Cut Polytope and its Lattices

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Abstract

We show that the cut polytope $PCut_n$ is an L-polytope of the lattice L_n, affinely generated by its vertices. We consider cut-sub-lattices of L_n generated by subsets of cuts. If n is even, L_n is generated by an odd system. We give a detailed description of L_n and $PCut_n$ for small n and sets of equiangular lines related to these polytopes. In particular, we give all 4 types of L-polytopes of the lattice $L_4 = \sqrt{2}D_6^{+2}$.

1 Introduction

Let V be a ground set of cardinality $|V| = n$. The **cut vector**, or, simply, the **cut** $\delta(S)$, $S \subseteq V$, is a vector of the space of all functions $d : V^2 \rightarrow \mathbf{R}$, defined on the set V^2 of all unordered pairs of the set V. The component $\delta_{ij}(S)$ is defined as follows. Let

$$(S,T) = \{(ij) \in V^2 : i \in S, j \in T\} \text{ and } D(S) = (S, V - S). \tag{1}$$

Then

$$\delta_{ij}(S) = \begin{cases} 1 & \text{if } (ij) \in D(S) \\ 0 & \text{otherwise.} \end{cases}$$

The dimension of the space spanned by all $\delta(S)$'s is equal to $N = |V^2| = n(n-1)/2$. Since $\delta(S) = \delta(V - S)$, there are 2^{n-1} cuts, including the zero cut $\delta(\emptyset) = 0$.

Let $\mathbf{Z}^N$ be the lattice of all integral N-vectors. Let $\mathcal{K}$ be a family of cuts. Clearly $\mathcal{K} \subset \mathbf{Z}^N$. Hence the lattice $L(\mathcal{K})$, linearly generated by all the cuts $\delta(S) \in \mathcal{K}$, i.e.

$$L(\mathcal{K}) = \{d : d = \sum_{\delta(S) \in \mathcal{K}} z_S \delta(S),\ z_S \in \mathbf{Z}\},$$

is a sublattice of $\mathbf{Z}^N$. Let $\mathcal{K}_n = \{\delta(S) : S \subseteq V - \{n\}\}$ be the set of all cut vectors, and, for even n, let $\mathcal{K}_n^{odd} = \{\delta(S) \in \mathcal{K}_n : |S| \text{ is odd}\}$, $\mathcal{K}_n^{ev} = \{\delta(S) \in \mathcal{K}_n : |S| \text{ is even}\}$. We set

$$L_n = L(\mathcal{K}_n),\ L_n^{odd} = L(\mathcal{K}_n^{odd}),\ L_n^{ev} = L(\mathcal{K}_n^{ev}).$$

It is proved in [1], $d \in L_n$ if and only if $d_{ij} + d_{jk} + d_{ki} \equiv 0 \pmod 2$ for all triples $\{ijk\}$. The condition is a special case of the following fact: cardinality of an intersection of a cut and a cycle in a graph is even. In fact, this is equivalent to the dual lattice of L_n is $\frac{1}{2}L(\mathcal{C})$, where $\mathcal{C}$ is the set of all cycles of the complete graph K_n. $\mathcal{C}$ itself is generated by triangles and trivial 2-circuits, $2e_{ij}$, containing 2 parallel edges (ij). Hence the dual lattice L_n^* contains the lattice $\mathbf{Z}$ of all integral vectors.

The lattice L_n^{ev}, for even n, is characterized in Section 4 of [5].

We call a set $\mathcal{V} \subseteq \mathbf{R}^N$ of vectors an **odd system** if the inner product of any pair of vectors $u, v \in \mathcal{V}$, denoted by juxtaposition uv, is an odd number. In particular, the **norm** v^2 of any $v \in \mathcal{V}$, i.e. the inner product of v with itself, is odd, too. Note that for a $(0, \pm 1)$-vector its norm equal to number of nonzero components.

A set of vectors $\mathcal{V}$ is called **symmetric** if $-v \in \mathcal{V}$ for all $v \in \mathcal{V}$. If $-v \notin \mathcal{V}$ for all $v \in \mathcal{V}$, then $\mathcal{V}$ is called **asymmetric**. Using the expression

$$\delta(S)\delta(T) = \frac{1}{2}[\delta^2(S) + \delta^2(T) - \delta^2(S\Delta T)],$$

it is easy to verify that, for even n, the set $\mathcal{K}_n^{odd}$ of all **odd** cuts is an asymmetric odd system such that norm of each odd cut is equal to $n-1 \pmod 4$.

A set of vectors is called **uniform** if norm of all its vectors is the same. In the case, the common norm is called **norm** of the set of vectors. A uniform odd system $\mathcal{V}$ of norm m is called **closed** if the set of all vectors of norm m of the lattice $L(\mathcal{V})$, generated by $\mathcal{V}$, coincides with $\mathcal{V}$.

It is proved in [4] the following facts about the lattice $L(\mathcal{V})$, generated by an odd system of vectors v of norm $4k(v) + p$, where $p = 1$ or $p = 3$ is the same for all $v \in \mathcal{V}$.

(1) $a^2 \equiv 0$ or $p \pmod 4$ for all $L(\mathcal{V})$.

(2) $L_0(\mathcal{V}) := \{a \in L(\mathcal{V}) : a^2 \equiv 0 \pmod 4\}$ is a double even sublattice of $L(\mathcal{V})$. (A **double even** lattice is a lattice of vectors with even inner products and having norm divisible by 4).

(3) $L_1(\mathcal{V}) := \{a \in L(\mathcal{V}) : a^2 \equiv p \pmod 4\} = L_0(\mathcal{V}) + a_1$, where a_1 is any vector of $L(\mathcal{V})$ of odd norm.

A special case of an odd system is represented by a set of vectors, spanning equiangular lines. A set of lines is **equiangular** if the acute angle between any pair of lines is the same. If there are sufficiently many of equiangular lines, then this angle is equal to $\arccos\frac{1}{m}$, where m is an odd integer. Hence the corresponding odd system is composed of vectors of norm m with inner products ± 1.

We give here more detailed description of the lattice L_n. In particular, we show that the cut polytope $PCut_n$, i.e. the convex hull of all cut vectors, including zero cut, is an L-polytope of the lattice L_n.

2 Some properties of the lattice L_n

Let $\{e_{ij} : 1 \le i \le j \le n\}$ be orthonormal basis of $\mathbf{R}^N$. The lattice generated by the basis is $\mathbf{Z}^N$. For $m \in \mathbf{Z}$, let $m\mathbf{Z}^N$ be the lattice of all integer vectors divisible by m. Then $\{me_{ij}\}$ is the basis of $m\mathbf{Z}^N$. Recall that $\mathcal{K}_n$ is an Abelian group with respect to symmetric difference of vectors, i.e. the sum of vectors modulo 2. Similarly, L_n is an Abelian group with respect to usual addition. In Proposition 1 below we give two simple but important properties of the lattice L_n.

Proposition 1 *(i)* $2\mathbf{Z}^N \subset L_n$.
(ii) $L_n/2\mathbf{Z}^N = \mathcal{K}_n$ *i.e.* $d \equiv \delta(S)$ *(mod 2) for some* $\delta(S) \in \mathcal{K}_n$ *for all* $d \in L_n$.

Proof. (i) The equality
$$2e_{ij} = \delta(i) + \delta(j) - \delta(ij) \tag{2}$$
shows that $2e_{ij} \in L_n$ for every pair $(ij) \in V^2$.

(ii) Let $d \in L_n$, $d = \sum_S z_S \delta(S)$. Then $d \equiv \sum_{S \in \mathcal{S}} \delta(S)$ (mod 2), where $\mathcal{S} = \{S : z_S \equiv 1 \text{ (mod 2)}\}$. Since $\delta(S) + \delta(T) \equiv \delta(S\Delta T)$ (mod 2), using induction on number elements of $\mathcal{S}$, we obtain that $d \equiv \delta(T)$ (mod 2) for some $T \subseteq V$. □

Proposition 1(ii) implies that every point $d \in L_n$ has the form
$$d = 2a + \delta(S), \text{ where } a \in \mathbf{Z}^N. \tag{3}$$
for some $S \subseteq V$.

Of course, the lattice L_n has infinitely many bases. Proposition 2 below gives an example of a basis of L_n, containing in $\mathcal{K}_n$.

Proposition 2 *The following set of* N *cuts forms a basis of* L_n
$B = \{\delta(i),\ \delta(ij) : i, j \in V - \{n\}\}$.

Proof. Using (2), we have $\delta(S) = \sum_{j \in S} \delta(j) - 2\sum_{i<j \in S} e_{ij} = \sum_{j \in S} \delta(j) - \sum_{i<j \in S}(\delta(i) + \delta(j) - \delta(ij))$, i.e. each cut $\delta(S)$, $S \subseteq V - \{n\}$ has the unique representation
$$\delta(S) = \sum_{i<j \in S} \delta(ij) - (|S| - 2)\sum_{i \in S} \delta(i). \square \tag{4}$$

Note that the set of points $B_a = \{\delta(\emptyset)\} \cup B$ is an affine basis of the lattice L_n. Let the origin does not belong to L_n. Then we distinguish **lattice points** $d \in L_n$ and **lattice vectors**. Any point $d \in L_n$ has the following **affine** representation
$$d = \sum_{a \in B_a} z_a a, \quad \sum_{a \in B_a} z_a = 1.$$
The lattice points do not form a group.

A lattice vector is a difference of two lattice points. Lattice vectors form the same Abelian group L_n and have the following affine representation in the basis B_a
$$d = \sum_{a \in B_a} z_a a, \quad \sum_{a \in B_a} z_a = 0.$$

Let $\gamma(S) = \delta(S) - a_0$, where a_0 is the new origin. Then $\delta(S) = \gamma(S) - \gamma(\emptyset)$. Using (4), we obtain the following affine representation of the vectors $\gamma(S)$ in the basis B_a.
$$\gamma(S) = \sum_{i<j \in S} \gamma(ij) - (|S| - 2)\sum_{i \in S} \gamma(i) + (\frac{|S|(|S| - 3)}{2} + 1)\gamma(\emptyset). \tag{5}$$

The representation is useful if we take the center of $PCut_n$ as origin.

