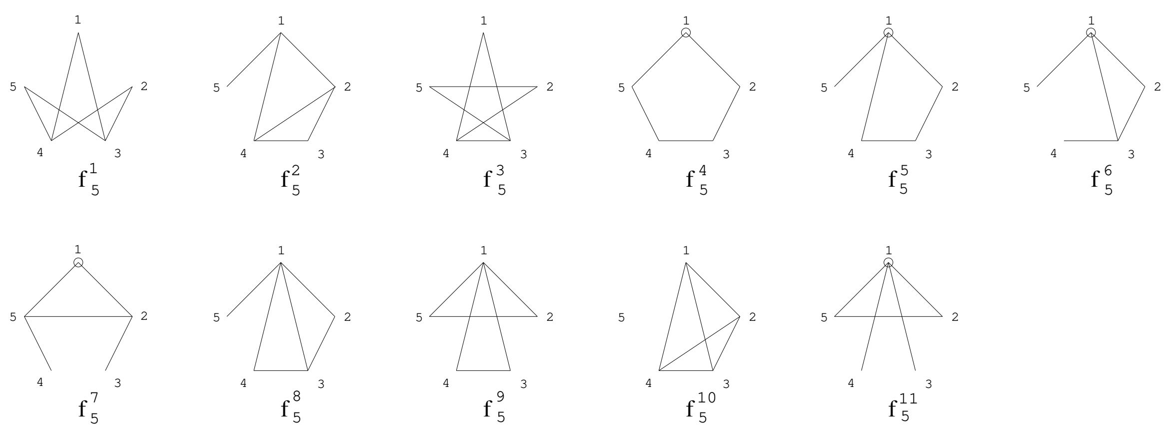

Figure 3.4: 5-faces of c_5

- the 4-faces are partitioned into the 8 orbits $C_4^1, \ldots, C_4^8$ respectively represented by the graphs given in Figure 3.5,

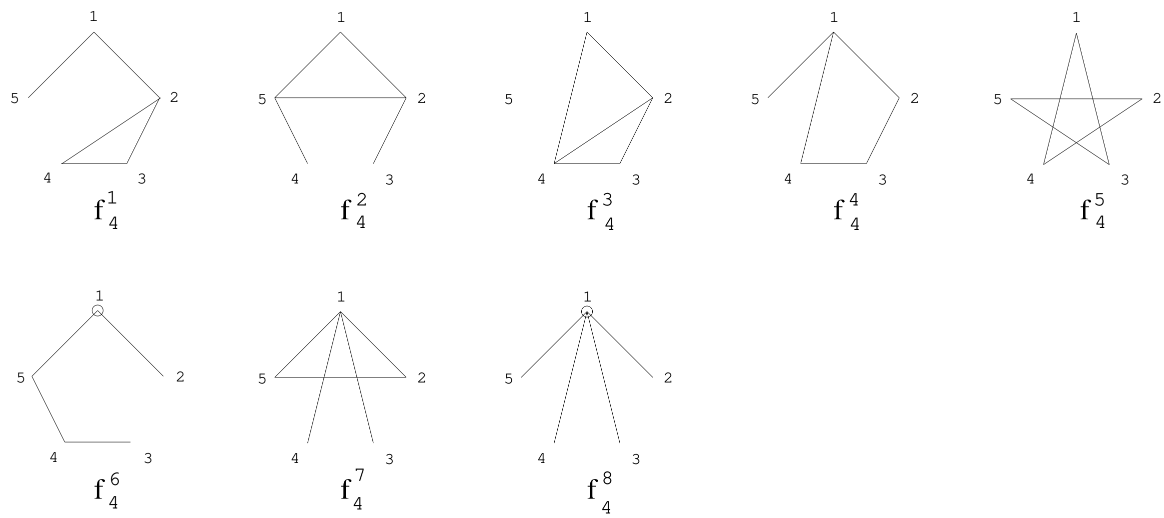

Figure 3.5: 4-faces of c_5

- the 1780 3-faces are partitioned into the 7 orbits $C_3^1, \ldots, C_3^7$ respectively represented by the graphs given in Figure 3.6. Actually, the only sets of 4 cuts which are not 3-faces of c_5 are the 40 members of the orbit of $\{\delta(\emptyset), \delta(1,2), \delta(1,3), \delta(2,3)\}$,

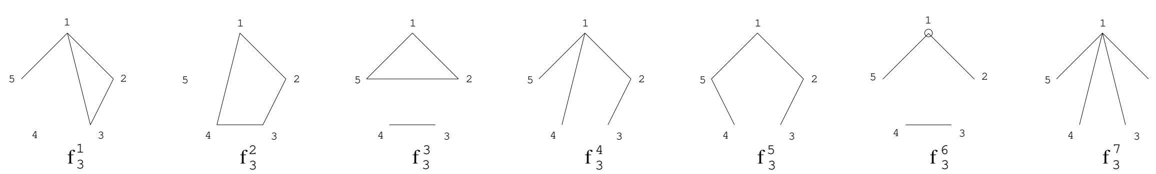

Figure 3.6: 3-faces of c_5

- the 560 2-faces are partitioned into the 3 orbits C_2^1, C_2^2 and C_2^3 of size 160, 160 and 240, and respectively represented by $f_2^1 = \{\delta(\emptyset), \delta(1), \delta(2)\}$, $f_2^2 = \{\delta(\emptyset), \delta(1,2), \delta(1,3)\}$ and $f_2^3 = \{\delta(\emptyset), \delta(1), \delta(2,3)\}$,

- the 120 edges are partitioned into the 2 orbits C_1^1 and C_1^2 respectively formed by the 40 edges $\{\delta(S), \delta(S')\}$ with $|S \triangle S'| = 1$ or 4 and the 80 ones with $|S \triangle S'| = 2$ or 3 (that is respectively represented by $f_1^1 = \{\delta(\emptyset), \delta(1)\}$ and $f_1^2 = \{\delta(\emptyset), \delta(1,2)\}$),

- the 16 vertices form the orbit C_0^1.

