Order Types and Visibility types of Configurations of Disjoint Convex Plane Sets

Extented Abstract

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Extended Abstract

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Abstract

We introduce the notions of order type (or: dual arrangement) and visibility type (or: tangent visibility graph) for configurations of disjoint convex sets in the plane. We develop optimal algorithms for computing and sweeping the order type, and also give a worst case optimal algorithm for computing the tangent visibility graph. The methods are based on a relation, introduced in this paper, between configurations of disjoint convex sets and arrangements of pseudolines. Finally we give enumeration results for the number of distinct order and visibility types.

1 Introduction

Configurations of points in the plane, or dually, arrangements of lines, have been studied extensively in discrete and computational geometry, during the last decade especially by Goodman and Pollack, see e.g. [3, 13] for background material on this topic. A characteristic feature of such a point set is its order type: with every ordered triple of distinct points one associates +1 (0, or -1) if the third point of the triple is to the left of (on, or to the right of) the directed line through the other points, directed from the first point to the second point.

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Every configuration of points corresponds, under duality, with an arrangement of lines. More generally, one may consider arrangements of pseudolines, viz. configurations of curves that intersect pairwise in exactly one point (and moreover satisfy some technical conditions justifying the name pseudoline). One of the main questions concerns realizability (see [3]) by arrangements of straight lines (also called stretchability): given a configuration of pseudolines, is it isomorphic to an arrangement of straight lines? It is known that 'most' arrangements of pseudolines are not stretchable. The realizability question is NP–hard, see [26]. Since pseudoline arrangements are identical with (reorientation classes of) rank 3 simple oriented matroids, see [3], pages 248–249, this implies that most rank 3 simple oriented matroids are not realizable by configurations of points (or, equivalently, by arrangements of straight lines).

In this paper we study, more generally, configurations of disjoint convex sets (objects, for short) in the plane. Such configurations arise naturally in the context of visibility and shortest path problems (which were actually our starting point, see [20]). For such configurations we also introduce the notion of *order type*, which generalizes the order type of configurations of points (in the sense that the order type depends only on the order types of triples) as well as the concept of *visibility type*, which generalizes the visibility graph of a configuration of line segments (where the order of visible segments around each endpoint is taken into account).

Section 2 is concerned with a *unifying presentation* of both concepts and their relation with arrangements of pseudolines. The order type is defined as the combinatorial dual arrangement, in the space of directed lines ($= \mathcal{S}^2$), of the curves of tangent lines to the objects. Similarly the visibility type is defined as the combinatorial arrangement, in a suitable space (a quotient of the space of free line segments), of the curves of free line segments tangent to the objects; this latter arrangement, called the *visibility complex*, was introduced in [20]. The complement (in their convex hull) of the union of n disjoint convex obstacles can be subdivided into finitely many so-called pseudotriangles, by inserting a maximal number of pairwise disjoint free bitangents of pairs of objects (a free bitangent is a line segment that is tangent to two obstacles at its endpoints, but whose relative interior is disjoint from any of the objects, see also figure 4). Representing a pseudotriangle by its set of tangent lines, we obtain a curve (the canonical image of the pseudotriangle) in the dual plane. Two distinct pseudotriangles are disjoint, and share exactly one tangent line. In other words: their canonical images intersect in exactly one point. Therefore the set of canonical images of all pseudotriangles is an arrangement of pseudolines, embedded in the dual arrangement of the objects. A careful study of this embedding yields generalizations of existing algorithms to the context of convex objects. More specifically, using the well-known techniques for constructing (as well as topologically sweeping) arrangements of lines (see [7]) and an amortization scheme we describe, in section 2, algorithms for the construction of the order type of a collection of n disjoint convex objects in optimal $O(n^2)$ time (and linear working space for the sweep). The same technique can be applied to obtain worst case optimal (viz. $O(n^2)$) and almost optimal output sensitive (viz. $O(k \log n)$, where k is the size of the output) algorithms for computing visibility types, cf. [19]. For an optimal algorithm we refer to [20].

A collection of pseudotriangles whose dual image is isomorphic to a given arrangement of pseudolines will be called a *realization* of this arrangement. If we allow the pseudotriangles to intersect without violating the property that every pair has exactly one common tangent line, the dual image of such a configuration is again an arrangement of pseudolines. We show, in section 3, that any arrangement of pseudolines can be realized by a collection of pseudotriangles (in fact by giving an algorithm). Our conjecture is that it can even be realized by a collection of *disjoint* pseudotriangles, but so far we have only been able to prove this for a large class, of size 2^{cn^2} for some constant c, of arrangements of n pseudolines. This result may be regarded as a new *geometric interpretation* for the class of all rank 3 acyclic matroids.