The equation (3) shows that the unique (0,1)-vectors $d \in L_n$ are cut vectors. Now we consider vectors of L_n modulo 3, i.e. vectors having $(0, \pm 1)$-components only. Let $d \in L_n$ has $(0, \pm 1)$-components. Since d (mod 2) is a cut vector, all such vectors have the form

$$d(S;X) = \delta(S) - 2\chi(X),\ X \subseteq D(S), \tag{6}$$

where

$$\chi(X) = \sum_{(ij)\in X} e_{ij},$$

and the set $D(S)$ is defined in (1). Note that $\delta(S) = \chi(D(S))$. There are $2^{k(n-k)}$, $k = |S|$, such subsets X. So between $3^{\binom{N}{2}}$ of $(0, \pm 1)$-vectors of $\mathbf{R}^N$, L_n contains $\sum_{k=0}^{n} \binom{n}{k} 2^{k(n-k)-1}$ (including $4^{n-1} - 2^{n-1} + 1$ of form $\delta(S) - \delta(S')$). Thus, the ratio (equal to 1, 13/27, 80/729 for $n = 2, 3, 4$) goes to 0 when $n \to \infty$; compare with $|L_n \cap [0,1]^N|/|[0,1]^N| = 2^{-\binom{n-1}{2}}$.

3 Cut-sub-lattices of L_n

We call a sublattice $L \subseteq L_n$ a **cut-lattice** if it is generated by a set $\mathcal{K}$ of cuts, i.e. if $L = L(\mathcal{K})$. Call a cut-lattice $L(\mathcal{K})$ **uniform** if $Sym_n \subseteq Aut(L(\mathcal{K}))$. If $\mathcal{K}$ is uniform, then $\mathcal{K}_n \cap L(\mathcal{K})$ is also uniform. Call a cut-sublattice $L(\mathcal{K})$ **maximal** if L_n is the only cut-lattice having $L(\mathcal{K})$ as a proper sublattice. Call a cut-lattice **minimal** if it has no proper full-dimensional cut sub-lattices. Call a set of cuts $\mathcal{K}$ **closed** if $\mathcal{K}_n \cap L(\mathcal{K}) = \mathcal{K}$.

We shall see that $PCut(\mathcal{K})$ is an L-polytope in $L(\mathcal{K})$ for closed $\mathcal{K}$.

Set
$\mathcal{K}_n^{j,k} = \{\delta(S) : |S| = j, k\}$, for $1 \le j < k \le \frac{n}{2} - 1$, and $L_n^{j,k} = L(\mathcal{K}_n^{j,k})$,
$\mathcal{K}_n^{mod3} = \{\delta(S) : |S| \le \frac{n}{2}$ and $|S| \not\equiv 1 \pmod 3\}$, and $L_n^{mod3} = L(\mathcal{K}_n^{mod3})$,
$\mathcal{K}_n^{\neq i} = \{\delta(S) : |S| \neq i, |S| \le \frac{n}{2}\}$, and $L_n^{\neq i} = L(\mathcal{K}_n^{\neq i})$,
$\mathcal{K}_n^{k} = \{\delta(S) : |S| = k\}$, and $L_n^{k} = L(\mathcal{K}_n^{k})$.
$\mathcal{K}_n^{ev(mod3)} = \{\delta(S) \in \mathcal{K}_n^{ev} : \frac{|S|}{2} \equiv 0, 1 \pmod 3\ , |S| \le \frac{n}{2}\}$, and $L_n^{ev(mod3)} = L(\mathcal{K}_n^{ev(mod3)})$.

Any proper generating subset of $\mathcal{K}$ (examples, beside $\mathcal{K}_n^{1,2}$, will be given in Theorem 8 and following remark) is not closed. Also it seems that $L_n^{i,j} = L_n$ if and only if $(ij) = (12)$.

Examples of closed $\mathcal{K}$ are $\mathcal{K} = \mathcal{K}_n, \mathcal{K}_n^{ev}, \mathcal{K}_n^{odd}$, $\mathcal{K}_T = \{\delta(S) : |S \cap T|$ is even for given $T \in V\}$, and $\mathcal{K} = \mathcal{K}_n \cap H$, where H is a hyperplane in $\mathbf{R}^N$, containing 0 (for example, $\mathcal{K} = \mathcal{K}^1$, $\mathcal{K}^{\lfloor \frac{n}{2} \rfloor}$, see Proposition 6 below, or $\mathcal{K}$ is a face of the cut cone $\mathbf{R}_+(\mathcal{K}_n)$).

We need the following

Lemma 3 *Assume $\delta(S) = \sum_{h=1}^{k} z_h \delta(S_h)$, where $z_h \in \mathbf{Z}$, $z_h \neq 0$, and $|S|, |S_h| \le \frac{n}{2}$. Then $\binom{|S|+1}{2} \equiv 0$ (mod g.c.d.$_{1 \le h \le k} \binom{|S_h|+1}{2}$).*

Proof. Set $d = \sum_{h=1}^{k} z_h \delta(S_h)$. Define $\pi(d) \in \mathbf{R}^N$ by:

$$\begin{array}{lll} \pi(d)_{ii} = & \sum_{h: S_h \ni i} z_h & i = 1, ..., n \\ \pi(d)_{ij} = & \frac{1}{2}[\pi(d)_{ii} + \pi(d)_{jj} - d_{ij}] & 1 \le i < j \le n. \end{array}$$

Note that, for $d = \delta(S)$, $\pi(\delta(S))_{ij} = 1$ if $i, j \in S$ and $= 0$ otherwise. We have

$$\pi(d) = \sum_{h=1}^{k} z_h \pi(\delta(S_h)).$$

Taking inner product with j_N of both sides, we get

$$\sum_{1 \le i \le j \le n} \pi(d)_{ij} = \sum_{h=1}^{k} z_h \binom{|S_h|+1}{2}.$$

Hence, taking $d = \delta(S)$, the result follows. $\square$

Lemma 3 is used in the proof of Proposition 6 below.

In what follows, we use the following characterization of the lattice L_n^k, $k \neq \frac{n}{2}$, given in Proposition 4.3 of [6] (see Proposition 4, below). This characterization was obtained as an adaptation of a theorem of R.M.Wilson.

For $d \in \mathbf{R}^N$, define

$$d_{i,n+1}^k := \frac{1}{n-2k}\Big(\sum_{1 \le j \le n, j \neq i} d_{ij} - \frac{1}{n-k} \sum_{1 \le r < s \le n} d_{rs}\Big) \text{ for } 1 \le i \le n. \tag{7}$$

Proposition 4 *[6] Given $d \in \mathbf{Z}^N$, then $d \in L_n^k$, $k \neq \frac{n}{2}$, if and only if*

(i) $\sum_{1 \le i < j \le n} d_{ij} \equiv 0 \pmod{k(n-k)}$,

(ii) $d_{i,n+1}^k \in \mathbf{Z}$ for all $1 \le i \le n$,

(iii) $d_{i,n+1}^k + d_{j,n+1}^k + d_{ij} \equiv 0 \pmod 2$ for all $1 \le i < j \le n$. $\square$

The characterization of L_n^k, given in Proposition 4 implies, for example, that $\frac{n+1}{4} L_n^1 \subset L_n^{\frac{n-1}{2}}$, $\frac{n^2-1}{8} L_n \subset L_n^{\frac{n-1}{2}}$ for $n \equiv 1 \pmod 4$.

Proposition 5 *Given $d \in \mathbf{Z}^N$, then*

a) $d \in L_n^{\frac{n}{2}}$, n is even, if and only if

(i) $\sum_{1 \le i < j \le n} d_{ij} \equiv 0 \pmod{\frac{n^2}{4}}$,

(ii) $\sum_{1 \le i < j \le n} d_{ij} = \frac{n}{2} \sum_{q=1}^{n} d_{pq}$ for any p, $1 \le p \le n$.

b) $d \in L_n^{\frac{n-1}{2}}$, n is odd, if and only if

$\sum_{1 \le i < j \le n} d_{ij} \equiv 0 \pmod{\frac{n^2-1}{4}}$.

Proof. The conditions (i) and (ii) of a) are clearly necessary for membership in $L_n^{\frac{n}{2}}$. Conversely, suppose that d satisfies both the conditions, and let d' denote its projection on the set $V - \{n\}$. From (ii), we obtain

$$\sum_{1 \le r < s \le n-1} d'_{rs} = \Big(\frac{n}{2} - 1\Big) \sum_{1 \le q \le n-1} d_{qn}. \tag{8}$$

This implies that $\sum_{1 \le r < s \le n-1} d'_{rs} \equiv 0 \pmod{\frac{n(n-2)}{4}}$, since $\sum_{1 \le i \le n-1} d_{in} \equiv 0 \pmod{\frac{n}{2}}$ by (i) and (ii). Using Proposition 4, we deduce that $d' \in L_{n-1}^{\frac{n}{2}}$. Hence $d' = \sum_{S \subseteq V - \{n\}, |S| = \frac{n}{2}} \lambda_S \delta(S)$ with $\lambda_S \in \mathbf{Z}$ for all S.

We show that $d = \sum_S \lambda_S \delta(S)$. As $\sum_{1 \le r < s \le n-1} d'_{rs} = \frac{n(n-2)}{4} \sum_S \lambda_S$, (8) yields: $\sum_{1 \le i \le n-1} d_{in} = \frac{n}{2} \sum_S \lambda_S$. Hence $\sum_{1 \le r < s \le n} d_{rs} = \sum_{1 \le r < s \le n-1} d'_{rs} + \sum_{1 \le i \le n-1} d_{in} = \frac{n^2}{4} \sum_S \lambda_S$ and by (i) $\sum_{1 \le j \le n} d_{ij} = \frac{n}{2} \sum_S \lambda_S$ for each $i \in V$.

