Hypercube Embeddings and Designs

Monique LAURENT
Michel DEZA

Laboratoire d'Informatique, URA 1327 du CNRS
Département de Mathématiques et d'Informatique
Ecole Normale Supérieure

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Michel Deza
Monique Laurent

CNRS, LIENS, Ecole Normale Superieure
45, rue d'Ulm
75230 Paris Cedex 05, France

Abstract

This is a survey on hypercube embeddable semimetrics and the link with designs. We investigate, in particular, the variety of hypercube embeddings of the equidistant metric. For some parameters, it is linked with the question of existence of projective planes or Hadamard matrices. The problem of testing whether a semimetric is hypercube embeddable is NP-hard in general. Several classes of semimetrics are described for which this problem can be solved in polynomial time. We also consider questions related to some necessary conditions for hypercube embeddability.

1 Introduction

In this paper, we survey hypercube embeddability of some classes of metrics and, in particular, the link with designs.

Let $t \geq 1$ be an integer. A very simple metric is the **equidistant metric** on n points, denoted by $2t\mathbb{1}_n$, which takes the same value $2t$ on each pair of points. The metric $2t\mathbb{1}_n$ is obviously hypercube embeddable. Indeed, a hypercube embedding of $2t\mathbb{1}_n$ is obtained by labeling the points by disjoint sets, each of cardinality t. It is shown in Section 2 that, if $n \geq t^2+t+3$, then this embedding is essentially the unique hypercube embedding of $2t\mathbb{1}_n$. In Section 3, we investigate how various hypercube embeddings of $2t\mathbb{1}_n$ arise from designs. We then consider in Section 4 some other classes of metrics for which we are able to characterize hypercube embeddability. Typically, these metrics have a small range of values so that one can still take advantage of the knowledge available for their equidistant submetrics. For instance, one can characterize the hypercube embeddable metrics with values in the set $\{1,2,3\}$, or in the set $\{3,5,8\}$. Moreover, this characterization yields a polynomial time algorithm for checking hypercube embeddability of such metrics. We recall that, for general semimetrics, it is NP-complete to check whether a given semimetric is hypercube embeddable. Several additional results related to the notion of hypercube embeddability are grouped in Section 5.

We now recall some definitions and terminology that we use in this paper. Given a subset S of $V_n := \{1,\ldots,n\}$, the **cut semimetric** $\delta(S)$ is the vector of $\mathbb{R}^{\binom{n}{2}}$ defined by $\delta(S)(i,j) = 1$ if $|S \cap \{i,j\}| = 1$ and $\delta(S)(i,j) = 0$ otherwise, for $1 \leq i < j \leq n$. Then, the cone in $\mathbb{R}^{\binom{n}{2}}$ generated by the cut semimetrics $\delta(S)$, for $S \subseteq V_n$, is called the **cut cone** and is denoted by CUT_n.

Let d be a distance on V_n. Then, d is said to be **hypercube embeddable** if there exist vectors $u_i \in \{0,1\}^m$ ($m \geq 1$), for $i \in V_n$, such that

$$(1.1) \qquad d(i,j) = \| u_i - u_j \|_1 \ (= \sum_{1 \leq h \leq m} |(u_i)_h - (u_j)_h|)$$

for all $i,j \in V_n$. Let M denote the $n \times m$ matrix whose rows are the vectors $u_1,\ldots,u_n$; M is called the **realization matrix** of the embedding $u_1,\ldots,u_n$ of d. Any matrix arising as the realization matrix of some hypercube embedding of d is called an h**-realization matrix** of d. Each vector u_i can be seen as the incidence vector of a subset A_i of $\{1,\ldots,m\}$. Hence, (1.1) can be rewritten as

$$(1.2) \qquad d(i,j) = |A_i \triangle A_j|$$

for all $i,j \in V_n$. We also say that the sets $A_1,\ldots,A_n$ form an h**-labeling** of d.

Note that, if M is an h-realization matrix of d, we can assume that a row of M is the zero vector. This amounts to assuming that one of the points is labeled by $\emptyset$ in the corresponding h-labeling of d.

Let $\mathcal{B}$ denote the collection of subsets of V_n whose incidence vectors are the columns of M; $\mathcal{B}$ is a multiset, i.e., it may contain several times the same member. Then, (1.1) is equivalent to

$$(1.3) \qquad d = \sum_{B \in \mathcal{B}} \delta(B).$$

This shows that a semimetric is hypercube embeddable if and only if it can be decomposed as a nonnegative integer combination of cut semimetrics. If (1.3) holds, we also say that $\sum_{B\in\mathcal{B}} \delta(B)$ is a $\mathbb{Z}_+$**-realization** of d. It will be convenient to use both representations (1.1) (or 1.2)), and (1.3) for a hypercube embeddable semimetric d; so we shall speak of a hypercube embedding (or of an h-labeling of d), and of a $\mathbb{Z}_+$-realization of d, which basically amounts to looking either to the rows, or to the columns of the matrix M.

Let d be a hypercube embeddable distance on V_n. Then, the quantities:

$$s_h(d) := \min(\sum_S \lambda_S \mid d = \sum_S \lambda_S \delta(S) \text{ with } \lambda_S \in \mathbb{Z}_+ \text{ for all } S) \tag{1.4}$$

$$s_{\ell_1}(d) := \min(\sum_S \lambda_S \mid d = \sum_S \lambda_S \delta(S) \text{ with } \lambda_S \geq 0 \text{ for all } S) \tag{1.5}$$

are called, respectively, the **minimum h-size** and the **minimum ℓ_1-size** of d.

Let M be a h-realization matrix of the hypercube embeddable distance d. Consider the following operations on the matrix M:
(i) Permute the columns of M.
(ii) Add to (or delete from) M a column with entries all equal to 0, or all equal to 1.
(iii) Add modulo 2 a vector $a \in \{0,1\}^m$ to all rows of M.
If we apply any of the operations $(i),(ii),(iii)$ to M, we obtain another matrix M' which is still an h-realization matrix of d. However, M' yields (via (1.3)) the same $\mathbb{Z}_+$-realization as M (indeed, (i) means permuting the terms in the sum $\sum_{B \in \mathcal{B}} \delta(B)$, (ii) means adding the vector $\delta(\emptyset) = \delta(V_n) = 0$, and (iii) means replacing the vector $\delta(B)$ by the same vector $\delta(V_n \setminus B)$). For this reason, two h-realization matrices are said to be **equivalent** if they can be obtained from one another via the operations $(i),(ii)$, or (iii). In the same way, two hypercube embeddings are equivalent if their realization matrices are equivalent. The distance d is said to be **h-rigid** if, up to equivalence, d has a unique hypercube embedding or, equivalently, if d has a unique $\mathbb{Z}_+$-realization.

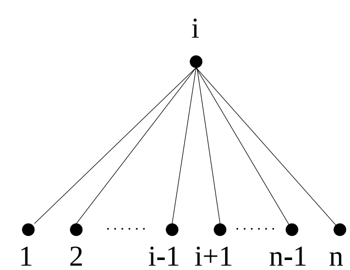
We refer, for instance, to [DL93b] for a survey on ℓ_1-metrics and hypercube embeddable metrics and their link with cut polyhedra.

2 Rigidity of the equidistant metric

In this section, we study h-rigidity of the equidistant metric $2t\mathbb{1}_n$. As was already mentioned, $2t\mathbb{1}_n$ is hypercube embeddable. Indeed, a hypercube embedding of $2t\mathbb{1}_n$ is obtained by labeling the n points by pairwise disjoint sets, each of cardinality t. This embedding is called the **star embedding** of $2t\mathbb{1}_n$; it corresponds to the following $\mathbb{Z}_+$-realization:

$$2t\mathbb{1}_n = \sum_{1 \leq i \leq n} t\delta(\{i\}), \tag{2.1}$$

called the **star realization** of $2t\mathbb{1}_n$. The word "star" is used since each cut semimetric $\delta(\{i\})$ takes nonzero values on the pairs (i,j) for $j \in \{1,\ldots,n\} \setminus \{i\}$, which are the edges of the following graph, commonly called a star in graph theory.



It will be useful to have the following matrix notation:
$0_{p,q}$ denotes the $p \times q$ matrix zero matrix, $J_{p,q}$ denotes the $p \times q$ matrix of all ones,
$C^{(i)}_{p,q}$ denotes the $p \times q$ matrix with 0's in column i and 1's elsewhere, and
$R^{(i)}_{p,q}$ denotes the $p \times q$ matrix with 1's in row i and 0's elsewhere.
We may omit the subscripts which indicate the size of the matrix.
For instance, the following matrix (with $m_0, m_1 \geq 0$) is an h-realization matrix of $2t1\!\!1_n$, which gives an embedding equivalent to the star embedding.

$R^{(1)}_{n,t}$	$R^{(2)}_{n,t}$	$\ldots$	$R^{(n)}_{n,t}$	$0_{n,m_0}$	J_{n,m_1}

For $n = 3$, the equidistant metric $2t1\!\!1_3$ is h-rigid. This follows from the fact that the cut cone CUT_3 is a simplex cone (indeed, CUT_3 is generated by the three linearly independent vectors $\delta(\{i\})$ for $i = 1, 2, 3$). For $n = 4$, $2t1\!\!1_4$ is not h-rigid. Indeed, besides the star realization from (2.1), $2t1\!\!1_4$ admits the following $\mathbb{Z}_+$-realization:

$$2t1\!\!1_4 = t(\delta(\{1,2\}) + \delta(\{1,3\}) + \delta(\{1,4\})); \tag{2.2}$$

$21\!\!1_4$ has no other $\mathbb{Z}_+$-realization. In fact, $2t1\!\!1_n$ is not ℓ_1-rigid for any $n \geq 4$ as, for instance, $2t1\!\!1_n = \frac{t}{n-2} \sum_{1 \leq i < j \leq n} \delta(\{i,j\})$ is a decomposition of $2t1\!\!1_n$ as a nonnegative sum of cut semimetrics which is distinct from the decomposition from relation (2.1). But, as we see below, if n is large with respect to t, then $2t1\!\!1_n$ is h-rigid. We now present the main results of this section.

Theorem 2.3. [Dez73] *If $n \geq t^2 + t + 3$, then $2t1\!\!1_n$ is h-rigid, i.e., the only $\mathbb{Z}_+$-realization of $2t1\!\!1_n$ is the star realization from* (2.1). *If there exists a projective plane of order t, then the metric $2t1\!\!1_{t^2+t+2}$ is not h-rigid.*

Theorem 2.4. [vL73] *Let $n = t^2 + t + 2$ with $t \geq 3$. If the metric $2t1\!\!1_n$ is not h-rigid, then there exists a projective plane of order t.*

Recall that a (finite) **projective plane of order** t, commonly denoted by $PG(2,t)$, consists of a collection $\mathcal{L}$ of subsets, called **lines**, of a set X of cardinality $|X| = t^2 + t + 1$, satisfying:
- each line $L \in \mathcal{L}$ has cardinality $t + 1$,
- each point of X belongs to $t + 1$ lines, and
- any two distinct points of X belong to exactly one common line.

We now give the proofs of Theorems 2.3 and 2.4.

Let M be a binary $n \times m$ matrix which is an h-realization matrix of $2t1\!\!1_n$. Without loss of generality, we can suppose that the first row of M is the zero vector. Then, each other row of M has $2t$ units and any two rows (other than the first one) have t units in common. We give some preliminary results on M.

Lemma 2.5. *Let r denote the number of units in a column of M. Then, $r(n - r) \leq nt$,* *implying that* $\min(r, n-r) \le \frac{1}{2}(n - \sqrt{n^2 - 4nt})$.

PROOF. Let w be a column of M, let r denote the number of 1's in w, and let ρ denote the number of columns of M identical to w. Let M' denote the $n \times (m-\rho)$ denote the submatrix obtained from M by deleting these ρ columns, and let d' denote the distance on n points defined by letting d'_{ij} denote the Hamming distance between the i-th and j-th rows of M'. We can suppose that the first $n-r$ entries of w are equal to 0 and its last r entries are equal to 1. Then,

$$\begin{cases} d'_{ij} = 2t & \text{if } 1 \le i < j \le n-r, \text{ or } n-r+1 \le i < j \le n, \\ d'_{ij} = 2t - \rho & \text{if } 1 \le i \le n-r < j \le n. \end{cases}$$

Consider the inequality:

$$\sum_{1 \le i < j \le n-r} r^2 x_{ij} + \sum_{n-r+1 \le i < j \le n} (n-r)^2 x_{ij} - \sum_{1 \le i \le n-r < j \le n} r(n-r) x_{ij} \le 0.$$

(It is an inequality of negative type; see [DL93b].) It is not difficult to check that, as d' is hypercube embeddable by construction, d' satisfies the above inequality. We deduce from it that $\rho r(n-r) \le nt$, which implies

$$r(n-r) \le nt.$$

From the latter relation follows immediately that

$$\min(r, n-r) \le \frac{1}{2}(n - \sqrt{n^2 - 4nt}).$$

∎

LEMMA 2.6. *Suppose that the number r of units in any column of M satisfies*

$$\min(r, n-r) \le \left\lfloor \frac{n+2t-1}{t+1} \right\rfloor - 1. \tag{2.7}$$

Then, M is the realization matrix of a hypercube embedding of $2t\mathbb{1}_n$ equivalent to the star embedding.

PROOF. Set $\alpha = \left\lfloor \frac{n+2t-1}{t+1} \right\rfloor$. By assumption, the number r of units in a column of M satisfies: $r \le \alpha - 1$, or $r \ge n - \alpha + 1$. Let $\mathcal{C}_1$ denote the set of columns of M whose number r of units satisfies $r \le \alpha - 1$, and let $\mathcal{C}_2$ denote the set of remaining columns, with at least $n - \alpha + 1$ units. We claim

$$|\mathcal{C}_2| \le t, \tag{2.8}$$

$$\text{each nonzero row of } M \text{ has at least } t \text{ units in the columns of } \mathcal{C}_2. \tag{2.9}$$

If (2.8) and (2.9) hold, then $|\mathcal{C}_2| = t$ and it is easy to see that M is the realization matrix of a hypercube embedding equivalent to the star embedding of $2t\mathbb{1}_n$. Suppose, for contradiction, that $|\mathcal{C}_2| \ge t+1$. Let Y denote the $(n-1) \times (t+1)$ submatrix of M formed by its last $n-1$ rows restricted to these $t+1$ columns. Each column of Y has at most $\alpha - 2$ zeros, which implies that the number of zeros in Y is less than or equal to $(t+1)(\alpha-2) \leq n-3$, by definition of α. Hence, at least two rows of Y have all their entries equal to 1, which contradicts the fact that two rows of M have t units in common. This shows (2.8). We now show (2.9). Let u denote a row of M, distinct from the first one, and let q denote the number of units of u in the columns of $\mathcal{C}_2$. Hence, u has $2t-q$ units in the columns of $\mathcal{C}_1$. Let Z denote the $(n-2) \times 2t$ submatrix of M consisting of the rows of M, other than u and the first one, restricted to the columns that have a unit in row u. Each row of Z has t units, which implies that Z has $t(n-2)$ units. On the other hand, Z has at most $\alpha - 2$ units in each of its columns belonging to $\mathcal{C}_1$, which implies that the number of units in Z is less than or equal to $(2t-q)(\alpha-2) + q(n-2)$. Therefore,

$$t(n-2) \leq (2t-q)(\alpha-2) + q(n-2),$$

which implies

$$q \geq t - t\frac{\alpha-2}{n-\alpha} > t-1,$$

since $t\frac{\alpha-2}{n-\alpha} < 1$ by definition of α. This shows (2.9). ∎

PROOF OF THEOREM 2.3. Let $n \geq t^2+t+3$. Let M be an h-realization matrix of $2t\mathbb{1}_n$, whose first row is equal to zero. We have

$$\frac{1}{2}(n - \sqrt{n^2-4nt}) < t+2,$$

since $n \geq t^2+t+3$. Therefore,

$$\min(r, n-r) \leq t+1 \leq \left\lfloor \frac{n+2t-1}{t+1} \right\rfloor - 1,$$

the first inequality following from Lemma 2.5 and the second one from the assumption $n \geq t^2+t+3$. Lemma 2.6 implies that M is the realization matrix of a hypercube embedding of $2t\mathbb{1}_n$ equivalent to the star embedding. This shows that $2t\mathbb{1}_n$ is h-rigid.
Let $n = t^2+t+2$ and suppose that there exists a projective plane of order t. Let $\mathcal{L}$ denote its set of lines and let Z be a set of size $t-1$ disjoint from the lines of $\mathcal{L}$. Then, for $L, L' \in \mathcal{L}$, $|L \cup Z| = 2t$ and $|(L \cup Z) \cap (L' \cup Z)| = t$. Therefore, the sets $L \cup Z$, for $L \in \mathcal{L}$, together with $\emptyset$, provide an h-labeling of $2t\mathbb{1}_n$. This shows that $2t\mathbb{1}_n$ is not h-rigid. ∎

PROOF OF THEOREM 2.4. Set $n = t^2+t+2$. Suppose that $2t\mathbb{1}_n$ is not h-rigid. Let M be an h-realization matrix of $2t\mathbb{1}_n$ which gives a hypercube embedding of $2t\mathbb{1}_n$ which is not equivalent to the star embedding. We assume that the first row of M is the zero vector. Let r denote the number of units in a column of M. By Lemma 2.5 and since $t \geq 3$,

$$\min(r, n-r) \leq \frac{1}{2}(n - \sqrt{n^2-4nt}) < t+2,$$

implying that

$$\min(r, n-r) \leq t+1.$$

As $\lfloor \frac{n+2t-1}{t+1} \rfloor = t+1$, we deduce from Lemma 2.6 that the number r of units in at least one of the columns of M satisfies

$$\min(r, n-r) = t+1.$$

The columns of M can be split into two classes $\mathcal{C}_I$ and $\mathcal{C}_{II}$, where $\mathcal{C}_I$ consists of the columns with $r \leq t+1$, and $\mathcal{C}_{II}$ of the columns with $r \geq n-t-1 = t^2+1$. We distinguish two cases, depending whether $|\mathcal{C}_{II}| \geq t+1$ or $|\mathcal{C}_{II}| \leq t$.

Case A: $|\mathcal{C}_{II}| \geq t+1$.
At most one row of M has all its entries equal to 1 in these $t+1$ columns of $\mathcal{C}_{II}$. Hence, the number of 1's in these $t+1$ columns is less than or equal to $t+1+t(n-2) = (t+1)(t^2+1)$. On the other hand, this number is greater or equal to $(t+1)(t^2+1)$ by definition of $\mathcal{C}_{II}$. Therefore, the number of 1's in these $t+1$ columns of $\mathcal{C}_{II}$ is equal to $(t+1)(t^2+1)$. Moreover, one row of M has all its entries equal to 1 in these $t+1$ columns of $\mathcal{C}_{II}$, while the other nonzero rows of M have t units in these $t+1$ columns, and each of these $t+1$ columns has exactly t^2+1 units. Hence, after a suitable permutation, the matrix M has the following form:

$$\begin{array}{|c|c|c|c|c|c|}
\hline
0\ldots0 & \multicolumn{4}{c|}{0\ldots\ldots\ldots\ldots\ldots0} & 0\ldots\ldots0 \\
\hline
C^{(1)}_{t,t+1} & R^{(1)}_{t,t} & R^{(2)}_{t,t} & \ldots & R^{(t)}_{t,t} & \\
\cline{1-5}
C^{(2)}_{t,t+1} & \multicolumn{4}{c|}{} & \\
\cline{1-1}
\vdots & \multicolumn{4}{c|}{M^*} & 0_{t^2+t,t-1} \\
\cline{1-1}
C^{(t+1)}_{t,t+1} & \multicolumn{4}{c|}{} & \\
\hline
1\ldots1 & \multicolumn{4}{c|}{0\ldots\ldots\ldots\ldots\ldots0} & 1\ldots\ldots1 \\
\hline
\end{array}$$

The rows of M^* satisfy:
- each row of M^* has t units, one below each of the matrices $R^{(1)}_{t,t}, \ldots, R^{(t)}_{t,t}$,
- two rows of M^* that follow $C^{(i)}_{t,t+1}$ and $C^{(j)}_{t,t+1}$ $(i \neq j)$ have one unit in common and two rows that follow the same $C^{(i)}_{t,t+1}$ have no unit in common.
Hence, M^* is the incidence matrix of a transversal system $T_t(t,t)$ (see [MH67], section 15.2). The existence of such a transversal system is equivalent to the existence of an orthogonal array $OA(t, t+1)$, which implies the existence of a $PG(2,t)$ ([MH67], section 13.2).