3.2.2 Face lattice of m_5

In Figure 3.8 all the 8 orbits of proper non-simplex faces of m_5 are given. As for c_5, each orbit is represented by the set of vertices belonging to one of its representative face. While a straight line links 2 incident faces, a dotted one links 2 faces incident up to a permutation. Besides the 16 cuts, the vertices of m_5 are the 16 anticuts $\hat{\delta}(S) = \frac{2}{3}(1,\ldots,1)$-$\frac{1}{3}\delta(S)$. A anticut $\hat{\delta}(i)$, respectively $\hat{\delta}(ij)$, is denoted by a grey circled point i and respectively by a grey edge $\{i,j\}$. The anticut $\hat{\delta}(\emptyset)$ is denoted by a grey $\emptyset$. Note that a face cannot contain both $\delta(S)$ and $\hat{\delta}(S)$.

The face lattice of m_5 is partitioned into the following orbits of $Is(m_5)$:

- the 40 triangle facets form the orbit M_9^1 represented by $g_9^1 = Tr_{123}$,

- the 480 ridges are partitioned into the 2 orbits M_8^1 and M_8^2, both of size 240 and respectively represented by $g_8^1 = Tr_{123} \cap Tr_{124}$ and $g_8^2 = Tr_{123} \cap Tr_{145}$,

- the 1880 7-faces are partitioned into the 6 orbits $M_7^1, M_7^2, \ldots, M_7^6$ of size 120, 960, 480, 160, 160 and 960 respectively represented by the graphs given in Figure 3.7. We have: $g_7^1 = g_8^1 \cap Tr_{24,3} \cap Tr_{13,4}$, $g_7^2 = g_8^1 \cap Tr_{35,4}$, $g_7^3 = g_8^1 \cap Tr_{345}$, $g_7^4 = g_8^1 \cap Tr_{134}$, $g_7^5 = g_8^1 \cap Tr_{125}$ and $g_7^6 = g_8^1 \cap Tr_{135}$,