We compute, for instance, d_{1n}. The above relations for $i = 1$ yield: $d_{1n} = \frac{n}{2} \sum_S \lambda_S - \sum_{2 \le j \le n-1} d_{1j}$. Using the value of $d_{1j} = d'_{1j}$ given by the decomposition of d', we obtain that $d_{1n} = \sum_{S: 1 \in S} \lambda_S$. This shows that $d = \sum_S \lambda_S \delta(S)$, i.e. that $d \in L_n^{\frac{n}{2}}$.

It is not difficult to verify that b) is implied by Proposition 4, which can be applied, since $\frac{n-1}{2} < \frac{n}{2}$. □

Proposition 6 *(i)* $L_n^{mod3} \subset L_n$ *strictly.*

(ii) $L_n^{j,j+1} = L_n$ *if and only if* $j = 1$.

(iii) $L_n^{ev(mod3)} \subset L_n^{ev}$ *strictly.*

In particular, $L_n^4 \subset L_n^{ev} = L_n^{2,4}$ *for* $n = 8, 10$.

(iv) $\mathcal{K}_n^k$ *is closed if* $k(n-k)$ *does not divide* $i(n-i)$ *for all* $i \ne k$, $1 \le i \le \lfloor \frac{n}{2} \rfloor$.

In particular, $\mathcal{K}_n^k$ *is closed if* $\lfloor \frac{n(2-\sqrt{2})}{4} \rfloor < k \le \lfloor \frac{n}{2} \rfloor$.

(v) $\mathcal{K}_n^1$ *is closed.*

Proof. (i) We check that no cut $\delta(i)$ belongs to L_n^{mod3}. Else, $\delta(i) = \sum_{S, |S| \not\equiv 1 (mod3)} z_S \delta(S)$, from which we deduce that g.c.d.$\binom{|S|+1}{2} = 1$ for $|S| \not\equiv 1$ (mod 3). But $\binom{|S|+1}{2} = \frac{1}{2}|S|(|S|+1) \equiv 0$ (mod 3) if $|S| \equiv 0, 2$ (mod 3).

(ii) If $j \ge 2$, no cut $\delta(i)$ belongs to $L_n^{j,j+1}$, since g.c.d.$(\binom{j+1}{2}, \binom{j+2}{2}) \ne 1$.

If $j = 1$, the equality (4) shows that $L_n = L_n^{1,2}$.

(iii) Since $\binom{|S|+1}{2}$ is divided by 3 if and only if $\frac{|S|}{2} \equiv 0, 1$ (mod 3), we conclude, that $\delta(S)$ with $\frac{|S|}{2} \equiv 2$ (mod 3) does not belong to $L_n^{ev(mod3)}$.

(iv) We use (i) of Proposition 4. Therefore, if cut $\delta(S) \in L_n^k$ with $|S| = i$, then $k(n-k)$ divides $i(n-i)$, since $\sum_{1 \le i < j \le n} \delta(S) = i(n-i)$.

(v) Any point $d \in L_n^1$ satisfies the following condition

$$d_{ij} - d_{jk} + d_{kl} - d_{li} = 0 \text{ for all distinct } i, j, k, l \in V, \qquad (9)$$

since this condition holds for all $\delta(i) \in \mathcal{K}_n^1$, generating L_n^1. Take any $\delta(S)$ with $|S| \ne 1, n-1$. Then we can find distinct $i, j, k, l \in V$ such that $i, j \in S$ and $k, l \notin S$. Since (9) is violated by this $\delta(S)$, we conclude that $\delta(S) \notin L_n^1$. This implies that L_n^1 is closed. □

We group in Theorems 7, 8 below some interesting facts on uniform cut-lattices.

Theorem 7 *(i)* $(\binom{n-3}{k-2} - \binom{n-3}{k-3})\delta(i) \in L_n^{k-1,k}$ *if* $n \equiv 0$ *(mod k) and so* $(n-4)L_n \subset L_n^{2,3}$ *if* $n \equiv 0$ *(mod 3).*

(ii) $(n-2)L_n \subset L_n^{2,3}$ *and, for even n,* $\frac{n-2}{2} L_n \subset L_n^{2,3}$.

(iii) $(\delta(i) - \delta(j)) \in L_n^{k-1,k}$, $(k-2)L_n^k \subset L_n^{1,k-1}$.

(iv) $(\delta(S) - \delta(S')) \in L_n^k$ *with* $|S| = |S'| = l$ *if and only if* $\frac{n-2l}{n-2k}$ *is an odd integer.*

Proof. (i) Recall that j_N is all ones N-vector. The following identity is true

$$(\binom{n-3}{k-2} - \binom{n-3}{k-3})\delta(i) + 2\binom{n-3}{k-3} j_N = \sum_{S: S \ni i, |S| = k-1} \delta(S).$$

If $n \equiv 0 \pmod{k}$, then we can partition V into $\frac{n}{k}$ disjoint k-sets A_j, $1 \le j \le \frac{n}{k}$. Then $-2j_N = \sum_{j=1}^{\frac{n}{k}}[(k-2)\delta(A_j) - \sum_{t\in A_j}\delta(A_j - \{t\})]$.

(ii) is implied by the equalities (10) and (11) in the proof of Theorem 8 below.

(iii) For any k-subset S of V, we have

$$(k-2)\delta(S) = \sum_{t\in S}(\delta(S-\{t\}) - \delta(t)).$$

Subtracting the equalities for $S = T\cup\{j\}$ and $S = T\cup\{i\}$, we obtain

$$\delta(i)-\delta(j) = (k-2)(\delta(T\cup\{j\}) - \delta(T\cup\{i\})) + \sum_{t\in T}(\delta(T\cup\{i\}-\{t\}) - \delta(T\cup\{j\}-\{t\})).$$

The equality implies (iii).

(iv) We use Proposition 4. Let $d = \delta(S)-\delta(T)$. It is not difficult to verify the following identities.

$$\sum_{1\le j\le n} d_{ij} = \pm(|S|-|T|) \text{ for } i\in V-(S\Delta T),$$

$$\sum_{1\le j\le n} d_{ij} = \pm(n-|S|-|T|) \text{ for } i\in S\Delta T, \text{ and}$$

$$\sum_{1\le i<j\le n} d_{ij} = (|S|-|T|)(n-|S|-|T|).$$

Suppose that $|S|=|T|=l$. Then $\sum_{1\le i<j\le n} d_{ij} = 0$, and substituting the above expressions into (7) we obtain

$$d^k_{i,n+1} = \begin{cases} \frac{n-2l}{n-2k} & \text{for } i\in S-T,\\ -\frac{n-2l}{n-2k} & \text{for } i\in T-S,\\ 0 & \text{for } i\in V-(S\Delta T).\end{cases}$$

Besides, for any integer k, $1\le k<\frac{n}{2}$, and integers i,j with $1\le i<j\le n$, we have

$$\frac{1}{2}(d^k_{i,n+1}+d^k_{j,n+1}-d_{ij}) = \begin{cases} \frac{n-2l}{n-2k} & \text{for } i,j\in S-T,\\ -\frac{n-2l}{n-2k} & \text{for} i,j\in T-S,\\ \frac{1}{2}(\frac{n-2l}{n-2k}+1) & \text{for } i\in S\cap T, j\in S \text{ or vice versa},\\ -\frac{1}{2}(\frac{n-2l}{n-2k}+1) & \text{for } i\in S\cap T, j\in T \text{ or vice versa},\\ \frac{1}{2}(\frac{n-2l}{n-2k}-1) & \text{for } i\in S, j\in V-(S\cup T) \text{ or vice versa},\\ -\frac{1}{2}(\frac{n-2l}{n-2k}-1) & \text{for } i\in T, j\in V-(S\cup T) \text{ or vice versa}.\end{cases}$$

We see that the conditions (i)–(iii) of Proposition 4 are satisfied if and only if $\frac{n-2l}{n-2k}$ is an odd integer. □

Any cut-lattice containing $L_n^{\frac{n-1}{2}}$ is uniform, because the condition (iv) of Theorem 7 holds for any l if n is odd and $k=\frac{n-1}{2}$. It implies that $L_n^{\neq i}$ is maximal for odd n, $i\neq\frac{n-1}{2}$.

Denote

$$t_n := min\{t\in \mathbf{Z}_+ : tL_n \subset L_n^{\neq 1}\}.$$

Theorem 8 *For $n \geq 5$, we have:*

(i) if $n-2=p^s$, s is an integer, then $t_n \in \{1,p\}$, and $t_n = 3,2,5$ for $n=5,6,7$, respectively.

(ii) if $n-2 \neq p^s$, then $t_n = 1$, i.e. $L_n^{\neq 1} = L_n$.

Proof. We supposed that $n \geq 5$ since $L_3^{\neq 1} = \emptyset$, and $L_4^{\neq 1} = L_4^2$ has dimension 3, i.e. it is not full dimensional, since dimension of L_4 is 6.

Clearly, the lattice $L_n^{\neq 1}$ contains elements

$$z_i = \sum_{S \subseteq V-\{1\}, |S|=i} \delta(S) \text{ for any } i\ , 2 \leq i \leq n-2.$$

We want to recognize when $\delta(1)$ is represented as sum of $\delta(S) \in L_n^{\neq 1}$. It is not difficult to verify that the following identity is true

$$z_i = \binom{n-2}{i-1}\delta(1) + 2\binom{n-3}{i-1}(j_N - \delta(1)).$$

Hence we have

$$\binom{n-2}{i-1}\delta(1) = iz_{i+1} - (n-2-i)z_i. \tag{10}$$

Moreover, for $n \equiv 2 \pmod i$, setting $f_i = \frac{\binom{n-2}{i-1}}{i}$, the above equality implies

$$f_i\delta(1) = z_{i+1} - (\frac{n-2}{i} - 1)z_i \tag{11}$$

with integer coefficients. Hence $f_i\delta(1) \in L_n^{\neq 1}$. We want to find g.c.d of numbers f_i.