Case B: $|\mathcal{C}_{II}| \leq t$
We claim that each nonzero row of M has at least $t-1$ units in the columns of $\mathcal{C}_{II}$. This statement is an analogue of relation (2.9) and can be proved in the same way. We now claim that $|\mathcal{C}_{II}| \leq t-1$. Suppose, for contradiction, that $|\mathcal{C}_{II}| = t$. Then, the matrix M is of the form:

$$\begin{array}{|c|c|c|c|c|}
\hline
0\ldots0 & \multicolumn{3}{c|}{0\ldots\ldots\ldots0} & 0\ldots0 \\
\hline
C^{(1)}_{m_1,t} & \multicolumn{3}{c|}{} & \\
\cline{1-1}
C^{(2)}_{m_2,t} & \multicolumn{3}{c|}{} & \\
\cline{1-1}
\vdots & \multicolumn{3}{c|}{M^*} & \\
\cline{1-1}
C^{(t)}_{m_t,t} & \multicolumn{3}{c|}{} & \\
\hline
J_{p,t} & R^{(1)}_{p,t} & \ldots & R^{(p)}_{p,t} & 0\ldots0 \\
\hline
\end{array}$$

for some integers $p \geq 0$, $0 \leq m_1, \ldots, m_t \leq t$. Moreover, $m_i \neq 0$ for some i (else, M would provide an embedding equivalent to the star embedding). Each row of M^* following some $C^{(i)}_{m_i,t}$ has one unit above each of the matrices $R^{(1)}_{p,t}, \ldots, R^{(p)}_{p,t}$. Hence, $p \leq t+1$. This implies that $m_1 = \ldots = m_t = t$ and $p = t+1$, since $m_1 + \ldots + m_t + p = n-1 = t^2+t+1$. Let us count in two ways the number of units in M which are below the 1's in the second row of M. As each row has t units in common with second row, this number is equal to $t(n-2) = t(t^2+t)$. On the other hand, this number is less than or equal to $t^2(t-1) + t(t+1) = t(t^2+1)$. Hence, $t(t^2+t) \leq t(t^2+1)$, contradicting the fact that $t \geq 3$. This shows that $|\mathcal{C}_{II}| \leq t-1$. As each nonzero row of M has at least $t-1$ units in the columns of $\mathcal{C}_{II}$, we deduce that $|\mathcal{C}_{II}| = t-1$. Hence, the matrix M has the following form:

$$\begin{array}{|c|c|}\hline 0\ldots\ldots0 & 0\ldots0 \\ \hline J_{t^2+t-1,t-1} & M_1 \\ \hline \end{array}$$

The matrix M_1 satisfies:
- each row of M_1 has $t+1$ units,
- two rows of M_1 have one unit in common,
- each column of M_1 has at most $t+1$ units, hence exactly $t+1$ units. Indeed, we know that at least one column of M_1 has exactly $t+1$ units. From this follows easily that M_1 has t^2+t+1 (nonzero) columns and, therefore, each column of M_1 has $t+1$ units.

Hence, M_1 is the incidence matrix of a $PG(2,t)$. This concludes the proof. ∎

The following result is a common extension of Theorems 2.3 and 2.4.

THEOREM 2.10. [Hal77] *Let $n \geq t^2 \geq 4$. The metric $2t\mathbb{1}_n$ is not h-rigid if and only if $n \leq t^2+t+2$ and there exists a projective plane of order t.* ∎

Another case of h-rigidity of the metric $2t\mathbb{1}_n$ is given in Corollary 3.19.

Consider, for instance, the case $t = 6$. By Theorems 2.3 and 2.4, the metric $12\mathbb{1}_n$ is h-rigid if $n \geq 44$ (as $PG(2,6)$ does not exist). It is, in fact, h-rigid for all $n \geq 33$, as stated in the next result, which was proved independently in [HJKvL77] and [MV75, VM77].

PROPOSITION 2.11. *The equidistant metric $12\mathbb{1}_n$ is h-rigid for all $n \geq 33$.* ∎

The h-rigidity result from Theorem 2.3 was extended to the class of metrics of the form $\sum_{1\leq i\leq n} t_i\delta(\{i\})$ for $t_1, \ldots, t_n \in \mathbb{Z}_+$, the case $t_1 = \ldots = t_n = t$ corresponding to the case of the equidistant metric $2t\mathbb{1}_n$.

THEOREM 2.12. ([DEF78], Theorem 7 (i)) *Let $t_1, \ldots, t_n$ be nonnegative integers. If n is* *large with respect to* $\max(t_1,\dots,t_n)$*, then the metric* $\sum_{1\le i\le n} t_i\delta(\{i\})$ *is h-rigid.* ∎

3 Hypercube embeddings of the equidistant metric

We show in this section how to construct various hypercube embeddings of the equidistant metric $2t\mathbb{1}_n$ from designs. A $\mathbb{Z}_+$-realization of $2t\mathbb{1}_n$ consists of a family $\mathcal{B}$ of (not necessarily distinct) subsets of V_n such that

$$\sum_{B\in\mathcal{B}} \delta(B) = 2t\mathbb{1}_n.$$

Given $i_0 \in V_n$, we can suppose without loss of generality that $i_0 \notin B$ for all $B \in \mathcal{B}$ (replacing if necessary B by $V_n \setminus B$). Then $\mathcal{B}$ is a collection of subsets of V_{n-1} satisfying:
- each point of V_{n-1} belongs to $2t$ members of $\mathcal{B}$, and
- any two distinct points of V_{n-1} belong to t common members of $\mathcal{B}$.

Such a set family $\mathcal{B}$ is known as a $(2t, t, n-1)$-design. Therefore, the hypercube embeddings of $2t\mathbb{1}_n$ are nothing but special classes of designs. We review in Section 3.1 some known results on designs and we state precisely the link with hypercube embeddings of the equidistant metric in Section 3.2. Results on the minimum h-size of the equidistant metric are grouped in Section 3.3. We describe all the hypercube embeddings of $2t\mathbb{1}_n$ for small n or t in Section 3.4.

Much of the exposition in this section follows from [DL93c].

3.1 Preliminaries on designs

3.1.1 (r,λ,n)-designs and BIBD's

Let $\mathcal{B}$ be a collection of (not necessarily distinct) subsets of V_n. The sets $B \in \mathcal{B}$ are called **blocks**. Let r, k, λ be positive integers. Consider the following properties:
(i) Each point of V_n belongs to r blocks.
(ii) Any two distinct points of V_n belong to λ common blocks.
(iii) Each block has cardinality k.
Clearly, if $(ii), (iii)$ hold, then (i) holds with

$$r = \lambda\frac{n-1}{k-1} \tag{3.1}$$

and the total number b of blocks in $\mathcal{B}$ (counting multiplicities) is given by

$$b = \frac{rn}{k} = \lambda\frac{n(n-1)}{k(k-1)}. \tag{3.2}$$

The multiset $\mathcal{B}$ is called a (r,λ,n)-**design** if $(i),(ii)$ hold with $0 < \lambda < r$. $\mathcal{B}$ is said to be **trivial** if $\mathcal{B}$ consists of the folowing blocks: V_n repeated λ times and, for each $i \in V_n$, the block $\{i\}$ repeated $r-\lambda$ times. In fact, if n is large with respect to r and λ, then every (r,λ,n)-design is trivial (this follows, e.g., from the rigidity results of Section 2).

The multiset $\mathcal{B}$ is called a (n,k,λ)-**BIBD** if $(i),(ii),(iii)$ hold with $\lambda > 0$, $1 < k < n-1$. (BIBD stands for balanced incomplete block design.) A (n,k,λ)-BIBD is said to be

symmetric if $r = k$ holds or, equivalently, the number of blocks b is equal to the number n of points.

Let $\mathcal{B}$ be a (n, k, λ)-BIBD. Then, the collection

$$\mathcal{B}^* := \{V_n \setminus B \mid B \in \mathcal{B}\}$$

is a $(n, k' := n - k, \lambda' := b - 2r + \lambda)$-BIBD, called the **dual** of $\mathcal{B}$. (Note that $1 < k' < n - 1$ and $(n-1)(b - 2r + \lambda) = (b - r)(n - k - 1)$, which permits to check that $\lambda' > 0$.) If $\mathcal{B}$ is symmetric, then $\mathcal{B}^*$ too is symmetric. For instance, the dual of $PG(2, t)$ is a symmetric $(t^2 + t + 1, t^2, t^2 - t)$-BIBD.

The following result is due to Ryser (see [Rys63], Chapter 8).

THEOREM 3.3. *Let $\mathcal{B}$ be a (r, λ, n)-design with b blocks. Then, $b \geq n$ holds, with equality if and only if $\mathcal{B}$ is a symmetric (n, r, λ)-BIBD.*

PROOF. Let A denote the incidence matrix of $\mathcal{B}$, i.e., A is the $n \times b$ matrix with entries $a_{i,B} = 1$ if $i \in B$ and $a_{i,B} = 0$ if $i \notin B$, for $i \in V_n$, $B \in \mathcal{B}$. Suppose that $b < n$. Let M denote the $n \times n$ matrix obtained by adding $n - b$ zero columns to A. Then,

$$MM^T = \lambda J + (r - \lambda)I,$$

where J is the all ones matrix and I the identity matrix. One can check that the eigenvalues of MM^T are $r + (n-1)\lambda$ and $r - \lambda$ (with multiplicity $n - 1$), which shows that M is nonsingular. This contradicts the fact that M has a zero column. Hence, we have shown that $b \geq n$. Suppose now that $b = n$. We show that each block of $\mathcal{B}$ has cardinality r. From the above argument, the matrix A is an $n \times n$ matrix satisfying

$$AA^T = \lambda J + (r - \lambda)I, \quad \text{and} \quad AJ = rJ. \tag{3.4}$$

Hence,

$$A^{-1}J = r^{-1}J \quad \text{and } AA^TJ = (\lambda n + r - \lambda)J,$$

implying

$$A^TJ = (\lambda n + r - \lambda)r^{-1}J, \text{ i.e., } JA = (\lambda n + r - \lambda)r^{-1}J. \tag{3.5}$$

Therefore,

$$JAJ = (\lambda n + r - \lambda)r^{-1}nJ.$$

But, $JAJ = rnJ$ from (3.4), which implies

$$r - \lambda = r^2 - \lambda n. \tag{3.6}$$

Substituting (3.6) in (3.5), we obtain $JA = rJ$. This shows that each block of $\mathcal{B}$ has size r. Hence, $\mathcal{B}$ is a symmetric (n, r, λ)-BIBD. ∎

Clearly, from (3.2), a necessary condition for the existence of a (n, k, λ)-BIBD is the following divisibility condition:

$$k - 1 \mid \lambda(n-1) \quad \text{and} \quad k(k-1) \mid \lambda n(n-1). \tag{3.7}$$

This condition is, in some cases, already sufficient for the existence of a (n, k, λ)-BIBD.

THEOREM 3.8. *(i)* [Wil75] *Suppose that* (3.7) *holds and that n is large with respect to k and λ. Then, there exists a (n, k, λ)*-BIBD.
(ii) [Han75] *For $k \leq 5$, a (n, k, λ)*-BIBD *exists whenever* (3.7) *holds with the single exception: $n = 15, k = 5, \lambda = 2$. For $k = 6$, $\lambda \geq 2$, a $(n, 6, \lambda)$*-BIBD *exists whenever* (3.7) *holds with the single exception: $n = 21, \lambda = 2$.*
(iii) [Mil90] *For $k = 6, \lambda = 1$, a $(n, 6, 1)$*-BIBD *exists whenever* (3.7) *holds with the possible exception of* 95 *undecided cases (including $n = 46, 51, 61, 81, 141, \ldots, 5391, 5901$).* ∎

Two important cases of parameters for a symmetric BIBD are:
- the $(t^2+t+1, t+1, 1)$-BIBD, which is nothing but the projective plane of order t, denoted by $PG(2, t)$,
- the $(4t-1, 2t, t)$-BIBD, also known as the **Hadamard design** of order $4t-1$.
Recall that Hadamard designs are in one-to-one correspondance with Hadamard matrices. Namely, a Hadamard matrix is an $n \times n$ ± 1-matrix A such that $AA^T = nI$. Its order n is equal to 1, 2 or $4t$ for some $t \geq 1$. We can suppose that all entries in the first row and in the first column of A are equal to 1. Replace each -1 entry of A by 0 and delete its first row and column. We obtain a $(4t-1) \times (4t-1)$ binary matrix whose columns are the incidence vectors of the blocks of a Hadamard design of order $4t-1$.
It is conjectured that Hadamard matrices of order $4t$ exist for all $t \geq 1$. This was proved for $t \leq 106$. (For more information on Hadamard matrices, see, e.g., [GS79, Wal88].)

REMARK 3.9. The parameters (k, λ) with $3 \leq k \leq 15$ for which there exists a symmetric (n, k, λ)-BIBD (then, $n = 1 + \frac{k(k-1)}{\lambda}$, by (3.1)) have been completely classified (with the exception of $k = 13, \lambda = 1$ corresponding to the question of existence of $PG(2, 12)$) (see [BI80]). Besides the parameters corresponding to a projective plane, or to a Hadamard design, or to a dual of them, a symmetric (n, k, λ)-BIBD exists if and only if (n, k, λ) is one of the following list: $(16, 6, 2), (37, 9, 2), (25, 9, 3), (16, 10, 6)$ (which is dual to the case $(16, 6, 2)$), $(56, 11, 2), (31, 10, 3), (45, 12, 3), (79, 13, 2), (40, 13, 4), (71, 15, 3)$ and $(36, 15, 6)$.
∎

A useful notion is that of extension of a design. Let $\mathcal{B}$ be a collection of subsets of V_n and let $i_0 \notin V_n$. Given an integer s, the s-**extension** of $\mathcal{B}$ is the collection $\mathcal{B}'$ whose blocks are the blocks of $\mathcal{B}$ together with the block $\{i_0\}$ repeated s times.

3.1.2 Intersecting systems

Let $\mathcal{A}$ be a collection of subsets of a finite set and let r, λ be positive integers. Then, $\mathcal{A}$ is called a (r, λ)-**intersecting system** if $|A| = r$ for all $A \in \mathcal{A}$ and $|A \cap B| = \lambda$ for all distinct $A, B \in \mathcal{A}$. The maximum cardinality of a (r, λ)-intersecting system consisting of subsets of V_b is denoted by $f(r, \lambda; b)$.

$\mathcal{A}$ is called a Δ-**system** with **kernel** K and **parameters** (r, λ) if $|K| = \lambda$, $|A| = r$ for all $A \in \mathcal{A}$, and $A \cap B = K$ for all distinct $A, B \in \mathcal{A}$. Clearly, if $\mathcal{A}$ consists of subsets of V_b, then $|\mathcal{A}| \leq \frac{b-\lambda}{r-\lambda}$.

REMARK 3.10. (r, λ, n)-designs and (r, λ)-intersecting systems are basically the same objects. Namely, let M be a $n \times b$ binary matrix, let $\mathcal{B}$ denote the family of subsets of V_n whose incidence vectors are the columns of M, and let $\mathcal{A}$ denote the family of subsets of V_b whose incidence vectors are the rows of M. Then, $\mathcal{B}$ is a (r, λ, n)-design if and only if $\mathcal{A}$ is a (r, λ)-intersecting system of cardinality n. Moreover, $\mathcal{B}$ is trivial if and only if $\mathcal{A}$ is a Δ-system. These two terminologies of (r, λ, n)-designs and intersecting systems are commonly used in the literature. ▮

Intersecting systems arise as the h-labelings of the equidistant metric. Namely,

PROPOSITION 3.11. *There is a one-to-one correspondance between the h-labelings of the equidistant metric $2t\mathbb{1}_n$ and the $(2t, t)$-intersecting systems of cardinality $n-1$.*

PROOF. Indeed, in any h-labeling of $2t\mathbb{1}_n$, we may asume that one of the points is labeled by $\emptyset$ and then the sets labeling the remaining $n-1$ points are the members of a $(2t, t)$-intersecting system. ▮

Hence, Theorem 2.3 can be reformulated as follows.

THEOREM 3.12. [Dez73] *Let $t \geq 1$ be an integer and let $\mathcal{A}$ be a $(2t, t)$-intersecting sytem. If $|\mathcal{A}| \geq t^2 + t + 2$, then $\mathcal{A}$ is a Δ-system.* ▮

As an application of Theorem 3.12, Deza proved the following result, solving a conjecture of Erdös and Lovász.

THEOREM 3.13. [Dez74] *Let $t \geq 1$ be an integer and let $\mathcal{A}$ be a collection of subsets of a finite set such that $|A \cap B| = t$ for all $A \neq B \in \mathcal{A}$. Set $k := \max(|A| \mid A \in \mathcal{A})$. If $|\mathcal{A}| \geq k^2 - k + 2$, then $\mathcal{A}$ is a Δ-system.* ▮

We conclude with an easy application, that will be needed later.

LEMMA 3.14. *Let $k, t \geq 1$ be integers such that $t < k^2 + k + 1$ and let $\mathcal{A}$ be a $(k+t, t)$-intersecting system. If $|\mathcal{A}| \geq k^2 + k + 3$, then $\mathcal{A}$ is a Δ-system.*

PROOF. Let $A_1 \in \mathcal{A}$ and set $\mathcal{A}' := \{A \triangle A_1 \mid A \in \mathcal{A} \setminus \{A_1\}\}$. One checks easily that $\mathcal{A}'$ is a $(2k, k)$-intersecting system with $|\mathcal{A}'| \geq k^2 + k + 2$. By Theorem 3.12, $\mathcal{A}'$ is a Δ-sytem. Let K denote its kernel, $|K| = k$. Let $A \in \mathcal{A}, A \neq A_1$. Set $\alpha := |A_1 \cap K|$, then $|A_1 \cap ((A \triangle A_1) \setminus K)| = k - \alpha$ since $A_1 \cap (A \triangle A_1) = A_1 \setminus A$ has cardinality k. If $\alpha \leq k-1$, then $k + t = |A_1| \geq \alpha + |\mathcal{A}'|(k-\alpha) \geq \alpha + (k^2+k+2)(k-\alpha)$, implying

$t \geq (k-\alpha)(k^2+k+1)$, contradicting the assumption on t. Hence, $\alpha = k$, i.e., $A_1 \setminus A = K$ and, thus, $A_1 \cap A = A_1 \setminus K$. This shows that $\mathcal{A}$ is a Δ-system. ∎

3.2 Embeddings of $2t\mathbb{1}_n$ and designs

Let $t, n \geq 1$ be integers. Every $\mathbb{Z}_+$-realization of $2t\mathbb{1}_n$ is of the form

$$2t\mathbb{1}_n = \sum_{B\in\mathcal{B}} \delta(B), \tag{3.15}$$

where $\mathcal{B}$ is a collection of (not necessarily distinct) subsets of V_n. Let $k \geq 1$ be an integer. The realization (3.15) is said to be k-**uniform** if $|B| = k, n-k$ for all $B \in \mathcal{B}$. It is very easy to construct $\mathbb{Z}_+$-realizations of the equidistant metric from designs.

For instance, let $\mathcal{B}$ be a (r, λ, n)-design. Then, $\sum_{B\in\mathcal{B}} \delta(B) = 2(r-\lambda)\mathbb{1}_n$. Moreover, if $r \geq 2\lambda$, then the $(r-2\lambda)$-extension of $\mathcal{B}$ yields a $\mathbb{Z}_+$-realization of $2(r-\lambda)\mathbb{1}_n$, namely, $\sum_{B\in\mathcal{B}} \delta(B) + (r-2\lambda)\delta(\{i_0\}) = 2(r-\lambda)\mathbb{1}_{n+1}$, where $V_{n+1} \setminus V_n = \{i_0\}$. In particular, each $(t+1, 1, n)$-design yields a $\mathbb{Z}_+$-realization of $2t\mathbb{1}_n$ and its $(t-1)$-extension yields a realization of $2t\mathbb{1}_{n+1}$. Also, the 0-extension of a $(2t, t, n-1)$-design gives a $\mathbb{Z}_+$-realization of $2t\mathbb{1}_n$.

If $\mathcal{B}$ is a (n, k, λ)-BIBD, then (3.15) is a $\mathbb{Z}_+$-realization of $2\lambda\frac{n-k}{k-1}\mathbb{1}_n$. In particular, if $\mathcal{B}$ is a Hadamard design of order $4t-1$, then (3.15) is a $\mathbb{Z}_+$-realization of $2t\mathbb{1}_{4t-1}$ and the 0-extension of $\mathcal{B}$ yields a $\mathbb{Z}_+$-realization of $2t\mathbb{1}_{4t}$. If $\mathcal{B}$ is $PG(2,t)$, then (3.15) is a $\mathbb{Z}_+$-realization of $2t\mathbb{1}_{t^2+t+1}$ and the $(t-1)$-extension of $\mathcal{B}$ yields a $\mathbb{Z}_+$-realization of $2t\mathbb{1}_{t^2+t+2}$.

The next result makes precise the correspondance between $\mathbb{Z}_+$-realizations of the equidistant metric and designs. The first assertion (i) is nothing but a reformulation of Proposition 3.11 (using the link between intersecting systems and designs, explained in Remark 3.10).

Proposition 3.16. *(i) There is a one-to-one correspondance between the $\mathbb{Z}_+$-realizations of $2t\mathbb{1}_n$ and the $(2t, t, n-1)$-designs.*
(ii) For $k \neq \frac{n}{2}$, there is a one-to-one correspondance between the k-uniform $\mathbb{Z}_+$-realizations of $2t\mathbb{1}_n$ and the $(n, k, \frac{t(k-1)}{n-k})$-BIBD's.