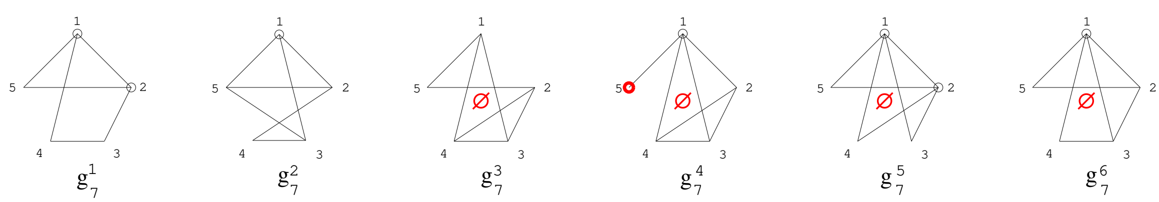

Figure 3.7: 7-faces of m_5

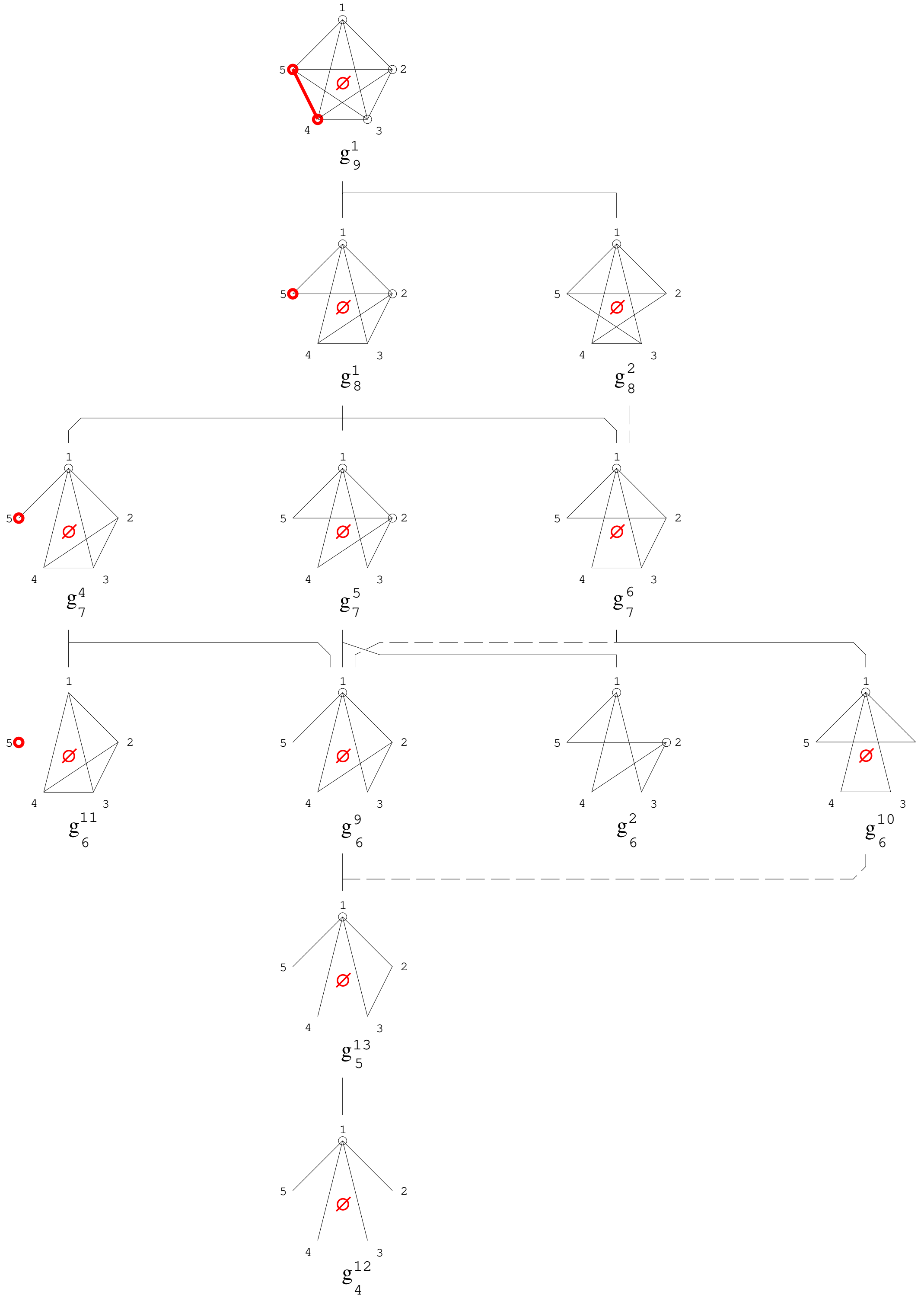

Figure 3.8: Non-simplices of the face lattice of m_5

- the 6-faces are partitioned into the 11 orbits $M_6^1, \ldots, M_6^{11}$ respectively represented by the graphs given in Figure 3.9,

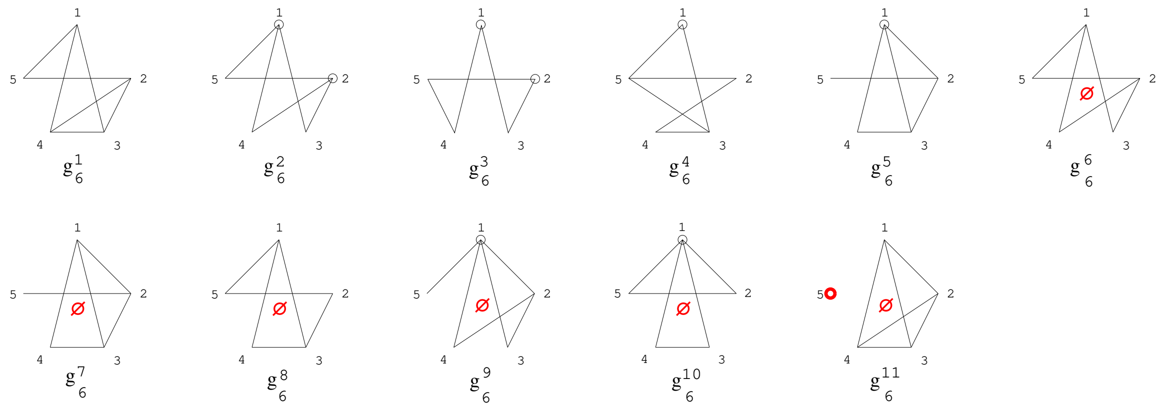

Figure 3.9: 6-faces of m_5

- the 5-faces are partitioned into the 13 orbits $M_5^1, \ldots, M_5^{13}$ respectively represented by the graphs given in Figure 3.10,

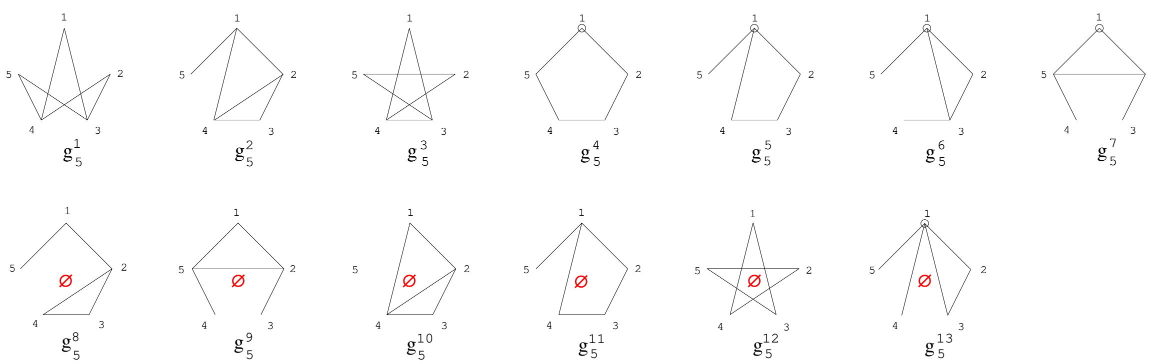

Figure 3.10: 5-faces of m_5

- the 4-faces are partitioned into the 12 orbits $M_4^1, \ldots, M_4^{12}$ respectively represented by the graphs given in Figure 3.11,