If $n \equiv 2 \pmod i$, and $n \not\equiv 2 \pmod{i^2}$, then $f_i = \frac{n-2}{i}\prod_{j=1}^{i-2}\frac{n-2-j}{i-j}$ is an integer and it is not divided by i, because $1 \leq j < i$ implies $(n-2)-j \not\equiv 0 \pmod i$.

By its definition, t_n divides any integer m such that $m\delta(1) \in L_n^{\neq 1}$.

Let us prove (ii) at first. Suppose that $n-2 = p_1^{s_1}\cdots p_r^{s_r}$ is the prime decomposition of $n-2$. Apply equality(11), in turn, for $i = p_1^{s_1}, \cdots, i = p_r^{s_r}$. Then 1 is only common divisor of r numbers f_i, proving (ii).

If $n-2=p^s$, $s \geq 2$, then apply (10) for $i=2$ and (11) for $i=p^{s-1}$. Then 1 and p are only possible common divisors of $n-2=p^s$ and f_i. Finally, $t_n \neq 1$ for $n=5,6,7$ because $L_n^{\neq 1} = L_n^{mod 3}$ for these n, and we can apply Proposition 6(i). □

Remark. The proof of Theorem 8(ii) above gives, actually, $\mathcal{K}^1 \subset L(\mathcal{K}_n^{a,a+1} \cup \mathcal{K}_n^{b,b+1})$ if $n-2$ has two divisors $a,b>1$ such that a^2, b^2 do not divide $n-2$ and g.c.d.$(a,b)=1$. Such a,b exist if and only if $n \neq 2+p^s$. For example, $L_n = L(\mathcal{K}_n^{2,3} \cup \mathcal{K}_n^{b,b+1})$ if n is even and $n \neq 2+2^s$, for any odd divisor b of $n-2$. In particular, $L_{4t} = L(\mathcal{K}_{4t}^{2,3} \cup \mathcal{K}_{4t}^{2t-1,2t})$ for $t \geq 2$, and $L_n = L(\mathcal{K}_n^{2,3} \cup \mathcal{K}_n^{3,4})$ for $n = 2+6m$ with $m \equiv 1,5 \pmod 6$ (i.e. $m \not\equiv 0$ (mod 2,3)).

By the same way as in Theorem 8(ii), one can check that $\mathcal{K}_n^2 \subset L(\mathcal{K}_n^{a,a+1} \cup \mathcal{K}_n^{b,b+1})$ if $n-3$ has two divisors $a,b>2$ such that g.c.d.$(a,b)=1$ and a^2, b^2 do not divide $n-3$. Hence, for $n-2, n-3 \neq p^s, 2p^s$ (for $n \leq 50$ all such numbers are 23,38,42 and 47), the set $\mathcal{K}_n^{a,a+1} \cup \mathcal{K}_n^{b,b+1}$ contains cut bases of L_n, which are disjoint with $\mathcal{K}_n^{1,2}$. For example, since

$23-2=3\cdot 7$ and $23-3=4\cdot 5$, we have $\mathcal{K}_{23}^1 \subseteq L(\mathcal{K}_{23}^{3,4}\cup\mathcal{K}_{23}^{7,8})$ and $\mathcal{K}_{23}^2 \subseteq L(\mathcal{K}_{23}^{4,5}\cup\mathcal{K}_{23}^{5,6})$. Similarly, since $38-2=4\cdot 9$ and $38-3=5\cdot 7$, we obtain $\mathcal{K}_{38}^1 \subseteq L(\mathcal{K}_{38}^{4,5}\cup\mathcal{K}_{38}^{9,10})$, and $\mathcal{K}_{38}^2 \subseteq L(\mathcal{K}_{38}^{5,6}\cup\mathcal{K}_{38}^{7,8})$. But $L_6^{\neq 2}=L_6^{odd}$, as well as $L_6^{\neq 1}$, are proper sub-lattices of L_6, for example.

We can generalize the assertion of Theorem 8 as follows. Set $S_m := \{n-m+2,\dots,n\}$. For any $m, (\lceil \frac{n}{2}\rceil \le m \le n)$ and $i \neq m-1, n-m+1$, $(1\le i\le m-1)$, define

$$z_{m,i} := \sum_{S\subseteq S_m, |S|=i} \delta(S).$$

The following identity is true

$$z_{m,i} = \binom{m-2}{i-1}\delta(S_m) + 2\binom{m-3}{i-1} j_N^{(m)},$$

where $j_N^{(m)}(k,l)=1$ if $n-m+2\le k<l\le n$ and $j_N^{(m)}(k,l)=0$, otherwise.

For $1\le i<k\le m-1$, let $g(i,k;m) =$g.c.d.$(\binom{k-1}{k-i}, \binom{m-i-2}{k-i})$. Using the above identity, we obtain

$$\frac{\binom{k-1}{k-i}}{g(i,k;m)} z_{m,k} - \frac{\binom{m-i-2}{k-i}}{g(i,k;m)} z_{m,i} = f_{i,k}\delta(S_m),$$

where $f_{i,k} = \frac{\binom{m-2}{i-1}\binom{m-i-2}{k-i-1}}{g(i,k;m)}$. If $m=n$ and $k=i+1$, then $f_{i,i+1}=f_i$, $g(i,k;m)=i$, $S_m = V-\{1\}$, and the identity coincides with (11).

Let $i+1<k<2(i+1)$ and $m-2$ is divided by i, but not by i^2. Then $\binom{m-2}{i-1}\binom{m-i-2}{k-i-1}$ is divided by i^2, but not by i^3. In fact,

$$\frac{\binom{m-2}{i-1}\binom{m-i-2}{k-i-1}}{i^2} = (\frac{m-2}{i})(\frac{m-2-i}{i})\prod_{j=1}^{i-2}\frac{m-2-j}{i-j}\prod_{j=1}^{k-i-2}\frac{m-2-i-j}{k-i-j}.$$

Note that $\frac{m-2-j}{i-j}$ is not divided by i, since $1\le j<i$, and $\frac{m-2-i-j}{k-i-j}$ is not divided by i, since $1\le j<k-i-2<i$.

If g.c.d.$(f_{i,k}, f_{r,s})=1$ for some 4-subset $\{i,k,r,s\}$ of $\{1,\dots,m-2\}-\{n-m+1\}$, then

$$\mathcal{K}_n^{n-m+1} \subset L(\mathcal{K}_n^i\cup\mathcal{K}_n^k\cup\mathcal{K}_n^r\cup\mathcal{K}_n^s).$$

Corollary 9 *For all $n\ge 5$ (except, possibly, $n=2+p^s$ with $p>2$), we have $2\mathbf{Z}^N \subset L_n^{\neq 1}$.*

Proof. In fact, for $n\neq 2+p^s$, $L_n^{\neq 1}=L_n$. For $n=2+2^s$, (i) of Theorem 8 implies that $2\delta(1)$ (and so, by uniformity of $L_n^{\neq 1}$, all $2\delta(j)$) belongs to $L_n^{\neq 1}$. By Theorem 7(i), $\delta(i)-\delta(j)\in L_n^{k-1,k}$, for $k=\frac{n}{2}$, say. Hence the equality

$$2e_{ij} = \delta(i)+\delta(j)-\delta(ij) = \delta(i)-\delta(j)+2\delta(j)-\delta(ij)$$

shows that $2\mathbf{Z}^N \subset L_n^{\neq 1}$. □

Some explicit decompositions for $\delta(1)$ are as follows:

(10) with $i=2$:

$$(n-2)\delta(1) = 2z_3 - (n-4)z_2 \text{ for } n-2 \text{ prime, including } n = 5, 7, 9, 13, 15, \cdots$$

(11) with $i = 2$:

$$\frac{n-2}{2}\delta(1) = z_3 - \frac{n-4}{2}z_2 \text{ for } n = 6, 12, \cdots$$

$$\delta(1) = -z_4 + 3z_3 - 4z_2 \text{ for } n = 8,\ 3\delta(1) = z_4 - 4z_3 + 7z_2 \text{ for } n = 11,$$

$$2\delta(1) = z_5 - 7z_3 + 16z_2 \text{ for } n = 10,\ 2\delta(1) = -z_4 + 7z_3 - 20z_2 \text{ for } n = 14.$$

3.1 Cut-sub-lattices of L_n for small n

For $n \le 4$, the lattice L_n has no proper full dimensional cut-sub-lattices, i.e. it is minimal.

For $n = 5$, L_5^2 is the unique full dimensional cut-sublattice of L_5. From Theorem 8(i), we have $3L_5 \subset L_5^2$.

Recall that the lattice L_6 has dimension 15. All full-dimensional proper uniform cut-sub-lattices of L_6 are the lattices

$$L_6^{ev} = L_6^2,\ L_6^{odd} = L_6^{1,3}, \text{ and } L_6^{\neq 1} = L_6^{(mod3)} = L_6^{2,3}.$$

In fact, $L_6^{1,2} = L_6$, and $\dim L_6^1 = 6$, $\dim L_6^3 = 10$. We know that $2\mathbf{Z}^N \subset L_n$. For what minimal $t \in \mathbf{Z}_+$ and cut-sub-lattices $L \in L_6$ the inclusion $2t\mathbf{Z}^{15} \subset L$ is true?

The characterization of L_n^{ev}, given in [5], implies that $t = 4$ for L_6^2. Similarly, the characterization of $L_6^{1,3}$, given in [8], implies that $t = 6$ for $L_6^{1,3}$.