Proof. (i) follows by assuming that all $B \in \mathcal{B}$ do not contain a given point i_0 of V_n (replacing, if necessary, B by $V_n \setminus B$).
(ii) It is immediate to check that (3.15) holds if $\mathcal{B}$ is a $(n, k, \frac{t(k-1)}{n-k})$-BIBD. Suppose now that (3.15) holds, with $|B| = k$ for all $B \in \mathcal{B}$, and $k \neq \frac{n}{2}$. By taking the scalar product of both sides of (3.15) with the all ones vector, we obtain that the number b of blocks satisfies

$$b = \frac{tn(n-1)}{k(n-k)}.$$

We show that each point belongs to the same number of blocks. For this, let r denote the number of blocks that contain the point 1 and denote by a_i the number of blocks containing both points 1 and i, for $i = 2, \ldots, n$. Then, $\sum_{2\leq i\leq n} a_i = r(k-1)$. Counting in two ways the total number of units in the incidence matrix of $\mathcal{B}$ (summing over the columns or over the rows), we obtain

$$bk = r + \sum_{2 \le i \le n} (2t - r + 2a_i),$$

implying $r = t\frac{n-1}{n-k}$. Hence, any two points of V_n belong to $r - t = t\frac{k-1}{n-k}$ common blocks. Therefore, $\mathcal{B}$ is a $(n, k, \frac{t(k-1)}{n-k})$-BIBD. ∎

THEOREM 3.17. [Hal77] *Suppose that $n \ge \frac{1}{2}(t+2)^2$ with $t \ge 3$ or $n > \frac{1}{2}(t+2)^2$ with $t = 2$. Let $\mathcal{B}$ be a family of subsets of V_n for which* (3.15) *holds. Then, either $\mathcal{B}$ is a $(t+1, 1, n)$-design, or $\mathcal{B}$ is the $(t-1)$-extension of a $(t+1, 1, n-1)$-design.* ∎

Take, for instance, $t = 3$ and $n = 12$ $(< \frac{1}{2}(t+2)^2)$. Then, $6\mathbb{1}_{12}$ has a $\mathbb{Z}_+$-realization which is *not* of the form indicated in Theorem 3.17; such a realization can be obtained from the 1-extension of a (5,2,11)-design.

THEOREM 3.18. [MV77] *Let α, t be integers such that $t > 2\alpha^2 + 3\alpha + 2$ (i.e., $\alpha < \frac{\sqrt{8t-7}-3}{4}$). Suppose that $PG(2,t)$ does not exist. Then, for $n \ge t^2 - \alpha$, each $(t+1, 1, n)$-design is trivial.* ∎

COROLLARY 3.19. *Suppose that $PG(2,t)$ does not exist. If $n > t^2 + 1 - \frac{\sqrt{8t-7}-3}{4}$, then the metric $2t\mathbb{1}_n$ is h-rigid.*

PROOF. Let $\mathcal{B}$ be a family of subsets of V_n for which (3.15) holds. By Theorem 3.17, $\mathcal{B}$ is a $(t+1, 1, n)$-design or the $(t-1)$-extension of a $(t+1, 1, n-1)$-design. By Theorem 3.18, such designs are trivial. Hence, $\mathcal{B}$ yields the star realization of $2t\mathbb{1}_n$. ∎

3.3 The minimum h-size of $2t\mathbb{1}_n$

Recall from (1.4) that the **minimum h-size** $s_h(2t\mathbb{1}_n)$ of $2t\mathbb{1}_n$ is defined as the smallest cardinality of a multiset $\mathcal{B} \subseteq 2^{V_n}$ satisfying (3.15). The following result is a reformulation of Ryser's result on the number of blocks of a $(2t, t, n-1)$-design.

THEOREM 3.20. *(i) $s_h(2t\mathbb{1}_n) \ge n-1$, with equality if and only if $n = 4t$ and there exists a Hadamard matrix of order $4t$.*
(ii) Suppose $n \ne 4t$. If $n = 2t + \lambda + \frac{t(t-1)}{\lambda}$ for some integer $\lambda \ge 1$ and if there exists a symmetric $(n, \lambda + t, \lambda)$-BIBD, then $s_h(2t\mathbb{1}_n) = n$.

PROOF. (i) By Proposition 3.16, the minimum h-size of $2t\mathbb{1}_n$ is equal to the minimum number of blocks in a $(2t, t, n-1)$-design, which is greater or equal to $n-1$, by Theorem 3.3.

If $s_h(2t\mathbb{1}_n) = n-1$, then there exists a $(2t, t, n-1)$-design $\mathcal{B}$ with $n-1$ blocks. Applying again Theorem 3.3, we deduce that $\mathcal{B}$ is a symmetric $(4t-1, 2t, t)$-design, i.e., a Hadamard design of order $4t-1$. (ii) is an easy check. ∎

As an application of Theorem 3.20 and Remark 3.9, we deduce that $s_h(2t\mathbb{1}_n) = n$ for the following parameters (t, n): (7,37), (6,25), (9,56), (7,31), (9,45), (11,79), (9,40), (12,71). Note also that $s_h(2t\mathbb{1}_n) = n-1$ for $(t, n) = (9, 36), (4, 16)$.

The implication from Theorem 3.20 (ii) is, in fact, an equivalence in the cases $\lambda = 1$ (i.e., $n = t^2 + t + 1$) and $\lambda = t$ (i.e., $n = 4t-1$).

PROPOSITION 3.21. *(i)* $s_h(2t\mathbb{1}_{t^2+t+1}) = t^2 + t + 1$ *if and only if there exists a projective plane of order* t.
(ii) $s_h(2t\mathbb{1}_{4t-1}) = 4t-1$ *or, equivalently,* $s_h(2t\mathbb{1}_{4t}) = 4t-1$ *if and only if there exists a Hadamard design of order* $4t-1$.
(iii) Suppose $PG(2, t)$ *exists. Then,* $s_h(2t\mathbb{1}_{t^2+t+2}) = t^2 + 2t$ *if* $t \geq 3$ *and* $s_h(2t\mathbb{1}_{t^2+t+2}) = t^2 + t + 1$ *if* $t = 1, 2$.
(iv) Suppose $PG(2, t)$ *does not exist. If* $n > t^2 + 1 - \frac{\sqrt{8t-7}-3}{4}$, *then* $s_h(2t\mathbb{1}_n) = nt$.

PROOF. (i) follows from Theorems 2.10 and 3.20.
(ii) Suppose $\mathcal{B}$ is a block family yielding a $\mathbb{Z}_+$-realization of $2t\mathbb{1}_{4t-1}$ with $|\mathcal{B}| = 4t-1$. Then, $|\mathcal{B}| = \frac{n(n-1)t}{\lceil\frac{n}{2}\rceil\lfloor\frac{n}{2}\rfloor}$ $(n = 4t-1)$, which implies that all blocks of $\mathcal{B}$ have size $2t$. Hence, $\mathcal{B}$ is a Hadamard design of order $4t-1$. The remaining of (ii) follows from Theorem 3.20.
(iii) For the case $t = 1, 2$, use Theorem 3.20. Suppose $t \geq 3$ and set $n = t^2 + t + 2$. The $(t-1)$-extension of $PG(2, t)$ yields a $\mathbb{Z}_+$-realization of $2t\mathbb{1}_n$ of size $t^2 + t$, implying $s_h(2t\mathbb{1}_n) \leq t^2 + 2t$. Let $\mathcal{B}$ be a block family yielding a $\mathbb{Z}_+$-realization of $2t\mathbb{1}_n$. We show that $|\mathcal{B}| \geq t^2 + 2t$. For this, we use Theorem 3.17. Either, $\mathcal{B}$ is a $(t+1, 1, n)$-design; then, its $(t-1)$-extension yields a $\mathbb{Z}_+$-realization of $2t\mathbb{1}_{t^2+t+3}$ distinct from the star realization, in contradiction with Theorem 2.3. Or, $\mathcal{B}$ is the $(t-1)$-extension of a $(t+1, 1, n-1)$-design and, then, $|\mathcal{B}| \geq n - 1 + t - 1 = t^2 + 2t$. This shows that $s_h(2t\mathbb{1}_n) = t^2 + 2t$.
(iv) is a reformulation of Corollary 3.19. ∎

Set
$$a_n^t := \left\lceil \frac{n(n-1)t}{\lfloor\frac{n}{2}\rfloor\lceil\frac{n}{2}\rceil} \right\rceil = \left\lceil 4t - \frac{2t}{\lceil\frac{n}{2}\rceil} \right\rceil.$$
By taking the scalar product of both sides of (3.15) with the all ones vector, we obtain the following bounds:
$$a_n^t \leq s_h(2t\mathbb{1}_n) \leq nt.$$
The equality
$$s_h(2t\mathbb{1}_n) = nt$$
holds if and only if the star realization (2.1) is the only $\mathbb{Z}_+$-realization of $2t\mathbb{1}_n$, i.e., if $2t\mathbb{1}_n$ is h-rigid. This is the case, for instance, if $n \geq t^2 + t + 3$ (by Theorem 2.3). Several other results about classes of parameters n, t for which $2t\mathbb{1}_n$ is h-rigid are given in Section 2. A

natural question is what are the parameters n, t for which the equality

$$s_h(2t\mathbb{1}_n) = a_n^t$$

holds. If $2t\mathbb{1}_n$ admits a $\mathbb{Z}_+$-realization $\sum_S \lambda_S \delta(S)$ where $\lambda_S > 0$ only if $\delta(S)$ is an equicut (i.e., satisfies $|S| = \lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil$), then the equality $s_h(2t\mathbb{1}_n) = a_n^t$ holds. For instance, $s_h(4\mathbb{1}_7) = a_7^2 = 7$ and $s_h(4\mathbb{1}_8) = a_8^2 = 7$ as each of $4\mathbb{1}_7$ and $4\mathbb{1}_8$ has a $\mathbb{Z}_+$-realization using only equicuts (see Proposition 3.30).

Clearly (from Theorem 3.20), the equality $s_h(2t\mathbb{1}_n) = a_n^t$ can occur only if $n \le 4t$. The case $n = 4t$ is well understood: equality holds if and only if there exists a Hadamard matrix of order $4t$. The following conjectures are proposed in [DL93c].

CONJECTURE 3.22. *Suppose $n \le 4t$ and that there exists a Hadamard matrix of order $4t$. Then, $s_h(2t\mathbb{1}_n) = a_n^t$.*

CONJECTURE 3.23. *Suppose $n \le 4t$ and that there exist Hadamard matrices of suitable orders. Then, $s_h(2t\mathbb{1}_n) = a_n^t$.*

Conjecture 3.23 is weaker than Conjecture 3.22. We refer to [DL93c] for partial results related to these conjectures. In particular, the following results are proved there.

PROPOSITION 3.24. *(i) Conjecture 3.22 holds for all n, t such that $n \le 4t$, and $\frac{2t}{3} < \lceil \frac{n}{2} \rceil$ or $\min(n,t) \le 20$.*
(ii) Conjecture 3.23 holds for all n, t such that n is even and satisfies $2\sqrt{2t} \le n \le 4t$. (It suffices to assume the existence of Hadamard matrices of orders $2n$, $4n$, and n (if $\frac{n}{2}$ is even) and $n+2$ (if $\frac{n}{2}$ is odd).)

COROLLARY 3.25. *If $n \le 4t \le 80$ then $s_h(2t\mathbb{1}_n) = a_n^t$.* ∎

EXAMPLE 3.26. As an example, let us consider the minimum h-size of the metric $2t\mathbb{1}_n$ for $t = 6$ and $n \ge 31$. We have
(i) $s_h(12\mathbb{1}_n) = 6n$ for all $n \ge 33$ (by Proposition 2.11),
(ii) $s_h(12\mathbb{1}_{32}) = 67$ and $s_h(12\mathbb{1}_{31}) \le 62$.
Indeed, let $\mathcal{B}$ be a block design on V_{32} for which (3.15) holds. By Theorem 3.17, $\mathcal{B}$ is a (7,1,32)-design, or the 5-extension of a (7,1,31)-design. Each (7,1,32)-design is trivial (as its 5-extension yields a $\mathbb{Z}_+$-realization of the h-rigid metric $12\mathbb{1}_{33}$). It is shown in [MMS$^+$76b, MMS$^+$76a] that the unique nontrivial (7,1,31)-design is the block family obtained by taking the blocks of $PG(2,5)$ together with the 31 singletons; it yields a $\mathbb{Z}_+$-realization of $12\mathbb{1}_{31}$ of size $31 + 31 = 62$. Its 5-extension yields a $\mathbb{Z}_+$-realization of $12\mathbb{1}_{32}$ of size $62 + 5 = 67$. This shows that $s_h(12\mathbb{1}_{32}) = 67$. ∎

3.4 All hypercube embeddings of $2t\mathbb{1}_n$ for small n, t

We list all the $\mathbb{Z}_+$-realizations of the equidistant metric $2t\mathbb{1}_n$ in the following cases: $t = 1$, $t = 2$, $n = 4$, and we give partial information in the case $n = 5$. The results are taken from [DL93c].

Let t, n be positive integers. For each integer s such that $t - \lfloor \frac{t}{n-3} \rfloor \leq s \leq t$, we have the following $\mathbb{Z}_+$-realization of $2t\mathbb{1}_n$:

$$2t\mathbb{1}_n = (t - (n-3)(t-s))\delta(\{n\}) + \sum_{1 \leq i \leq n-1} (t-s)\delta(\{i,n\}) + s\delta(\{i\}). \tag{3.27}$$

Its size is equal to $(n-3)s + 3t$ and (3.27) coincides with the star realization (2.1) for $s = t$.

PROPOSITION 3.28. **(Case $n = 4$)** *The metric $2t\mathbb{1}_4$ has $t + 1$ $\mathbb{Z}_+$-realizations, given by* (3.27) *for $0 \leq s \leq t$.*

PROOF. This follows from the fact that the restriction to V_3 of any $\mathbb{Z}_+$-realization of $2t\mathbb{1}_4$ coincides with the star realization of $2t\mathbb{1}_3$. ∎

PROPOSITION 3.29. **(Case $t = 1$)** *For $n \neq 4$,* (2.1) *is the only $\mathbb{Z}_+$-realization of the metric $2\mathbb{1}_n$ and, for $n = 4$, $2\mathbb{1}_4$ has two $\mathbb{Z}_+$-realizations: the star realization* (2.1) *and* (3.27) *for $s = 0$, namely, $2\mathbb{1}_4 = \sum_{1 \leq i \leq 4} \delta(\{i\}) = \delta(\{1,4\}) + \delta(\{2,4\}) + \delta(\{3,4\})$.* ∎

PROPOSITION 3.30. **(Case $t = 2$)**
(i) For $n \geq 9$, (2.1) *is the only $\mathbb{Z}_+$-realization of $4\mathbb{1}_n$.*
(ii) For $n = 4$, $4\mathbb{1}_4$ has three $\mathbb{Z}_+$-realizations: (2.1) *and* (3.27) *for $s = 0, 1$, namely, $4\mathbb{1}_4 = 2(\sum_{1 \leq i \leq 4} \delta(\{i\})) = 2(\sum_{1 \leq i \leq 3} \delta(\{i,4\})) = \sum_{1 \leq i \leq 4} \delta(\{i\}) + \sum_{1 \leq i \leq 3} \delta(\{i,4\})$.*
(iii) For $n = 5$, $4\mathbb{1}_5$ has (up to permutation) three $\mathbb{Z}_+$-realizations: the star realization (2.1), (3.27) *for $s = 1$, i.e., $4\mathbb{1}_5 = \sum_{1 \leq i \leq 4} \delta(\{i,5\}) + \delta(\{i\})$, and $4\mathbb{1}_5 = \delta(\{5\}) + \sum_{1 \leq i < j \leq 4} \delta(\{i,j\})$.*
(iv) For $n = 6$, $4\mathbb{1}_6$ has (up to permutation) three $\mathbb{Z}_+$-realizations: the star realization (2.1), *$4\mathbb{1}_6 = \delta(\{2\}) + \delta(\{3\}) + \delta(\{4,6\}) + \delta(\{5,6\}) + \delta(\{1,4\}) + \delta(\{1,5\}) + \delta(\{1,2,6\}) + \delta(\{1,3,6\})$, and $4\mathbb{1}_6 = \delta(\{1,2\}) + \delta(\{3,4\}) + \delta(\{5,6\}) + \delta(\{1,3,6\}) + \delta(\{2,4,6\}) + \delta(\{1,4,5\}) + \delta(\{2,3,5\}) + \delta(\{1,3,6\})$.*
(v) For $n = 7$, $4\mathbb{1}_7$ has (up to permutation) three $\mathbb{Z}_+$-realizations: the star realization (2.1), *$4\mathbb{1}_7 = \delta(\{7\}) + \delta(\{1,2\}) + \delta(\{3,4\}) + \delta(\{5,6\}) + \delta(\{1,3,6\}) + \delta(\{2,4,6\}) + \delta(\{1,4,5\}) + \delta(\{2,3,5\})$, and $4\mathbb{1}_7 = \delta(\{1,2,7\}) + \delta(\{3,4,7\}) + \delta(\{5,6,7\}) + \delta(\{1,3,6\}) + \delta(\{2,4,6\}) + \delta(\{1,4,5\}) + \delta(\{2,3,5\})$.*
(vi) For $n = 8$, $4\mathbb{1}_8$ has (up to permutation) three $\mathbb{Z}_+$-realizations: the star realization (2.1), *$4\mathbb{1}_8 = \delta(\{8\}) + \delta(\{1,2,7\}) + \delta(\{3,4,7\}) + \delta(\{5,6,7\}) + \delta(\{1,3,6\}) + \delta(\{2,4,6\}) + \delta(\{1,4,5\}) + \delta(\{2,3,5\})$, and $4\mathbb{1}_8 = \delta(\{1,2,7,8\}) + \delta(\{3,4,7,8\}) + \delta(\{5,6,7,8\}) + \delta(\{1,3,6,8\}) + \delta(\{2,4,6,8\}) + \delta(\{1,4,5,8\}) + \delta(\{2,3,5,8\})$ (corresponding to a Hadamard design).* ∎

It seems a quite difficult task to list all the $\mathbb{Z}_+$-realizations of the metric $2t\mathbb{1}_n$ in the case $n=5$. Note that we already have the realizations (3.27) for $t-\lfloor\frac{t}{2}\rfloor \le s \le t$. For t odd and $t\ge 3$, we also have

$$2t\mathbb{1}_5 = \delta(\{1,5\}) + \delta(\{2\}) + \delta(\{3\}) + \delta(\{4,5\}) + \frac{t+1}{2}\delta(\{1,4\}) + \frac{t-3}{2}\delta(\{2,3\}) + \frac{t-1}{2}(\delta(\{5\})+\delta(\{1,2\})+\delta(\{1,3\})+\delta(\{2,4\})+\delta(\{3,4\})). \tag{3.31}$$

The following is also a $\mathbb{Z}_+$-realizations of $2t\mathbb{1}_5$:

$$2t\mathbb{1}_5 = p\delta(\{5\}) + q\delta(\{1\}) + (s-q)\delta(\{1,5\}) + \alpha\sum_{2\le i\le 4}\delta(\{i,5\}) + (s-\alpha)\sum_{2\le i\le 4}\delta(\{i\}) + \beta\sum_{2\le i\le 4}\delta(\{1,i\}) + (t-s-\beta)\sum_{2\le i\le 4}\delta(\{1,i,5\}), \tag{3.32}$$

where α,β,p,q,s are integers satisfying

$$\begin{cases} 0\le s\le t, \\ 0\le \alpha \le \min(s,\frac{t}{2}), \\ \max(0, s-2\alpha, \frac{t-3\alpha}{2}) \le p \le \min(t-2\alpha, \frac{t-3\alpha+s}{2}), \\ \beta = t-2\alpha-p, \\ q = 3\alpha+2p-t. \end{cases}$$

Let $\lambda(s,t,\alpha,p)$ denote the realization from (3.32).

For $t=3$, the feasible parameters for (3.32) are (s,α,p) =(1,0,2), (1,1,0), (2,0,2), (2,1,0), (2,1,1), (3,0,3), (3,1,1), (3,0,3), and (3,1,1). Note, however, that $\lambda(3,3,0,3)$ coincides with the star realization (2.1), $\lambda(3,3,1,1)$ reads

$$6\mathbb{1}_5 = \delta(\{5\}) + \sum_{1\le i\le 4}\delta(\{i,5\}) + 2\delta(\{i\}) \tag{3.33}$$

(this is (3.27) in the case $t=3, n=5, s=2$), $\lambda(2,3,0,2)$ is a permutation of (3.33), and $\lambda(2,3,1,1)$ coincides with $\lambda(1,3,0,2)$ (up to permutation).

PROPOSITION 3.34. **(Case $t=3, n=5$)** *The metric $6\mathbb{1}_5$ has five distinct (up to permutation) $\mathbb{Z}_+$-realizations: the star realization* (2.1), (3.33), (3.31) *(with $t=3$), and* (3.32) *for the parameters (s,α,p) =$(2,1,1),(2,1,0),(1,1,0)$ which read, respectively,*

$$6\mathbb{1}_5 = \delta(\{5\}) + 2\delta(\{1\}) + \sum_{2\le i\le 4}\delta(\{i,5\}) + \delta(\{i\}) + \delta(\{1,i,5\}),$$

$$6\mathbb{1}_5 = 2\delta(\{1,5\}) + \sum_{2\le i\le 4}\delta(\{i,5\}) + \delta(\{i\}) + \delta(\{1,i\}),$$

$$6\mathbb{1}_5 = \delta(\{1,5\}) + \sum_{2\le i\le 4}\delta(\{i,5\}) + \delta(\{1,i\}) + \delta(\{1,i,5\}).$$

∎

4 Recognition of hypercube embeddable metrics

In this section, we consider the following problem, called the **hypercube embeddability problem**:

Given a distance d on V_n, test whether d is hypercube embeddable.

When restricted to the class of path metrics of connected graphs, this is the problem of testing whether a graph can be isometrically embedded into a hypercube. Such graphs have a good characterization and can be recognized in polynomial time as the next result shows.

THEOREM 4.1. [Djo73, Avi81] *Let $G = (V, E)$ be a connected graph with shortest path metric d_G. The following assertions are equivalent.*
(i) G can be isometrically embedded into a hypercube.
(ii) G is bipartite and the set $\{i \in V \mid d_G(i,a) < d_G(i,b)\}$ is convex for each edge (a,b) of G.
(iii) G is bipartite and d_G satisfies the following 5-gonal inequality:

$$d(i_1,i_2) + d(i_1,i_3) + d(i_2,i_3) + d(i_4,i_5) - \sum_{\substack{h=1,2,3\\k=4,5}} d(i_h,i_k) \leq 0 \tag{4.2}$$

for all nodes $i_1, \ldots, i_5 \in V$.