In section 4 we derive upper and lower bounds for the number of order types and visibility types of configurations of n convex obstacles: both are of the form $2^{cn^2(1+o(1))}$, for some positive constant c. If we only consider convex objects whose boundaries are algebraic curves of degree at most d, both upper and lower bounds are of the form $2^{c(d)(n \log n + o(1))}$, where $c(d)$ depends on the complexity d of the objects.

Arrangements have been applied to a wide range of by now classical problems in discrete and computational geometry, especially to configurations of points, line segments and polygons, see e.g. [14]. The visibility type (or more precisely its geometric version, the visibility complex) can be applied to many problems arising in the context of convex objects, like computing the view (visibility polygon) from a point and computing shortest paths, see [20] and the references therein. We mention a related application, viz. the characterization of minimal tangent visibility graphs (cf. [24]). We finally apply the key concept of pseudo–triangulation to give a new, simple solution of the Fejes-Tóth illumination problem (see [10]).

2 Order types, visibility types and dual arrangements

Terminology and notation

We identify the point (a, b, c), with $c \neq \pm 1$, on the 2-sphere $\mathcal{S}^2 = \{(x, y, z) \in \mathcal{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ with the directed line with equation $ax + by + c = 0$ and direction $u = (-b/(a^2+b^2)^{1/2}, a/(a^2+b^2)^{1/2}) \in \mathcal{S}^1 (\approx \mathcal{R}/2\pi\mathcal{Z})$, see figure 1. The north pole $(0,0,1)$ and south pole $(0,0,-1)$ of $\mathcal{S}^2$ are called the two lines at infinity.

A bounded convex subset O of $\mathcal{R}^2$ is called a convex *object* if it is closed and has interior points. Two disjoint convex objects O and O' have exactly 8 common (directed) tangent lines. We denote by $L(\epsilon O, \epsilon' O')$, with $\epsilon, \epsilon' \in \{+,-\}$, the tangent line that is directed from O to O' and contains the objects O and O' in their left or right half–planes according to the sign ϵ and ϵ' in front of O and O'. We denote by $B(\epsilon O, \epsilon' O')$ the bitangent line segment, supported by the line $L(\epsilon O, \epsilon' O')$ (in other words: the endpoints of the bitangent are the points of tangency on $L(\epsilon O, \epsilon' O')$).

Consider a collection $\mathcal{O} = O_1, \ldots, O_n$ of $n \geq 2$ pairwise disjoint convex objects. Each object is *strictly* convex, and has a *smooth* boundary. We assume that the objects are in *general position*, in the sense that no three objects share a common tangent line. The closure $\mathcal{F}$ of the complement of the union of the objects is called *free space*; a subset of $\mathcal{F}$ is called free. Two points p and q are said to be (mutually) *visible* (along the direction u) if the line segment $[p,q]$ is free (and is parallel to u). Two objects are *weakly visible* if two of their points are visible. The endpoints of the free bitangents subdivide the boundaries of the convex sets into a sequence of arcs, which are, with the (undirected) bitangents, the edges of the so-called *tangent visibility graph*. The *weak visibility graph* of the collection of objects is the graph whose nodes are the objects and whose arcs are pairs of weakly visible objects.

Order type of a configuration

We denote by γ_i (γ_{-i}) the set of tangent lines of object O_i, that contain O_i in their left (right) half plane; obviously $\gamma_{\pm i}$ is homeomorphic to $\mathcal{S}^1$. Note that the two intersection points of the curves $\gamma_{\epsilon i}$, $\gamma_{\epsilon' j}$, $i \neq j$, are the lines $L(\epsilon O_i, \epsilon' O_j)$ and $L(\epsilon' O_j, \epsilon O_i)$.