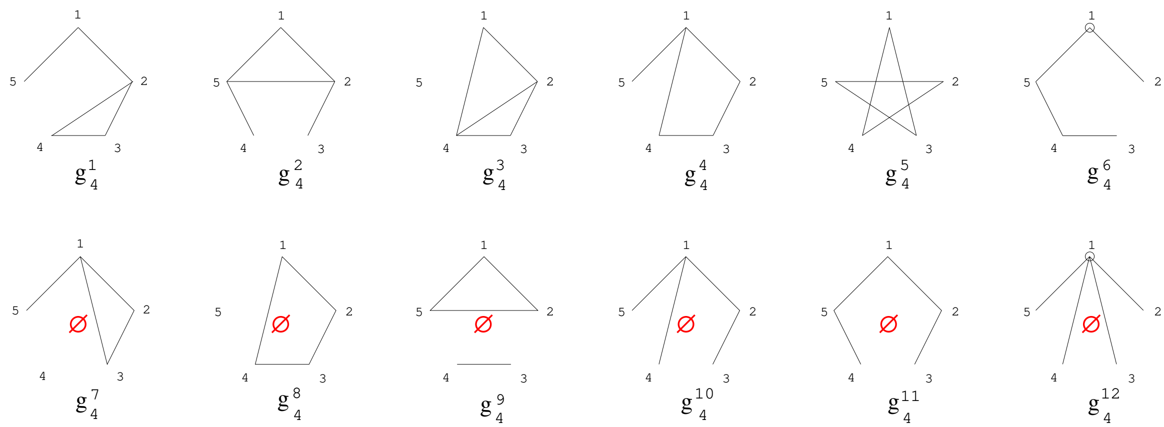

Figure 3.11: 4-faces of m_5

- the 3620 3-faces are partitioned into the 10 orbits $M_3^1, \ldots, M_3^{10}$ respectively represented by the graphs given in Figure 3.12,

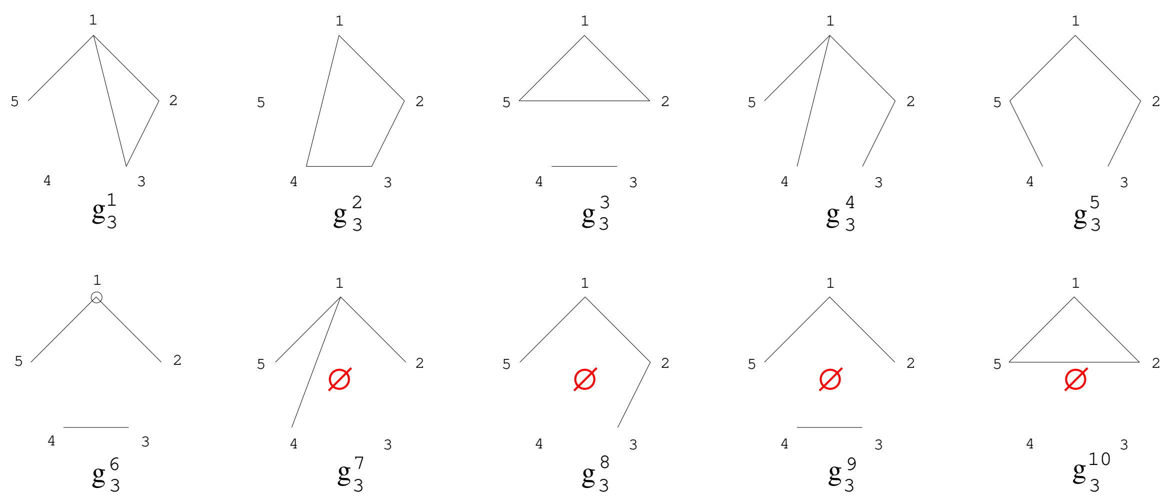

Figure 3.12: 3-faces of m_5

- the 1280 2-faces are partitioned into the 5 orbits M_2^1, $M_2^2, \ldots, M_2^5$ of size 160, 160, 240, 480 and 240, and respectively represented by $g_2^1 = \{\delta(\emptyset), \delta(1), \delta(2)\}$, $g_2^2 = \{\delta(\emptyset), \delta(1,2), \delta(1,3)\}$, $g_2^3 = \{\delta(\emptyset), \delta(1), \delta(2,3)\}$, $g_2^4 = \{\hat{\delta}(\emptyset), \delta(1,2), \delta(1,3)\}$ and $g_2^5 = \{\hat{\delta}(\emptyset), \delta(1,2), \delta(3,4)\}$,

- the 280 edges are partitioned into the 3 orbits M_1^1, M_1^2 respectively formed by the 40 edges $\{\delta(S), \delta(S')\}$ with $|S \triangle S'| = 1$ or 4 and the 80 ones with $|S \triangle S'| = 2$ or 3, and the orbit M_1^3 formed by the 160 edges $\{\hat{\delta}(S), \delta(S')\}$ with $|S \triangle S'| = 2$ or 3 (that is respectively represented by $g_1^1 = \{\delta(\emptyset), \delta(1)\}$, $g_1^2 = \{\delta(\emptyset), \delta(1,2)\}$ and $g_1^3 = \{\hat{\delta}(\emptyset), \delta(1,2)\}$).

