Odd Systems of Vectors and Related Lattices

M. DEZA
V.P. GRISHUKHIN*

Laboratoire d'Informatique, URA 1327 du CNRS
Département de Mathématiques et d'Informatique
Ecole Normale Supérieure
*CEMI RAN, Moscow

LIENS - 94 - 5

April 1994

Odd systems of vectors and related lattices

M.Deza
CNRS-LIENS, Ecole Normal Supérieure, Paris

V.P.Grishukhin
CEMI RAN, Moscow

April 1994

Abstract

We consider uniform odd systems, i.e. sets of vectors of constant odd norm with odd inner products, and the lattice $L(\mathcal{V})$ linearly generated by a uniform odd system $\mathcal{V}$ of odd norm $2t+1$. If $u^2 \equiv p \pmod 4$ for all $u \in \mathcal{V}$, one has $v^2 \equiv p \pmod 4$ if v^2 is odd and $v^2 \equiv 0 \pmod 4$ if v^2 is even, for any vector $v \in L(\mathcal{V})$. The vectors of even norm form a double even sublattice $L_0(\mathcal{V})$ of $L(\mathcal{V})$, i.e. $\frac{1}{\sqrt{2}}L_0(\mathcal{V})$ is an even lattice. The closure of $\mathcal{V}$, i.e. all vectors of $L(\mathcal{V})$ of norm $2t+1$, are minimal vectors of $L(\mathcal{V})$ for $t=1$, and they are almost always minimal for $t=2$. For these t's, the convex hull of vectors of the closure of $\mathcal{V}$ is an L-polytope of $L_0(\mathcal{V})$ and the contact polytope of $L(\mathcal{V})$. As an example, we consider closed uniform odd systems of norm 5 spanning equiangular lines.

1 Introduction

We study here **odd** systems, i.e. sets of vectors with odd inner (scalar) products. An odd system is a special case of an **integral** system, which is a set of vectors with integral inner products. In particular, vectors of an integral system have integral norm (squared length), which is the inner product of a vector with itself.

Similar to an odd system, an **even** system is defined. But an even system is nothing else but an integral system multiplied by $\sqrt{2}$.

Clearly, an integral system generates an integral lattice. Such lattices arise naturally in various contexts.

The integral lattice $L(\mathcal{V})$ generated by a **uniform** integral system $\mathcal{V}$ (i.e. system of vectors of some fixed norm m) determines naturally a **closure** of the system $\mathcal{V}$, cl$\mathcal{V}$, i.e. the set of all vectors of norm m of the lattice $L(\mathcal{V})$. In other words, cl$\mathcal{V}$ is the set of all vectors of norm m which are integral combinations of vectors of $\mathcal{V}$.

The classical example of an integral system is a root system. For our aims, uniform root systems are important. Only they occur in integral lattices. A uniform root system is a set of vectors of norm 2 with integral inner products. Each root system is a direct sum of irreducible root systems. All irreducible root systems are closed. They are completely classified. Namely, a uniform irreducible root system is one of A_n, D_n, E_6, E_7 and E_8.

It is natural to consider the next case of an integral system of vectors of norm 3. Several authors aproached those systems, see, for example, [9], [7]. This case is much more complicated. But closed integral systems of vectors of norm 3 with inner products ± 1 are classified completely. Note that such a system is an odd system.

It turns out that the odd condition is restricting enough. The above mentioned odd system of norm 3 is a special case of a system of vectors spanning equiangular lines. It is known that if a set of equiangular lines has sufficiently many lines, then spanning vectors can be chosen such that they have odd norm and inner products ± 1. This odd system lead to study general odd systems.

An important invariant of an odd system spanning equiangular lines is the property to be closed or not.

For an integral lattice it is important to find minimal nonzero norm of its vectors. An integral lattice generated by an odd system of norm 3 has minimal norm 3. Similarly, "most" of integral lattices generated by an odd system of norm 5 have minimal norm 5.

2 Lattices generated by an odd system

Definition 1 An integral (even, odd) system $\mathcal{V}$ *is a set of vectors with integral (even, odd, respectively) mutual inner products. In particular, the* **norm** $v^2 = vv$ *of* $v \in \mathcal{V}$, *i.e. the inner product of* v *with itself, is integral (even, odd), too.*

An integral system is called **irreducible** *if it cannot be partitioned into two subsystems such that inner product of vectors from different subsystems is equal to 0. Otherwise, the system is called* **reducible**.

Clearly, each odd system is irreducible. Of course, an odd or an even system is a special case of an integral system. Moreover, the division of vectors of an even system by $\sqrt{2}$ makes an isomorphism of the even system to an integral system.

Dimension dim$\mathcal{V}$ of an integral system $\mathcal{V}$ is dimension of span$\mathcal{V}$, the space spanned by $\mathcal{V}$.

Let $\mathcal{V}$ be an integral system, and let

$$L(\mathcal{V}) = \{u : u = \sum_{v \in \mathcal{V}} z_v v, z_v \in \mathbf{Z}\}.$$

Obviously, $L(\mathcal{V})$ is an integral lattice, and if $\mathcal{V}$ is an even system, then $L(\mathcal{V})$ is an even system, too. Therefore $L(\mathcal{V})$ is a special case of an even lattice. Recall that an integral lattice is called **even** if all its vectors have even norm.

Proposition 2 *Let $\mathcal{V}$ be an odd system, $u \in L(\mathcal{V})$, and $u = \sum_{v\in\mathcal{V}} z_v v$. Then*

$$\sum_{v\in\mathcal{V}} z_v \equiv \sum_{v\in\mathcal{V}} |z_v| \equiv u^2 \ (mod\ 2)\ .$$

Proof. Since for $v, v' \in \mathcal{V}$, $vv' \equiv 1 \pmod 2$, we have

$$u^2 = \sum_{v,v'\in\mathcal{V}} z_v z_{v'} vv' \equiv \sum_{v,v'\in\mathcal{V}} z_v z_{v'} = (\sum_{v\in\mathcal{V}} z_v)^2 \pmod 2\ .$$

This implies $\sum_{v\in\mathcal{V}} z_v \equiv u^2 \pmod 2$. The comparision $\sum_{v\in\mathcal{V}} z_v \equiv \sum_{v\in\mathcal{V}} |z_v| \pmod 2$ is obvious. □

Proposition 3 *Let $\mathcal{V}$ be an odd system, and let $u_i \in L(\mathcal{V})$, $u_i = \sum_{v\in\mathcal{V}} z_v^i v$, i=1,2. Then*

$$u_1u_2 \equiv u_1^2u_2^2 \ (mod\ 2)\ .$$

Proof. Using Proposition 2, we obtain

$$u_1u_2 = \sum z_v^1 z_{v'}^2 vv' \equiv (\sum z_v^1)(\sum z_v^2) \equiv u_1^2u_2^2 \pmod 2. \ \square$$

For $q = 0, 1$,consider the following subsets of $L(\mathcal{V})$.

$$L_q(\mathcal{V}) = \{u \in L(\mathcal{V}) : u^2 \equiv q \pmod 2\ \},$$

$$L^q(\mathcal{V}) = \{u : u = \sum_{v\in\mathcal{V}} z_v v,\ z_v \in \mathbf{Z},\ \sum_{v\in\mathcal{V}} z_v = q\}.$$

We call the vector $-v$ an **opposite** of the vector v and denote it v^*. $\mathcal{V}$ is called **symmetric** if for each $v \in \mathcal{V}$ its opposite belongs to $\mathcal{V}$, too. $\mathcal{V}$ is called **asymmetric** if $\mathcal{V}$ has no pair of opposite vectors.

Proposition 4 *Let $\mathcal{V}$ be an odd system. Then*

(i) $L_0(\mathcal{V})$ is an even sublattice of $L(\mathcal{V})$,

(ii) $L_1(\mathcal{V}) = v + L_0(\mathcal{V})$ for any $v \in \mathcal{V}$, i.e. $L_1(\mathcal{V})$ is an affine sublattice of $L(\mathcal{V})$,

(iii) if $\mathcal{V}$ contains a pair of opposite vectors, then $L_q(\mathcal{V}) = L^q(\mathcal{V})$. In particular, this is true for a symmetric $\mathcal{V}$.