The curves $\gamma_{\pm i}$ induce a 2-dimensional cell decomposition of the space of lines ($= \mathcal{S}^2$), called the dual arrangement of the collection of objects and denoted by Γ. The vertices of Γ are the $4n(n-1)$ lines $L(\epsilon O_i, \epsilon' O_j)$ with $\epsilon, \epsilon' \in \{+,-\}$. We define the label of the vertex $L(\epsilon O_i, \epsilon' O_j)$ to be the symbol $(\epsilon i, \epsilon' j)$, and the *cycle* $C(i)$ is the circular sequence of labels of the vertices lying on the curve γ_i; for example if $n = 2$ then $C(1) = (1,2)(1,-2)(-2,1)(2,1)$ and $C(2) = (2,1)(2,-1)(-1,2)(1,2)$. In figure 2 we have depicted a configuration of 3 objects. Here e.g. $C(1) = (1,2)(1,3)\ (1,-2)(-2,1)\ (1,-3)(-3,1)\ (2,1)(3,1)$. Obviously $C(-i)$ and $C(+i)$ are related by the transformation $(k,l) \mapsto (-l,-k)$. All lines in a face of Γ pierce the same collection of objects in the same order; this ordered sequence of (indices of) pierced objects is called the label of the face. Let $P(\Gamma)$ be the poset of cells of Γ, partially ordered by defining $\sigma_i \leq \sigma_j$ if $\overline{\sigma}_i \subseteq \overline{\sigma}_j$. This poset inherits the orientation from the underlying space of lines (hence it makes sense to speak of the face *above/below* an edge or to speak of the *left/right* endpoint of an edge). An edge lying on the curve $\gamma_{\pm i}$ is labeled by $\pm i$, which is called its *canonical label*.

Definition 1 *The order type of a configuration is its oriented poset $P(\Gamma)$, that is augmented with the canonical labeling of edges.*

Precisely because an order type is (by definition) oriented it has only one orientation-preserving embedding in the 2–sphere. It turns out that its orientation depends only on the set of cycles.

Proposition 2 *The order type of a collection of n disjoint convex objects is uniquely determined by its set of cycles, and conversely. Furthermore the labels of the faces of the dual arrangement are uniquely determined by the order type.* □

Our definition of order type generalizes the notion of order type for configurations of point sets (see [12, 13, 15]); indeed for point sets the curves γ_{+i} and γ_{-i} collapse and the dual arrangement Γ reduces, via the classical duality which maps the point (a,b) onto the line $y = ax - b$, to the arrangement of *dual lines*, provided we restrict directions to $[0,\pi)$ in stead of $\mathcal{S}^1$ It is well-known that this arrangement is uniquely determined by the set of signs of determinants of triples of points, and conversely. This last (obvious) result has the following (less obvious) counterpart in the context of convex objects.

Theorem 3 *The order type of a collection of n disjoint convex objects is uniquely determined by the $\binom{n}{3}$ order types of triples of objects.*

Proof (sketch). First we prove that the order type depends only of the $\binom{n}{4}$ order types of 4–tuples of convex objects. Then we prove by a case analysis that the order type of 4 objects depends only on the order types of triples of objects. See the full version for a complete proof.□

Remark In the full version we show that there are 118 order types of triples of objects. If we forget about the labeling and the orientation, there are 16 different order types.

Visibility type of a configuration

The definition of *visibility type* is very similar to that of the order type. Since the underlying space (viz. the set V of maximal free line segments, see below) has a slightly more complicated topology than the space of lines, our rather concise description may cause intuitive difficulties to the uninitiated reader; we refer to [20] for a more detailed presentation. However most of the rest of the paper can be understood if the reader is willing to accept the alternative representation by the collection of visible cycles, see proposition 5.

The space of *maximal free directed line segments*, denoted by V, is a topological space that is defined as follows. Let X denote the cartesian product $\mathcal{F} \times \mathcal{S}^1$. We then form a quotient space of X by identifying the points (p, u) and (p', u) if the points p and p' are visible along the direction u. The set of pairs that are equivalent with $(p, u) \in X$, will be called a *maximal free line segment* (through p with direction u). Note that a maximal free line segment corresponds with a well defined geometric line segment in the plane, that does not intersect the interior of any object, but whose endpoints lie on the boundary of some object. Occasionally we shall abuse language by calling the equivalence class itself a maximal free line segment. Similarly we shall speak of the directed line supporting a maximal free line segment, etc.

There is a canonical map $\pi : V \to \mathcal{S}^2$, mapping the equivalence class of (p, u) onto the directed line $p + \mathcal{R}u$ (recall that $\mathcal{S}^2$ is the set of directed lines). Note that π is a kind of branched covering map: the pre–image $\pi^{-1}(l)$ of a line $l \in \mathcal{S}^2$ consists of the maximal free line segments in l.