- the 32 vertices are partitioned into the 2 orbits M_0^1 and M_0^2 respectively formed by the 16 cuts and the 16 anticuts.

Proposition 3.2
(i) The maximal (under inclusion) simplex faces of c_5 are the faces belonging to the orbits C_9^2, C_8^2, C_7^1 and C_7^5.
(ii) The maximal (under inclusion) simplex faces of m_5 are the faces belonging to the orbits M_7^1, M_7^2 and M_7^3.
(iii) The simplicial faces of c_5 and m_5 are respectively the faces the belonging to the orbit C_6^2 and M_6^{11}, M_6^2 and M_4^{12}.
(iv) The faces belonging to C_6^2 and M_6^2 are combinatorially equivalent to $m_4 = c_4$ and so to the 6-dimensional cyclic polytope with 8 vertices.

Proposition 3.3
(i) The maximal (under inclusion) simplex faces of m_5 containing an anticut are the faces belonging to the orbit M_7^3.
(ii) The number of simplex i-faces of m_5 containing an anticut is, for each anticut, $\hat{f}_i = 1, 10, 45, 120, 205, 222, 130, 30, 0, 0$ for $i = 0, 1, \ldots, 9$.

PROOF. Let g be a i-face of m_5, with $i \leq 6$, containing an anticut, for example $\hat{\delta}(\emptyset)$, and denote $g' := \bigcap_{\delta(ijk) \notin g} Tr_{ijk}$. Clearly, we have $g \subseteq g'$ with equality if and only if g is a simplex. Suppose that $\delta(i) \in g' - g$, it will mean that i is a universal vertex, that is, the graph $G(g)$ representing g is one of the following graphs:

It turns out that those 4 graphs are respectively subgraphs (we require only inclusion of the edge-set) of the graphs representing the non-simplex faces: g_6^9, $g_6^{10}4$ g_5^{13}, g_4^{12} (see Figure 3.8). Now, since g' is the intersection of homogeneous triangle facets, $\hat{\delta}(ij) \in g'$ implies that $\hat{\delta}(i), \hat{\delta}(j) \in g'$. Then $\hat{\delta}(i) \in g' - g$ if and only if $G(g)$ contains the clique K_4 which turns out to be a subgraph of the non-simplex face: g_6^{11}. □

3.3 Wrapping of c_n by m_n

Let call *extra i-face* of c_n (respectively m_n) a i-face of c_n (resp. m_n) which is not a i-face of m_n (resp. c_n). We recall that all i-faces of c_n are also i-faces of m_n for $i = 0, 1$ and 2; moreover, it was conjectured in [14] that for n large enough ($n > 2^i$) all i-faces of c_n are also i-faces of m_n. We disprove this conjecture by exhibiting an extra 3-face of c_n. For $n \geq 5$, let consider the following face of c_n:

$$f_3 = \{\delta(12), \delta(13), \delta(14), \delta(15)\}.$$

Proposition 3.4 *For $n \geq 5$, the face f_3 is an extra 3-face of c_n.*

Proof. For $n = 5$, the face f_3 is the 3-face f_3^7 of c_5 which is itself a 10-face of c_n ($c_5 = \bigcap_{i=6}^n (Tr_{12,i} \cap Tr_{2i,1})$), that is, f_3 is a 3-face of c_n. Now let suppose that f_3 is also a face of m_n, it would implies that f_3 is a face of the following face of m_n:

$$g = (\bigcap_{2 \leq i < j \leq 5} Tr_{1ij}) \cap (\bigcap_{i=6}^{n} (Tr_{12,i} \cap Tr_{2i,1})),$$

where the triangle facets are seen as facets of m_n. For $n = 5$, we have $g = g_4^{12}$. One can easily check that g contains, besides the 5 cuts $\delta(1), \delta(12), \delta(13), \delta(14)$ and $\delta(15)$, the vertex x which coordinates $x_{ij} : 1 \leq i \leq j \leq n$ are: $x_{ij} = 0$ for $i \neq 2, 3, 4, 5 < j$ and $\frac{2}{3}$ otherwise. Then, to remove x we should intersect g with some $Tr_{ij,k}$ with $1 \leq i, j, k \leq 5$, but doing so will also eliminate one of the 4 cuts $\delta(12), \delta(13), \delta(14)$ or $\delta(15)$ of f_5 as well, which implies that f_3 cannot be a face of m_n. □

Corollary 3.5 *For $n \geq 5$, all i-faces of c_n are also i-faces of m_n for exactly $i = 0, 1$ and 2.*

Proof. Let suppose that all i_0-faces of c_n are also i_0-faces of m_n with $i_0 \geq 3$. Then, the face g_{i_0} of m_n equal to a face f_{i_0} of c_n containing f_3 will contain a face g_3 equal to f_3, which contradicts Proposition 3.4. □

Proposition 3.6

(i) The representative extra i-faces of c_5 *belong to* $F = \{f_9^2, f_8^3, f_7^3, f_7^4, f_6^6, f_6^7, f_6^8, f_5^8, f_5^9, f_5^{10}, f_4^7, f_3^7\}$ *(that is, all extra faces belonging to* f_9^2*) and to* $F' = \{c_5 = f_{10}^1, f_9^1, f_8^1, f_8^2, f_7^5, f_7^6, f_7^7, f_6^9, f_6^{10}, f_5^{11}, f_4^8\}$.