The hypercube embeddability problem is NP-complete for general distances; it remains NP-complete for the class of distances with values in the set $\{2,3,4,6\}$ (see Theorem 4.10).

However, the hypercube embeddability problem can be shown to be polynomial for some classes of metrics, having a restricted range of values. For instance, it is polynomial for the class of distances with range of values $\{1,2,3\}$, or $\{3,5,8\}$ or, more generally, $\{x, y, x+y\}$ where x, y are two positive integers such that, either x, y are odd, or x is even and y is odd. This class is discussed in Section 4.3. We also consider generalized bipartite metrics, which are the metrics d on V_n for which there exists a subset $S \subseteq V_n$ such that $d(i,j) = 2$ for all $i \neq j \in S$ and for all $i \neq j \in V_n \setminus S$. The hypercube embeddable generalized bipartite metrics can also be recognized in polynomial time; see Section 4.2. The basic idea that is used for characterizing the hypercube embeddable metrics within the above classes is the existence of equidistant submetrics, which are h-rigid if they are defined on sufficiently many points (by the results of Section 2). We present in Section 4.4 the following result of Karzanov [Kar85]: Let d be a metric whose extremal graph is K_4, C_5, or a union of two stars; then, d is hypercube embeddable if and only if d satisfies the parity condition (4.3). We group in Section 4.1 some preliminary results.

Let us point out that no characterization is known for the hypercube embeddable metrics taking two or three values, all of them even. For instance, the complexity of the hypercube embeddability problem for the class of distances with range of values $\{2,4\}$, or $\{2,4,6\}$, is not known. (Compare with the results of Proposition 4.9 and Theorem 4.10.)

4.1 Preliminary results

Let d be a distance on the set V_n. A first easy observation is that we may assume that no pair of distinct points is at distance 0. Indeed, if $d(i,j) = 0$ for some distinct $i, j \in V_n$, then d is hypercube embeddable if and only if its restriction to the set $V_n \setminus \{j\}$ is hypercube embeddable (as the points i and j should be labeled by the same set in any hypercube embedding of d).

If d is hypercube embeddable, then

$$d(i,j) + d(i,k) + d(j,k) \in 2\mathbb{Z} \text{ for all } i,j,k \in V_n. \tag{4.3}$$

(Indeed, if $A_1, \ldots, A_n$ are sets forming an h-labeling of d, then $d(i,j) + d(i,k) + d(j,k) = 2(|A_i| + |A_j| + |A_k| - |A_i \cap A_j| - |A_i \cap A_k| - |A_j \cap A_k|) \in 2\mathbb{Z}$.) The condition (4.3) is called the **parity condition**; it was first introduced in [Dez61]. This condition expresses the fact that each hypercube embeddable distance d on V_n can be decomposed as an *integer* combination of cut semimetrics, i.e., belongs to the cut lattice $\mathcal{L}_n$ (in fact, (4.3) characterizes membership in $\mathcal{L}_n$, see Proposition 5.2). As an application, we deduce that each hypercube embeddable distance has some bipartite structure, namely, the set of pairs at an odd distance forms a complete bipartite graph.

LEMMA 4.4. *Let d be a distance on V_n. If d satisfies the parity condition* (4.3)*, then V_n can be partitioned into $V_n = S \cup T$ in such a way that $d(i,j)$ is even if $i,j \in S$ or $i,j \in T$, and $d(i,j)$ is odd if $i \in S, j \in T$.* ∎

This simple fact will be central in our treatment. For instance, the generalized bipartite metrics, considered in Section 4.2, have only one even distance equal to 2, i.e., they satisfy $d(i,j) = 2$ for $i \neq j \in S$, $i \neq j \in T$, for some bipartition (S,T) of V_n.

Obviously, every hypercube embeddable distance d on V_n is ℓ_1-embeddable, i.e., belongs to the cut cone CUT_n. In other words, d can be decomposed as a *nonnegative* combination of cut semimetrics.

So, we have the implication:

$$d \text{ is hypercube embeddable} \implies d \in \mathrm{CUT}_n \text{ and } d \text{ satisfies (4.3).}$$

In general, this implication is strict. But, for some classes of distances, this implication is, in fact, an equivalence; this is the case, for instance, for the distances with range of values $\{1,2\}$, or $\{1, 2\alpha, 2\alpha+1\}$ ($\alpha \geq 2$) (see Propositions 4.11 and 4.12), or for the distances considered in Proposition 4.52 or in Theorem 4.55. This is also the case for the distances on $n \leq 5$ points:

THEOREM 4.5. [Dez61, Dez82] *Let d be a distance on $n \leq 5$ points. Then, d is hypercube embeddable if and only if $d \in CUT_n$ and d satisfies the parity condition* (4.3). ∎

We will consider in Section 5 the quasi h-points, which are the distances that belong to CUT_n and satisfy (4.3) but are not hypercube embeddable.

Each valid inequality for the cut cone yields therefore a necessary condition for hypercube embeddability. It turns out that the hypermetric inequalities will play a crucial role for the characterization of certain classes of hypercube embeddable distances; see Propositions 4.11, 4.36, 4.37, 4.52. Let d be a distance on V_n and $k \geq 1$ be an integer. Recall that d is said to be $(2k+1)$-**gonal** if, for all (not necessarily distinct) points $i_1, \ldots, i_k, i_{k+1}, j_1, \ldots, j_k \in V_n$, the following inequality holds:

$$\sum_{1 \leq r < s \leq k+1} d(i_r, i_s) + \sum_{1 \leq r < s \leq k} d(j_r, j_s) - \sum_{\substack{1 \leq r \leq k+1 \\ 1 \leq s \leq k}} d(i_r, j_s) \leq 0. \tag{4.6}$$

Equivalently, d is $(2k+1)$-gonal if, for all $b \in \mathbb{Z}^n$ with $\sum_{1 \leq i \leq n} b_i = 1$ and $\sum_{1 \leq i \leq n} |b_i| = 2k+1$,

$$\sum_{1 \leq i < j \leq n} b_i b_j d(i,j) \leq 0. \tag{4.7}$$

Moreover, d is **hypermetric** if d is $(2k+1)$-gonal for all $k \geq 1$. The inequality (4.6) is called the $(2k+1)$-gonal inequality.

We now recall the link existing between hypercube embeddable distances and intersection patterns. A vector $p \in \mathbb{R}^{V_n \cup E_n}$ is called an **intersection pattern** if there exist n sets $A_1, \ldots, A_n$ such that

$$p_{ij} = |A_i \cap A_j| \ \text{ for all } 1 \leq i \leq j \leq n. \tag{4.8}$$

Hypercube embeddable distances are in one-to-one correspondance with intersection patterns, via the correspondance $p = \xi(d)$ defined below.

Namely, let d be a distance on V_{n+1} and let $p = (p_{ij})_{1 \leq i \leq j \leq n}$ be defined by

$$\begin{cases} p_{ii} = d(i, n+1) & \text{for } 1 \leq i \leq n, \\ p_{ij} = \frac{1}{2}(d(i,n+1) + d(j,n+1) - d(i,j)) & \text{for } 1 \leq i < j \leq n. \end{cases}$$

The mapping $\xi : d \mapsto p$ is known as the **covariance mapping**. Then, d is hypercube embeddable if and only if its image $p = \xi(d)$ under the covariance mapping is an intersection pattern (indeed, the sets $A_1, \ldots, A_n, A_{n+1} = \emptyset$ form an h-labeling of d if and only if $A_1, \ldots, A_n$ satisfy (4.8)). This correspondance permits to show:

PROPOSITION 4.9. [Chv80] *The hypercube embeddability problem is polynomial for the class of distances with range of values* $\{2, 4\}$ *and having a point at distance* 2 *from all other points.*

PROOF. Let d be a distance on V_{n+1} such that $d(i, n+1) = 2$ for all $i \in V_n$ and $d(i,j) \in \{2,4\}$ for all $i \neq j \in V_n$. Its image $p = \xi(d)$ satisfies $p_{ii} = 2$ for all $i \in V_n$ and $p_{ij} \in \{0,1\}$ for all $i \neq j \in V_n$. Let H denote the graph on V_n with edges the pairs (i,j) such that $p_{ij} = 1$. Then, d is hypercube embeddable if and only if p is an intersection pattern which, in turn, is equivalent to H being a line graph. The result now follows from the fact that line graphs can be recognized in polynomial time [Bei70]. ∎

THEOREM 4.10. [Chv80] *The hypercube embeddability problem is NP-complete for the class of distances having a point at distance* 3 *from all other points and with distances between those points belonging to* $\{2, 4, 6\}$.

PROOF. We sketch the proof. Let d be a distance as in the theorem. Hence, its image $p = \xi(d)$ satisfies $p_{ii} = 3$ for all $i \in V_n$ and $p_{ij} \in \{0, 1, 2\}$ for all $i \neq j \in V_n$. Let H denote the multigraph with node set V_n and having p_{ij} parallel edges between nodes i and j. It is easy to see that d is hypercube embeddable, i.e., p is an intersection pattern, if and only if the edge set of H can be partitioned into cliques in such a way that each node belongs to three of these cliques. Chvátal [Chv80] shows that this problem can be reduced to the problem of testing whether a 4-regular graph is 3-colourable, which is NP-complete. ∎

There are some classes of distances for which hypercube embeddability is very easy to characterize. Two examples are given below.

PROPOSITION 4.11. [AD80] *Let d be a distance on V_n with values in $\{1, 2\}$. The following assertions are equivalent.*
(i) d is hypercube embeddable.
(ii) d is 5-gonal and satisfies the parity condition (4.3).
(iii) d is the path metric of the complete bipartite graphs $K_{1,n-1}$ or $K_{2,2}$ (with $n = 4$), or $d = 2d(K_n)$.

PROOF. We check $(ii) \Longrightarrow (iii) \Longrightarrow (i)$. By Lemma 4.4, the set of pairs (i, j) at distance 1 forms a complete bipartite graph $K_{S,T}$ for some bipartition (S, T) of V_n with, e.g., $|T| \leq |S|$. If $|T| \geq 2$ and $|S| \geq 3$, then d violates the 5-gonal inequality (indeed, let $i_1, i_2, i_3 \in S$, $j_1, j_2 \in T$, and $k = 2$, then the left hand side of (4.6) is equal to $8-6 = 2 > 0$). If $|T| = 2$ and $|S| = 2$, then d is hypercube embeddable; indeed, if $S = \{i_1, i_2\}$ and $T = \{j_1, j_2\}$ then $d = \delta(\{i_1, j_1\}) + \delta(\{i_1, j_2\})$. If $|T| = 1$ then d is also hypercube embeddable as $d = \sum_{i \in S} \delta(\{i\})$. ∎

PROPOSITION 4.12. [DL94] *Let d be a metric on V_n with range of values $\{1, 2\alpha, 2\alpha + 1\}$, for some integer $\alpha \geq 2$. Then, d is hypercube embeddable if and only if d satisfies the parity condition (4.3).*

PROOF. Suppose that d satisfies (4.3). Hence, the set of pairs at odd distance forms a complete bipartite graph $K_{S,T}$ for some bipartition (S, T) of V_n. As $\alpha \geq 2$, the pairs at distance 1 form a matching, say, $d(i_1, j_1) = \ldots = d(i_k, j_k) = 1$ for $i_1, \ldots, i_k \in S$ and $j_1, \ldots, j_k \in T$. Then, $d = \delta(S) + \sum_{1 \leq h \leq k} \alpha\delta(\{i_h, j_h\}) + \sum_{\substack{i \in S \setminus \{i_1, \ldots, i_k\} \\ j \in T \setminus \{j_1, \ldots, j_k\}}} \alpha\delta(\{i\})$, showing that d is hypercube embeddable. ∎

The case $\alpha = 1$, i.e., the case of distances with values 1,2,3, is more complicated and will be treated in Section 4.3.

We close this section with a result on the number of distinct hypercube embeddings of a given distance. Given a hypercube embedable distance d on V_n and an integer $s \geq 0$, let $N_n(d, s)$ denote the number of distinct $\mathbb{Z}_+$-realizations $d = \sum_S \lambda_S \delta(S)$ (with $\lambda_S \in \mathbb{Z}_+$) of d with size $\sum_S \lambda_S = s$. Set

$$M_n(x) := \sum N(d, s)$$

where the sum is taken over all $s \in \mathbb{Z}_+$ and all distances d on V_n with $\sum_{1 \le i < j \le n} d(i,j) = x$. It is shown in [DCS90] that the function $x \in \mathbb{Z}_+ \mapsto M_n(x)$ is quasipolynomial. In other words, there exist an integer $t \ge 1$ and polynomials $f_0, f_1, \ldots, f_{t-1}$ such that

$$M_n(x) = f_i(x) \text{ if } x \equiv i \pmod{t}, \text{ for } 0 \le i \le t-1.$$

In particular, $M_n(x)$ is bounded by a polynomial in x. Therefore, the number of distinct $\mathbb{Z}_+$-realizations of d is bounded by a polynomial in $x = \sum_{1 \le i < j \le n} d(i,j)$.

4.2 Generalized bipartite metrics

Let d be a metric on V_n such that $d(i,j) = 2$ for all $i \ne j \in S$ and $i \ne j \in T$, for some bipartition (S,T) of V_n. Such a metric is called a **generalized bipartite metric**. The $|S| \times |T|$ matrix D with entries $d(i,j)$ for $i \in S, j \in T$ is called the **(S,T)-distance matrix** of d. For instance, the path metric of a complete bipartite graph is a generalized bipartite metric. In this section, we prove the following result.

Theorem 4.13. [DL94] *The hypercube embeddability problem is polynomial for the class of generalized bipartite metrics.*

We start with an easy observation.

Lemma 4.14. *Let d be a generalized bipartite metric with bipartition (S,T). If d is hypercube embeddable, then there exists an integer α such that $d(i,j) \in \{\alpha, \alpha+2, \alpha+4\}$ for all $i \in S, j \in T$.*

Proof. Let α, β denote the smallest and largest value taken by $d(i,j)$ for $i \in S, j \in T$; say $\alpha = d(i,j), \beta = d(i',j')$ for $i,i' \in S$, $j,j' \in T$. Using the triangle inequality, we obtain $\beta = d(i',j') \le d(i', +d(i,j) + d(j,j') \le 4 + \alpha$. Moreover, α, β have the same parity by (4.3). ∎

We will see below what are the possible configurations for the pairs at distance $\alpha, \alpha + 2, \alpha + 4$.

Set $s := |S|$ and $t := |T|$. Let d_S (resp. d_T) denote the restriction of d to the set S (resp. T). Then, $d_S = 2\mathbb{1}_s$ and $d_T = 2\mathbb{1}_t$ are equidistant metrics. Recall (from Proposition 3.29) that the equidistant metric $2\mathbb{1}_n$ is h-rigid if $n \ne 4$ and that $2\mathbb{1}_4$ has exactly two $\mathbb{Z}_+$-realizations, namely, its star realization: $2\mathbb{1}_4 = \sum_{1 \le i \le 4} \delta(\{i\})$, and an additional realization:

$$2\mathbb{1}_4 = \delta(\{1,2\}) + \delta(\{1,3\}) + \delta(\{1,4\}),$$

called the **special realization**.

The proof of Theorem 4.13 is based on the following simple observation. Let $d = \sum_{A \subseteq V_n} \lambda_A \delta(A)$ be a $\mathbb{Z}_+$-realization of d. Then, its projection on S: $\sum_{A \subseteq V_n} \lambda_A \delta(A \cap S)$, is a $\mathbb{Z}_+$-realization of d_S. Hence, if $s \ne 4$, then it must coincide with the star realization of $2\mathbb{1}_s$ and, if $s = 4$, it must coincide with the star realization or with the special realization of $2\mathbb{1}_4$. The same holds for d_T.

The following definitions will be useful in the sequel. A $\mathbb{Z}_+$-realization of d is called a **star-star** realization if both its projections on S and on T are the star realizations of $2\mathbb{1}_s$ and $2\mathbb{1}_t$, respectively. A realization of d is called a **star-special** realization if its projection on S is the star realization of $2\mathbb{1}_s$, but $t = 4$ and its projection on T is the special realization of $2\mathbb{1}_4$. Finally, a realization of d is called a **special-special** realization if $s = t = 4$ and both its projections on S and T are the special realization of $2\mathbb{1}_4$.

We now analyze the structure of the hypercube embeddable generalized bipartite metrics admitting a star-star realization.

Proposition 4.15. *Let d be a generalized bipartite metric with bipartition (S,T). Then, d admits a star-star realization if and only if there exist a partition $\{A,B,C,D\}$ of S and a partition $\{A',B',C',D'\}$ of T (with possibly empty members) with $|A| = |A'|$ and $|B| = |B'|$ and there exist one-to-one mappings $\sigma : A \longrightarrow A'$ and $\tau : B \longrightarrow B'$ and an integer $f \geq |B| + |D| + |D'|$ such that*

$$(4.16)\quad d(i,j) = \begin{cases} f & \text{for } (i,j) \in \quad ((A \cup C) \times (B' \cup D')) \cup ((B \cup D) \times (A' \cup C')) \\ & \qquad\qquad\quad \cup \{(k,\sigma(k)) \mid k \in A\} \cup \{(k,\tau(k)) \mid k \in B\}, \\ f+2 & \text{for } (i,j) \in \quad ((A \cup C) \times (A' \cup C')) \setminus \{(k,\sigma(k)) \mid k \in A\}, \\ f-2 & \text{for } (i,j) \in \quad ((B \cup D) \times (B' \times D')) \setminus \{(k,\tau(k)) \mid k \in B\}. \end{cases}$$

Figure 4.17 shows the (S,T)-distance matrix of the metric d defined by (4.16). We use the following notation in Figures 4.17 and 4.18: I_a denotes the $a \times a$ identity matrix, J_a the $a \times a$ all ones matrix, and a block marked, say, with f, has all its entries equal to f. As a rule, we denote the cardinality of a set by the same lower case letter; e.g., $a = |A|, a' = |A'|$, etc.

	A'	C'	B'	D'
A	$(f+2)J_a$ $-2I_a$	$f+2$	f	f
C	$f+2$	$f+2$	f	f
B	f	f	$(f-2)J_b$ $+2I_b$	$f-2$
D	f	f	$f-2$	$f-2$

Figure 4.17

	a	b	c	d	c'	d'	m
A	I_a	0	0	0	0	0	0
B	0	I_b	0	0	0	0	0
C	0	0	I_c	0	0	0	0
D	0	0	0	I_d	0	0	0
A'	I_a	1	0	1	0	1	1
B'	0	$J_b - I_b$	0	1	0	1	1
C'	0	1	0	1	$I_{c'}$	1	1
D'	0	1	0	1	0	$J_{d'} - I_{d'}$	1

Figure 4.18

PROOF OF PROPOSITION 4.15. Let d be a generalized bipartite metric admitting a star-star realization: $d = \sum_{U \in \mathcal{U}} \delta(U)$, where $\mathcal{U}$ is a collection (allowing repetition) of nonempty subsets of V. Hence, $|U \cap S| \in \{0, s, 1, s-1\}$ and $|U \cap T| \in \{0, t, 1, t-1\}$ for all $U \in \mathcal{U}$. We can suppose without loss of generality that $|U \cap S| \in \{0, 1\}$ for all $U \in \mathcal{U}$. Let M denote the matrix whose columns are the incidence vectors of the members of $\mathcal{U}$. Combining the above mentioned two possibilities for $U \cap S$ with the four possibilities for $U \cap T$, we obtain that M has the form shown in Figure 4.18. Hence the sets A, B, C, D and A', B', C', D' form the desired partitions of S and T. We can now compute $d(i,j)$ for $(i,j) \in S \times T$ and verify that they satisfy relation (4.16), after setting $f := |B| + |D| + |D'| + m$.
Conversely, suppose that d is defined by (4.16). Set $A = \{x_1, \ldots, x_n\}$ and $B = \{y_1, \ldots, y_n\}$. One can easily check that d satisfies:

$$\begin{aligned} d = \; & \textstyle\sum_{1 \le i \le |A|} \delta(\{x_i, \sigma(x_i)\}) + \sum_{1 \le i \le |B|} \delta(T \setminus \{\tau(y_i)\} \cup \{y_i\}) + \sum_{x \in C \cup C'} \delta(\{x\}) \\ & + \textstyle\sum_{x \in D} \delta(T \cup \{x\}) + \sum_{x \in D'} \delta(T \setminus \{x\}) + (f - |B| - |D| - |D'|)\delta(T). \end{aligned}$$

This realization is clearly a star-star realization. ∎

It is quite clear that the description from Proposition 4.15 permits to test in polynomial time whether a generalized bipartite metric has a star-star realization and to find one if one exists (see [DL94] for details). Actually, this can be done in $O(n^2)$ if the metric is on n points.