Now consider the curves $\gamma_{\pm i} \subset \mathcal{S}^2$. The pre–image $\pi^{-1}(l)$ of a directed line $l \in \gamma_{\pm i}$ consists of the maximal free line segments contained in l, exactly one of which, denoted by $\varphi_{\pm i}(l)$, is tangent to O_i. $\varphi_{\pm i}(l)$ ranges over a curve $\varphi_{\pm i} \subset V$ as l ranges over $\gamma_{\pm i}$. This curve is homeomorphic to $\mathcal{S}^1$. The curves $\varphi_{\pm i}$ induce a 2-dimensional cell decomposition of the space V, called the *visibility complex* of the collection of objects (introduced in [20]), which is denoted by Γ_V. A maximal free line segment lies on a face (edge, vertex) if it is tangent to 0 (1, at least 2) objects. An edge lying on the curve $\varphi_{\pm i}$ is labeled with $\pm i$. Obviously π maps the 1–skeleton of Γ_V onto the 1-skeleton of Γ: if a maximal free segment is tangent to at least one object, then so is its supporting line.

Every edge of the visibility complex is incident with three faces. To see this, consider an edge e, of Γ_V, say with label $\pm i$. Let $l \in e$, a maximal free line segment tangent to O_i. If we perturb the line supporting l slightly so that it intersects the interior of O_i, the perturbed line contains two maximal free line segments, l_1 and l_2 say, that are incident with O_i at one of their endpoints, and are not tangent to any object. Therefore l_1 and l_2 belong to distinct faces of Γ_V, that are incident upon e. If we perturb the line supporting l slightly so that it is disjoint from O_i, the perturbed line contains one maximal free line segment l_3 near l, that is not tangent to any object, and hence belongs to a third face of Γ_V that is incident upon e. In the same way one may prove that every vertex is incident with 4 edges and 6 faces, see figure 3, and also [20] for further details.

Observe that the vertices of Γ_V are in 1-1 correspondence with the set of directed arcs of the tangent visibility graph that correspond to free bitangents. Similarly the edges of the visibility complex are in 1-1 correspondence with the directed arcs of the tangent visibility graph, that are contained in the boundaries of the objects. Each face of the visibility complex induces a directed edge of the weak visiblity graph, corresponding to the pair of objects incident with any maximal free line segment belonging to this face (of course several faces can define the same edge of the weak visibility graph).

We define the label of a vertex $B(\epsilon O_i, \epsilon' O_j)$ to be the pair $(\epsilon i, \epsilon' j)$, and the *visible cycle* $C_V(\pm i)$ to be the circular sequence of labels of vertices lying on the curve $\varphi_{\pm i}$; obviously $C_V(i)$ is a subword of the cycle $C(i)$ and $C_V(-i)$ and $C_V(+i)$ are related by the mapping $(k,l) \mapsto (-k,-l)$. In figure 2 we have e.g. $C_V(1) = (1,2)(1,-2)$ $(-2,1)(1,-3)$ $(-3,1)(3,1)$ (we have to remove $(1,3)$ and $(2,1)$ from $C(1)$).

Let $P(\Gamma_V)$ be the poset of cells of Γ_V, ordered by inclusion, and edowed with the orientation inherited from the underlying space of maximal free line segments.

Definition 4 *The visibility type of a configuration is its oriented poset $P(\Gamma_V)$, augmented with the canonical labeling of edges.*

Proposition 5 *The visibility type of a configuration of n disjoint convex objects is uniquely determined by its set of visible cycles, and conversely.* □

Since $C_V(i)$ is a subword of $C(i)$ the visibility type can be considered as a "partial" order type. In particular it depends only on the order type of the collection of objects.

Pseudotriangles, pseudo–triangulations, and their relation with arrangements of pseudolines

A *pseudotriangle* is a simply connected bounded subset T of $\mathcal{R}^2$ such that (i) the boundary ∂T is a sequence of three convex curves that are tangent at their endpoints, and (ii) T is contained in the triangle formed by the three endpoints of these convex curves. Clearly there exists a unique tangent line to the boundary of T with a given direction; we denote by T^* the set of (directed) tangent lines to the boundary of T. A crucial observation is the following.