(ii) The representative extra i-faces of m_5 *belong to* $G = \{g_7^3, g_6^6, g_6^7, g_6^8, g_5^8, g_5^9, g_5^{10}, g_5^{11}, g_5^{12}, g_4^7, g_4^8, g_4^9, g_4^{10}, g_4^{11}, g_3^7, g_3^8, g_3^9, g_3^{10}, g_2^4, g_2^5, g_1^3, g_0^2\}$ *(that is, all simplex extra faces, i.e. all extra faces belonging to* g_7^3*) and* $G' = \{m_5 = g_{10}^1, g_9^1, g_8^1, g_8^2, g_7^4, g_7^5, g_7^6, g_6^9, g_6^{10}, g_6^{11}, g_5^{13}, g_4^{12}\}$ *(that is, all faces given in Figure 3.8 except* $g_6^2 = f_6^2$ *which is the unique non-simplex common face of* c_5 *and* m_5*).*

(iii) All remaining faces, besides the ones given in (i) and (ii) are common faces of c_5 *and* m_5*, we ordered them so that* $f_j^i = g_j^i$*. Their representative i-faces belong to* $H = \{f_6^1, f_5^1, f_5^2, f_5^3, f_4^1, f_4^2, f_4^3, f_4^4, f_4^5, f_3^1, f_3^2, f_3^3, f_3^4, f_3^5, f_2^1, f_2^2, f_2^3, f_1^1, f_1^2, f_0^1, f_{-1}^1 = \emptyset\}$ *(that is, all common faces belonging to* f_6^1*) and to* $H' = \{f_7^1, f_7^2, f_6^2, f_6^3, f_6^4, f_6^5, f_5^4, f_5^5, f_5^6, f_5^7, f_4^6, f_3^6\}$.

Remark 3.7 *We observed that* $F = \{g \cap f_9^2 : g \in G'\}$, $F' = \{g \cap c_5 : g \in G' - \{g_6^{11}\}\}$, $H = \{g \cap c_5 : g \in G\}$ *and* $F - \{f_5^{10}\} = \{f \cap f_9^2 : f \in F'\}$ *and that the dimension of those intersections is one less than the one of the corresponding face. The four above equalities,* $g_3^7 \cap c_5 \simeq g_3^{10} \cap c_5$ *and* $g_6^{11} \cap c_5 = f_5^{10}$ *give the following four bijections:* $F \leftrightarrow G'$, $F' \leftrightarrow G' - \{g_6^{11}\}$, $H \leftrightarrow G - \{g_3^7\}$ *and* $F' \leftrightarrow F - \{f_5^{10}\}$.

dimension	-1	0	1	2	3	4	5	6	7	8	9	10
total # of orbits in c_5	1	1	2	3	7	8	11	10	7	3	2	1
# of orbits in F					1	1	3	3	2	1	1	
# of orbits in F'						1	1	2	3	2	1	1
# of orbits in H	1	1	2	3	5	5	3	1				
# of orbits in H'					1	1	4	4	2			
# of orbits in G		1	1	2	4	5	5	3	1			
# of orbits in G'						1	1	3	3	2	1	1
total # of orbits in m_5	1	2	3	5	10	12	13	11	6	2	1	1

Figure 3.13: Number of orbits of i-faces

Remark 3.8

(i) All minimal (by inclusion) faces from F *belong to the orbits of* f_3^7 *or* $f_5^{10} = f_9^2 - f_3^7$.

(ii) All minimal (by inclusion) faces from H' *belong to the orbits of* f_3^6 *or* f_5^7.

(iii) Besides f_3^7 *and the cofacet* $f_3^6 = Tr_{134} \cap Tr_{13,4} \cap Tr_{14,3} = \bar{T}r_{34,1}$*, any of the* 5 *remaining representative 3-faces of* c_5 *belongs to exactly either 4 or 5 triangle facets.*

3.4 Some orbits of faces and nofaces of c_n and m_n

In Figure 3.14, we present 5 orbits of c_n and m_n. For each orbit a representative facet, the codimension, the size and the number of cuts (and anticuts for orbits of m_n) belonging to a face of the orbit are given. The orbit O_2^5 (respectively O_3^5, O_4^5 and O_5^5) corresponds to C_8^1 and M_8^1 (respectively C_8^2 and M_8^2, C_7^1 and M_7^1, and C_6^2 and M_6^2). In m_n, the faces belonging to O_5^n are combinatorially equivalent to m_{n-1}.