Proof. (i) and (ii). Using Proposition 2 we obtain

$$L_q(\mathcal{V}) = \{u : u = \sum_{v\in\mathcal{V}} z_v v,\ z_v \in \mathbf{Z},\ \sum_{v\in\mathcal{V}} z_v \equiv q \pmod 2\ \}.$$

It easy to see that, if $u, u' \in L_1(\mathcal{V})$, then $u_0 = u - u' \in L_0(\mathcal{V})$, i.e. $u = u' + u_0$, and $L_1(\mathcal{V})$ is an affine sublattice of $L(\mathcal{V})$.

(iii). Let $u \in L_q(\mathcal{V})$, $u = \sum_{v\in\mathcal{V}} z_v v$, $\sum_{v\in\mathcal{V}} z_v \equiv q \pmod 2$. Then $\sum_{v\in\mathcal{V}} z_v = 2z + q$ for some integer z. If $\mathcal{V}$ is symmetric, then, adding to $\sum_{v\in\mathcal{V}} z_v$ the item $-z(v_0 + v_0^*)$ for some $v_0 \in \mathcal{V}$, we obtain $\sum_{v\in\mathcal{V}} z_v = q$. □

If $\mathcal{V}$ is asymmetric, then it is possible that the lattice $L^q(\mathcal{V})$ is a layer of the lattice $L_q(\mathcal{V})$.

3 Uniform integral systems

Let L be an integral lattice. For $k = 0, 1, 2, ...,$ we set

$$\mathcal{M}_k(L) = \{u \in L : u^2 = k\}.$$

Let $\mathcal{V}$ be an integral system. Call $\mathcal{V}$ **uniform** if all vectors of $\mathcal{V}$ have the same norm, which is called **norm** of $\mathcal{V}$. Consider uniform subsets of $L(\mathcal{V})$. Let

$$\mathrm{cl}_k(\mathcal{V}) = \mathcal{M}_k(L(\mathcal{V})).$$

We cal the operator cl_k the k-**closure**. If $\mathcal{V}$ is an even system, then $\mathrm{cl}_{2k+1}(\mathcal{V}) = \emptyset$. If $\mathcal{V}$ is an odd system, then, by Proposition 3, $\mathrm{cl}_{2k+1}(\mathcal{V})$ is an odd system for any integer $k \geq 0$. We have

$$L_0(\mathcal{V}) = \cup_{k=0}^{\infty}\mathrm{cl}_{2k}(\mathcal{V}),\ L_1(\mathcal{V}) = \cup_{k=0}^{\infty}\mathrm{cl}_{2k+1}(\mathcal{V}),$$

and $L_1(\mathcal{V})$ is an odd system.

Let $\mathcal{V}$ be a uniform integral system of norm k. We call $\mathcal{V}$ **closed** if $\mathcal{V} = \mathrm{cl}_k(\mathcal{V})$. Obviously, a closed system is symmetric.

A uniform integral system is called **maximal** if it cannot be enlarged without augmenting its dimension. Clearly, maximal uniform integral system is closed. In general, the converse is not true.

For $\alpha \in \mathbf{R}$, denote by $\alpha\mathcal{V}$ the set of vectors αv for $v \in \mathcal{V}$. Obviously, $m\mathrm{cl}_k\mathcal{V} \subseteq \mathrm{cl}_t\mathcal{V}$, where $t = m^2k$, and $(-1)\mathrm{cl}_k\mathcal{V} = \mathrm{cl}_k\mathcal{V}$. Hence cardinality $|\mathrm{cl}_k\mathcal{V}|$ is an even integer for all k. We set

$$n(\mathcal{V}) = \begin{cases} \frac{1}{2}|\mathcal{V}| & \text{if } \mathcal{V} \text{ is symmetric} \\ |\mathcal{V}| & \text{if } \mathcal{V} \text{ is asymmetric.} \end{cases}$$

Proposition 5 *Let $\mathcal{V}$ be an odd system. Then $n(cl_1\mathcal{V}) \leq 1$.*

Proof. Let $v_1, v_2 \in \mathrm{cl}_1\mathcal{V}$. Since $\mathrm{cl}_1\mathcal{V}$ is an odd system, $v_1v_2 = \pm 1$. This implies that $v_2 = \pm v_1$. □

Following to [6], call an odd system $\mathcal{V}$ **pillar** if there is a vector e of norm 1 such that $ve \in \{\pm 1\}$ for all $v \in \mathcal{V}$. The vector e is called the **shaft** of $\mathcal{V}$.

Proposition 6 *Let $\mathcal{V}$ be a pillar odd system with a shaft e. If $cl_1\mathcal{V} \neq \emptyset$, then $cl_1\mathcal{V} = \{\pm e\}$.*

Proof. Let $\mathrm{cl}_1\mathcal{V} \neq \emptyset$, i.e. $\mathrm{cl}_1\mathcal{V} = \{\pm e_1\}$. Then $e_1 = \sum_{v\in\mathcal{V}} z_v v$, where $\sum_{v\in\mathcal{V}} z_v$ is odd. Obviously, $ee_1 \leq 1$. But $ee_1 = \sum_v z_v(ev) = \sum_v \pm z_v$ is an odd integer. Hence $ee_1 \in \{\pm 1\}$, i.e. $e_1 = \pm e$. □

Recall that a set of vectors is a **frame** if any two vectors of the set are either orthogonal or opposite.

Proposition 7 *If $\mathcal{V}$ is an odd system, then $cl_2\mathcal{V}$ is a frame.*

Proof. By Proposition 3, $v_1 v_2$ is an even integer for any $v_1, v_2 \in \mathrm{cl}_2\mathcal{V}$. Hence if $v_1 \neq \pm v_2$, $v_1 v_2 = 0$. $\square$

Closed uniform integral systems of norm 2 are classified in [2]. Such a system is a direct sum of irreducible root systems of type A_n, D_n, E_6, E_7, E_8. The index n is dimension of the corresponding root system. They are closed, and they are maximal, apart from $A_7 \subset E_7$, $A_8 \subset E_8$, $D_8 \subset E_8$. A frame, up to a multiple, is the special root system A_1^r, the direct sum of r root systems A_1, for some r. Detailed description of the root systems is given, for example, in [2].

Below a uniform integral system $\mathcal{V}$ of norm 2 is called **root system** if it is closed. Otherwise, $\mathcal{V}$ is called a **set of roots**.

Let $\mathcal{V}_4$ be an even system of norm 4. Set $\beta = \frac{1}{\sqrt{2}}$. Since $\beta\mathcal{V}_4$ has integral inner products, we have

Proposition 8 *Let $\mathcal{V}_4$ be a uniform even system of norm 4. Then $\beta\mathcal{V}_4$ is a root system if $\mathcal{V}_4$ is closed, and it is a set of roots if it is not closed.* $\square$

We end this section with two useful lemmas.

Lemma 9 *Let $\mathcal{V}$ be an odd system, and let $v_i \in cl_{2i+1}\mathcal{V}$, i is an integer. Let $v_i v_{i'} = 2r+1 > 0$ and $i \neq i'$. Then $r < (i+i')/2$.*

Proof. Obviously, $v_i \neq v_{i'}$. Hence $0 < (v_i - v_{i'})^2 = 2(i+i') + 2 - 2v_i v_{i'} = 2(i+i'-2r)$, i.e. $r < (i+i')/2$. $\square$

For any set $\mathcal{X}$ of vectors, we set

$$a(\mathcal{X}) = \sum_{v \in \mathcal{X}} v. \tag{1}$$

Lemma 10 *Let $\mathcal{V}$ be a uniform integral system of norm m, and $\mathcal{K} \subseteq \mathcal{V}$ be a maximal subset of vectors with mutual inner products -1. Then $\mathcal{K}$ contains at most $m+1$ vectors, and if $|\mathcal{K}| = m+1$, then $a(\mathcal{K}) = 0$.*

Proof. Set $k = |\mathcal{K}|$. We have

$$0 \leq (a(\mathcal{K}))^2 = \sum_{v \in \mathcal{K}} v^2 + \sum_{v,v' \in \mathcal{K}, v \neq v'} vv' = km + k(k-1)(-1) = k(m+1-k),$$

i.e. $k \leq m+1$, and if $k = m+1$, then $a(\mathcal{K}) = 0$. $\square$

We call such $\mathcal{K}$ of cardinality $m+1$ a **star**. By Lemma 10, $a(\mathcal{K}) = 0$ for a star $\mathcal{K}$.