The representation (strangely asymmetric) of $2e_{ij}$, given below, illustrates explicitly the fact, proved in Theorem 8, that $t = 1$ for $L = L_6^{2,3}$). Let $V = \{ijkpqr\}$. Then

$$2e_{ij} = 2\delta(ijk)+\delta(ikp)-\delta(iqr)+\delta(jp)+\delta(iq)+\delta(ir)-(\delta(ij)+\delta(ik)+\delta(pq)+\delta(pr)+\delta(qr)).$$

Proposition 6 implies that $L_n^{\neq 1}$ is maximal proper cut-sublattice of L_n for $n = 5, 6, 7$. For example, adding a cut $\delta(i)$ to $L_6^{2,3}$, will give L_6, because

$$\delta(j) = \delta(i) + (\delta(ikp) - \delta(jkp)) + (\delta(jp) - \delta(ip)) + (\delta(jk) - \delta(ik)).$$

Using that $\sum_{2 \le i<j<k \le 6} \delta(ijk) = 3\sum_{1 \le i \le 6} \delta(i)$ is the unique linear dependency on the set $\mathcal{K}_6^{1,3}$, one can see that $L_6^{1,3}$ has exactly 6 proper full-dimensional cut-sub-lattices (obtained by removing a cut $\delta(i)$, $1 \le i \le 6$.)

Remark. It is clear that $\mathbf{Z}_+(\mathcal{K}_n)$ is the set of all integer-valued semi-metrics on V, which are embeddable isometrically into a hypercube $\{0,1\}^m$. Clearly also, $L_n = \{a - b : a, b \in \mathbf{Z}_+(\mathcal{K}_n)\}$.

Denote by M_n the set $\{a - b : a, b$ are any integer-valued semi-metrics on n points$\}$. Using description of all extreme rays of the cone of all semi-metrics on 6 and 7 points, one can check that $M_n = L_n$ for $n \le 6$ and $M_n = \mathbf{Z}^N$ for $n \ge 7$.

Remark. Consider the **covariance map** $\pi^1 : \mathbf{R}^N \to \mathbf{R}^N$, defined by

$$\begin{array}{ll} \pi^1(d)_{ii} = d_{1i} & \text{for } 2 \le i \le n, \\ \pi^1(d)_{ij} = \frac{1}{2}(d_{1i} + d_{1j} - d_{ij}) & \text{for } 2 \le i < j \le n. \end{array}$$

This linear map of $\mathbf{R}^N$ into itself is important, because the **boolean quadric cone** $\pi^1(\mathbf{R}_+(\mathcal{K}_n)) = \mathbf{R}_+(\pi^1(\mathcal{K}_n))$ and **boolean quadric polytope** conv$\pi^1(\mathcal{K}_n))$ have many applications (combinatorial optimization, quantum mechanics etc.). But the lattice $L(\pi^1(\mathcal{K}_n))$ is nothing but $\mathbf{Z}^N$, because $e_{ii} = \pi^1(\delta(i))$, $e_{ij} = \pi^1(\delta(i) + \delta(j) - \delta(ij))$ for $2 \le i < j \le n$. Hence conv$\pi^1(\mathcal{K}_n)$ is not an L-polytope in $L(\pi^1(\mathcal{K}_n))$.

The map $\pi(d)$, given in the proof of Lemma 3, is just $\pi^1(\tilde{d})$, where $\tilde{d} = (\tilde{d}_{ij} : 1 \le i < j \le n+1)$, $\tilde{d}_{1j} = d_{1,j-1}$ for $2 \le j \le n+1$, $\tilde{d}_{ij} = d_{i-1,j-1}$ for $2 \le i < j \le n+1$.

Compare, finally, evident $\pi^1(\delta(S))\pi^1(\delta(T)) = \pi^1(\delta(S\Delta T))$ with $\delta(S)\delta(T) = \frac{1}{2}(\delta^2(S) + \delta^2(T) - \delta^2(S\Delta T))$ given in Introduction.

4 L-polytopes of the lattice L_n

Since $\delta(S)$ is a (0,1)-vector, it is a vertex of the N-dimensional cube Q_N. Let S_N be the sphere circumscribing the cube Q_N. The squared diameter of S_N is equal to N. Hence the squared radius of S_N is equal to $\frac{N}{4}$.

Clearly $PCut_n \subset Q_N$. Take the center of S_N as the new origin. Let j_N be the all-one vector of dimension N. Then each cut vector takes the form $\delta(S) - \frac{1}{2}j_N$ and has $\pm\frac{1}{2}$ coordinates. Norm of all the vectors is equal to $\frac{N}{4}$.

An L-polytope of a lattice is the convex hull of all lattice points lying on an empty sphere of the lattice, and the lattice points on the empty sphere have full rank. A sphere is called empty if there is no lattice point strictly inside it. An L-polytope P is called **basic** if the set of its vertices contains an affine basis of the lattice affinely generated by vertices of P. An L-polytope P is called **symmetric** or **asymmetric** according to whether the set of vectors, representing the vertices of P when the center of P is origin, is symmetric or asymmetric, respectively.

Proposition 10 *The cut polytope $PCut_n$ is a basic asymmetric L-polytope of the lattice L_n.*

Proof. By definition, $PCut_n$ generates the lattice L_n. Hence it is sufficient to prove, that the sphere S_N, circumscribing $PCut_n$ is empty. All integral points inside S_N are vertices of Q_N. The vertices lies on the sphere. Besides, by Proposition 1(ii), all (0,1)-vertices of L_n are cut vectors. The expression (5) shows that $PCut_n$ is basic. It is easy to see that the polytope is asymmetric. □

Let x be the center of an L-polytope P of L_n, whose set of vertices $V(P)$ contains the origin $0 = \delta(\emptyset)$. Then, for all lattice points $d \in L_n$ we have

$$(d - x)^2 \ge r^2,$$

with equality for $d \in V(P)$. Here r is the radius of the sphere circumscribing P. Since $0 \in V(P)$, we have $r^2 = x^2$, and the above inequality takes the form

$$2dx \le d^2, \text{ for all } d \in L_n. \tag{12}$$

We say that x is of **full rank** if the system of inequalities (12) satisfied by x as equalities uniquely determines x. Clearly, x of full rank is the center of an L-polytope $P(x)$ of the lattice L_n.

Proposition 11 *Let $x \in \mathbf{R}^N$ satisfies (12) for $d = 2e_{ij}$ for all $(ij) \in V^2$, and for $d = \delta(S)$ for all $S \subseteq V$. Then*

(i) $|x_{ij}| \leq 1$ for all $(ij) \in V^2$,

(ii) $x_{ij} = 1$ if $2e_{ij}$ satisfies (12) as equality,

(iii) if $x \geq 0$, then x satisfies (12) for all $d \in L_n$.

Proof. (i) is implied by the inequality (12) for $d = \pm 2e_{ij} \in L_n$.

(ii) If $d = 2e_{ij}$ satisfies (12) as equality, then the equality takes the form $x_{ij} = 1$.

(iii) If $d \in L_n$, then, according to (3), $d = 2a + \delta(S)$ for $a \in \mathbf{Z}^N$ and $S \subseteq V$. Recall that (12) is equivalent to $(d-x)^2 \geq x^2$. Set $y = \delta(S) - x$. Since $x \geq 0$ and $x_{ij} \leq 1$, $|y_{ij}| \leq 1$ for all (ij). We have $(d-x)^2 = (2a+y)^2 = 4a(a+y) + y^2$. But $a(a+y) = \sum_{(ij):a_{ij}\neq 0} |a_{ij}||a_{ij} + y_{ij}| \geq 0$. Hence $(d-x)^2 \geq y^2 = (\delta(S)-x)^2 \geq x^2$. □

Corollary 12 *Let $x = \chi(E) + \frac{\varepsilon}{2}\delta(T)$, where $\varepsilon = 0$ or 1, and $E \subseteq V^2$ be such a set that*

$$2|E \cap D(S)| + \varepsilon|D(T) \cap D(S)| \leq |S|(n - |S|) \text{ for all } S \subseteq V. \tag{13}$$

If x is of full rank, then x is the center of a symmetric L-polytope $P(x)$ of the lattice L_n with the set of vertices

$$V(P(x)) = \{2\chi(A^+) + d(S; A^-) : \ A^+ \subseteq E - D(S), A^- \subseteq D(S) - (E \cup D(T))\},$$

where S satisfies (13) as equality. In particular, $2\chi(E) \in V(P(x))$ and $\delta(T) \in V(P(x))$ if $\varepsilon = 1$.

Proof. We have to verify that x satisfies all inequalities (12). The inequality (13) implies that x satisfies (12) with $d = \delta(S)$ for all $S \subseteq V$. Since (13) is satisfied for $S = T$, we have $E \cap D(T) = \emptyset$. Hence x satisfies (12) for $d = 2e_{ij}$ for all pairs (ij). Now, by Proposition 11(iii), x satisfies (12) for all $d \in L_n$.

Let $d = 2a + \delta(S)$. In the proof of Proposition 11(iii) we see that

$$(d-x)^2 = (2a + \delta(S) - x)^2 = 4a(a + \delta(S) - x) + (\delta(S) - x)^2 \geq (\delta(S) - x)^2 \geq x^2,$$

and $a(a + \delta(S) - x) \geq 0$. Hence $d \in V(P(x))$ if and only if $\delta(S) \in V(P(x))$ and $a(a + \delta(S) - \chi(E) - \frac{\varepsilon}{2}\delta(T)) = 0$. The last equality can hold if and only if the supports of the vectors a and $a + \delta(S) - \chi(E) - \frac{\varepsilon}{2}\delta(T)$ do not intersect. Hence $|a_{ij}| \leq 1$ (i.e. $a = \chi(A^+) - \chi(A^-)$) and $A^+ \subseteq E - D(S)$, $A^- \subseteq D(S) - (E \cup D(T))$. Note that $\delta(S) \in V(P(x))$ if and only if S satisfies (13) as equality. It is easy to verify that $\delta(T) \in V(P(x))$. This implies that $2\chi(E) \in V(P(x))$.

Recall that an L-polytope P of a lattice L with the vertex set $V(P)$, containing origin 0, is symmetric if and only if the antipode of 0 in the sphere circumscribing P is a vertex of P, too. Clearly, the point $2x$ is the antipode of 0 for $P(x)$. Hence $P(x)$ is symmetric if and only if $2x \in L_n$. In our case $2x = 2\chi(E) + \varepsilon\delta(T) \in L_n$. □

Now we describe L-polytopes of L_n, which are contiguous to the cut L-polytope. The type of the new polytopes depends on the facet by that it is contiguous to $PCut_n$.