orbit	representative	codimension	size	# cuts	# anticuts
O_1^n	$Tr_{135} \cap Tr_{246}$	2	$160\binom{n}{6}$	$9 \times 2^{n-5}$	2^{n-5}
O_2^n	$Tr_{123} \cap Tr_{124}$	2	$48\binom{n}{4}$	$5 \times 2^{n-4}$	2^{n-4}
O_3^n	$Tr_{123} \cap Tr_{145}$	2	$240\binom{n}{5}$	$9 \times 2^{n-5}$	2^{n-5}
O_4^n	$Tr_{135} \cap Tr_{45,3}$	3	$24\binom{n}{4}$	2^{n-2}	0
O_5^n	$Tr_{135} \cap Tr_{15,3}$	$n-1$	$2\binom{n}{2}$	2^{n-2}	0

Figure 3.14: All pairwise intersections of facets of m_n

We then consider the following set of cuts: $A_n = \{\delta(i) : 1 \leq i \leq n\}$. It was remarked in [14] that no triangle facet contains A_n. Moreover, we have:

Proposition 3.9

(i) No proper face of c_n contains A_n.

(ii) For $n \geq 5$, any $(n-1)$-subset of A_n forms an extra $(n-2)$-face of c_n.

(iii) The size of the orbit represented by A_n is $|O(A_n)| = 1, 2, 2^{n-1}$ for $n = 3, 4, \geq 5$.

Proof. Let F be a facet induced by the inequality: $\sum_{1 \leq i < j \leq n} f_{ij} x_{ij} \leq a$ and $F(S)$ the value of the left hand side of the inequality on the cut $\delta(S)$. Since we have $F(\delta(ij)) = F(\delta(i)) + F(\delta(j)) - 2f_{ij} \leq a$, A_n belongs to F will implies that $f_{ij} \geq \frac{a}{2}$. So, we have $a \geq F(\delta(S)) \geq \frac{a|S|(n-|S|)}{2}$, this implies $a = 0$ and therefore, for $n \geq 4$ all $f_{ij} \geq \frac{a}{2} = 0$ which contradicts $F(\delta(1)) \leq a = 0$. To prove (ii), remark that $A_n - \{\delta(1)\}$ is the following face of c_n: $A_n - \{\delta(1)\} = E_{\delta(1)} \cap (\bigcap_{2 \leq i < j \leq n} Tr_{ij,1})$ where $E_{\delta(1)}$ is the switching by the cut $\delta(1)$ of the face defined by the inequality: $\sum_{1 \leq i < j \leq n} x_{ij} \leq \lceil \frac{n}{2} \rceil \cdot \lfloor \frac{n}{2} \rfloor$. On the other hand, $\bigcap_{2 \leq i < j \leq n} Tr_{ij,1}$ contains $\delta(\emptyset)$ which can be removed only by intersecting $A_n - \{\delta(1)\}$ with a non-homogeneous triangle facet, but this will also eliminate some cuts $\delta(i)$, that is, $A_n - \{\delta(1)\}$ cannot be a face of m_n. Since for $n \geq 5$ all switchings of A_n are different and any permutation amounts to a switching, $|O(A_n)| = 2^{n-1}$; cases $n = 3$ and 4 are clear. □

Remark 3.10 *Proposition 3.9 means that A_n is a minimal (by inclusion) blocker, that is $A_n \cap \bar{f} \neq \emptyset$, for the clutter $T_c := \{\bar{f} : f$ is a facet of $c_n\}$ of complements of facets of c_n and, a fortiori, for the clutter $T_m := \{\bar{g} : g$ is a facet of $m_n\}$. Perhaps, A_n is also a minimum, that is of minimal cardinality, blocker for T_c. Remark that minimum blockers for T_m are exactly the pairs $\{\delta(S), \hat{\delta}(S)\}$.*

We call *noface* of c_n a set of cuts which does not form a face of c_n.

Proposition 3.11
(i) For $n \geq 4$, any set containing member of $O(A_n)$ is a noface (see Proposition 3.9).
(ii) For $n = 4$, any noface contains a member of $O(A_4)$; the only nofaces which are cofaces belong to the orbits represented by $\bar{f}_1^2$ and $\bar{f}_2^2$.
(iii) For $n = 5$, there are exactly $1, 2, 3, 8, 13$ *orbits of i-sets of cuts in c_5 for* $i = 1, 2, 3, 4, 5$ *and among there is exactly 6 orbits of nofaces represented by the following graphs:*

One can check that the following holds for any $n \geq 3$:

- the 2^{n-1} vertices of c_n form the orbit represented by $\{\delta(\emptyset)\}$,
- the $\binom{2^{n-1}}{2}$ edges of c_n are partitioned into the $\lfloor \frac{n}{2} \rfloor$ orbits respectively represented by $f_1^i = \{\delta(\emptyset), \delta(1, \ldots, i)\}$ for $i = 1, \ldots, \lfloor \frac{n}{2} \rfloor$ with $|O_1^i| = 2^{n-2}\binom{n}{i}$ for $1 \leq i < \lfloor \frac{n}{2} \rfloor$ and $|O_1^i| = 2^{n-3}\binom{n}{\frac{n}{2}}$ for n even,
- the $\binom{2^{n-1}}{3}$ 2-faces of c_n are partitioned into the orbits respectively represented by $f_2^{r,s,t} = \{\delta(\emptyset), \delta(1, \ldots, r+s), \delta(r+1, \ldots, r+s+t)\}$ for all triplets of integers $\{r, s, t\}$ such that $1 \leq r \leq \lfloor \frac{n}{3} \rfloor$, $0 \leq s \leq r$, $r \leq t \leq min(\lfloor \frac{n-r}{2} \rfloor, \lfloor \frac{n}{2} \rfloor - s, n - 2r - s)$,
- for $n \geq 6$, the $\frac{16(n^2-7)\binom{n}{4}}{3}$ ridges, that is $(\binom{n}{2} - 2)$-face of m_n, are partitioned into the 3 orbits O_1^n, O_2^n and O_3^n given in Figure 3.14 (see [6]),
- the $4\binom{n}{3}$ facets of m_n form the orbit represented by Tr_{123}.