If $\mathcal{V}$ is an even system of norm $2m$, then $\beta\mathcal{V}$ is an integral system of norm m. If $\beta\mathcal{V}$ has a star $\mathcal{K}$ (of cardinality $m+1$), then $\beta^{-1}\mathcal{K}$ is a set of $m+1$ vectors of norm $2m$ with mutual inner products -2. We call the set a **star** of the even system $\mathcal{V}$.

Lemma 10 implies

Corollary 11 *Let $\mathcal{V}$ be an even system of norm $4t$. Then $|\mathcal{K}| = 2t+1$ for any star $\mathcal{K} \subseteq \mathcal{V}$.* $\square$

A star is a special case of a dependent set in an integral system $\mathcal{V}$. If $|\mathcal{V}| >$dim$\mathcal{V}$, then there are dependencies between vectors of $\mathcal{V}$. Let $C \subseteq \mathcal{V}$ be a dependency, i.e. a set of linearly dependent vectors of $\mathcal{V}$. Then

$$\sum_{v \in C} z_v v = 0. \qquad (2)$$

The equation (2) implies $\sum_{v\in C} z_v vu = 0$ for all $u \in \mathcal{V}$. Let A be Gram matrix of the set $\mathcal{V}$. Then $z' \in \mathbf{R}^{\mathcal{V}}$ such that $z'_v = z_v$ for $v \in C$ and $z'_v = 0$ for $v \in \mathcal{V} - C$ is a solution to the system of equation $zA = 0$. Since A is integral matrix, all solutions, up to a multiple, are rational, and therefore can be taken integral.

Call the dependency (2) **affine** if $\sum_{v\in C} z_v = 0$. By Proposition 4 (iii), each dependency of $\mathcal{V}$ can be transformed into an affine dependency if $\mathcal{V}$ is symmetric.

Call the dependency (2) **minimal** if $\sum_{v\in\mathcal{V}} y_v v \neq 0$ for all y such that $|y_v| \le |z_v|$ and there is a $w \in C$ with $|y_w| < |z_w|$. As in matroid theory, call a minimal dependency **circuit**. If $v \in C$ and $|z_v| = 1$ in the corresponding dependency, then the set $C - \{v\}$ is called **broken circuit**.

We shall consider only uniform circuits and broken uniform circuits.

Proposition 12 *Let $\mathcal{V}$ be a uniform integral system. Then $\mathcal{V}$ is closed if and only if $\mathcal{V}$ does not contain a broken circuit.*

Proof. Any vector $u \in (\mathrm{cl}\mathcal{V} - \mathcal{V})$ has the form $u = \sum_{v\in C_b} z_v v$, where $C_b \subseteq \mathcal{V}$ is a broken circuit. □

4 Sets of vectors with constant norm modulo 4

Consider an odd system $\mathcal{V}$ with vectors of norm $v^2 = 4m(v) + p$, where $m(v)$ is an integer, and $p \equiv 1$ or 3 is the same for all $v \in \mathcal{V}$. So, $v^2 \equiv p \pmod 4$ for all $v \in \mathcal{V}$.

Theorem 13 *Let $\mathcal{V}$ be an odd system, and let $v^2 \equiv p \pmod 4$ for all $v \in \mathcal{V}$, p =1 or 3. Then $u^2 \equiv 0 \pmod 4$ for $u \in L_0(\mathcal{V})$, and $u^2 \equiv p \pmod 4$ for $u \in L_1(\mathcal{V})$.*

Proof. Let $u = \sum_{v\in\mathcal{V}} z_v v$. We use induction over the number $s(u) = \sum_{v\in\mathcal{V}} |z_v|$. The assertion obviously is true if $s(u) = 1$. Let $u = \pm 2v_1$ or $u = v_1 \pm v_2$, and $v_1v_2 = 2k + 1$. Note that $2p \equiv 2 \pmod 4$. Hence $u^2 = v_1^2 + v_2^2 \pm 2v_1v_2 \equiv 2 \pm 2(2k + 1) \equiv 0 \pmod 4$. Plainly, $(\pm 2v_1)^2 \equiv 0 \pmod 4$, i.e. the assertion is true for $s(u) = 2$.

Let the above assertion is true for all u' with $s(u') < s$, and Let $s(u) = s$. Let $z_{v'} \neq 0$. Without loss of generality suppose that $z_{v'} > 0$. Then $u = u_1 + v'$, where $u_1 = \sum_{v\neq v'} z_v v + (z_{v'} - 1)v'$, and $s(u_1) = s - 1$. If s is even, then $u \in L_0(\mathcal{V})$, $u_1 \in L_1(\mathcal{V})$, and, by induction, $u_1^2 \equiv p \pmod 4$. Hence $u^2 = u_1^2 + v'^2 + 2u_1v' \pmod 4$. Since $u_1, v' \in L_1(\mathcal{V})$, Proposition 3 implies, that $u_1v' = 2k + 1$ for an integer k, i.e. $u^2 \equiv 2 + 4k + 2 \equiv 0 \pmod 4$. If s is odd, then $u \in L_1(\mathcal{V})$, $u_1 \in L_0(\mathcal{V})$, and, by induction, $u_1^2 \equiv 0 \pmod 4$. Proposition 3 implies $u_1v' = 2l$, i.e. $u^2 \equiv p + 2 \times 2l \equiv p \pmod 4$. □

Theorem 13 implies the following

Corollary 14 *If $v^2 \equiv p \pmod 4$ for all $v \in \mathcal{V}$, then $cl_k\mathcal{V} = \emptyset$ for $k \equiv 2, 4-p \pmod 4$.* □

Call a lattice L **double even** if L is an even system and $v^2 \equiv 0 \pmod 4$ for all $v \in L$. Obviously, L is double even if and only if βL is an even lattice ($\beta = 1/\sqrt{2}$).

Corollary 15 *If $\mathcal{V}$ is a uniform odd system, then the lattice $L_0(\mathcal{V})$ is a duble even lattice.* □

We can find out a uniform odd system $\mathcal{U}$ in a double even lattice as follows. Let L be a double even lattice, and $v_0 \in \mathcal{M}_{8k-4}(L)$, where $\mathcal{M}_k(L)$ is the set of all vectors of L of norm k. It is defined in Section 3. Set

$$\mathcal{M}_{4k}(v_0, L) = \{v \in \mathcal{M}_{4k}(L) : vv_0 = 4k-2\},$$

$$u(v) = v - \frac{1}{2}v_0, \; v(u) = u + \frac{1}{2}v_0,$$

$$\mathcal{U}_{2k+1}(v_0, L) = \{u(v) : v \in \mathcal{M}_{4k}(v_0, L)\}.$$

Note that $u(v)$ is orthogonal to v_0 for all $v \in \mathcal{M}_{4k}(v_0, L)$.

Theorem 16 *Let $\mathcal{U}$ be a set of vectors. The following assertions are equivalent:*
(i) $\mathcal{U} = \mathcal{U}_{2k+1}(v_0, L)$ for a double even lattice L,
(ii) $\mathcal{U}$ is a closed uniform odd system of norm $2k+1$.

Proof. $(i) \Rightarrow (ii)$. Let $v, v' \in \mathcal{M}_{4k}(v_0, L)$. Then vv' is an even integer, and $|vv'| \leq 4k$. Since $u(v)u(v') = vv' - 2k + 1$ is odd, $\mathcal{U} = \mathcal{U}_{2k+1}(v_0, L)$ is a uniform odd system of norm $2k+1$. It is symmetric, because $v_0 - v \in \mathcal{M}_{4k}(v_0, L)$ and $u(v_0 - v) = -u(v)$.