One can check whether a generalized bipartite metric has a star-special realization in the following way. Suppose $|T| = 4$. Let $z' \in T$ and let d' denote the restriction of d to the set $V \setminus \{z'\}$. If d has a star-special realization then d' has a star-star realization. We see easily that there are $O(1)$ possible star-star realizations for d' and all of them can be found in polynomial time. One then checks whether one of them can be extended to a star-special realization of d. (If a star-star realization of d' is as in Figure 4.18, there is a unique way to complete it to a star-special realization of d, namely, by adjoining the following row as a last row to Figure 4.18.)

	a	b	c	d	c'	d'	m
z'	1	0	0	1	1	0	1

Finally, a generalized bipartite metric d has a special-special realization if and only if, for some $m \in \mathbb{Z}_+$, the (S,T)-distance matrix of the semimetric $d - m\delta(T)$ is one of the nine matrices from Figure 4.19 (up to permutation on S and T). (This fact can be checked, using a characterization of the generalized bipartite metrics admitting a special-special realization analogous to that of Proposition 4.15, see [DL94].)

$$\begin{array}{|cccc|}\hline 3&1&1&1\\1&3&1&1\\1&1&3&1\\1&1&1&3\\\hline\end{array}\quad
\begin{array}{|cccc|}\hline 0&2&2&2\\2&0&2&2\\2&2&0&2\\2&2&2&0\\\hline\end{array}\quad
\begin{array}{|cccc|}\hline 1&1&1&3\\1&3&3&3\\1&3&3&3\\3&3&3&5\\\hline\end{array}$$

$$\begin{array}{|cccc|}\hline 4&4&4&2\\4&2&2&2\\4&2&2&2\\2&2&2&0\\\hline\end{array}\quad
\begin{array}{|cccc|}\hline 3&1&1&1\\1&3&1&1\\3&3&3&1\\3&3&1&3\\\hline\end{array}\quad
\begin{array}{|cccc|}\hline 0&2&2&2\\2&0&2&2\\2&2&2&4\\2&2&4&2\\\hline\end{array}$$

$$\begin{array}{|cccc|}\hline 2&2&2&4\\ 2&2&2&4\\ 2&2&2&4\\ 4&4&4&6\\ \hline\end{array}\quad \begin{array}{|cccc|}\hline 4&4&4&2\\ 4&4&4&2\\ 4&4&4&2\\ 2&2&2&0\\ \hline\end{array}\quad \begin{array}{|cccc|}\hline 3&3&3&1\\ 3&3&3&1\\ 3&3&3&1\\ 5&5&5&3\\ \hline\end{array}$$

Figure 4.19

EXAMPLE 4.20. Given an integer $k \geq 5$, let d denote the metric defined on $2k$ points by: $d(i, i+k) = 4$ for any $1 \leq i \leq k$ and $d(i,j) = 2$ for all other pairs $(i,j), 1 \leq i \neq j \leq 2k$. Hence, d_{2k} is a generalized bipartite metric with bipartition $(\{1,2\ldots,k\},\{k+1,k+2,\ldots,2k\})$. It is an easy exercise to verify, for instance using the above procedure, that d_{2k} is not hypercube embeddable and also that d_{2k} belongs to the cut cone $\mathcal{C}_{2k}$ and to the cut lattice $\mathcal{L}_{2k}$. ∎

The same technique could be used for testing hypercube embeddability for other metrics than generalized bipartite metrics. Let d be a semimetric on V_n. Suppose that there exists a bipartition (S,T) of V such that the projections d_S and d_T of d on S and T are of the form:

$$d_S = \sum_{x\in S} \alpha_x \delta(\{x\}), \quad d_T = \sum_{x\in T} \beta_x \delta(\{x\}) \tag{4.21}$$

for some positive integers α_x, β_x. From Theorem 2.12, we know that d_S and d_T are h-rigid if $|S|$ is big enough with respect to $\max_{x\in S}\alpha_x$ and if $|T|$ is big enough with respect to $\max_{x\in T}\beta_x$. So, theoretically, one could use the same technique as the one used in Proposition 4.15 for studying hypercube embeddability of these metrics. However, a precise analysis of the structure of the distance matrix of such metrics seems technically much more involved than in the case where all α_x, β_x are equal to 1, considered above.

The next simplest case to consider after the case of generalized bipartite metrics would be the class of metrics d for which $d(x,y) = 4$ for $x \neq y \in S$ and $d(x,y) = 2$ for $x \neq y \in T$ (i.e., all α_x's are equal to 2 and all β_x's to 1). One can characterize h-embeddability of these metrics by a similar reasoning as was applied to generalized bipartite metrics and, as a consequence, recognize them in polynomial time. Indeed, the metric $4\mathbb{1}_n$ is rigid for $n=3$ and $n \geq 9$ and $4\mathbb{1}_n$ has exactly three $\mathbb{Z}_+$-realizations: its star realization and two special ones, for each $n \in \{4,5,6,7,8\}$ [DL93c].

We give below a complete characterization of the hypercube embeddable metrics satisfying (4.21) in the case $|T| \leq 2$. We state the results without proofs; the proofs can be found in [DL94]. We first consider the case $|T| = 1$. We introduce some notation.

Let d be defined on the set $\{1,\ldots,n,n+1\}$ and let β, $\alpha_x \in \mathbb{Z}$ for $x \in S := \{1,\ldots,n\}$. For $x \in S$, set

$$\sigma_x := \frac{1}{2}\Big(\sum_{y\in S} d(y,n+1) - \alpha_y\Big) - \frac{n-2}{2}(d(x,n+1) - \alpha_x), \tag{4.22}$$

$$\beta_x := \frac{\sigma_x - \beta}{n-2}, \tag{4.23}$$

$$\begin{aligned} &\sigma := \min(\sigma_x \mid x \in S),\\ &\tau := \min\Big(\frac{1}{2}(d(x,n+1) - d(y,n+1) + d(x,y)) \mid x \neq y \in S\Big). \end{aligned} \tag{4.24}$$

PROPOSITION 4.25. *Let d be a semimetric on the set $\{1,\ldots,n,n+1\}$ which satisfies the parity* *condition* (4.3)*. Suppose that the projection d_S of d on the subset $S := \{1, \ldots, n\}$ satisfies: $d_S = \sum_{1\leq x\leq n} \alpha_x \delta(\{x\})$ for some positive integers $\alpha_1, \ldots, \alpha_n$ and that d_S is h-rigid. Then, d is hypercube embeddable if and only if $\sigma_x \geq 0$ for all $x \in S$. Moreover, the $\mathbb{Z}_+$-realizations of d are all the realizations of the form:*

$$d = \beta\delta(\{n+1\}) + \sum_{x\in S} \beta_x \delta(\{x, n+1\}) + (\alpha_x - \beta_x)\delta(\{x\}) \tag{4.26}$$

where β_x $(x \in S)$ are given by (4.23) *and β is a nonnegative integer satisfying*

$$\sigma - (n-2)\tau \leq \beta \leq \sigma \ \text{ and } \ \frac{\sigma - \beta}{n-2} \in \mathbb{Z} \tag{4.27}$$

(with σ, σ_x, τ being given by (4.22),(4.24)*). In particular, d is h-rigid whenever d satisfies some triangle inequality at equality.* ∎

COROLLARY 4.28. *Let d be defined on the set $\{1, \ldots, n, n+1\}$. Suppose that its projection d_S on the subset $S := \{1, \ldots, n\}$ satisfies $d_S = \sum_{1\leq x\leq n} \alpha_x \delta(\{x\})$ for some positive integers $\alpha_1, \ldots, \alpha_n$ and that d_S is h-rigid. Set $\beta := d(1, n+1)$ and suppose that $d(x, n+1) = \beta - d(1, x)$ for $2 \leq x \leq n$.*
(i) d is a semimetric if and only if $\beta \geq \alpha_1 + max(\alpha_x + \alpha_y : 2 \leq x < y \leq n)$.
(ii) d satisfies the parity condition (4.3) *if and only if β is an integer.*
(iii) d is hypercube embeddable if and only if β is an integer and $\beta \geq \sum_{x\in S} \alpha_x$; moreover, d is h-rigid. ∎

Suppose now that $|T| = 2$. Let d be defined on the set $\{1, \ldots, n, n+1, n+2\}$. let d_S,d',d'' denote the projections of d on the subsets $S := \{1, \ldots, n\}$, $S\cup\{n+1\}$, $S\cup\{n+2\}$, respectively. We suppose that $d_S = \sum_{x\in S} \alpha_x \delta(\{x\})$ for some positive integers α_x and that d_S is rigid. Hence, we can apply Proposition 4.25 for testing whether d' and d'' are hypercube embeddable. Let $\sigma'_x, \beta'_x, \sigma', \tau'$ be defined by relations (4.22), (4.23) and (4.24) (where β' is to be determined) when considering the semimetric d' instead of d. Similarly, let $\sigma''_x, \beta''_x, \sigma'', \tau''$ be defined by (4.22), (4.23) and (4.24) (where β'' is to be determined) when considering the semimetric d'' instead of d and the point $n+2$ instead of $n+1$.

PROPOSITION 4.29. *Let d be a semimetric on $V := \{1, \ldots, n, n+1, n+2\}$ that satisfies the parity condition* (4.3)*. Suppose that its projection d_S on the subset $S := \{1, \ldots, n\}$ is of the form: $d_S = \sum_{x\in S} \alpha_x \delta(\{x\})$ for some positive integers α_x and that d_S is h-rigid. Then d is hypercube embeddable if and only if (i),(ii) hold.*
(i) The projection d' (resp. d'') of d on $S\cup\{n+1\}$ (resp. on $S\cup\{n+2\}$) is hypercube embeddable.

$$(ii) \begin{cases} d(n+1, n+2) \leq \beta' + \beta'' + \sum_{x\in S} \min(\beta'_x + \beta''_x, 2\alpha_x - \beta'_x - \beta''_x), \\ d(n+1, n+2) \geq \max(\beta', \beta'') - \min(\beta', \beta'') + \sum_{x\in S} \max(\beta'_x, \beta''_x) - \min(\beta'_x, \beta''_x), \end{cases}$$

where β', β'' are nonnegative integers satisfying $\sigma' - (n-2)\tau' \leq \beta' \leq \sigma'$, $\frac{\sigma'-\beta'}{n-2} \in \mathbb{Z}$ and $\sigma'' - (n-2)\tau'' \leq \beta'' \leq \sigma''$, $\frac{\sigma''-\beta''}{n-2} \in \mathbb{Z}$. ∎

4.3 Metrics with few values

In this section, we consider the distances taking two values with distinct parities, and the distances taking three values, not all even and one of them being the sum of the other

two. Namely, given $a, b \in \mathbb{Z}_+$, we consider the following classes of distances d:
(a) d takes the values $2a, b$ with b odd,
(b) d takes the values $a, b, a+b$ with a, b odd,
(c) d takes the values $2a, b, 2a+b$ with b odd and $b < 2a$, and
(d) d takes the values $2a, b, 2a+b$ with b odd and $2a < b$.
We have the following result.

THEOREM 4.30. [Lau93b] *For fixed a, b, the hypercube embeddability problem within each of the classes $(a), (b), (c), (d)$ can be solved in polynomial time.*

We sketch the proof of Theorem 4.30 in the rest of the section. Each of the classes $(a), (b), (c), (d)$ has to be treated separately. Actually, the instance $a = b = 1$ of the class (c) was considered in [Avi90], where it is shown that hypercube embeddable distances with range of values $\{1, 2, 3\}$ can be recognized in polynomial time. The proof for the class (c) is essentially the same as in the subcase $a = b = 1$.

The basic steps of the proof are as follows. Let d be a distance on V_n from one of the classes $(a), (b), (c), (d)$. One first checks whether d satisfies the parity condition (4.3). If not, then d is not hypercube embeddable. Otherwise, let (S, T) be the partition of V_n provided by Lemma 4.4, with $|S| \geq |T|$. Set $n(a, b) := a^2 + a + 3$ if d belongs to the classes (a),(c), or (d), and $n(a, b) := (\frac{a+b}{2})^2 + \frac{a+b}{2} + 3$ if d belongs to the class (b).
If $n < 2n(a, b) - 1$, one can test directly whether d is hypercube embeddable, for instance, by brute force enumeration (the number of operations in this step depends only on a, b but may be exponential in a, b).
If $n \geq 2n(a, b) - 1$, then $|S| \geq n(a, b)$. Hence, the restriction of d to the set S is an h-rigid equidistant metric and, therefore, the points of S should be labeled by the star embedding (or an equivalent of it) in any h-labeling of d. For the classes $(a), (b), (c), (d)$, this information enables us to completely characterize the hypercube embeddable distances on $n \geq 2n(a, b) - 1$ points by a set of conditions that can be checked in polynomial time; see Propositions 4.35, 4.41, 4.43, 4.52, and 4.53.

We have some partial results for the characterization of the hypercube embeddable distances on n points, for n arbitrary. See Propositions 4.36, 4.37, and 4.42.

4.3.1 Distances with values $2a, b$ (b odd)

Let d be a distance on V_n with range of values $\{2a, b\}$, where a, b are positive integers with b odd. Suppose that d is a semimetric and satisfies the parity condition (4.3). Then, $b \geq a$ and let (S, T) be the partition of V_n provided by Lemma 4.4. Then an h-labeling of d consists of two set families $\mathcal{A}$ and $\mathcal{B}$ such that

$$(4.31) \qquad \begin{cases} \mathcal{A} \text{ is a } (b, b-a)\text{-intersecting system,} \\ \mathcal{B} \text{ is a } (2a, a)\text{-intersecting system,} \\ |A \cap B| = a \text{ for all } A \in \mathcal{A}, B \in \mathcal{B}, \\ |\mathcal{A}| = |S|, |\mathcal{B}| = |T| - 1. \end{cases}$$

(Indeed, label a point $j_0 \in T$ by $\emptyset$, the remaining points of T by the members of $\mathcal{B}$, and the points of S by the members of $\mathcal{A}$.) For instance, it is easy to see that such families $\mathcal{A}, \mathcal{B}$ can be constructed if $|T| = 1$, or if $b \geq 2a$, or if $b < 2a$ and $2 \leq |T| \leq |S| \leq \frac{a}{2a-b} + 1$. Note also that, for $b < 2a$, $\min(|T|, |S| - 1) \leq \lfloor \frac{b}{2a-b} \rfloor$ holds if d is hypercube embeddable (else, d violates a $(2k+1)$-gonal inequality, for $k := \min(|T|, |S| - 1)$).

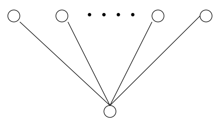


Figure 4.32

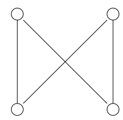


Figure 4.33

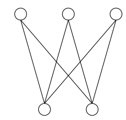
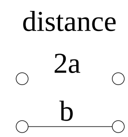


Figure 4.34



PROPOSITION 4.35. *Let $a \leq b$ be positive integers with b odd. Let d be a distance on n points with range of values $\{2a, b\}$. If $n \geq 2a^2 + 2a + 5$, then d is hypercube embeddable if and only if d satisfies* (4.3) *and $b \geq 2a$, or d is the distance from Figure* 4.32.

PROOF. Remains to show the "if" part. Suppose that d is hypercube embeddable and $b < 2a$. Let $\mathcal{A}$ and $\mathcal{B}$ satisfying (4.31). By assumption, we have $|S| \geq a^2 + a + 3$. Hence, $\mathcal{A}$ is a $(b, b-a)$-intersecting system with $|\mathcal{A}| \geq a^2 + a + 3$. By Lemma 3.14, $\mathcal{A}$ is a Δ-system; let A_0 be its kernel, $|A_0| = b - a$. If $|T| \geq 2$, then $|\mathcal{B}| \geq 1$. Let $B \in \mathcal{B}$ and set $\alpha := |B \cap A_0|$. Then, $|B \cap (A \setminus A_0)| = a - \alpha$ for all $A \in \mathcal{A}$. Therefore, $2a = |B| \geq \alpha + |\mathcal{A}|(a - \alpha) = a|\mathcal{A}| - \alpha(|\mathcal{A}| - 1) \geq a|\mathcal{A}| - (b-a)(|\mathcal{A}| - 1) = (2a - b)|\mathcal{A}| + b - a$, which implies $|\mathcal{A}| \leq \frac{3a-b}{2a-b} = \frac{a}{2a-b} + 1$. This contradicts the fact that $|\mathcal{A}| = |S| \geq a^2 + a + 3$. Therefore, $|T| = 1$, i.e., d is the distance from Figure 4.32. ∎

PROPOSITION 4.36. *Let $a \leq b$ be positive integers with b odd and let d be a distance on n points with range of values $\{2a, b\}$. If $b \geq 2a$, then d is hypercube embeddable if and only if d satisfies* (4.3)*. If $b < \frac{4}{3}a$, then the following assertions $(i), (ii), (iii)$ are equivalent.*
(i) d is hypercube embeddable.
(ii) d satisfies the parity condition (4.3) *and the* 5*-gonal inequality (i.e., d does not contain as substructure the distance from Figure* 4.34*).*
(iii) d is one of the distances from Figures 4.32 *and* 4.33*.* ∎

Note that Proposition 4.11 is the case $a = b = 1$ of Proposition 4.36. So, we have a complete characterization of the hypercube embeddable distances with values in $\{2a, b\}$ (b odd) except when a, b satisfy: $\frac{4}{3}a \leq b < 2a$.

4.3.2 Distances with values $a, b, a+b$ (a,b odd)

Let d be a distance on V_n with range of values $\{a, b, a+b\}$, where a, b are positive odd integers with $a < b$. Suppose that d is a semimetric and satisfies the parity condition (4.3). Let (S, T) be the bipartition of V_n provided by Lemma 4.4 with $|S| \geq |T|$. Then, the pairs ij with $d(i,j) = a$ form a matching.

PROPOSITION 4.37. *If there are at least two pairs at distance a, then the following assertions are equivalent.*
(i) d is hypercube embeddable.
(ii) d satisfies (4.3) *and the* 5*-gonal inequality.*
(iii) d is the distance from Figure 4.38*.* ∎

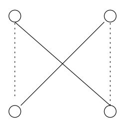


Figure 4.38

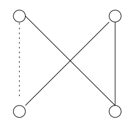


Figure 4.39

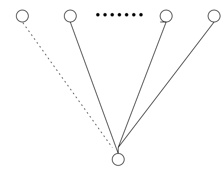
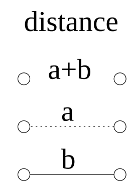


Figure 4.40



We now suppose that there is exactly one pair (i_0, j_0) at distance a, where $i_0 \in S$, $j_0 \in T$. In an h-labeling of d, we can suppose that j_0 is labeled by $\emptyset$ and, then, i_0 should be labeled by a set A_0 of cardinality a. Therefore, an h-labeling of d exists if and only if there exist two set families $\mathcal{A}$ and $\mathcal{B}$ such that

$$\begin{cases} \mathcal{A}, \mathcal{B} \text{ are } (b, \frac{b-a}{2}) - \text{intersecting systems}, \\ |A \cap B| = \frac{a+b}{2} \text{ for all } A \in \mathcal{A}, B \in \mathcal{B}, \\ A \cap A_0 = B \cap A_0 = \emptyset \text{ for all } A \in \mathcal{A}, B \in \mathcal{B}, \\ |\mathcal{A}| = |S| - 1, |\mathcal{B}| = |T| - 1. \end{cases}$$

PROPOSITION 4.41. *Let $a < b$ be odd integers and let d be a distance on $n \geq 2(\frac{a+b}{2})^2 + a + b + 7$ points with range of values $\{a, b, a+b\}$ which is not the distance from Figure 4.38. Then, d is hypercube embeddable if and only if d is the distance from Figure 4.40.*

PROOF. The distance from Figure 4.40 is clearly hypercube embeddable (take for $\mathcal{A}$ a Δ-system). Conversely, suppose that d is hypercube embeddable. Then, $\mathcal{A}$ is a $(b, \frac{b-a}{2})$-intersecting system with $|\mathcal{A}| \geq (\frac{a+b}{2})^2 + \frac{a+b}{2} + 3$. By Lemma 3.14, $\mathcal{A}$ is a Δ-system; let A_1 be its kernel, $|A_1| = \frac{b-a}{2}$. Suppose that $|T| \geq 2$ and let $B \in \mathcal{B}$. Then, $|B \cap (A \setminus A_1)| \geq a$ for all $A \in \mathcal{A}$, implying $b = |B| \geq a|\mathcal{A}|$, in contradiction with the above assumption on $|\mathcal{A}|$. Therefore, $|T| = 1$, i.e., d is the distance from Figure 4.40. ∎

PROPOSITION 4.42. *Let a, b be odd integers such that $a < b < 2a$. Let d be a distance with range of values $\{a, b, a+b\}$. Then, d is hypercube embeddable if and only if d is one of the distances from Figures 4.38, 4.39, and 4.40.*

PROOF. Suppose that d is hypercube embeddable and that d is not the distance from Figure 4.38. Set $k := \min(|T|, |S| - 1)$. If $k \geq 2$, then $k \leq \lfloor \frac{b}{a} \rfloor$ (else, d violates a $(2k+1)$-gonal inequality). Hence, $k = 1$, which implies that d is the distance from Figures 4.40 or 4.39. ∎

4.3.3 Distances with values $b, 2a, b + 2a$ (b odd, $b < 2a$)

PROPOSITION 4.43. *Let a, b be positive integers with b odd and $b < 2a$. Let d be a distance on $n \geq 2a^2 + 2a + 5$ points with range of values $\{2a, b, 2a + b\}$. The assertions (i),(ii) are equivalent.*
(i) d is hypercube embeddable.
(ii) d is a semimetric, d satisfies (4.3) and d does not contain as substructure any of the distances from Figures 4.44-4.51.
In particular, if $b < a$, then d is hypercube embeddable if and only if d is a semimetric and satisfies (4.3).