Denote by $P_n(F)$ the L-polytope which is adjacent to $PCut_n$ by a facet F. Let Q be the support of the facet F, i.e. the coefficients of the inequality defining F are nonzero only for (ij) such that $i,j \in Q$. Let $\mathcal{S}(F)$ be the set of all $S \subseteq Q$ such that $\delta(S)$ belongs to the facet F. We suppose that $\emptyset \in \mathcal{S}(F)$. Note that $Q-S \in \mathcal{S}(F)$ for $S \in \mathcal{S}(F)$. Then the cut $\delta(T)$ belong to the facet F if and only if $T \in \mathcal{T}(F)$, where $\mathcal{T}(F) = \{S \cup T' : S \in \mathcal{S}(F) \text{ and } T' \subseteq V-Q\}$.

Let x be the center of $P_n(F)$. Then the inequality (12) holds as equality for $\delta(T)$ for all $T \in \mathcal{T}(F)$, i.e.

$$\delta(T)x = \frac{1}{2}\delta^2(T),\ T \in \mathcal{T}(F). \tag{14}$$

Since F is a facet, rank of the matrix $(\delta_{ij}(T) : (ij) \in V^2, T \in \mathcal{T}(F))$ of order $|\mathcal{T}(F)| \times N$ is equal to $N-1$. Hence the equation (14) determines all coordinates x_{ij} of the vector x up to a parameter t. The value of t is uniquely determined by a vertex of $P_n(F)$ not belonging to $PCut_n$.

Proposition 13 *The system of equations (14) is equivalent to the following system*

$$\begin{array}{ll} x_{ij} = \frac{1}{2} & \text{for } i,j \in V-Q \\ \sum_{i\in S} x_{ip} = \frac{|S|}{2} & \text{for } p \in V-Q,\ S \in \mathcal{S}(F) \\ \sum_{(ij)\in(S,Q-S)} x_{ij} = \frac{1}{2}|S|(|Q|-|S|) & \text{for } S \in \mathcal{S}(F). \end{array}$$

Proof. Note that $\delta(T) \in \mathcal{T}(F)$ for all $T \subseteq V-Q$, since $\emptyset \in \mathcal{S}(F)$. Consider equation (14) for $T = \{i\}, \{j\}$ and $\{ij\}$, $i,j \in V-Q$. Using the equality (2), we obtain

$$2x_{ij} = (\delta(i) + \delta(j) - \delta(ij))x = \frac{1}{2}((n-1)+(n-1)-2(n-2)) = 1,$$

i.e. the first system of equations.

Now consider equation (14) for $T = S$ and $T = S \cup \{p\}$, where $S \in \mathcal{S}(F)$ and $p \in V-Q$. Let $|S| = s$. Then $\delta^2(S) = s(n-s)$ and $\delta^2(S \cup \{p\}) = (s+1)(n-s-1)$. We have

$$\delta(S)x = \sum_{(ij)\in D(S)} x_{ij} = \frac{1}{2}s(n-s), \tag{15}$$

$$\delta(S\cup\{p\}) = \sum_{(ij)\in D(S)} x_{ij} - \sum_{i\in S} x_{ip} + \sum_{j\in V-(S\cup\{p\})} x_{jp} = \frac{1}{2}(s+1)(n-s-1).$$

Substituting the first equality in the second, and using the equality $x_{jp} = \frac{1}{2}$ for $j \in V-(Q\cup\{p\})$, we obtain the second system of equations.

We can rewrite the equation (15) as follows

$$\sum_{(ij)\in(S,Q-S)} x_{ij} + \sum_{i\in S, p\in V-Q} x_{ip} = \frac{1}{2}[s(q-s)+s(n-q)],$$

where $q = |Q|$. Using the second system of equations, we obtain the third system. □

Now, as an example, we consider a pure $(2k+1)$-gonal facet $F = F_k$, $k \in \mathbf{Z}_+$, $1 \le k < \frac{n}{2}$, defined by the equation $\sum_{1\le i<j\le q} b_i b_j d_{ij} = 0$. Here $Q = Q^+ \cup Q^-$, $|Q| = q = 2k+1$, $|Q^+| = k+1$, $|Q^-| = k$, and $b_i = 1$ for $i \in Q^+$, $b_i = -1$ for $i \in Q^-$. Besides,

$$\mathcal{S}(F_k) = \{S^+ \cup S^-, Q-(S^+\cup S^-) : S^+ \subseteq Q^+, S^- \subseteq Q^-, |S^+| = |S^-| = s, 0 \le s \le k\}.$$

The second system of equations of Proposition 13 implies $x_{ip} = \frac{1}{2}$ for all $i \in Q$, $p \in V - Q$. Using the symmetry of third system under permutations in Q^+ and Q^-, we can suppose that

$$x_{ij} = x \text{ for all } i, j \in Q^+, \; x_{ij} = y \text{ for all } i, j \in Q^-, \; x_{ij} = z \text{ for all } i \in Q^+, j \in Q^-.$$

Then the third system takes the form

$$s(k-s+1)x + s(k-s)y + s(2(k-s)+1)z = s(2(k-s)+1), \; 0 \le s \le k.$$

The solution of the system is $x = y = t$ and $z = 1 - t$, where t is a parameter.

Denote $P_n(k) := P_n(F_k)$. Suppose that $2e_{ij} \in V(P_n(k))$ for some $(ij) \in V^2$. Then, according to Proposition 11(ii), $x_{ij} = 1$. Hence $i, j \in Q$.

The case $i \in Q^+, j \in Q^-$ is impossible. In fact, then $z = 1$, $x = y = 0$ and for $i \in Q^-$, we have $2\delta(i)x = 2(k+1) + (n-q) > \delta^2(i) = 2k + (n-q)$. This contradicts to (12).

Let $i, j \in Q^+$. Then $x = y = 1$, $z = 0$. In this case, $x = \sum_{(ij) \in Q^{+2}} e_{ij} + \sum_{(ij) \in Q^{-2}} e_{ij} + \frac{1}{2}\sum_{(ij) \in V^2 - Q^2} e_{ij}$. Note that $V^2 - Q^2 = D(Q) \cup (V-Q)^2$. Hence

$$2x = 2\chi(Q^{+2}) + 2\chi(Q^{-2}) + \delta(Q) + \chi((V-Q)^2). \tag{16}$$

Since $0 \le x_{ij} \le 1$ for all (ij), Proposition 11(iii) implies that $P_n(k)$ is an L-polytope, i.e. the inequality $(d-x)^2 \ge x^2$ is valid for all $d \in L_n$.

Proposition 14 *The L-polytope $P_n(k)$, $n > 2$, is symmetric if and only if $n = 2k+1+\varepsilon$ for $\varepsilon = 0$ or 1. For $n = 2k+1+\varepsilon$, the vertices of the polytope $P_n(k)$ are*

$$2\chi(X) + d(T;Y),$$

where $T = S$ if $\varepsilon = 0$ and $T = S$ and $S \cup \{n\}$ if $\varepsilon = 1$, and

$$S = S^+ \cup S^-, \; S^+ \subseteq Q^+, \; S^- \subseteq Q^-, \; |S^+| = |S^-| = s, \; 0 \le s \le k,$$

$$X \subseteq (Q^{+2} - (Q^+ - S^+, S^+)) \cup (Q^{-2} - (Q^- - S^-, S^-)),$$
$$Y \subseteq (S^+, Q^- - S^-) \cup (S^-, Q^+ - S^+).$$

Proof. Note that $(V-Q)^2 \ne \delta(S)$ for all S if $(V-Q)^2 \ne \emptyset$, and $(V-Q)^2 = \emptyset$ if $|V-Q| \le 1$. Comparing (3) and (16), we see $2x \in L_n$ if and only if $(V-Q)^2 = \emptyset$. This implies that the L-polytope $P_n(k)$ is symmetric if and only if $n = 2k+1+\varepsilon$.

Let $V = Q \cup \varepsilon\{n\}$. In this case $\delta(Q) \equiv \delta(V-Q) = \varepsilon\delta(n)$ and $x = \chi(E) + \frac{\varepsilon}{2}\delta(n)$ with $E = Q^{+2} \cup Q^{-2}$. We can apply Corollary 12. We obtain that the vertices of $P_n(k)$ are as in the assertion of this proposition. □

Let $PCut(\mathcal{K})$ be convex hulls of all cuts, contained in the lattice $L(\mathcal{K})$.

Since, for $\mathcal{K} \subseteq \mathcal{K}_n$, $L(\mathcal{K}) \subset L_n$, Proposition 10 implies

Corollary 15 *The polytope $PCut(\mathcal{K})$ is an asymmetric (basic?) L-polytope of the lattice $L(\mathcal{K})$.* □

The symmetry group of $PCut_n$ consists of reflections $r_{\delta(T)}$ and of transformations generated by all permutations of the set V. The reflection $r_{\delta(T)}$ reflects $PCut_n$ simultaneously in all the hyperplanes, which contain the center of $PCut_n$ and are orthogonal to the vectors e_{ij} for $(ij) \in D(T)$.

Since $r_{\delta(T)}(\delta(S)) = \delta(T\Delta S)$, for odd T, $r_{\delta(T)}$ transforms $PCut(\mathcal{K}_n^{ev})$ and $PCut(\mathcal{K}_n^{odd})$ each into other. Hence $PCut(\mathcal{K}_n^{ev})$ and $PCut(\mathcal{K}_n^{odd})$ are congruent. Clearly, the symmetry group of $PCut(\mathcal{K}_n)$ ($PCut(\mathcal{K}_n^{ev})$) is the symmetry group of L_n (L_n^{ev}, respectively).

Remark.

Note that, since $PCut_n$ is embedded into N-dimensional cube, we can define Hamming distance $d_n(\delta(S), \delta(T)) = \|\delta(S) - \delta(T)\|_{l_1}$ between vertices of $PCut_n$. We have $d_n(\delta(S) - \delta(T)) = (\delta(S) - \delta(T))^2 = \delta^2(S\Delta T)$.