Let $u_1 \in \mathrm{cl}_{2k+1}\mathcal{U}$. Since $u_1 \in L_1(\mathcal{U})$, by Proposition 4(iii),

$$u_1 = \sum_{v \in \mathcal{M}_{4k}(v_0, L), \sum z_v = 1} z_v u(v).$$

Hence $u_1 = v_1 - \frac{1}{2}v_0$, where $v_1 = \sum_{v \in \mathcal{M}_{4k}(v_0, L), \sum z_v = 1} z_v v$. Since $\sum z_v = 1$, we have $v_1 v_0 = \sum z_v(vv_0) = (4k-2)\sum z_v = 4k-2$. Since u_1 is orthogonal to v_0, we have $v_1^2 = u_1^2 + (\frac{1}{2}v_0)^2 = 4k$. This implies that $v_1 \in \mathcal{M}_{4k}(v_0, L)$ and $u_1 = u(v_1) \in \mathcal{U}$. This means that $\mathcal{U}$ is closed.

$(ii) \Rightarrow (i)$. Let u_0 be a vector of norm $2k-1$ which is orthogonal to the space spanned by $\mathcal{U}$. Let $\mathcal{V}_{4k} = \{v(u) : u \in \mathcal{U}\}$, with $v_0 = 2u_0$ in definition of $v(u)$. Since $v(u)v(u') = uu' + u_0^2$ is an even integer, $\mathcal{V}_{4k}$ is an even system of vectors of norm $4k$.

Let $L = L(\mathcal{V}_{4k})$ be the lattice linearly generated by $\mathcal{V}_{4k}$. Obviously, $v_0 = 2u_0 = (u+u_0)+(-u+u_0) \in L$, $v_0^2 = 8k-4$, and $\mathcal{V}_{4k} \subseteq \mathcal{M}_{4k}(v_0, L)$, since $v_0 v(u) = 2u_0^2 = 4k-2$. Besides, $\mathcal{U} \subseteq \mathcal{U}_{2k+1}(v_0, L)$, since $u(v(u)) = u$. We have to prove that $\mathcal{U} = \mathcal{U}_{2k+1}(v_0, L)$. Let $u' \in \mathcal{U}_{2k+1}(v_0, L)$. Then $u' = v - \frac{1}{2}v_0$ for $v \in \mathcal{M}_{4k}(v_0, L)$, and $v = \sum_{u \in \mathcal{U}} z_u v(u)$, since L is generated by $v(u)$ for $u \in \mathcal{U}$, i.e. $u' = \sum_{u \in \mathcal{U}} z_u v(u) - v_0 = \sum_{u \in \mathcal{U}} z_u u + (\sum_{u \in \mathcal{U}} z_u - 1)v_0$. Since $u'v_0 = uv_0 = 0$, we have $\sum_{u \in \mathcal{U}} z_u = 1$. This implies that $u' \in L(\mathcal{U})$. Because $(u')^2 = 2k+1$, we have $u' \in \mathrm{cl}_{2k+1}\mathcal{U} = \mathcal{U}$. Hence $\mathcal{U} = \mathcal{U}_{2k+1}(v_0, L)$. □

The following theorem has an application to odd systems with inner products equal to ± 1.

Theorem 17 *Let $\mathcal{U}$ be a set of vectors. The following assertions are equivalent:*

(i) $\mathcal{U} = \mathcal{U}_{2k+1}(v_0, L)$ for a double even lattice L with minimal norm $4k$,

(ii) $\mathcal{U}$ is a closed uniform odd system of norm $2k+1$, and $cl_1\mathcal{U} = cl_{2r+1}\mathcal{U} = cl_{4r}\mathcal{U} = \emptyset$ for $0 < r < k$.

Proof. $(i) \Rightarrow (ii)$. By Theorem 16, $\mathcal{U}$ is closed uniform odd system of norm $2k+1$. Let $\mathcal{V} = \mathcal{M}_{4k}(v_0, L)$, $u' \in L_1(\mathcal{U})$, i.e. $u' = \sum_{v\in\mathcal{V}} z_v u(v)$ with $\sum_{v\in\mathcal{V}} z_v = 1$. Then $u' + \frac{1}{2}v_0 = \sum_{v\in\mathcal{V}} z_v v \in L^1(\mathcal{V}) \subseteq L$.

If $u' \in \mathrm{cl}_{2r+1}\mathcal{U}$, then, using $u'v_0 = 0$ and $v_0^2 = 8k-4$, we obtain $(u' + \frac{1}{2}v_0)^2 = 2(r+k) < 4k$ for $r < k$, a contradiction, since minimal norm of L is $4k$.

Let $u' \in L_0(\mathcal{U})$. By Proposition 4, $u' = \sum_{v\in\mathcal{V}} z_v u(v)$ with $\sum_{v\in\mathcal{V}} z_v = 0$. Hence $u' = \sum_{v\in\mathcal{V}} z_v v \in L^0(\mathcal{V}) \subseteq L$, i.e. $u'^2 \geq 4k$. This means that $\mathrm{cl}_{4r}\mathcal{U} = \emptyset$ for $r < k$.

$(ii) \Rightarrow (i)$. By Theorem 16, $\mathcal{U} = \mathcal{U}_{2k+1}(v_0, L)$ for a double even lattice L. We have to prove that minimal norm of L is $4k$, i.e. that $\mathcal{M}_{4r}(L) = \emptyset$ for $0 < r < k$. According to proof of $(ii) \Rightarrow (i)$ of Theorem 16, we can take $L = L(\mathcal{V})$, where $\mathcal{V} = \{u + \frac{1}{2}v_0 : u \in \mathcal{U}\}$.

Let $v \in L(\mathcal{V})$, then

$$v = \sum_{u\in\mathcal{U}} z_u(u + \frac{v_0}{2}) = (\sum z_u)\frac{v_0}{2} + \sum z_u u = z\frac{v_0}{2} + u_1,$$

where $z = \sum z_u$, $u_1 = \sum z_u u \in L(\mathcal{U})$.

Let $z = 0$, then $u_1 \in L_0(\mathcal{U})$. By Corollary 15, $L_0(\mathcal{U})$ is a double even lattice. Since $\mathrm{cl}_{4r}\mathcal{U} = \emptyset$ for $0 < r < k$, minimal norm of $L_0(\mathcal{U})$ is $\geq 4k$.

Let $z = 1$. Then $u_1 \in L_1(\mathcal{U})$. Since $\mathrm{cl}_{2r+1}\mathcal{U} = \emptyset$ for $0 < r < k$, we have

$$v^2 = (\frac{v_0}{2})^2 + u_1^2 = 2k - 1 + u_1^2 \geq 2k - 1 + (2k+1) = 4k.$$

It is easy to see that norm of v with $|z| > 1$ is greater than $4k$. Since $v^2 = 4k$ for $v \in \mathcal{V}$, minimal norm of $L(\mathcal{V})$ is equal to $4k$. □

The following lemma sais that the set $\mathcal{U}_{2k+1}(v_0, L)$ of Theorem 17 spans a set of equiangular lines at angle $arccos\frac{1}{2k+1}$.

Lemma 18 *Let $\mathcal{U}$ be a closed uniform odd system of norm $2k+1$. Then $uu' = \pm 1$ for all $u, u' \in \mathcal{U}$ such that $u \neq \pm u'$ if and only if $cl_{4r}\mathcal{U} = \emptyset$ for $0 < r < k$.*

Proof. Suppose that there is a pair $u, u' \in \mathcal{U}$ such that $uu' = 2r+1$ with $0 < r < k$. Then $(u - u')^2 = 4(k-r)$, i.e. $u - u' \in \mathrm{cl}_{4(k-r)}\mathcal{U}$. □

Lemma 18 and Theorem 17 imply the following fact proved in [6].