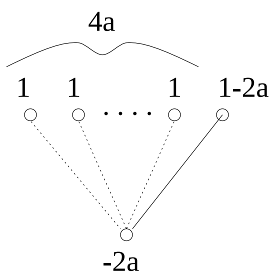


Figure 4.44

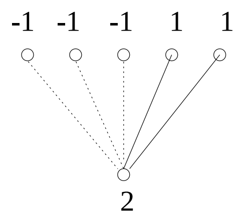


Figure 4.45

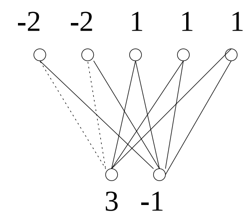


Figure 4.46

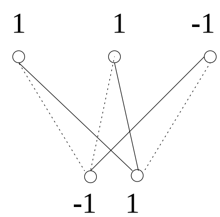


Figure 4.47

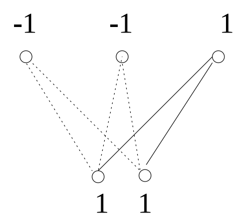


Figure 4.48

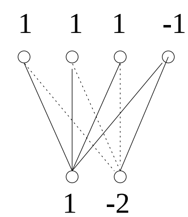


Figure 4.49

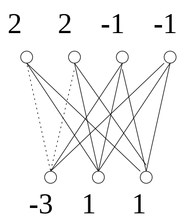


Figure 4.50

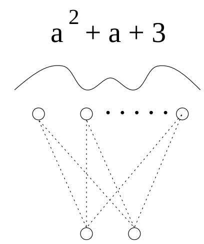


Figure 4.51

In Figures 4.44-4.51, a plain edge represents distance $2a + b$, a dotted edge distance b and no edge means distance $2a$.

Proof. For the implication $(i) \Longrightarrow (ii)$, we check that none of the distances from Figures 4.44-4.51 is hypercube embeddable. Indeed, the distances from Figures 4.44-4.50 violate some hypermetric inequality. The numbers assigned to the nodes in Figures 4.44-4.50 indicate a choice of integers b_i's for which the hypermetric inequality (4.7) is violated. For instance, for the distance from Figure 4.44, $\sum_{i,j \in V_n} b_i b_j d(i,j) = 4a(2a(2a-b)-b) \geq 4a > 0$ since $2a - b \geq 1$. The distance from Figure 4.51 is not hypercube embeddable by Proposition 4.35 (and its proof).

We show the implication $(ii) \Longrightarrow (i)$. As d satisfies the parity condition, V_n is partitioned into $S \cup T$ with $|S| \geq |T|$, $d(i,j) = 2a$ for $(i,j) \in S^2 \cup T^2$, $d(i,j) \in \{b, b+2a\}$ for $(i,j) \in S \times T$. Set $s := |S|$. For $j \in T$, set

$$N_b(j) := \{i \in S \mid d(i,j) = b\}.$$

For $v \in \{0, 1, 2, \ldots, s-1, s\}$, set

$$T_v := \{j \in T \mid |N_b(j)| = v\}.$$

We group below several observations on the sets T_v.

(i) $T_{s-1} = \emptyset$ (since d does not contain the configuration from Figure 4.44).
(ii) $|T_s| \leq 1$ (since d does not contain the configuration from Figure 4.51).
(iii) All T_v are empty except maybe T_0, T_1, T_2, T_s (indeed, $|N_b(j)| \leq 2$ or $|N_b(j)| \geq s-1$ for all $j \in T$, since d does not contain the substructure from Figure 4.45).
(iv) At most one of T_0 and T_2 is not empty (since d does not contain the substructure from Figure 4.46).
(v) If $|T_1| \geq 2$, then
$(v1)$ either all $N_b(j)$ $(j \in T_1)$ are equal,
$(v2)$ or all $N_b(j)$ $(j \in T_1)$ are distinct
(since d does not contain the substructure from Figure 4.47).
(vi) If $j \neq j' \in T_2$, then $|N_b(j) \cap N_b(j')| = 1$ (use Figures 4.47 and 4.48).
(vii) If $j \in T_1$ and $j' \in T_2$, then $N_b(j) \cap N_b(j') \neq \emptyset$ (by Figure 4.47).
$(viii)$ If $b < a$, then $T_2 = T_s = \emptyset$ (by the triangle inequality).

We show how to construct an h-labeling of d. Let A_i $(i \in S)$ be disjoint sets of cardinality a. Set $A := \cup_{i \in S} A_i$. Label the elements of S by the A_i's.

Suppose first that $b < a$. Then, by $(viii)$, $d(i_1, j_1) = \ldots = d(i_r, j_r) = b$ for some $i_1, \ldots, i_r \in S$, $j_1, \ldots, j_r \in T$, $1 \leq r \leq |T|$. Let X, B_j $(j \in T \setminus \{j_1, \ldots, j_r\})$ be pairwise disjoints sets that are disjoint from A and satisfy $|X| = b$, $|B_j| = a$. Label $j_1, \ldots, j_r$ by $A_{i_1} \cup X, \ldots, A_{i_r} \cup X$, respectively, and $j \in T \setminus \{j_1, \ldots, j_r\}$ by $X \cup B_j$. This gives an h-labeling of d.

We now suppose that $b \geq a$. Let X be a set disjoint from A with $|X| = b - a$.
- If $T_s \neq \emptyset$ then $T_s = \{x\}$ (by (i)); label x by X.
- Label each element $j \in T_2$ by $\bigcup_{i \in N_b(j)} A_i \cup X$ (this gives already an h-labeling of the projection of d on $S \cup T_s \cup T_2$ (by (vi))).
- Suppose that all $N_b(j)$ $(j \in T_1)$ are equal to, say, $\{i_0\}$, as in $(v1)$. Let Y_j $(j \in T_1)$ be pairwise disjoint sets that are disjoint from A and X and have cardinality a. Label $j \in T_1$ by $A_{i_0} \cup X \cup Y_j$. If all $N_b(j)$ $(j \in T_1)$ are distinct as in $(v2)$, then label $j \in T_1$ by $\bigcup_{i \in N_b(j)} A_i \cup X \cup Y$, where Y is a set disjoint from A and X with $|Y| = a$.
(In both cases, we have obtained an h-labeling of the projection of d on $S \cup T_s \cup T_2 \cup T_1$ (by (vii)).)
- Suppose that $T_0 \neq \emptyset$. Then, $T_2 = \emptyset$ by (iv). Let Z_k $(k \in T_0)$ be pairwise disjoint sets that are disjoint from all the sets constructed so far and have cardinality a.
If we are in case $(v1)$, then $|T_1| \leq 1$ or $(|T_1| \leq 2$ and $|T_0| = 1)$. (Indeed, if $|T_1|, |T_2| \geq 2$, then d contains the substructure from Figure 4.50 and, if $|T_1| \geq 3$, $|T_0| = 1$, then we have the substructure from Figure 4.49.) If $|T_1| = 1$, $T_1 = \{j\}$, label $k \in T_0$ by $X \cup Y_j \cup Z_k$. If $|T_1| = 2$, $T_1 = \{j, j'\}$, then label the unique element $k \in T_0$ by $X \cup Y_j \cup Y_{j'}$.
Else, we are in case $(v2)$. Then, label $k \in T_0$ by $X \cup Y \cup Z_k$.
In both cases, we have constructed an h-labeling of d. ∎

Observe that the exclusion of the distance from Figure 4.51 is used only for showing that $|T_s| \leq 1$, i.e., that at most one point is at distance b from all points of S. Consider the distance d_s on $s+2$ points which has the same configuration as in Figure 4.51 but with s nodes on the top level instead of a^2+a+3. Let $s(a,b)$ denote the largest integer s such that d_s is hypercube embeddable. Then, Proposition 4.43 remains valid if we exclude the distance $d_{s(a,b)+1}$ instead of excluding the distance d_{a^2+a+3} from Figure 4.51. Note that $2 \leq \frac{a}{2a-b} + 1 \leq s(a,b) \leq a^2+a+2$, with $s(a,b) = 2$ if $b < \frac{4}{3}a$ (use Proposition 4.36). This implies the following result, which is a direct extension of the result given in [Avi90] for the subcase $a = b = 1$.

PROPOSITION 4.52. *Let a, b be positive integers with b odd and $b < \frac{4}{3}a$. Let d be a distance on $n \geq 2a^2+2a+5$ points with range of values $\{2a, b, 2a+b\}$. The following assertions are equivalent.*
(i) d is hypercube embeddable.
(ii) d is ℓ_1-embeddable and satisfies (4.3).

(iii) d is hypermetric and satisfies (4.3).
(iv) d satisfies (4.3) *and the* $(2k+1)$*-gonal inequalities for* $2k+1 = 5, 7, 11, 8a-1$.
(v) d is a semimetric, d satisfies (4.3), *and d does not contain as substructure any of the distances from Figures* 4.34 *and* 4.44-4.50. ∎

4.3.4 Distances with values $2a, b, 2a+b$ (b odd, $2a < b$)

Let a, b be positive integers such that b is odd and $2a < b$. Let d be a distance on n points with range of values $\{2a, b, 2a+b\}$. We suppose that d satisfies the parity condition (4.3). Then, V_n is partitioned into $V_n = S \cup T$ with $d(i,j) = 2a$ for $(i,j) \in S^2 \cup T^2$, $d(i,j) \in \{b, b+2a\}$ for $(i,j) \in S \times T$, and $|T| \leq |S|$. Set

$$I := \{j \in T \mid d(i,j) = b + 2a \text{ for all } i \in S\},\ U := \{j \in T \mid d(i,j) = b \text{ for all } i \in S\},$$

and $M := T \setminus I \cup U$. For $j \in T$, set

$$N_b(j) := \{i \in S \mid d(i,j) = b\}.$$

Two distinct elements $j, j' \in M$ are said to be **twins** (resp. **pseudotwins**, **symmetric**) if $N_b(j) = N_b(j')$ (resp. $|N_b(j) \triangle N_b(j')| = 1$, $|N_b(j) \setminus N_b(j')| = |N_b(j') \setminus N_b(j)| = 1$). A subset $M' \subseteq M$ is called a **twin class** (resp. a **symmetric class**) if any two distinct elements of M' are twins (resp. symmetric).

PROPOSITION 4.53. *With the notation above, suppose d is a distance on* $n \geq 2a^2 + 2a + 5$ *points. Then, d is hypercube embeddable if and only if (i) or (ii) holds.*
(i) $M = \emptyset$ *and* $|U| \leq \frac{b}{a}$ *if* $|I| \geq 2$, $|U| \leq f(2a, a; a+b)$ *if* $|I| = 1$.
(ii) $M = T$, *any two elements of T are twins, pseudotwins, or symmetric, and*
- either $|N_b(j)| = v$ *for all* $j \in T$ *for some* $1 \leq v \leq \frac{b}{a} + 1$ *and T is a twin class or a symmetric class,*
- or $|N_b(j)| \in \{v, v+1\}$ *for all* $j \in T$ *for some* $1 \leq v \leq \frac{b}{a}$. *Set* $T' = \{j \in T : |N_b(j)| = v\}$ *and* $T'' = T \setminus T'$. *Then, either* $|T'| = 1$, T'' *is a symmetric class, or* T'' *is a twin class with* $|T''| \leq \frac{b}{a} - v + 1$); *or* T' *is a twin class with* $|T'| \geq 2$ *and* T'' *is a symmetric class; or* T' *is a symmetric class with* $|T'| = 2$ *and* T'' *is a twin class with* $|T''| \leq \frac{b}{a} - v + 1$.

We refer to [Lau93b] for the proof. Recall that $f(2a, a; a+b)$ denotes the maximum cardinality of a $(2a, a)$-intersecting system consisting of subsets of V_{a+b}. Hence, the condition $|U| \leq f(2a, a; a+b)$ occurring in Proposition 4.53 (i) is equivalent to the existence of a $(2a, a)$-intersecting system of cardinality $|U|$ on V_{a+b}. This is equivalent to the existence of a $(2a, a, |U|)$-design with $a+b$ blocks (recall Remark 3.10). Hence, by Theorem 3.3, such a design exists only if $|U| \leq a+b$. Therefore, its existence can be checked, e.g., by brute force enumeration.

Consider, for instance, the distance d from Figure 4.54. If $|S| \geq a^2 + a + 3$, then d is hypercube embeddable if and only if $|U| \leq f(2a, a; a+b)$, i.e., there exists a $(2a, a)$-intersecting system on V_{a+b} of cardinality $|U|$.

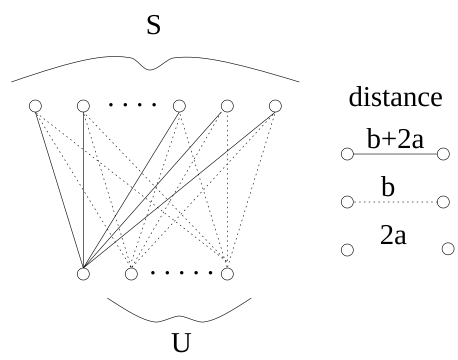


Figure 4.54

4.4 Metrics with restricted extremal graph

let d be a metric on V_n. Given distinct $i, j \in V_n$, the pair ij is said to be **extremal** for d if there does not exist $k \in V_n \setminus \{i, j\}$ such that $d(i,k) = d(i,j) + d(j,k)$ or $d(j,k) = d(i,j) + d(i,k)$. Then, the **extremal graph** of d is defined as the subgraph of K_n induced by the set of extremal edges of d.

The notion of extremal graph turns out to be useful when studying the metrics that can be decomposed as a nonnegative (integer) sum of cut semimetrics.

THEOREM 4.55. *Let d be a metric on V_n whose extremal graph is either K_4, or C_5, or a union of two stars. Then,*
(i) [Pap76] *d is ℓ_1-embeddable, i.e., $d \in \mathrm{CUT}_n$.*
(ii) [Kar85] *d is hypercube embeddable if and only if d satisfies the parity condition* (4.3).
(A graph is a union of two stars if its edges can be covered by two nodes.)

Note that it suffices to show Theorem 4.55 (ii), as it implies (i). The proof we present was given by Schrijver [Sch91]. It is much shorter than Karzanov's original proof, but it is nonconstructive. Karzanov's proof yields an algorithm permitting to construct a $\mathbb{Z}_+$-realization of d in $O(n^3)$ time (if one exists). Schrijver shows the following result, from which Theorem 4.55 will then follow easily.

THEOREM 4.56. *Let $G = (V, E)$ be a connected bipartite graph and, for $W \subseteq V$, let $H = (W, F)$ be a graph which is either K_4, C_5, or a union of two stars. Then, there exist pairwise edge disjoint cuts $\delta_G(S_1), \ldots, \delta_G(S_t)$ in G such that, for each $(r,s) \in F$, the number of cuts $\delta_G(S_h)$ $(1 \leq h \leq t)$ separating r and s is equal to the distance $d_G(r,s)$ from r to s in G.* (Here, the symbol $\delta_G(S)$ denotes the cut in G which consists of the edges of G having one endnode in S and the other endnode in $V \setminus S$.)

PROOF. Let G be a counterexample with smallest value of $|E|$. Then,

(4.57) for each $\emptyset \neq S \subset V$, there exist $(r,s) \in F$ and a path P connecting r and s in G such that $|P \setminus \delta_G(S)| \leq d_G(r,s) - 2$

(where P denotes the edge set of the path). Suppose S is a subset of V for which (4.57) does not hold. Then, for each $(r,s) \in F$, $|P \cap \delta_G(S)| = 1$ (resp. 0) for each shortest rs-path P if $\delta_G(S)$ separates (resp. does not separate) r and s. Let G' denote the connected bipartite graph obtained from G by contracting the edges of $\delta_G(S)$. Hence, for $(r,s)\in F$, $d_{G'}(r,s)=d_G(r,s)-1$ if $\delta_G(S)$ separates r,s and $d_{G'}(r,s)=d_G(r,s)$ otherwise. As G' has fewer edges than G, by Theorem 4.56, we can find paiwise edge disjoint cuts $\delta_{G'}(S'_1),\ldots,\delta_{G'}(S'_t)$ in G' such that $d_{G'}(r,s)$ is equal to the number of cuts $\delta_{G'}(S'_h)$ separating r and s. These t cuts yield t cuts $\delta_G(S_h)$ in G which, together with the cut $\delta_G(S)$, are pairwise disjoint and satisfy: for $(r,s)\in F$, the number of cuts $\delta_G(S_h),\delta_G(S)$ separating r and s is equal to $d_G(r,s)$. This contradicts our assumption that G is a counterexample to Theorem 4.56.

Claim 4.58. *For all $i\neq j\in V$, there exists $(r,s)\in F$ such that $\{i,j\}\cap\{r,s\}=\emptyset$ and $d_G(i,j)+d_G(r,s)\geq\max(d_G(i,r)+d_G(j,s),d_G(i,s)+d_G(j,r))$.*

Proof of Claim 4.58. Let $i\neq j\in V$. Set $X:=\{k\in V\mid d_G(i,j)=d_G(i,k)+d_G(j,k)\}$.

Suppose first that $X=V$. By (4.57) applied to $\{i\}$, we find $(r,s)\in F$ and a rs-path P such that $|P\setminus\delta_G(\{i\})|\leq d_G(r,s)-2$. Hence, P is a shortest rs-path and i is an internal node of P and, thus, $i\not\in\{r,s\}$. Using the fact that $X=V$, one obtains that $j\not\in\{r,s\}$ and $d_G(i,j)+d_G(r,s)=d_G(i,r)+d_G(j,r)+d_G(r,s)\geq d_G(r,i)+d_G(s,j)$; the other inequality of Claim 4.58 follows in the same way.

Suppose now that $X\neq V$. Let G' denote the graph obtained from G by contracting the edges of $\delta_G(X)$. By (4.57) applied to X, there exists $(r,s)\in F$ such that

$$d_{G'}(r,s)\leq d_G(r,s)-2.$$

Moreover,

$$(4.59)\qquad \begin{cases} d_{G'}(i,s)\geq d_G(i,s)-1,\ d_{G'}(r,j)\geq d_G(r,j)-1,\\ d_{G'}(j,s)\geq d_G(j,s)-1,\ d_{G'}(r,i)\geq d_G(r,i)-1.\end{cases}$$

We show that $d_{G'}(i,s)\geq d_G(i,s)-1$; the other inequalities of (4.59) can be proved in the same way. Let P be a path connecting i and s in G such that $|P\setminus\delta_G(X)|=d_{G'}(i,s)$ and with smallest value of $|P\cap\delta_G(X)|$. Suppose that $|P\cap\delta_G(X)|\geq 2$. Let P' denote the smallest subpath of P starting at i and such that $|P'\cap\delta_G(X)|=2$. Let k denote the other endnode of P', so $k\in X$, and set $P'':=P\setminus P'$. As P' is not contained in X, we have $d_G(i,k)\leq|P'|-1$ and, as G is bipartite, $d_G(i,k)\leq|P'|-2$. Let Q' be a shortest path from i to k in G. Then, $|P'|-2=d_{G'}(i,k)\leq|Q'\setminus\delta_G(X)|\leq|P'|-2-|Q'\cap\delta_G(X)|$, which implies $Q'\cap\delta_G(X)=\emptyset$ and $|Q'|=d_G(i,k)=|P'|-2$. Consider the path Q from i to s obtained by juxtaposing Q' and P''. Then, $|Q\setminus\delta_G(X)|=|P\setminus\delta_G(X)|$ and $|Q\cap\delta_G(X)|=|P\cap\delta_G(X)|-2$, contradicting our choice of P. Therefore, $|P\cap\delta_G(X)|\leq 1$. This shows that $d_{G'}(i,s)=|P\setminus\delta_G(X)|\geq|P|-1\geq d_G(i,s)-1$.

From $d_{G'}(r,s)\leq d_G(r,s)-2$ and (4.59), we deduce that $\{i,j\}\cap\{r,s\}=\emptyset$. Moreover, there exists a rs-path P in G such that $|P\setminus\delta_G(X)|=d_{G'}(r,s)$ and P contains a node $k\in X$. Hence,

$$\begin{aligned} d_G(r,s)+d_G(i,j) &\geq d_{G'}(r,s)+2+d_G(i,j)\\ &= d_{G'}(r,k)+d_{G'}(s,k)+2+d_G(i,k)+d_G(j,k)\\ &\geq d_{G'}(r,i)+d_{G'}(s,j)+2\geq d_G(r,i)+d_G(s,j)\end{aligned}$$

(using (4.59) for the last inequality). The other inequality from Claim 4.58 follows in the same way. ∎

From Claim 4.58, we deduce, in particular, that H is not a union of two stars. Hence, H is either K_4 or C_5.

Suppose first that $H = K_4$. From Claim 4.58, we obtain

$$d_G(i,j) + d_G(h,k) = d_G(i,h) + d_G(j,k) \text{ for all distinct } i,j,h,k \in W. \tag{4.60}$$

For $i \in W$, set $f(i) := \frac{1}{2}(d_G(i,h)+d_G(i,k)-d_G(h,k))$ where $h \neq k \in W\setminus\{i\}$; the definition does not depend on the choice of h,k by (4.60). Then, $d_G(i,j) = f(i)+f(j)$ for $i \neq j \in W$. Suppose $f(i) \neq 0$. By (4.57) applied to $\{i\}$, there exists $(r,s) \in F$ and a rs-path P such that $|P \setminus \delta_G(\{i\})| \leq d_G(r,s)-2$. Hence, P is a shortest rs-path passing through i. Thus, $|P| = d_G(r,s) = f(r)+f(s)$, and $|P| = d_G(i,r)+d_G(i,s) = f(r)+f(s)+2f(i)$, implying $f(i) = 0$. We obtain a contradiction.