On the other hand, since $\delta(S) = \delta(V - S)$, the vertices of $PCut_n$ relate to bipartitions of the set V. There is the well known distance-regular graph, the **folded cube** $\Box_n$, defined on all bipartitions. Two bipartitions are adjacent if its common refinement contains a one-element set, i.e. $\delta(S)$ is adjacent to $\delta(T)$ if $|S\Delta T| = 1$. The graphic distance $d[\Box_n](x,y)$ and the distance $d_n(x,y)$ are related as follows (we set $e(n) := 2^n$)

$$d_n(x,y) = d[\Box_n](x,y)(n - d[\Box_n](x,y)),$$

$$\text{i.e. } d_n = d[\Box_n](nd[K_{e(n-1)}] - d[\Box_n]) \in \mathbf{Z}_+(\mathcal{K}_{e(n-1)}^{e(n-2)}).$$

where K_m is the complete graph on m vertices.

Similarly, the graph on all even cuts is the halved folded cube $\frac{1}{2}\Box_n$. The vertices $\delta(S)$ and $\delta(T)$ are adjacent in $\frac{1}{2}\Box_n$ if $|S\Delta T| = 2$. We have

$$d_n^{ev} = 2d[\frac{1}{2}\Box_n](nd[K_{e(n-2)}] - 2d[\frac{1}{2}\Box_n] \in \mathbf{Z}_+(\mathcal{K}_{e(n-2)}^{e(n-3)}).$$

Actually, we have $\Box_3 = \frac{1}{2}\Box_4 = K_4$, $\frac{1}{2}\Box_6 = K_{16}$, $\Box_4 = K_{4,4}$, where $K_{n,m}$ is the complete bipartite graph on $n+m$ vertices. The complement of $\Box_5$, is the Clebsh graph, i.e. the halved 5-cube $\frac{1}{2}H(5)$. The halved folded cubes $\frac{1}{2}\Box_n$ for $n = 8, 10$ have diameter 2, and therefore they are strongly regular. For small n we have

$$d_3 = 2d[K_4];\ d_4 = 2d[K_8] + d[K_{4,4}];\ d_4^{ev} = 4d[K_4];\ d_6^{ev} = 6d[K_{16}];$$

$$d_5 = d[\Box_5](2d[K_{16}] + d[\frac{1}{2}H(5)]) = 2d[K_{16}] + 2d[\Box_5].$$

So d_3 and d_4 belong to the interior of the hypermetric cone $Hyp_n := \{x \in \mathbf{R}^N : \sum_{1\le i<j\le n} b_i b_j x_{ij} \le 0$ for all $b \in \mathbf{Z}^n, \sum_{1\le i\le n} b_i = 1\}$ with $n = 4$. Clearly, $PCut_3$ and $PCut_4^{ev}$ are regular 3-simplices in L_3, and L_4^{ev}, respectively. One can check that the minimal face of Hyp_8, containing d_4 is the 5-face obtained as the intersection of 8 facets $\sum_{1\le i<j\le n} b_i^a b_j^a x_{ij} = 0$ for each cut a. Namely, $b_i^a = 0$ for the index i corresponding to a, $b_i^a = -1$ for 3 other cuts of the same parity and $b_i^a = 1$ for 4 remaining cuts. Actually, $d_4 = \delta(1256) + \delta(1278) + \delta(1357) + \delta(1368) + \delta(1458) + \delta(1467)$, where the numbers 1,2,3,4 correspond to 4 even cuts, and this representation in a 5-simplex seems to be the unique representation of d_4 in Cut_8.

5 Odd systems, related to the lattice L_n

Denote $d(S;X)$, defined in (6), by $d(k;X)$ if $S=\{k\}$.

Let

$$\mathcal{V}_n=\{d(k;X):X\subseteq D(k), k\in V\}.$$

Proposition 16 *The set $\mathcal{V}_n$ generates the lattice L_n.*

Proof is implied by the equality

$$\delta(S)=\sum_{k\in S}d(k;X_k),$$

where $X_k=\{(kj):j\in S, j<k\}$. □

Proposition 17 *$\mathcal{V}_n$ is a uniform odd system of norm $n-1$ if n is even.*

Proof. We have $d(i;X)d(j;Y)=\pm 1$ if $i\neq j$, $d(i;X)d(i;Y)=n-1-2|X\Delta Y|\equiv n-1 \pmod 2$. □

For n even, we can recognize in the odd system $\mathcal{V}$ subsystems spanning equiangular lines. The simplest such a system is $\mathcal{K}_n^1=\{\delta(i):i\in V\}$. A maximal set of vectors with mutual inner products ± 1 is constructed as follows.

If $k\in V$, then $|D(k)|=n-1$ is odd. Consider on the set $D(k)$ such a maximal by inclusion family $\mathcal{X}_k$ of subsets $X\subseteq D(k)$ of cardinality $|X|=\frac{n}{2}-1$, that the inner product $i(X,X'):=d(k;X)d(k;X')=\pm 1$ for any two subsets X,X' of the family. The conditions $i(X,X')=\pm 1$ implies that either $|X\cap X'|=\frac{n}{4}-1$ or $|X\cap X'|=\frac{n-2}{4}$. Since n is even, we have exactly one of these cases, i.e.

$$|X\cap X'|=\begin{cases}\frac{n}{4}-1 & \text{and } i(X,X')=-1 \quad \text{if } n\equiv 0 \pmod 4,\\ \frac{n-2}{4} & \text{and } i(X,X')=1 \quad \text{if } n\equiv 2 \pmod 4.\end{cases}$$

Then the set

$$\mathcal{M}_n=\{d(k,X):X=\emptyset \text{ or } X\in\mathcal{X}_k,\ k\in V\}$$

spans a set of equiangular lines at angle $arccos\frac{1}{n-1}$. Since $d(k,\emptyset)=\delta(k)$, $\mathcal{K}_n^1\subset\mathcal{M}_n$.

So, we have $L_n=L(\mathcal{V}_n)$. For even n, denote the sublattice $L_0(\mathcal{V}_n)$ by L_n^0. According to (3),

$$L_n^0=\{d\in L_n: d=2a+\delta(S) \text{ where } \delta(S)\in\mathcal{K}_n^{ev}\}.$$

Besides, the lattice L_n has the double even sub-lattices L_n^{ev}, $2\mathbf{Z}^N$.

There is another double even lattice related to odd cuts:

$$L_0(\mathcal{K}_n^{odd})=\{d:d=\sum_{\delta(S)\in\mathcal{K}_n^{odd}}z_S\delta(S),\ \sum_{\delta(S)\in\mathcal{K}_n^{odd}}z_S=0,\ z_S\in\mathbf{Z}\}.$$

Proposition 18 *L_n^{ev} is isomorphic but not equal to $L_0(\mathcal{K}_n^{odd})$.*

Proof. Let $d\in L_0(\mathcal{K}_n^{odd})$ has a representation $d=\sum_S z_S\delta(S)$. Since $\sum_S z_S=0$, $d=\sum_S z_S(\delta(S)-\delta(T))$ for some odd T. The expression means that the lattice $L_0(\mathcal{K}_n^{odd})$ is affinely generated by vertices of the polytope $PCut_n^{odd}$. But this polytope is congruent to the polytope $PCut_n^{ev}$, which affinely generate the lattice L_n^{ev}. Hence the lattice $L_n^{ev}=L(\mathcal{K}_n^{ev})$ is isomorphic (in fact, congruent) to the lattice $L(\mathcal{K}_n^{odd})$. Clearly, these lattices are distinct. □

6 Cut polytopes $PCut_n$ for small n

Consider $PCut_n$ and corresponding lattices for small n in details.

(1) **n=2**, $N=1$. $PCut_2$ is a unit segment, $PCut_n^{ev}$ and $PCut_2^{odd}$ are zero-dimensional points, ends of the segment. $L_2 = L_n^{odd} = \mathbf{Z}$, $L_2^{ev} = \emptyset$.

(2) **n=3**, $N=3$. $PCut_3$ is the regular tetrahedron α_3 with norms of edges (squared lengths) 2. The lattice L_3 is the 3-dimensional face-centered lattice $A_3 = D_3$. The second type of the L-polytopes of L_3 is known, and it is described also in Proposition 14. They are the cross-polytopes $P_3(1) = \beta_3$ (octahedrons) for $\varepsilon = 0$ centered at the points e_{ij} and having edges of norm 2.

(3) **n=4**, $N=6$. $PCut_4$ is the 6-dimensional repartitioning L-polytope with 8 vertices. It is combinatorially equivalent to the cyclic 6-polytope with 8 vertices. $PCut_4$ relates to a pure 7-gonal facet F_3 of the hypermetric cone Hyp_7 of hypermetrics on 7 points.

By Proposition 16, L_4 is generated by the odd system $\mathcal{V}_4$. The odd system $\mathcal{V}_4$ coincides with $\mathcal{M}_4$, which is the maximal closed uniform odd system of norm 3 and dimension 6 with $n(\mathcal{M}_4) = 16$. This odd system is related to the root system E_7. It is described in [4]. Table 1 of [4] shows that $L_4 = \sqrt{2}D_6^{+2}$, where D_6^{+2} is described in [3]. The root lattice D_6 has (up to signs) 30 roots $e_i - e_j$, $e_i + e_j$, $1 \le i < j \le 6$. If we add 32 vectors of the shape $(\pm 1/2)^6$ with even number of minus signs, we obtain D_6^+. Now take new orthonormal basis $\{f_i : 1 \le i \le 6\}$ of the space R^6. Let $g = \sqrt{2}$, then $f_1 = (e_1 - e_2)/g, f_2 = (e_1 + e_2)/g, f_3 = (e_3 - e_4)/g, f_4 = (e_3 + e_4)/g, f_5 = (e_5 - e_6)/g, f_6 = (e_5 + e_6)/g$. In this basis the lattice gD_6^+ takes the form L_4. We obtain 30 (up to signs) vectors of norm 4, and 32 vectors of norm 3.