References

[1] P. Assouad and M. Deza, "Metric subspaces of L^1", *Publications mathématiques d'Orsay* **3**, (1982).

[2] D. Avis , "On the extreme rays of the metric cone", *Canadian Journal of Mathematics XXXII* **1**, (1980) 126-144.

[3] F. Barahona and R. Mahjoub, "On the cut polytope", *Mathematical Programming* **36** (1986) 157-173.

[4] M. Bayer and C. Lee, "Combinatorial aspects of convex polytopes ", in P. Gruber and J. Wills (eds.) *Handbook on Convex Geometry*, North Holland (1994) 485-534.

[5] A. Brouwer, A. Cohen and A. Neumaier, *Distance-Regular Graphs*, Springer-Verlag, Berlin, 1989.

[6] A. Deza and M. Deza, "The ridge graph of the metric polytope and some relatives", in T. Bisztriczky, P. McMullen, R. Schneider & A. Ivic Weiss (eds.) *Polytopes: Abstract, Convex and Computational* (1994) 359-372.

[7] A. Deza and M. Deza, "On the skeleton of the dual cut polytope", to appear in *AMS Series Contemporary Mathematics*, H. Barcelo & G. Kalai (eds.), Proceedings of *Jerusalem Combinatorics 93*.

[8] A. Deza and M. Deza, "Geometry of cut and metric polytopes", *in preparation.*

[9] M. Deza, "Matrices de formes quadratiques non négatives pour des arguments binaires", *Comptes rendus de l'Académie des Sciences de Paris* **277**, (1973) 873-875.

[10] M. Deza, "Isometries of hypergraphs", *Proceedings of the International Conference on the Theory of Graphs*, in A. Rao (ed.), McMillan, Calcutta (1979) 174-189.

[11] M. Deza, V. Grishukhin and M. Laurent, "The symmetries of the cut polytope and of some relatives", in P. Gritzmann and B. Sturmfels (eds.) *Applied Geometry and Discrete Mathematics, the "Victor Klee Festschrift" DIMACS Series in Discrete Mathematics and Theoretical Computer Science* **4** (1991) 205-220.

[12] M. Deza and M. Laurent, "Facets for the cut cone I", *Mathematical Programming* **56 (2)**, (1992) 121-160.

[13] M. Deza and M. Laurent, "Applications of cut polyhedra" Rapport de Recherche du LIENS 92-18, Ecole Normale Supérieure, Paris, to appear in *Journal of Computational and Applied Mathematics*, (1992).

[14] M. Deza, M. Laurent and S. Poljak, "The cut cone III: on the role of triangle facets", *Graphs and Combinatorics* **8**, (1992) 125-142.

[15] V. P. Grishukhin, "Computing extreme rays of the metric cone for seven points", *European Journal of Combinatorics* **13**, (1992) 153-165.

[16] M. Iri, "On an extension of maximum-flow minimum-cut theorem to multicommodity flows", *Journal of the Operational Society of Japan* **13**, (1970-1971) 129-135.

[17] M. Laurent, "Graphic vertices of the metric polytope", Research report No. 91737-OR, Institut für Diskrete Mathematik, Universität Bonn, (1991).

[18] M. Laurent and S. Poljak, "The metric polytope", in E. Balas, G. Cornuejols & R. Kannan (eds.), *Integer Programming and Combinatorial Optimization*, Carnegie Mellon University, GSIA, Pittsburgh, (1992) 274-286.

[19] K. G. Murty and S. J. Chung, "Segments in enumerating faces", to appear in *Mathematical Programming.*

[20] M. Padberg, "The boolean quadric polytope: some characteristics, facets and relatives", *Mathematical Programming* **45**, (1989) 139-172.

[21] S. Poljak and Z. Tuza, "Maximum Cuts and Large Bipartite Subgraphs", in W. Cook, L. Lovasz and P. D. Seymour (eds.) DIMACS volume, *in print.*

[22] G. Rote, personal communication.

[23] V. Trubin, "On a method of solution of integer linear problems of a special kind", *Soviet Mathematics Doklady* **10**, (1969) 1544-1546.

[24] G. M. Ziegler, *Lectures on Polytopes*, Graduate Texts in Mathematics **152**, Springer-Verlag, New York, Berlin, Heidelberg, 1995.

ANTOINE DEZA

Tokyo Institute of Technology, department of mathematical and computing sciences, Meguro-ku, Ookayama, Tokyo, Japan. E-mail: deza@is.titech.ac.jp

MICHEL DEZA

CNRS, Ecole Normale Supérieure, département mathématiques et informatique, 45 rue d'Ulm, Paris, France. E-mail: deza@dmi.ens.fr