Corollary 19 *[6] Let L be a double even lattice of minimal norm $4k$, and $v_0 \in \mathcal{M}_{8k-4}(L)$. Then the set $\mathcal{U}_{2k+1}(v_0, L)$ spans a set of equiangular lines at angle $arccos\frac{1}{2k+1}$.* □

We denote the lattice $L(\mathcal{M}_{4k})$ with $\mathcal{M}_{4k} = \{v(u) : u \in \mathcal{U}\}$ constructed in the proof of $(ii) \Rightarrow (i)$ of Theorem 16 by $L_2(\mathcal{U})$.

In the case of Theorem 17 the lattice $L_0(\mathcal{U})$ is the section of $L_2(\mathcal{U})$ by the affine hyperplane $H = \{x : xv_0 = 4k-2\}$. The lattice $L(\mathcal{U})$ is the projection of $L_2(\mathcal{U})$ on H.

5 Contact polytopes and L-polytopes of the lattice $L_0(\mathcal{V})$

The convex hull of all minimal vectors of a lattice is called **contact polytope** of the lattice.

Let $\mathcal{V}$ be an odd system. Let H_k be the space spanned by $\mathrm{cl}_k\mathcal{V}$.

Set

$$m = \min\{k > 0 : \mathrm{cl}_{2k}\mathcal{V} \neq \emptyset\}.$$

Then minimal norm of the lattice $L_0(\mathcal{V})$ is $2m$, and $\mathrm{conv}(\mathrm{cl}_{2m}\mathcal{V})$ is the contact polytope of $L_0(\mathcal{V})$.

An L-polytope of a lattice is the convex hull of all lattice points lying on an empty sphere of the lattice, and the lattice points on the empty sphere have full affine rank.

Let

$$t = \min\{k : \mathrm{cl}_{2k+1}\mathcal{V} \neq \emptyset\}.$$

Then the sphere circumscribing $\mathrm{cl}_{2t+1}\mathcal{V}$ with squared radius $2t+1$ is empty in $L_1(\mathcal{V})$. Recall that squared Euclidean distance of any point of $L_1(\mathcal{V})$ from origin is odd, and $L_1(\mathcal{V})$ is a translation of the lattice $L_0(\mathcal{V})$. Any vector of $\mathrm{cl}_{2t+1}\mathcal{V}$ can be taken as the translation vector. Hence the translation of $\mathrm{conv}(\mathrm{cl}_{2t+1}\mathcal{V})$ is an L-polytope of the sublattice $L_0(\mathcal{V}) \cap H_{2t+1}$.

Proposition 20 *Let $\mathcal{V}$ be an odd system. Then*

(i) If $v^2 \equiv 3 \pmod 4$ for all $v \in \mathcal{V}$, then conv($cl_3\mathcal{V}$) is an L-polytope of the lattice $L_0(\mathcal{V}) \cap H_3$, and the contact polytope of the lattice $L(\mathcal{V})$.

(ii) If $v^2 \equiv 1 \pmod 4$ for all $v \in \mathcal{V}$, and $cl_1\mathcal{V} = \emptyset$, then conv($cl_5\mathcal{V}$) is an L-polytope of the lattice $L_0(\mathcal{V}) \cap H_5$.

(iii) If $\mathcal{V}$ is a uniform odd system of norm 5 with $cl_1\mathcal{V} = \emptyset$, and $vv' = \pm 1$ for distinct $v, v' \in cl_5\mathcal{V}$, then conv($cl_5\mathcal{V}$) is an L-polytope of $L_0(\mathcal{V})$ and the contact polytope of $L(\mathcal{V})$, and conv($cl_8\mathcal{V}$) is the contact polytope of $L_0(\mathcal{V})$.

Proof. (i) and (ii) are implied by Corollaries 14 and 15.

(iii) Since $\mathcal{V} \subseteq \mathrm{cl}_5\mathcal{V}$, $L(\mathcal{V}) \subset H_5$. By (ii), $\mathrm{conv}(\mathrm{cl}_5\mathcal{V})$ is an L-polytope of $L_0(\mathcal{V})$. By Corollary 14 and Lemma 18, $\mathrm{cl}_k\mathcal{V} = \emptyset$ for $k = 2, 3, 4, 6, 7$. Hence $\mathrm{cl}_5\mathcal{V}$ and $\mathrm{cl}_8\mathcal{V}$ are sets of minimal vectors of the lattices $L(\mathcal{V})$ and $L_0(\mathcal{V})$, respectively. □

6 Closed uniform odd systems of norm 3

Note that uniform systems of norms 3 and 5 are the first members of the sequences of odd systems with norm 3 (mod 4) and norm 1 (mod 4), respectively.

Note also that, for $k = 1$, Theorems 16 and 17 coincide. In fact, on the one hand, norm of any vector of a double even lattice is a multiple of 4. Hence if $\mathcal{U}_3(v_0, L) \neq \emptyset$, then minimal norm of L is equal to 4. On the other hand, there is no r satisfying (ii) of Theorem 17, and $\mathrm{cl}_1\mathcal{U} = \emptyset$ by Corollary 14.

For a uniform odd system of norm 3, Proposition 20(i) gives the following analogue of Proposition 1 of [7].

Corollary 21 *Let $\mathcal{U}$ be a unioform odd system of norm 3. Then minimal norm of the lattice $L(\mathcal{U})$, generated by $\mathcal{U}$, is 3. i* □

Let $\mathcal{U}$ be a uniform odd system of norm 3. Then the lattices $L_0(\mathcal{U})$ and $L_2(\mathcal{U})$ are double even lattices of minimal norm 4, and $L_2(\mathcal{U})$ is generated by a uniform even system $\mathcal{M}_4$ of norm 4. Hence $L_2(\mathcal{U})$ is, up to multiple $\beta = \frac{1}{\sqrt{2}}$, a root lattice.

In contrast to $\beta L_2(\mathcal{U})$, the lattice $\beta L_0(\mathcal{U})$ is not always generated by the root system $\beta \mathrm{cl}_4\mathcal{U}$. Theorem 22 below shows that $\beta L_2(\mathcal{U})$ is an irreducible root lattice. Since all irreducible root lattices are known, this theorem allows to classify closed uniform odd systems of norm 3.

Theorem 22 *Let $\mathcal{U}$ be a closed uniform odd system of norm 3. Then the lattice $\beta L_2(\mathcal{U})$ is an irreducible root lattice.*

Proof. Recall that $\mathcal{U} = \mathcal{U}_3(v_0, L)$ for $L = L_2(\mathcal{U})$. Here $v_0 \in L$ is a vector of norm 4, i.e. $v_0 \in \mathcal{M}_4(L)$. Hence the set of vectors $\mathcal{M}_4(v_0, L) \cup \{v_0\}$, generating L, is irreducible, since $vv_0 \neq 0$ for all $v \in \mathcal{M}_4(v_0, L)$. This means that the lattice βL is generated by an irreducible subset of roots. Hence $\beta L = \beta L_2(\mathcal{U})$ is an irreducible root lattice. □

Theorems 17 and 22 imply the following known fact (see, for example, Theorem 1 of [7]).

Corollary 23 *There is one-to-one correspondence between closed uniform systems of norm 3 and irreducible root systems.* □

It is obvious that a uniform odd system of norm 3 spans a set of equiangular lines at angle $arccos\frac{1}{3}$.

The closed uniform odd systems $\mathcal{U}$ of norm 3 and corresponding lattices $L(\mathcal{U})$, $L_0(\mathcal{U})$ and $L_2(\mathcal{U})$, if they have known names, are given in the following Table 1.

The lattices A_5^{+2}, D_6^{+2} and E_7^{+2} of Table 1 are described in [4] and [5].

Table 1

$\beta L_2(\mathcal{U}) =$	A_n	D_n	E_6	E_7	E_8
$\dim\mathcal{U}$	$n-2$	$n-1$	5	6	7
$n(\mathcal{U})$	$n-2$	$2(n-2)$	10	16	28
$\beta L_0(\mathcal{U})$			A_5	D_6	E_7
$\beta L(\mathcal{U})$			A_5^{+2}	D_6^{+2}	E_7^{+2}

Recall that an odd system $\mathcal{U}$ is called pillar if there is a vector e of norm 1 such that $ve \in \{\pm 1\}$ for all $v \in \mathcal{U}$.