Suppose now that $H = C_5$. Say, $W := \{r_1,r_2,r_3,r_4,r_5\}$ and $F := \{(r_i,r_{i+1}) \mid 1 \leq i \leq 5\}$, where the indices are taken modulo 5. Applying Claim 4.58 to r_i, r_{i+2}, we obtain that

$$d_G(r_i,r_{i+2}) + d_G(r_{i+3},r_{i+4}) \geq d_G(r_i,r_{i+3}) + d_G(r_{i+2},r_{i+4}),$$

$$d_G(r_i,r_{i+2}) + d_G(r_{i+3},r_{i+4}) \geq d_G(r_i,r_{i+4}) + d_G(r_{i+2},r_{i+3})$$

for $1 \leq i \leq 5$ (as (r_{i+3},r_{i+4}) is the only edge of C_5 disjoint from r_i and r_{i+2}). Adding up these ten inequalities, we obtain the same sum on both sides of the inequality sign. Hence, each of the above inequalities is, in fact, an equality. Hence, (4.60) holds again, yielding a contradiction as above. ∎

PROOF OF THEOREM 4.55. Let d be a integral metric on V_n satisfying the parity condition (4.3) and whose extremal graph $H := (W,F)$ is either K_4, or C_5, or a union of two stars. We show that d can decomposed as a nonnegative integer sum of cut semimetrics. Consider the complete graph K_n on V_n. We construct a connected bipartite graph G by subdividing the edges of K_n in the following way: For all distinct $i,j \in V_n$, replace the edge ij by a path P_{ij} consisting of $d(i,j)$ edges. The fact that G is bipartite follows from the parity condition. By Theorem 4.56, there exist edge disjoint cuts $\delta_G(S_h)$ $(1 \leq h \leq t)$ in G such that, for each $(r,s) \in F$, $d_G(r,s)$ is equal to the number of cuts $\delta_G(S_h)$ separating r and s. Setting $T_h := S_h \cap V_n$, we obtain that, for each $(r,s) \in F$,

$$d(r,s) = d_G(r,s) = \sum_{1 \leq h \leq k} \delta(T_h)(r,s). \tag{4.61}$$

Moreover, for all $i \neq j \in V_n$, we have

$$d(i,j) \geq \sum_{1 \leq h \leq t} \delta(T_h)(i,j). \tag{4.62}$$

Indeeed, the number of cuts $\delta_G(S_h)$ separating r and s is less than or equal to the number of cuts $\delta_G(S_h)$ intersecting the path P_{ij} which, in turn, is less than or equal to the length $d(i,j)$ of P_{ij} since the cuts $\delta_G(S_h)$ are pairwise edge disjoint. In fact, equality holds in (4.62). To see it, let $i \neq j \in V_n$ and let $P := (i_0,\ldots,i_k)$ be a path in K_n which contains the edge (i,j) and is a geodesic for d (i.e., P is a shortest - with respect to the length function d - path between its extremities i_0 and i_k, that is, $d(i_0,i_k) = \sum_{0 \leq m \leq k-1} d(i_m,i_{m+1})$). Choose such a path P having maximum number of edges. Then, the pair (i_0,i_k) is extremal for d. For, if not, there exists $x \in V_n \setminus \{i_0,i_k\}$ such that, e.g., $d(i_0,x) = d(i_0,i_k) + d(x,i_k)$ and, then, $(i_0, \ldots, i_k, x)$ is a geodesic containing (i,j) and longer than P. Then, using (4.62), we have

$$d(i_0, i_k) = \sum_{m=0}^{k-1} d(i_m, i_{m+1}) \geq \sum_{m=0}^{k-1} \sum_{h=1}^{t} \delta(T_h)(i_m, i_{m+1}).$$

But,

$$\sum_{m=0}^{k-1} \sum_{h=1}^{t} \delta(T_h)(i_m, i_{m+1}) = \sum_{h=1}^{t} \sum_{m=0}^{k-1} \delta(T_h)(i_m, i_{m+1}) \geq \sum_{h=1}^{t} \delta(T_h)(i_0, i_k) = d(i_0, i_k),$$

where the last equality follows from (4.61) as the edge (i_0, i_k) belongs to F. Therefore, equality holds in (4.62) for each of the edges (i_m, i_{m+1}) of P and, in particular, for the edge (i,j). This shows that equality holds in (4.62) for all $i \neq j \in V_n$. Therefore, $d = \sum_{1 \leq h \leq t} \delta(T_h)$, showing that d is hypercube embeddable. ∎

Remark 4.63. One can check that a graph H with no isolated node is K_4, C_5, or a union of two stars if and only if H does not contain as a subgraph the two graphs from Figure 4.64. The exclusion of these two graphs is necessary for the validity of Theorem 4.55. Indeeed, let d_1 be the path metric of the complete bipartite graph $K_{2,3}$; then, d is not hypercube embeddable (as d does not satisfy the 5-gonal inequality) and its extremal graph is the graph (a) from Figure 4.64. Let d_2 be the path metric of the graph $K_{3,3} \setminus e$; then its extremal graph is the graph (b) from Figure 4.64 and d_2 is not hypercube embeddable (as it contains d_1 as a subdistance). (In fact, both d_1 and d_2 lie on extreme rays of the metric cone.) ∎

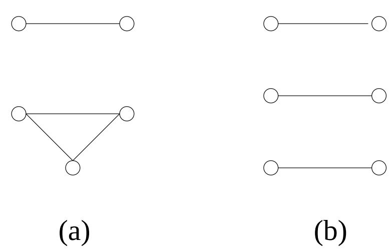


Figure 4.64

5 Cut lattices, quasi h-distances and Hilbert bases

We consider in this section several questions related to the notion of hypercube embedding. A possible way of relaxing this notion is to look for *integer* combinations rather than *nonnegative integer* combinations of cut semimetrics. In other words, one considers the lattice $\mathcal{L}_n$ generated by all cut semimetrics on V_n. We recall in Section 5.1 the characterization of $\mathcal{L}_n$, which is an easy result, namely, $\mathcal{L}_n$ consists of the integer distances satisfying the parity condition. We also present the characterization of two sublattices of $\mathcal{L}_n$, namely, of the sublattice generated by all even cut semimetrics and of the sublattice generated by all k-uniform cut semimetrics.

Clearly, for a distance d on V_n,

(5.1) $$d \text{ is hypercube embeddable } \Longrightarrow d \in \mathrm{CUT}_n \cap \mathcal{L}_n.$$

We consider in Section 5.2 quasi h-distances, which are the distances d that belong to $\mathrm{CUT}_n \cap \mathcal{L}_n$ but are not hypercube embeddable. As was mentioned in Theorem 4.5, the implication (5.1) is an equivalence for any distance d on $n \leq 5$ points. This fact can be reformulated as saying that, for $n \leq 5$, the family of cut semimetrics on V_n is a Hilbert base. We consider in Section 5.3 the more general question of characterizing the graphs whose family of cuts is a Hilbert base.

5.1 Cut lattices

Set

$$\mathcal{L}_n := \{ \sum_{S \subseteq V_n} \lambda_S \delta(S) \mid \lambda_S \in \mathbb{Z} \text{ for all } S \subseteq V_n \};$$

$\mathcal{L}_n$ is called the **cut lattice**. The next result gives a characterization of $\mathcal{L}_n$.

Proposition 5.2. [Ass82] *Let $d \in \mathbb{Z}^{E_n}$. Then, $d \in \mathcal{L}_n$ if and only if d satisfies the parity condition* (4.3).

Proof. The parity condition is clearly a necessary condition for membership in $\mathcal{L}_n$. Conversely, suppose d is integral and satisfies the parity condition. Then, V_n can be partitioned into $V_n = S \cup T$ in such a way that $d(i,j)$ is odd if $i \in S, j \in T$ and $d(i,j)$ is even otherwise. Set $d' := d + \delta(S)$. Then, all components of d' are even. As $d' = \sum_{1 \leq i<j \leq n} \frac{d'(i,j)}{2}(\delta(\{i\})+\delta(\{j\})-\delta(\{i,j\}))$, we deduce that $d' \in \mathcal{L}_n$ and, thus, $d = d' - \delta(S)$ belongs to $\mathcal{L}_n$ too. ∎

Complete characterizations are also known for several sublattices of $\mathcal{L}_n$. Given an integer k, the **k-uniform cut lattice** $\mathcal{L}_n^k$ is defined as the sublattice of $\mathcal{L}_n$ generated by the cut semimetrics $\delta(S)$ for $S \subseteq V_n$ with $|S| \in \{k, n-k\}$. The following characterization of the k-uniform cut lattice is given in [DL92], based on a result of Wilson [Wil73].

Proposition 5.3. *Let k be an integer such that $2 \leq k \leq n$ and $k \neq \frac{n}{2}$ and let $d \in \mathbb{Z}^{E_n}$. Then, d belongs to the k-uniform cut lattice $\mathcal{L}_n^k$ if and only if d satisfies (i),(ii),(iii):*
(i) $\sum_{1 \leq i<j \leq n} d(i,j) \equiv 0 \ (\mathit{mod}\ k(n-k))$,
(ii) $D_i := \frac{1}{n-2k}\left(\sum_{1 \leq j \leq n, j \neq i} d(i,j) - \frac{1}{n-k}\sum_{1 \leq r<s \leq n} d(r,s)\right) \in \mathbb{Z}$ *for all* $i \in V_n$,
(iii) $D_i + D_j + d(i,j) \equiv 0 \ (\mathit{mod}\ 2)$ *for all* $i,j \in V_n$.

In the case $k = \lfloor \frac{n}{2} \rfloor$, we have the following result.

Proposition 5.4. *Let $d \in \mathbb{Z}^{E_n}$.*
(i) If $n = 2k+1$, then $d \in \mathcal{L}_n^k$ if and only if $\sum_{1 \leq i<j \leq n} d(i,j) \equiv 0 \ (\mathit{mod}\ k(n-k))$.
(ii) If $n = 2k$, then $d \in \mathcal{L}_n^k$ if and only if (iia),(iib) hold:
(iia) $\sum_{1 \leq r<s \leq n} d(r,s) = k(\sum_{1 \leq j \leq n, j \neq i} d(i,j))$ *for each* $1 \leq i \leq n$,

(iib) $\sum_{1\leq r<s\leq n} d(r,s) \equiv 0 \ (mod\ k^2)$.

PROOF. For (i), observe that the conditions $(ii),(iii)$ from Proposition 5.3 are implied by the condition (i). The conditions $(iia),(iib)$ are clearly necessary for membership in $\mathcal{L}_n^k$. Conversely, suppose that d satisfies $(iia),(iib)$ and let d' denote its projection on the set $\{1,\ldots,n-1\}$. From (iia), we obtain

$$\sum_{1\leq r<s\leq n-1} d'(r,s) = (k-1)\sum_{1\leq i\leq n-1} d(i,n). \tag{5.5}$$

This implies that $\sum_{1\leq r<s\leq n-1} d'(r,s) \equiv 0 \ (\text{mod } k(k-1))$ since $\sum_{1\leq i\leq n-1} d(i,n) \equiv 0 \ (\text{mod } k)$ by $(iia,)(iib)$. Using (i), we deduce that $d' \in \mathcal{L}_{n-1}^k$. Hence, $d' = \sum_{S\subseteq\{1,\ldots,n-1\},|S|=k} \lambda_S\delta(S)$ with $\lambda_S \in \mathbb{Z}$ for all S. We show that $d = \sum_S \lambda_S\delta(S)$. As $\sum_{1\leq r<s\leq n-1} d'(r,s) = k(k-1)(\sum_S \lambda_S)$, (5.5) yields: $\sum_{1\leq i\leq n-1} d(i,n) = k(\sum_S \lambda_S)$. Then, by (iia), $\sum_{1\leq r<s\leq n} d(r,s) = k^2(\sum_S \lambda_S)$ and $\sum_{1\leq j\leq n, j\neq i} d(i,j) = k(\sum_S \lambda_S)$ for each $i = 1,\ldots,n$. We compute, for instance, $d(1,n)$. The above relations yield: $d(1,n) = k(\sum_S \lambda_S) - \sum_{2\leq j\leq n-1} d(1,j)$. Using the value of $d(1,j) = d'(1,j)$ given by the decomposition of d', we obtain that $d(1,n) = \sum_{S|1\in S} \lambda_S$. This shows that $d = \sum_S \lambda_S\delta(S)$, i.e., that $d \in \mathcal{L}_n^k$. ∎

Suppose n is even. Then, the **even cut lattice** $\mathcal{L}_n^{ev}$ is defined as the sublattice of $\mathcal{L}_n$ generated by the cut semimetrics $\delta(S)$ for $S \subseteq V_n$ with $|S|$ even. Similarly, the **odd cut lattice** $\mathcal{L}_n^{od}$ is the lattice generated by the cut semimetrics $\delta(S)$ for $S \subseteq V_n$ with $|S|$ odd. We give a characterization of the even cut lattice.

PROPOSITION 5.6. [DLP92] *Let $n \geq 6$ be an even integer and let $d \in \mathbb{Z}^{E_n}$. Then, d belongs to the even cut lattice $\mathcal{L}_n^{ev}$ if and only if d satisfies the parity condition* (4.3) *and (i),(ii):*
(i) $\sum_{1\leq i<j\leq n} d(i,j) \equiv 0 \ (mod\ 4)$,
(ii) $\sum_{i<j, i,j\in V_n\setminus\{k\}} d(i,j) - \sum_{i\in V_n\setminus\{k\}} d(i,k) \equiv 0 \ (mod\ 8)$ *for all* $k \in V_n$ *if* $n \equiv 0 (mod\ 4)$, *and* $d(h,k) + \sum_{i<j, i,j\in V_n\setminus\{h,k\}} d(i,j) - \sum_{i\in V_n\setminus\{h,k\}} (d(i,h) + d(i,k)) \equiv 0 \ (mod\ 8)$ *for all* $h \neq k \in V_n$ *if* $n \equiv 2 (mod\ 4)$. ∎

A characterization of the odd cut lattice is known only in the case $n = 6$; then, $\mathcal{L}_6^{od}$ is the lattice in $\mathbb{R}^{15}$ generated by the 16 cut semimetrics $\delta(\{i\})$ $(1 \leq i \leq 6)$ and $\delta(\{1,i,j\})$ $(2 \leq i < j \leq 6)$. We need the following notation. Given distinct $a,b,c \in V_6$, let $v^{a,bc} \in \mathbb{R}^{E_6}$ be the vector defined by

$$\begin{cases} v_{ab}^{a,bc} = v_{ac}^{a,bc} = 1, \ v_{bc}^{a,bc} = 2, & \\ v_{ij}^{a,bc} = 2 & \text{for } i \neq j \in V_6 \setminus \{a,b,c\}, \\ v_{ai}^{a,bc} = -2, \ v_{bi}^{a,bc} = v_{ci}^{a,bc} = -1 & \text{for } i \in V_6 \setminus \{a,b,c\}. \end{cases}$$

Consider the conditions:

$$(v^{a,bc})^T x \leq 0 \ \text{ for all distinct } a,b,c \in V_6, \tag{5.7}$$

$$(v^{a,bc})^T x \equiv 0 \ (mod\ 4) \text{ for all distinct } a,b,c \in V_6, \tag{5.8}$$

$$(5.9) \qquad (v^{1,bc})^T x - (v^{1,b'c'})^T x \equiv 0 \ (mod\ 12) \text{ for } 2 \leq b < c \leq 6,\ 2 \leq b' < c' \leq 6.$$

The next result gives the characterization of the odd cut lattice and also of the cone and integer cone generated by the odd cut semimetrics on V_6. As a consequence, it shows that the family of odd cut semimetrics on V_6 is a Hilbert base.

PROPOSITION 5.10. [DL93a] *(i) Let* $d \in \mathbb{R}_+^{E_6}$. *Then,* $d \in \{ \sum_{1 \leq i \leq 6} \lambda_i \delta(\{i\}) + \sum_{2 \leq i < j \leq 6} \lambda_{ij} \delta(\{1,i,j\}) \mid \lambda_i, \lambda_{ij} \geq 0$ *for all* $i,j \in V_6\}$ *if and only if* d *satisfies* (5.7).
(ii) Let $d \in \mathbb{Z}^{E_6}$. *Then,* $d \in \mathcal{L}_6^{od}$ *if and only if* d *satisfies* (5.8),(5.9).
(iii) Let $d \in \mathbb{Z}_+^{E_6}$. *Then,* $d \in \{ \sum_{1 \leq i \leq 6} \lambda_i \delta(\{i\}) + \sum_{2 \leq i < j \leq 6} \lambda_{ij} \delta(\{1,i,j\}) \mid \lambda_i, \lambda_{ij} \in \mathbb{Z}_+$ *for all* $i,j \in V_6\}$ *if and only if* d *satisfies* (5.7),(5.8),(5.9). ∎

5.2 Quasi h-distances

Let d be a distance on V_n. Then, d is called a **quasi h-distance** if $d \in \mathrm{CUT}_n \cap \mathcal{L}_n$ and d is not hypercube embeddable. In other words, d can be decomposed both as a *nonnegative* combination of cut semimetrics and as an *integer* combination of cut semimetrics, but not as a *nonnegative integer* combination of cut semimetrics. The smallest integer η such that ηd is hypercube embeddable is called the minimum scale of d and is denoted by $\eta(d)$.

As stated in Theorem 4.5, there are no quasi h-distances on $n \leq 5$ points. There are several ways of constructing quasi h-distances on $n \geq 6$ points.

Quasi h-distances can be constructed, for instance, using the antipodal extension operation. Let d be a distance on V_n and let $\alpha \in \mathbb{R}_+$. Then, its **antipodal extension** $ant_\alpha(d)$ is the distance on V_{n+1} defined by $ant_\alpha(d)(1,n+1) = \alpha$, $ant_\alpha(d)(i,n+1) = \alpha - d(1,i)$ for $1 \leq i \leq n$, and $ant_\alpha(d)(i,j) = d(i,j)$ for $1 \leq i < j \leq n$. One can check (see [DL92]) that, if d is hypercube embeddable and $\alpha \in \mathbb{Z}_+$ such that $s_{\ell_1}(d) \leq \alpha < s_h(d)$, then $ant_\alpha(d)$ is a quasi h-distance (see (1.4) and (1.5) for the definition of $s_h(d), s_{\ell_1}(d)$). As an example, for $n \geq 6$, the distance

$$d_n^* := 2d(K_n \backslash e) = ant_4(2\mathbb{1}_{n-1})$$

(taking value 2 on all pairs except value 4 on the pair of nodes of the edge e) is a quasi h-distance.

The gate extension operation permits also to construct quasi h-distances. If d is a distance on V_n and $\alpha \in \mathbb{R}_+$, its **gate extension** $gat_\alpha(d)$ is the distance on V_{n+1} defined by $gat_\alpha(d)(1,n+1) = \alpha$, $gat_\alpha(d)(i,n+1) = \alpha + d(1,i)$ for $1 \leq i \leq n$, and $gat_\alpha(d)(i,j) = d(i,j)$ for $1 \leq i < j \leq n$. Then, for $\alpha \in \mathbb{Z}_+$, $gat_\alpha(d)$ is a quasi h-distance if and only if d is a quasi h-distance. This implies, in particular, that there is an infinity of quasi h-distances on n points for all $n \geq 7$. Indeed, all gate extensions of $d_6^* = 2d(K_6 \backslash e)$ are quasi h-distances.

Other examples of quasi h-distances on 6 points can be constructed, for instance, as follows.

LEMMA 5.11. [Lab]*Let e be an edge of K_6 and let v be a node of K_6 which is not adjacent* *to e. Then, the distance $2d(K_6\backslash e)+m\delta(\{v\})$ is a quasi h-distance for each integer $m \geq 0$.*

PROOF. Suppose K_6 is the complete graph on $V_6 = \{1,\ldots,6\}$, e is the edge $(1,6)$ and v is the node 2. Set $d := 2d(K_6\backslash e) + m\delta(\{v\})$. Let $d = \sum_S \alpha_S \delta(S)$ be a $\mathbb{Z}_+$-realization of d, with $\alpha_S \in \mathbb{Z}_+$. As d satisfies the triangle equality: $d_{16} = d_{1i} + d_{i6}$ for $i = 3,4,5$, we deduce that $\alpha_S = 0$ if S is one of the sets: 3, 4, 5, 16, 23, 24, 25, 34, 35, 45, 126, 136, 146, and 156. Hence, $d = \sum_{S\in\mathcal{S}} \alpha_S\delta(S)$, where $\mathcal{S}$ may contain the sets: 1, 2, 6, 12, 13, 14, 15, 26, 36, 46, 56, 123, 124, 125, 134, 135, 145. By computing d_{12}, d_{26}, and d_{16}, we obtain, respectively,

$$m + 2 = \alpha_1 + \alpha_2 + \alpha_{13} + \alpha_{14} + \alpha_{15} + \alpha_{26} + \alpha_{134} + \alpha_{135} + \alpha_{145},$$

$$m + 2 = \alpha_2 + \alpha_6 + \alpha_{12} + \alpha_{36} + \alpha_{46} + \alpha_{56} + \alpha_{123} + \alpha_{124} + \alpha_{125},$$

$$4 = \sum_{S\in\mathcal{S}} \alpha_S - \alpha_2.$$

Adding the first two relations and substracting the third one, we obtain that $\alpha_2 = m$. Therefore, if d is hypercube embeddable, then so is $d - m\delta(\{2\})$. This contradicts the fact that $2d(K_6\backslash e)$ is a quasi h-distance. ∎

Hence, there is also an infinity of quasi h-distances on 6 points. However, we have the following conjecture:

CONJECTURE 5.12. *Every quasi h-distance on V_6 is a nonnegative integer sum of cuts and of the distances $2d(K_6\backslash e)$, for e edge of K_6.*

In fact, if this conjecture holds, then the only quasi h-distances on V_6 are those constructed in Lemma 5.11.