We apply Corollary 12 to $V = \{1234\}$, $E = \{(12), (34)\}$ and $\varepsilon = 0$. The only S satisfying (13) as equality are, up to complement, $S = \emptyset, \{13\}, \{14\}$. It is easy to verify that $x = \chi(E) = e_{12} + e_{34}$ is of full rank, and the set of vertices of $P(x)$ is the set

$$\mathcal{D}_4 = \{2\chi(X) : X \subseteq E\} \cup \{d(S;X) : X \subseteq D(S) - E, \text{ and } S = (13) \text{ or } (14)\}.$$

Denote the polytope by P_4.

If we take the origin in the center of P_4, then the vertices of P_4 are represented by vectors

$$\pm(e_{12} \pm e_{34}), \quad \pm(e_{13} \pm e_{24}), \quad \pm(e_{14} \pm e_{23}).$$

Now it is easy to see that P_4 is the symmetric 6-dimensional cross-polytope β_6 with edges of norm 4.

Since all facets of $PCut_4$ are 0-extensions of triangle facets, the L-polytopes contiguous to facets of $PCut_4$ all have the same type $P_4(1)$ described in Proposition 14. In the case $\varepsilon = 1$, $|Q^+| = 2$, $|Q^-| = 1$. Let $Q^+ = \{12\}$, $Q^- = \{3\}$. Then $x = e_{12} + \frac{1}{2}\delta(4)$. The vertices of $P_4(1)$ are

$$\delta(S) \text{ and } 2e_{12} + \delta(\{4\}) - \delta(S) \text{ for } S \in \mathcal{S}(F_1) = \{\emptyset, \{1\}, \{2\}, \{4\}, \{13\}, \{23\}\}.$$

Since the vectors $\delta(S)$ for $S \in \mathcal{S}(F_1)$, $S \neq \emptyset$, and $2e_{12}$ form the basis of L_4, $P_4(1)$ is basic.

If we take origin in the center of $P_4(1)$, then the vertices of $P_4(1)$ are represented by vectors

$$\pm\frac{1}{2}(2e_{12} \pm (e_{14} + e_{24} + e_{34})), \pm\frac{1}{2}(2e_{13} \pm (-e_{14} + e_{24} + e_{34})), \pm\frac{1}{2}(2e_{23} \pm (e_{14} - e_{24} + e_{34})).$$

It is easy to see that the vectors multiplied by 2 have norm 7 and mutual inner products ± 1. Hence they span 6 equiangular lines at angle $arccos\frac{1}{7}$.

Let e_0 be a unit vector orthogonal to the space spanned by L_4. If we add two pairs of vectors

$$\pm\frac{1}{2}(2e_0 \pm (e_{14} + e_{24} - e_{34})),$$

we obtain 8 pairs of vectors spanning in 7-dimensional space 8 equiangular lines at angle $arccos\frac{1}{7}$. The convex hull of all these vectors is, up to the multiple $\sqrt{2}$, the unique (basic) L-polytope of the lattice E_7^* (see, for example, [2]). Hence $\frac{1}{\sqrt{2}}L_4 = D_6^{+2}$ is a section of the lattice E_7^*.

There is the forth L-polytope of the lattice L_4, a 6-dimensional simplex Σ. The norms of its edges are equal to 3 and 4. The 6 edges of norm 3 are adjacent to the same vertex.

The adjacencies between these 4 types of L-polytopes of the lattice L_4 are as follows. $PCut_4$ is adjacent only to L-polytopes of the type $P_4(1)$. A cross-polytope $P_4 = \beta_6$ is adjacent to L-polytopes of the types $P_4(1)$ and Σ. The simplex Σ is adjacent to L-polytopes of the types β_6 and $P_4(1)$. The polytope $P_4(1)$ is adjacent to L-polytopes of all the 4 types.

The norms of vectors x of centers of polytopes $PCut_4$, Σ, $P_4(1)$ and β_6 are, respectively, $\frac{3}{2} < \frac{27}{16} < \frac{7}{4} < 2$. So, the deep hole is the cross-polytope β_6.

We call L-polytopes P and P' **lattice equivalent** if either $P' = -P$ or $P' = P + a$ for some lattice vector a. Note that if P is symmetric and $0 \in V(P)$, then $-P = P - 2x$, where x is the center of P. In this case, $2x \in V(P)$.

For an L-polytope P, we denote

the center of P by $x(P)$,

the number of vertices of P by $v(P)$,

the ratio of the volume of P to the volume of a basic simplex by $V_r(P)$,

the number of lattice nonequivalent L-polytopes of type P in the star at 0 by $N(P)$.

Let $s(P) = 1$ if P is symmetric, and $s(P) = 2$ if P is asymmetric. Then the number of L-polytopes congruent to P in a star is equal to $s(P)v(P)N(P)$. Besides we have $\sum_P s(P)N(P)V_r(P) = N!$, where $N = dimL_n$

The star at $0 \in L_4$ and corresponding Voronoi's polytope of L_4 are described by the following table. All indexes in the table are distinct.

P	V_r (P)	s(P)	v(P)	N(P)	x(P)
$PCut_4$	4	2	8	10	$\frac{1}{2}\sum_{(ij)\in V^2} \pm e_{ij}$, $\pm e_{ij} + \frac{1}{2}(\pm e_{ik} \pm e_{jk})$
$P_4(1)$	32	1	12	16	$\pm e_{ij} + \frac{1}{2}(\pm e_{ik} \pm e_{jk} \pm e_{kl})$
β_6	64	1	12	1	$\pm e_{ij} \pm e_{kl}$
Σ	2	2	7	16	$\pm\frac{3}{4}e_{ij} \pm e_{kl} + \frac{1}{4}(\pm e_{ik} \pm e_{jk})$, $\frac{3}{4}(\pm e_{ij} \pm e_{jk} \pm e_{ki})$

The centers off all L-polytopes of type P can be obtained from $x(P)$ by taking $i, j, k, l \in \{1, 2, 3, 4\}$ and taking signs + or - in $\pm$ independently. For example, $x(P) = \pm e_{ij} \pm e_{kl}$ has 4 distinct patterns of signs and 3 distinct partitions of $V = \{1, 2, 3, 4\}$ into equal parts. Hence there are 3.4=12 cross-polytopes β_6 in the star.

Note that there are $\sum_P s(P)v(P)N(P) = 588$ L-polytopes in a star, and therefore the Voronoi's polytope has 588 vertices.

The Voronoi's polytope P_V has 60+32=92 facets. The 60 facets of $P_V(0)$ with center in $0 \in L_4$ are orthogonal to the 60 vectors of norm 4 and 32 facets are orthogonal to 32 vectors of norm 3. Each facet contains the middle point of the corresponding vector. A facet orthogonal to a vector of norm 4 contains 42 vertices. A facet orthogonal to a vector of norm 3 contains 56 vertices.

$PCut_4^{ev}$ and $PCut_4^{odd}$ are 3-dimensional simplexes with edge length 2 (norm 4), spanning orthogonal 3-spaces. These simplexes intersect in the center of both, which is the center of the sphere S_6 circumscribing $PCut_4$. The squared distance between vertices of different simplexes is 3. $L_4^{ev} = \sqrt{2}A_3$, where A_3 is the 3-dimensional root lattice. The 4-dimensional lattice L_4^{odd} can be obtained from the root lattice $\sqrt{2}A_5$ as its section by a hyperplane orthogonal to an arbitrary root and going through the midpoint of the root.

(4) **n=5**, $N = 10$. $PCut_5$ is a 10-dimensional L-polytope.

Note that there are only 2 values of norms of edges of $PCut_5$, namely 4 and 6. Hence if we take origin in the center of $PCut_5$, then the vectors $\sqrt{2}(\delta(S) - \frac{1}{2}j_{10})$, representing vertices of $\sqrt{2}PCut_5$, have norm 5 and inner products ± 1, i.e. they span equiangular lines at angle $\arccos\frac{1}{5}$.

The graph on the vertices of $PCut_5$ with edges of norm 6 is the Clebsh graph, i.e. the Halved cube $\frac{1}{2}H(5)$.

(5) **n=6**, $N = 15$. $PCut_6^{ev}$ and $PCut_6^{odd}$ are 15-dimensional simplexes with squared length 8. They intersect in the center of $PCut_6$. The lattice L_6^{ev} is $2A_{15}$, the root lattice A_{15} multiplied by 2.

The set $\mathcal{M}_6$ contains 30 vectors spanning 30 equiangular lines at angle $arccos\frac{1}{5}$ in 15-dimensional space. A maximal set of such lines contains 36 lines.

(6) **n=8**, $N = 28$. The 28-dimensional L-polytopes $PCut_8^{ev}$ and $PCut_8^{odd}$ have only 2 squared Euclidean distances between vertices: 12 and 16. Hence if we take origin in the common center $\frac{1}{2}j_{28}$ of these polytopes, then the vectors $\delta(S) - \frac{1}{2}j_{28}$, (say, $|S|$ is even), have norm 7 and inner products ± 1, i.e. they span $2^{8-2} = 64$ equiangular lines at angle $arccos\frac{1}{7}$. Since 64 is the maximal number of lines at angle $arccos\frac{1}{7}$ in 28-dimensional space, we conclude that this graph on vertices of $PCut_8^{ev}$ (or $PCut_8^{odd}$) with edges of norm 16 is a strongly regular graph of the Pseudo Latin square type with parameters (64,35,18,20). The complement of the graph is the halved folded 8-cube. The graph (and its complement) has 2-transitive automorphism group.

The similar set of 64 equiangular lines with the graph with the same parameters of strongly regular graph as above spans the odd system $\mathcal{M}_8$. In this case, each family $\mathcal{X}_k$, $k \in \{1, 2, ..., 8\}$, is the unique Steiner triple system on 7 points, containing 7 triples. We don't know whether the graphs are isomorphic.

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