Denote by $\mathcal{U}_n^Q$ the system $\mathcal{U}$ corresponding to the root lattice Q_n where $Q = A, D$ or E. Note that $\mathcal{U}_n^A$ and $\mathcal{U}_n^D$ are pillar with $e = \beta(e_1 + e_2)$ where $\{e_i : 1 \leq i \leq n+1\}$ is an orthonormal basis such that the roots of A_n (D_n) are, $\pm(e_i - e_j)$ ($\pm(e_i \pm e_j)$), respectively, $1 \leq i, j \leq n+1$. The unit vector e does not belong to $\mathrm{span}\mathcal{U}_n^A$ and belongs to $\mathrm{span}\mathcal{U}_n^D$. The vectors $u \in \mathcal{U}_n^A$ have the form $u = \pm\beta(e_1 + e_2 - 2e_i)$, $3 \leq i \leq n+1$, and $u = \pm\beta(e_1 + e_2 \pm 2e_i)$, $3 \leq i \leq n+1$, for $u \in \mathcal{U}_n^D$.

The lattice $\beta L_0(\mathcal{U}_n^Q)$ is the section of Q_n by the affine hyperplane $H = \{x : x(e_1+e_2) = 2\}$. The lattice $\beta L(\mathcal{U}_n^Q)$ is the projection of Q_n onto the affine hyperplane H. The lattices $\beta L_0(\mathcal{U}_n^A)$ and $\beta L_0(\mathcal{U}_n^D)$ are **not** root lattices, because every its basis contains a vector of norm at least 4.

7 Pillar uniform odd systems of norm 5

Now we consider closed uniform odd systems of norm 5. Let $\mathcal{U}$ be such a system. According to Proposition 8, $\beta\text{cl}_4\mathcal{U}$ is a root system. We try to understand, when the root system $\mathcal{V} = \beta\text{cl}_4\mathcal{U}$ determines uniquely the system $\mathcal{U}$. For example, if e is a vector of norm 1, which is orthogonal to all roots of $\mathcal{V}$, then the set of vectors $\mathcal{U} = \{\sqrt{2}v \pm e : v \in \mathcal{V}\}$ is a closed uniform odd system of norm 5 with $\text{cl}_1\mathcal{U} = \{\pm e\}$ and $\text{cl}_4\mathcal{U} = \sqrt{2}\mathcal{V} \cup \{\pm 2e\}$. Moreover, $\mathcal{U}$ is pillar, and, as we show below, every closed odd system $\mathcal{U}$ of norm 5 with $\text{cl}_1\mathcal{U} \neq \emptyset$ has such a form.

Recall that a uniform odd system $\mathcal{U}$ is called pillar if there is a vector e of norm 1 such that $ue \in \{\pm 1\}$ for all $u \in \mathcal{U}$. By Proposition 6, $\text{cl}_1\mathcal{U} = \{\pm e\}$ if $\mathcal{U}$ is pillar with the shaft e and $\text{cl}_1\mathcal{U} \neq \emptyset$. The assertion of Proposition 6 can be reversed for a uniform odd system of norm 5. Moreover, an odd system $\mathcal{U}$ of norm 5 with $\text{cl}_1\mathcal{U} \neq \emptyset$ is a special case of a pillar odd system.

Proposition 24 *Let $\mathcal{U}$ be a uniform odd system of norm 5, and let e be a vector of norm 1. If $\mathcal{U}' = \mathcal{U} \cup \{e\}$ is an odd system, then $\mathcal{U}$ is pillar.*

Proof. By definition of $\mathcal{U}'$, $e \in\text{cl}_1\mathcal{U}'$. Let $u \in\text{cl}_5\mathcal{U}'$ be such that $ue > 0$. By Lemma 9, $ue = 2r + 1$, where r is an integer and $0 \leq r < \frac{1}{2}$, i.e. $r = 0$. Hence $ue = \pm 1$ for all $u \in\text{cl}_5\mathcal{U}'$, in particular, for all $u \in \mathcal{U}$. □

Note that Proposition 24 is not true for odd systems of norm > 5.

In this section we show that a pillar odd system of norm 5 is related to a set of roots. Let $\mathcal{U}$ be a pillar uniform odd system of norm 5 with the shaft e. For $u \in \mathcal{U}$, set

$$v(u) = u - (ue)e, \tag{3}$$

$$\mathcal{V}_4(\mathcal{U}) = \{v(u) : u \in \mathcal{U}\}. \tag{4}$$

It is easy to verify that $\mathcal{V}_4(\mathcal{U})$ is a uniform even system of norm 4, and $ve = 0$ for all $v \in \mathcal{V}_4(\mathcal{U})$. If $\mathcal{U}$ is closed, than $\mathcal{V}_4(\mathcal{U})$ is symmetric, since $v(-u) = -v(u)$. The equality (3) defines a linear map $v : \mathcal{U} \to \mathcal{V}_4(\mathcal{U})$ which is the projection of $\mathcal{U}$ on the hyperplane which is orthogonal to the vector e.

Obviously each vector $u \in \mathcal{U}$ has the form $u = (ue)e + v(u)$, where $v(u)$ has norm 4. If $\text{cl}_1\mathcal{U} \neq \emptyset$, then $v(u) \in\text{cl}_4\mathcal{U}$.

By Corollary 11, any star $\mathcal{K} \subseteq \mathcal{V}_4(\mathcal{U})$ contains 3 vectors with mutual inner products -2.

Lemma 25 *Let $\mathcal{V}_4(\mathcal{U})$ be closed. Then $\mathcal{V}_4(\mathcal{U})$ contains no star if and only if it is a frame.*

Proof. If $\mathcal{V}_4(\mathcal{U})$ is a frame, then it contains no pair of vectors with inner product ± 2, and therefore contains no star. Let $\mathcal{V}_4(\mathcal{U})$ is not a frame and contains 2 vectors v, v' with inner product $vv' = -2$. Then $(v+v')^2 = 4$, and $v + v' \in \mathcal{V}_4(\mathcal{U})$, since $\mathcal{V}_4(\mathcal{U})$ is closed, and $\{v, v', -(v+v')\}$ is a star. □

Below we consider closed pillar uniform odd systems of norm 5 with the shaft e. We have the following 3 cases.

(1) $\mathrm{cl}_1\mathcal{U} \neq \emptyset$,

(2) $\mathrm{cl}_1\mathcal{U} = \emptyset$, and $\mathcal{V}_4(\mathcal{U})$ is closed.

(3) $\mathrm{cl}_1\mathcal{U} = \emptyset$, and $\mathcal{V}_4(\mathcal{U})$ is not closed.

We consider the cases (1) and (2) in details.

Recall that $\mathcal{V}_4(\mathcal{U})$ and $\mathrm{cl}_4\mathcal{U}$ consist of vectors of norm 4. Proposition 26 below describes a relation between these sets.

Proposition 26 *Let $\mathcal{U}$ be a closed uniform pillar odd system of norm 5 with the shaft e. Then*

(i) $2e \in \mathrm{cl}_4\mathcal{U}$ if and only if there is $v \in \mathcal{V}_4(\mathcal{U})$ such that $v \pm e \in \mathcal{U}$, and if $2e \in \mathrm{cl}_4\mathcal{U}$ then $\mathcal{U} = \{v \pm e : v \in \mathcal{V}_4(\mathcal{U})\}$.

(ii) $\mathrm{cl}_1\mathcal{U} = \emptyset$ if and only if $\mathcal{V}_4(\mathcal{U}) \cap \mathrm{cl}_4\mathcal{U} = \emptyset$.

Proof. (i) Let $u \in \mathcal{U}$, then by (3) $u = v(u) + (ue)e$. Now, if $2e \in \mathrm{cl}_4\mathcal{U}$, then the vector $u' = v(u) - (ue)e$ belongs also to $\mathcal{U}$, since $u' = u - (ue)2e$ has norm 5 and $\mathcal{U}$ is closed. Conversely, if $u^{\pm} = v \pm e \in \mathcal{U}$, then $u^+ - u^- = 2e \in \mathrm{cl}_4\mathcal{U}$.