PROPOSITION 5.13. [Lab] *If Conjecture* 5.12 *holds, then the only quasi h-distances on V_6 are of those of the form: $2d(K_6\backslash e) + m\delta(\{v\})$, where e is an edge of K_6, v is a node of K_6 not adjacent to e, and $m \in \mathbb{Z}_+$.*

The proof uses the identities (a)-(i) below, which show that all pertubations of $2d(K_6\backslash e)$ (obtained by adding a cut semimetric), other than the one considered in Lemma 5.11, are hypercube embeddable. For $1 \leq i < j \leq n$, let e_{ij} denote the edge ij of K_6. Then,

(a) $2d(K_6\backslash e_{12}) + \delta(\{1\}) = \delta(\{2\}) + \delta(\{1,3\}) + \delta(\{1,4\}) + \delta(\{1,5\}) + \delta(\{1,6\})$,

(b) $2d(K_6\backslash e_{12}) + \delta(\{1,2\}) = 2\delta(\{1\}) + 2\delta(\{2\}) + \delta(\{3\}) + \delta(\{4\}) + \delta(\{5\}) + \delta(\{6\})$,

(c) $2d(K_6\backslash e_{12}) + \delta(\{1,3\}) = \delta(\{2\}) + \delta(\{1,3\}) + \delta(\{3,4,5\}) + \delta(\{3,4,6\}) + \delta(\{4,5,6\})$,

(d) $2d(K_6\backslash e_{12}) + \delta(\{3,4\}) = \delta(\{1\}) + \delta(\{3\}) + \delta(\{4\}) + \delta(\{2,5\}) + \delta(\{2,6\}) + \delta(\{2,3,4\})$,

(e) $2d(K_6\backslash e_{12}) + \delta(\{1,2,3\}) = \delta(\{1\}) + \delta(\{2\}) + \delta(\{4\}) + \delta(\{5\}) + \delta(\{6\}) + \delta(\{1,3\}) + \delta(\{2,3\})$,

(f) $2d(K_6\backslash e_{12}) + \delta(\{1,3,4\}) = \delta(\{1,3\}) + \delta(\{1,4\}) + \delta(\{2,5\}) + \delta(\{2,6\}) + \delta(\{1,5,6\})$,

(g) $2d(K_6\backslash e_{12}) + 2d(K_6\backslash e_{23}) = \delta(\{1\}) + \delta(\{2,3\}) + \delta(\{2,4\}) + \delta(\{2,5\}) + \delta(\{3,6\}) + \delta(\{1,2,6\}) + \delta(\{1,3,4\}) + \delta(\{1,3,5\})$,

(h) $$2d(K_6\backslash e_{12}) + 2d(K_6\backslash e_{34}) = \begin{array}{l}\delta(\{1\}) + \delta(\{2,3\}) + \delta(\{2,4\}) + \delta(\{3,5\}) + \delta(\{4,6\}) \\ +\delta(\{1,3,4\}) + \delta(\{1,3,6\}) + \delta(\{1,4,5\}),\end{array}$$

(i) $$2d(K_6\backslash e_{12}) + \delta(\{3\}) + \delta(\{4\}) = \begin{array}{l}\delta(\{1,3\}) + \delta(\{2,4\}) + \delta(\{3,4\}) + \delta(\{1,4,5\}) \\ +\delta(\{1,4,6\}).\end{array}$$

PROOF OF PROPOSITION 5.13. Let d be a quasi h-distance on V_6. Then, d can be written as
$$d = \sum_S \alpha_S \delta(S) + \sum_{1\le i<j\le 6} \beta_{ij} 2d(K_6\backslash e_{ij})$$
with α_S, $\beta_{ij} \in \mathbb{Z}_+$, as Conjecture 5.12 holds by assumption. We can suppose that $\beta_{ij} \in \{0,1\}$ for all i,j, because $4d(K_6\backslash e_{ij})$ is hypercube embeddable. Using (g) and (h), we can rewrite d as
$$d = \sum_S \alpha'_S \delta(S) + 2d(K_6\backslash e),$$
where $\alpha'_S \in \mathbb{Z}_+$ and, for instance, e is the edge $(1,2)$. From relations (a)-(f), we deduce that $\alpha_S = 0$ if $S = \{1\}$, or $\{2\}$, or if $|S| = 2$, or 3. Therefore, using relation (i), we obtain that $d = 2d(K_6\backslash e_{12}) + m\delta(\{i\})$, where $i \in \{3,4,5,6\}$ and $m \in \mathbb{Z}_+$. ∎

As we just saw, there is an infinity of quasi h-distances on V_n, for any $n \geq 6$. However, the next result shows the existence of an integer η_n which is a common scale for all quasi h-distances on V_n.

PROPOSITION 5.14. [DG94] *There exists an integer η_n such that $\eta_n d$ is hypercube embeddable for each quasi h-distance d on V_n.*

PROOF. The set $Y_n := \mathcal{L}_n \cap \{\sum_S \lambda_S \delta(S) \mid 0 \leq \lambda_S \leq 1 \text{ for all } S\}$ is finite. Let η_n denote the lowest common multiple of the minimum scales $\eta(d)$ for $d \in Y_n$. Hence, $\eta_n d$ is hypercube embeddable for each $d \in Y_n$. Let d be a quasi h-distance on V_n, $d = \sum_S \alpha_S \delta(S)$ with $\alpha_S \geq 0$. Set $d_1 := \sum_S \lfloor \alpha_S \rfloor \delta(S)$ and $d_2 := d - d_1 = \sum_S (\alpha_S - \lfloor \alpha_S \rfloor)\delta(S)$. Then, d_1 is hypercube embeddable, and $d_2 \in \mathcal{L}_n$ since $d, d_1 \in \mathcal{L}_n$. Therefore, $d_2 \in Y_n$ and, hence, $\eta_n d_2$ is hypercube embeddable. This implies that $\eta_n d = \eta_n d_1 + \eta_n d_2$ is hypercube embeddable. ∎

For the class of graphic distances, the following results are shown in [Shp93]: The minimum scale of the path metric of a connected graph on n nodes is equal to 1, or is an even integer less than or equal to $n-2$. Moreover, for an ℓ_1-rigid graph, the minimum scale is equal to 1 or 2.

Much of the treatment of Section 3 can be reformulated in terms of minimum scales. Indeed, consider the metric $d_n := ant_2(\mathbb{1}_n)$ (this is the path metric of the graph $K_{n+1}\backslash e$)). Then, $2td_n = 2t\ ant_2(\mathbb{1}_n) = ant_{4t}(2t\mathbb{1}_n)$ is hypercube embeddable if and only if $4t \geq s_h(2t\mathbb{1}_n)$. Therefore, the minimum scale $\eta(d_n)$ can be expressed as
$$\eta(d_n) = 2\min(t \in \mathbb{Z}_+ \mid 4t \geq s_h(2t\mathbb{1}_n)).$$
In particular, Theorem 3.20 (i) implies:
(a) $\eta(d_{4t}) \geq 2t$ with equality if and only if there exists a Hadamard matrix of order $4t$.

Compare (a) with the next statement (b), which follows from Theorems 2.3 and 2.4.
(b) $\eta^1(\mathbb{1}_{t^2+t+2}) \geq 2t$ with equality if and only if there exists a projective plane of order t, where, for a hypercube embeddable distance d, $\eta^1(d)$ denotes the smallest integer λ (if any) such that λd is not h-rigid, i.e., has at least two distinct $\mathbb{Z}_+$-realizations.

Some quasi h-distances can also be constructed using the spherical extension operation. If d is a distance on V_n and $t \in \mathbb{R}_+$, its **spherical t-extension** is the distance $sph_t(d)$ on V_{n+1} defined by $sph_t(d)(i, n+1) = t$ for all $1 \leq i \leq n$, and $sph_t(d)(i,j) = d(i,j)$ for all $1 \leq i < j \leq n$. If $d \in \mathrm{CUT}_n$ and $2t \geq s_{\ell_1}(d)$, then $sph_t(d) \in \mathrm{CUT}_{n+1}$. As a first example, consider the distance

$$\theta_n^t := ant_{2t}(sph_t(2\mathbb{1}_{n-2})),$$

where n, t are positive integers, i.e., θ_n^t is the distance on V_n defined by

$$\begin{cases} \theta_n^t(n-1, n) & = 2t, \\ \theta_n^t(i, n-1) = \theta_n^t(i,n) & = t \quad \text{for } 1 \leq i \leq n-2, \\ \theta_n^t(i,j) & = 2 \quad \text{for } 1 \leq i < j \leq n-2. \end{cases}$$

Clearly, θ_n^t admits the following decompositions:

$$\theta_n^t = \sum_{1 \leq i \leq n-2} \delta(\{i,n\}) + (t-1)\delta(\{n-1\}) + (t-n+3)\delta(\{n\}),$$

$$\theta_n^t = \frac{1}{2}\left(\sum_{1 \leq i \leq n-2} (\delta(\{i, n-1\}) + \delta(\{i,n\}))\right) + (2t-n+2)\left(\delta(\{n-1\}) + \delta(\{n\})\right).$$

This shows that θ_n^t is hypercube embeddable if $t \geq n-3$ and that $2\theta_n^t$ is hypercube embeddable if $t \geq \frac{n-2}{2}$.

LEMMA 5.15. [DG94] *Let $t \geq 1$ be an integer.*
(i) If $n \neq 6$, then θ_n^t is hypercube embeddable if and only if $t \geq n-3$.
(ii) For $n \geq 6$, if $\lceil \frac{n-2}{2} \rceil \leq t \leq n-4$, then θ_n^t is a quasi h-distance.

PROOF. (i) Suppose that θ_n^t is hypercube embeddable. Then, in any hypercube embedding of θ_n^t, we can suppose that each point $i \in \{1, \ldots, n-2\}$ is labeled by the singleton $\{i\}$ (as the metric $2\mathbb{1}_{n-2}$ is h-rigid if $n \neq 6$). This implies that one of the points $n-1, n$ should be labeled by a set A containing $\{1, \ldots, n-2\}$ and, thus, $|A| - 1 = t \geq n-3$.
(ii) If $t \geq \lceil \frac{n-2}{2} \rceil$, then θ_n^t is ℓ_1-embeddable. Hence, if $n \neq 6$ and $\lceil \frac{n-2}{2} \rceil \leq t \leq n-4$, then θ_n^t is a quasi h-distance. If $n = 6$ and $t = 2$, then θ_n coincides with the distance d_6^*, which is known to be a quasi h-distance. ∎

Given $n \geq 6$, let μ_n denote the distance on V_n defined by

$$\mu_n := \delta(\{1\}) + \delta(\{2\}) + \sum_{3 \leq i < j \leq n-1} \delta(\{1,2,i,j\}), \text{ i.e.,}$$

$$\begin{cases} \mu_n(1,2) & = 2, \\ \mu_n(1,n) = \mu_n(2,n) & = 1 + \binom{n-3}{2}, \\ \mu_n(1,i) = \mu_n(2,i) & = 1 + \binom{n-4}{2} \quad \text{for } 3 \leq i \leq n-1, \\ \mu_n(i,n) & = n-4 \quad \text{for } 3 \leq i \leq n-1, \\ \mu_n(i,j) & = 2(n-5) \quad \text{for } 3 \leq i < j \leq n. \end{cases}$$

For instance, for $n = 6$, μ_6 coincides with the path metric of the graph $K_6\backslash P$, where $P := (1,6,2)$ is a path on three nodes.

LEMMA 5.16. [DG94] *Let t, n be integers such that $n \geq 6$, $n \equiv 2$ (mod 4), and $2t \geq 2 + \binom{n-3}{2}$. Then, $sph_t(\mu_n)$ is a quasi h-distance.*

PROOF. It is easy to see that the condition $n \equiv 2$ $(mod\ 4)$ ensures that all components of μ_n are even integers, which implies that $sph_t(\mu_n) \in \mathcal{L}_{n+1}$. Let F denote the face of the cone CUT_n defined by the hypermetric inequality $Q(b)^T x := \sum_{1 \leq i < j \leq n} b_i b_j x_{ij} \leq 0$, where $b := (1, 1, -1, \ldots, -1, n-4) \in \mathbb{R}^n$ (with $n-3$ components -1). Set

$$\mathcal{S} := \{1, 2, 1i, 2i, 12i(3 \leq i \leq n-1), 12ij(3 \leq i < j \leq n-1)\},$$

(where we denote the sets $\{1\}, \{1, i\}$ by the strings $1, 1i$, etc.). The nonzero cut semimetrics satisfying the equation $Q(b)^T x = 0$ are $\delta(S)$ for $S \in \mathcal{S}$, which are linearly independent. Hence, the face F is a simplex face of CUT_n. As the distance μ_n lies on F, we deduce that μ_n is ℓ_1-rigid and $s_{\ell_1}(\mu_n) = 2 + \binom{n-3}{2}$. Let G denote the face of the cone CUT_{n+1} defined by the hypermetric inequality $Q(b,0)^T x \leq 0$; the nonzero cut semimetrics lying on G are $\delta(S)$, $\delta(S \cup \{n+1\})$ for $S \in \mathcal{S}$ and $\delta(\{n+1\})$. As $2t \geq s_{\ell_1}(\mu_n)$, $sph_t(\mu_n)$ is ℓ_1-embeddable and, in fact, $sph_t(\mu_n)$ lies on the face G. Suppose that $sph_t(d)$ is hypercube embeddable. Then, there exist nonnegative integers $\gamma, \alpha_S, \beta_S$ $(S \in \mathcal{S})$ such that

$$sph_t(\mu_n) = \gamma\delta(\{n+1\}) + \sum_{S \in \mathcal{S}} \alpha_S \delta(S) + \beta_S \delta(S \cup \{n+1\}).$$

Then, $\sum_{S \in \mathcal{S}} (\alpha_S + \beta_S)\delta(S) = d$, which implies that $\alpha_S = \beta_S = 0$ if S is not one of the sets $\{1\}, \{2\}, \{1,2,i,j\}$, and

$$\begin{cases} \alpha_i + \beta_i & = 1 \quad \text{for } i = 1, 2, \\ \alpha_{ij} + \beta_{ij} & = 1 \quad \text{for } 3 \leq i < j \leq n-1. \end{cases}$$

(setting $\alpha_{ij} = \alpha_{12ij}, \beta_{ij} = \beta_{12ij}$). Looking at the component of $sph_t(\mu_n)$ indexed by the pairs $(1, n+1)$ and $(2, n+1)$, we obtain: $\alpha_1 + \beta_2 + \sum_{i,j} \alpha_{ij} + \gamma = t$, $\alpha_2 + \beta_1 + \sum_{i,j} \alpha_{ij} + \gamma = t$, which implies

$$\alpha_1 = \alpha_2, \beta_1 = \beta_2, \gamma = t - \sum_{i,j} \alpha_{ij} - 1.$$

Looking at the component indexed by $(i, n+1)$ $(3 \leq i \leq n-1)$, we obtain: $\sum_j \alpha_{ij} + \beta_1 + \beta_2 + \sum_{i,j} \beta_{ij} - \sum_j \beta_{ij} + \gamma = t$. Therefore, $2\sum_j \alpha_{ij} + 2\beta_1 - 2\sum_{i,j} \alpha_{ij} + \binom{n-3}{2} - n + 3 = 0$. Summing over $i = 3, \ldots, n-1$ yields

$$4(n-5)\sum_{i,j} \alpha_{ij} = (n-3)(4\beta_1 + (n-3)(n-6)).$$

Looking finally at the component indexed by the pair $(n, n+1)$ yields: $\beta_1 + \beta_2 + \sum_{i,j} \beta_{ij} + \gamma = t$ and, thus,

$$2\sum_{i,j} \alpha_{ij} - 2\beta_1 - \binom{n-3}{2} + 1 = 0.$$

Using the fact that $2\sum_{i,j} \alpha_{ij} = \frac{n-3}{2(n-5)}(4\beta_1 + (n-3)(n-6))$, we deduce that $2\beta_1 = 1$, contradicting the fact that β_1 is integer. This shows that $sph_t(\mu_n)$ is not hypercube embeddable and, therefore, is a quasi h-distance. ∎

5.3 Hilbert bases of cuts

Let X be a finite set of vectors in $\mathbb{R}^k$. Set

$$\mathbb{R}_+(X) := \{\sum_{x \in X} \lambda_x x \mid \lambda_x \geq 0 \text{ for all } x \in X\},$$

$$\mathbb{Z}(X) := \{\sum_{x \in X} \lambda_x x \mid \lambda_x \in \mathbb{Z} \text{ for all } x \in X\},$$

$$\mathbb{Z}_+(X) := \{\sum_{x \in X} \lambda_x x \mid \lambda_x \in \mathbb{Z}_+ \text{ for all } x \in X\}.$$

So, $\mathbb{R}_+(X)$ is the cone generated by X, $\mathbb{Z}(X)$ is the lattice generated by X and $\mathbb{Z}_+(X)$ is the integer cone generated by X. Clearly, the following inclusion holds:

$$\mathbb{Z}_+(X) \subseteq \mathbb{R}_+(X) \cap \mathbb{Z}(X).$$

The set X is said to be a **Hilbert base** if equality holds, i.e.,

$$\mathbb{Z}_+(X) = \mathbb{R}_+(X) \cap \mathbb{Z}(X).$$

Clearly, if X is linearly independent, then X is a Hilbert base. We consider here the question of determining the graphs whose family of cuts is a Hilbert base.

Given a graph G, and $S \subseteq V$, the cut $\delta_G(S)$ consists of the edges $e \in E$ with one end node in S and the other in $V \setminus S$. Let $\mathcal{K}_G \subseteq \{0,1\}^E$ denote the family of the incidence vectors of the cuts of G. Then, $\mathbb{R}_+(\mathcal{K}_G)$ is the cut cone $\mathrm{CUT}(G)$ of G. Let $\mathcal{H}$ denote the collection of graphs G whose family of cuts $\mathcal{K}_G$ is a Hilbert base. So the question is to determine which graphs belong to $\mathcal{H}$.

By Theorem 4.5, the graphs K_3, K_4, K_5 belong to $\mathcal{H}$. On the other hand, the graph K_6 does not belong to $\mathcal{H}$ (as the distance $2d(K_6 \backslash e)$ belongs to $\mathbb{R}_+(\mathcal{K}_{K_6}) \cap \mathbb{Z}(\mathcal{K}_{K_6})$ but not to $\mathbb{Z}_+(\mathcal{K}_{K_6})$). We summarize some of the known results.

Proposition 5.17. *(i)* [FG] *Every graph not contractible to K_5 belongs to $\mathcal{H}$.*
(ii) [Lau93a] *Every graph on at most six nodes and distinct from K_6 belongs to $\mathcal{H}$.*
(iii) [Lau93a] *If G belongs to $\mathcal{H}$, then G is not contractible to K_6.*

The proof of the above result uses, in particular, the fact that the class $\mathcal{H}$ is closed under certain operations. Namely,
- $\mathcal{H}$ is closed under the k-sum ($k = 0, 1, 2, 3$).
- If $G \in \mathcal{H}$ and e is an edge of G, then the graph G/e (obtained by contracting the edge e) belongs to $\mathcal{H}$.
- If $G \in \mathcal{H}$, e is an edge of G for which each inequality $v^T x \leq 0$ defining a facet of the cut cone $\mathrm{CUT}(G)$ satisfies:

$$v_e \in \{0, 1, -1\}, \sum_{f \in \delta_G(S)} v_f \in 2\mathbb{Z} \text{ for all cuts } \delta_G(S),$$

then the graph $G\backslash e$ (obtained by deleting the edge e) belongs to $\mathcal{H}$.

For instance, Proposition 5.17 (*iii*) can be checked as follows. Suppose G is a graph that contains K_6 as a subgraph. Let $x \in \mathbb{R}^E$ be defined by $x_e = 2$ for all edges of G except $x_e = 4$ for one edge belonging to the subgraph K_6. Then, $x \in \mathbb{R}_+(\mathcal{K}_G) \cap \mathbb{Z}(\mathcal{K}_G)$ (as x can be extended to a point of $\mathrm{CUT}_n \cap \mathcal{L}_n$) and $x \notin \mathbb{Z}_+(\mathcal{K}_G)$ (because the projection of x on K_6 does not belong to $\mathbb{Z}_+(\mathcal{K}_{K_6})$).

The characterization of the class $\mathcal{H}$ seems a hard problem. This is due, partly, to the fact that the linear description of the cut cone is not known for general graphs. Many questions are yet unsolved.

For instance, is the class $\mathcal{H}$ closed under the ΔY-operation ? A first example to check is whether the following graph belongs to $\mathcal{H}$ (this is the graph obtained by applying once the ΔY-operation to K_6, i.e., replacing a triangle by a claw $K_{1,3}$).

Is the class $\mathcal{H}$ closed under the deletion of edges ? (As mentioned above, this could be proved only if a technical assumption is made on the facets of the cut cone.)

Another question is to determine a Hilbert base for the cut cone on 6 points; this is the smallest case when the cuts do not form a Hilbert base. The following conjecture is made; it is easily seen to be equivalent to Conjecture 5.12.

CONJECTURE 5.18. *The* 31 *nonzero cut semimetrics on* V_6 *together with the* 15 *metrics* $2d(K_6\backslash e)$ *(for* $e \in E(K_6)$*) form a Hilbert base.*

We also recall Proposition 5.10 which implies that the 16 odd cuts of K_6 form a Hilbert base.

On the other hand, the dual problem, which consists of characterizing the graphs whose family of cycles is a Hilbert base, is completely solved. Namely, the family of cycles of G is a Hilbert base if and only if G is not contractible to the Petersen graph [AGZ90]. Clearly, one may ask, more generally, what are the binary matroids whose family of cycles is a Hilbert base.

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