(ii) $v(u) \in \mathcal{V}_4(\mathcal{U})$ belongs to $\mathrm{cl}_4\mathcal{U}$ if and only if $\pm e = \pm(u - v(u)) \in \mathrm{cl}_1\mathcal{U}$. □

Proposition 27 below completely characterizes $\mathcal{U}$ with $\mathrm{cl}_1\mathcal{U} \neq \emptyset$. Recal that $\beta = \frac{1}{\sqrt{2}}$.

Proposition 27 *Let $\mathcal{U}$ be a closed uniform odd system of norm 5. The following statements are equivalent.*

(i) $\mathcal{U}$ contains 3 vectors u_1, u_2, u_3 such that $u_1u_2 = -1$, $u_1u_3 = u_2u_3 = 3$,

(ii) $\mathrm{cl}_1\mathcal{U} \neq \emptyset$,

(iii) $\mathcal{U}$ is pillar with the shaft e, and $\mathrm{cl}_4\mathcal{U} = \mathcal{V}_4(\mathcal{U}) \cup \{\pm 2e\}$,

(iv) $\mathcal{U}$ is pillar with the shaft e, $2e \in \mathrm{cl}_4\mathcal{U}$ and $\mathcal{V}_4(\mathcal{U})$ contains a star.

Proof. $(i) \Rightarrow (ii)$. It is easy to see that the vector $u_1 + u_2 - u_3$ has norm 1, i.e. $u_1 + u_2 - u_3 \in \mathrm{cl}_1\mathcal{U}$.

$(ii) \Rightarrow (iii)$. Since $e \in \mathrm{cl}_1\mathcal{U}$, $\mathcal{U}$ is pillar and clearly $\mathcal{V}_4(\mathcal{U}) \cup \{\pm 2e\} \subseteq \mathrm{cl}_4\mathcal{U}$. Let $v \in \mathrm{cl}_4\mathcal{U}$. Then $ve \in \{0, \pm 2\}$ (see Proposition 3). Let $ve = 2$. Then $(v-e)^2 = 1$, i.e. $v - e \in \mathrm{cl}_1\mathcal{U}$. This implies that $v = 2e$. Hence if $v \neq \pm 2e$, $ve = 0$. It follows that $u(v) = v + e \in \mathrm{cl}_5\mathcal{U}$. Since $\mathcal{U}$ is closed, $v + e = u$ for some $u \in \mathcal{U}$, and $ue = 1$, i.e. $v = v(u)$, where $v(u)$ is from (3). Hence $\{v(u) : u \in \mathcal{U}\} = \mathrm{cl}_4\mathcal{U} - \{\pm 2e\}$.

$(iii) \Rightarrow (iv)$. Since $\mathcal{V}_4(\mathcal{U}) \subseteq \mathrm{cl}_4\mathcal{U}$ and e is orthogonal to all $v(u)$, $\mathcal{V}_4(\mathcal{U})$ is closed. By Proposition 8, $\mathcal{V}_4(\mathcal{U})$ is a root system. Suppose that $\mathcal{V}_4(\mathcal{U})$ is a frame, i.e. $\mathcal{V}_4(\mathcal{U}) = \cup_i\{\pm 2e_i\}$. Then, by Proposition 26(i), $\mathcal{U} = \cup_i\{\pm(e \pm 2e_i)\}$. The inclusion $e \in L(\mathcal{U})$ implies

$$e = \sum_i z_i^+(e + 2e_i) + \sum_i z_i^-(e - 2e_i) = \sum_i (z_i^+ + z_i^-)e + \sum_i (z_i^+ - z_i^-)2e_i.$$

Since all e_i and e are mutually orthogonal, $z_i^+ - z_i^- = 0$. Hence $z_i^+ = z_i^-$, and $2\sum_i z_i^+ = 1$. But this is a contradiction, since z_i^+ is an integer. Now, by Lemma 25, $\mathcal{V}_4(\mathcal{U})$ contains a star.

$(iv) \Rightarrow (i)$. Since $2e \in \mathrm{cl}_4\mathcal{U}$, by Proposition 26, $v \pm e \in \mathcal{U}$ for all $v \in \mathcal{V}_4(\mathcal{U})$. Let $\mathcal{K} = \{v_1, v_2, v_3\} \subseteq \mathcal{V}_4(\mathcal{U})$ be a star. Then the vectors $u_1 = e + v_1$, $u_2 = e + v_2$, $u_3 = e - v_3$ belong to $\mathcal{U}$. It is easy to see that these vectors satisfy (i). □

As we saw at the beginning of this section, any root system determines an odd system of norm 5 with $\mathrm{cl}_1\mathcal{U} \neq \emptyset$. Since $\beta\mathcal{V}_4(\mathcal{U})$ is a root system, Proposition 26 implies the following characterization of odd systems of norm 5 with $\mathrm{cl}_1\mathcal{U} \neq \emptyset$.

Theorem 28 *There is one-to-one correspondence between closed uniform odd systems $\mathcal{U}$ of norm 5 with $cl_1\mathcal{U} \neq \emptyset$ and root systems $\beta\mathcal{V}$. This correspondence is such that if $cl_1\mathcal{U} = \{\pm e\}$, then $cl_4\mathcal{U} = \{\pm 2e\} \cup \mathcal{V}$, and $\mathcal{U} = \{v \pm e : v \in \mathcal{V}\}$, where e is orthogonal to the space spanning $\mathcal{V}$.* □

Now we consider the case $\mathrm{cl}_1\mathcal{U} = \emptyset$ and distinguish the case when $\mathcal{V}_4(\mathcal{U})$ is closed.

Proposition 29 *Let $\mathcal{U}$ be a closed uniform pillar odd system of norm 5 with the shaft e and $cl_1\mathcal{U} = \emptyset$. If $\mathcal{V}_4(\mathcal{U})$ is closed, then $\mathcal{V}_4(\mathcal{U})$ is a frame $\cup_{i=1}^k\{\pm 2e_i\}$, and exactly one of (i), (ii) below is true.*

(i) $\mathcal{U} = \cup_{i=1}^k\{\pm(e \pm 2e_i\}$, and $cl_4\mathcal{U} = \{\pm 2e\}$,

(ii) $\mathcal{U} = \cup_{i=1}^k\{\pm(e + 2e_i\}$ and $cl_4\mathcal{U} = \emptyset$.

Proof. If $\mathcal{V}_4(\mathcal{U})$ is closed, then $\beta\mathcal{V}_4(\mathcal{U})$ is a root system. According to Proposition 27(iv) and Lemma 25, the equality $\mathrm{cl}_1\mathcal{U} = \emptyset$ implies that $\mathcal{V}_4(\mathcal{U})$ is a frame, say $\mathcal{V}_4(\mathcal{U}) = \cup_1^k\{\pm 2e_i\}$.

For each $v \in \mathcal{V}_4(\mathcal{U})$ we have either $v + e \in \mathcal{U}$ and $v - e \in \mathcal{U}$, or only one of these vectors belongs to $\mathcal{U}$.

If there is $v \in \mathcal{V}_4(\mathcal{U})$ such that $v + e, v - e \in \mathcal{U}$, then by Proposition 26 $\mathcal{U} = \cup_1^k\{\pm(e \pm 2e_i)\}$.

Conversely, if $\mathrm{cl}_4\mathcal{U} \neq \emptyset$, and $v \in \mathrm{cl}_4\mathcal{U}$, then $v = \sum_i z_i^+(e + 2e_i) + \sum_i z_i^-(e - 2e_i) = \sum_i(z_i^+ + z_i^-)e + \sum_i(z_i^+ - z_i^-)2e_i$ and $4 = v^2 = (\sum_i(z_i^+ + z_i^-))^2 + 4\sum_i(z_i^+ - z_i^-)^2$. This implies that $(z_i^+ - z_i^-)^2 = 0$, i.e. $z_i^+ = z_i^-$. Hence $v = 2\sum_i z_i^+ e = \pm 1$, i.e. $v = \pm 2e$.

Now, let $\mathrm{cl}_4\mathcal{U} = \emptyset$. Then above reasoning implies that only one vector from $v \pm e$ belongs to $\mathcal{U}$. Since $-u \in \mathcal{U}$ if $u \in \mathcal{U}$ and $-v \in \mathcal{V}_4(\mathcal{U})$ if $v \in \mathcal{V}_4(\mathcal{U})$, reversing, if necessary, the sign of v, we can suppose that $v + e \in \mathcal{U}$ for all $v \in \mathcal{V}_4(\mathcal{U})$. This means that $\mathcal{U} = \cup_{i=1}^k\{\pm(2e_i + e)\}$. □

The case of pillar $\mathcal{U}$ with $\mathrm{cl}_1\mathcal{U} = \emptyset$ and not closed set $\mathcal{V}_4(\mathcal{U})$ is much more complcated. In the case, $\beta\mathcal{V}_4(\mathcal{U})$ is a set of roots, but not a root system which is closed.

8 Closed odd systems of norm 5 spanning equiangular lines

Let $\mathcal{U}$ be a closed uniform odd system of norm 5 such that $uu' = \pm 1$ for distinct $u, u' \in \mathcal{U}$. Since $\mathrm{cl}_3\mathcal{U} = \emptyset$ by Collorary 14, and $\mathrm{cl}_4\mathcal{U} = \emptyset$ by Lemma 18, Theorem 17 can be reformulated as follows.

Theorem 30 *Let $\mathcal{U}$ be a set of vectors. The following assertions are equivalent*

(i) $\mathcal{U} = \mathcal{U}_5(v_0, L)$ for a double even lattice L of minimal norm 8 with $v_0 \in L$ of norm 12,

(ii) $\mathcal{U}$ is closed uniform odd system of norm 5 such that $uu' = \pm 1$ for $u, u' \in \mathcal{U}$, $u \neq \pm u'$, and $cl_1\mathcal{U} = \emptyset$, i.e. $\mathcal{U}$ spans a set of equiangular lines at angle $\arccos\frac{1}{5}$. □

As an example, we consider regular uniform odd systems of norm 5 spanning equiangular lines. These systems are in one-to-one correspondence with regular two-graphs with minimal eigenvalue -5.

In Table 2 below we give dimensions for which such regular two-graphs are known (see [8]).

In each dimension, we know only one closed odd system. This closed system corresponds to the two-graph with a doubly-transitive automorphism group. The corresponding lattices $L(\mathcal{U})$, $L_0(\mathcal{U})$ and $L_2(\mathcal{U})$, which can be identified up to a multiple γ with known lattices, are given in Table 2. The Q-lattices are described in [4]. The lattices Λ_{16} and Λ_{24} are Barnes-Wall and Leech lattices, respectively (see [3]). For $\dim\mathcal{U} = 5$, $\gamma = \frac{1}{\sqrt{6}}$, and $\gamma = \beta = \frac{1}{\sqrt{2}}$ for other $\dim\mathcal{U}$.

The lattice $\gamma L_2(\mathcal{U})$, for $\dim\mathcal{U} = 10$, is a sublattice of the Barnes-Wall lattice Λ_{16}. Similarly, the lattices $\gamma L_2(\mathcal{U})$ for $\dim\mathcal{U} = 21, 22$ are sublattices of the Leech lattice Λ_{24}.

If $\dim\mathcal{U} = 5$, then $\mathcal{U}$ is a star. Let $\mathcal{U}(t)$ be the star of norm $2t + 1$. In Table 2 we include the lattices related to $\mathcal{U}(t)$, too. For $\mathcal{U} = \mathcal{U}(t)$, $\gamma = \frac{1}{\sqrt{2t+2}}$.

Table 2

$\dim\mathcal{U}$	$2t+1$	5	10	13	15	21	22	23
$n(\mathcal{U})$	$2t$	6	16	26	36	126	176	276
$\gamma L(\mathcal{U})$	$A^*_{2t+1} = A^{+(2t+2)}_{2t+1}$	$A^*_5 = A^{+6}_5$		$Q_{13}(2)^{+2}$				$Q_{23}(6)^{+2}$
$\gamma L_0(\mathcal{U})$	$A^{+(t+1)}_{2t+1}$	A^{+3}_5		$Q_{13}(2)$				$Q_{23}(6)$
$\gamma L_2(\mathcal{U})$					Λ_{16}			Λ_{24}

Baranovskii in [1] studies the L-polytope affinely spanned by $\mathcal{U}(t)$ and the lattice $\gamma L_0(\mathcal{U}(t))$. In particular, he proves that the lattice $\gamma L_0(\mathcal{U}(4))$ has minimal covering radius between all integral lattices of dimension 9.

Note that other regular uniform odd systems with ± 1 inner products (=regular two-graphs) are known only for dimensions 13 and 15. There are exactly 3 nonisomorphic systems of dimension 13 and, as it is asserted in [8], one knows 227 systems of dimension 15. These odd systems are partitioned into classes of systems with the same closure. Besides, for all these systems $\mathcal{U}$, acording to Lemma 18, $\mathrm{cl}_4\mathcal{U} \neq \emptyset$. Hence odd systems with distinct closure are distinguished by the root system $\beta\mathrm{cl}_4\mathcal{U}$. We denote below, following to [3], by A^k_n the direct sum of k root systems A_n.

The 3 odd systems of dimension 13 have 2 closeness with $\beta\mathrm{cl}_4\mathcal{U} = A_{12}$ and $\beta\mathrm{cl}_4\mathcal{U} = A^3_4$.

We find 5 different closeness of odd systems $\mathcal{U}$ of dimension 15:

for $\mathcal{U}$ related to Latin squares, $\beta\mathrm{cl}_4\mathcal{U} = A^3_5$, and A^9_1,

for $\mathcal{U}$ related to Steiner triple systems, $\beta\mathrm{cl}_4\mathcal{U} = A_{14}$ and A^7_1,

for $\mathcal{U}$ founded by E.Spence (see [8]), $\beta\mathrm{cl}_4\mathcal{U} = A^7_1$ and $A^9_1 \oplus D_4$.

These odd systems will be described in details elsewere.

References

[1] E.P.Baranovskii, *Perfect lattices* $\Gamma(\mathcal{A}^n)$*, and covering density of* $\Gamma(\mathcal{A}^9)$, to appear in *Europ. J. of Combinatorics* (1994)

[2] P.J.Cameron, J.M.Goethals, J.J.Seidel, E.E.Shult, *Line graphs, Root systems, and Elliptic Geometry*, Journal of Algebra **43** (1976) 305–327.

[3] J.H.Conway, N.J.A.Sloane, *Sphere packing, lattices and groups*, Grundlagen der Mathematischen Wissenschaften 290, Springer-Verlag, Berlin et al., 1987.

[4] J.H.Conway, N.J.A.Sloane, *Low-dimensional lattices. II. Subgroups of GL(n,Z)*, *Proc. Roy. Soc. Lond.* A**419** (1988) 29–68.

[5] P.J.Conway, N.J.A.Sloane, *The cell structures of certain lattices*, in: *Miscellanea Mathematica* (eds.P.Hilton et al.), Springer-Verlag, 1991, pp. 71–107.

[6] M.Deza, V.P.Grishukhin, *L-polytopes and equiangular lines*, *Rapport de Recherche du LIENS*, **LIENS-92-26**, 1992, to appear in *Discrete Mathematics*.

[7] A.Neumaier, *On norm three vectors in integral Euclidean lattices*, *Math. Zeitschrift* **183** (1983) 565–574.

[8] J.J.Seidel, *More about two-graphs*, in: Fourth Czechoslovakian Symposium on Combinatorics, Graphs and Complexity, eds. J.Nesetril and M.Fiedler, *Annals of Discrete Mathematics*, Elsevier Sci. Publ. B.V. 1992, 297–308.

[9] E.Shult, A.Yanushka, *Near n-gons and line systems*, *Geometriae Dedicata* **9** (1980) 1–72.