Lemma 6 *Let $T_1, \ldots, T_n$ be a family of pairwise disjoint pseudotriangles. Then the T_i^* are pseudocircles (i.e., simple closed curves in $\mathcal{S}^2$) and the family $(T_i^*)_{1\ldots n}$ is an arrangement of pseudocircles (i.e. T_i^* and T_j^* intersect transversally in exactly two points, and the two points in $T_i^* \cap T_j^*$ are separated by T_k^*, for some $k \neq i,j$).*

Proof. Indeed T_i^* is centrally symmetric (two tangent lines with opposite directions have the same supporting undirected line), and two tangent lines to the boundary of a pseudotriangle cross inside the pseudotriangle. □

A *pseudo–triangulation* of the set of objects is a subdivision of the plane induced by a maximal (with respect to inclusion) family of pairwise non–crossing free bitangents, see figure 4. It is clear that a pseudo–triangulation always exists and that it contains the bitangents of the convex hull of the objects. Pseudo–triangulations are interesting because they decompose free space into pseudotriangles.

Lemma 7 *Let B be a family of pairwise non-crossing free bitangents of a collection of n objects, and let S be the subdivision of the plane induced by B and the set of objects. The following assertions are equivalent*

1. B is maximal (with respect to inclusion);

2. each free bounded face of S is a pseudotriangle;

3. the number of free bounded faces of S is $2n-2$;

4. the number of bitangents in B is $3n-3$. □

Finally we mention the following result from [20].

Theorem 8 *[20].*
There is a pseudo–triangulation of a collection of n disjoint convex objects that can be computed in $O(n \log n)$ time. □

Remark. Note that the boundary of a pseudotriangle can be "illuminated" by at most 2 points (two of the intersection points of the tangent lines at the vertices); therefore we can deduce that $4n-7$ is the maximal number of points (in free space) required to illuminate the boundary of $n \geq 4$ disjoint convex objects. This result is due to Fejes-Tóth [10]. The use of a pseudo–triangulation provides an alternative proof and, furthermore, gives an $O(n \log n)$ time algorithm to find a placement of the illuminating points (cf. [10, 27]).

Construction of dual arrangements
We now give an optimal algorithm to compute the dual arrangement of a collection of disjoint convex objects, assuming that the common tangents of a pair of objects are computable in $O(1)$ time.

Theorem 9 *The dual arrangement of n disjoint convex objects can be constructed in $O(n^2)$ time and space.*

Proof. We first compute in $O(n \log n)$ time a pseudo–triangulation of the objects. Let $T_1, \ldots, T_{2n-2}$ be the $2n-2$ pseudotriangles of this pseudo–triangulation. Clearly the arrangement of the curves T_i^* coincides with the arrangement of the curves $\gamma_{\pm i}$ (up to some trivial details concerning the convex hull). To compute the arrangement of the curves T_i^* we use the optimal incremental technique which have been developed for constructing arrangement of (pseudo)lines [2, 6, 9] (we omit trivial details concerning the $3n-3$ touching points between the curves T_i^* and T_j^* for adjacent pseudotriangles T_i and T_j in the pseudo–triangulation); however we have to be careful because the intersection of two pseudocircles T_i^* and T_j^* is not computable in $O(1)$ time unless the complexities of the pseudotriangles T_i and T_j are $O(1)$. Let n_i be the complexity (number of objects which contribute to the boundary of the pseudotriangle) of the pseudotriangle T_i; we note that $\sum_i n_i = O(n)$. It follows that the complexity of the zone of a curve T_i^* is still linear and that the curve T_i^* can be inserted in time $O(n + n_i)$. Consequently the incremental algorithm is still quadratic. □

At this point it is an open probem whether the zone of a curve γ_i is linear in size. A positive answer to this question will give a fully incremental quadratic algorithm. However, the relation with arrangements of pseudolines allows us to adapt the topological sweep technique of [7, 8]. In the companion paper [22] we show how, for a configuration of n objects, this sweep can be performed consistently in $O(n^2)$ time and linear storage. Several applications of this sweep technique are described in [7]; they can be translated into similar results in the context of disjoint convex objects. In particular we get a worst case optimal algorithm to compute the tangent visibility graph of n disjoint convex objects which uses only linear working storage. Thus we obtain:

Corollary 10 *The tangent visibility graph of a collection of n disjoint convex objects is computable in $O(n^2)$ time and linear storage.* □

Similarly we can extend the $O(k \log n)$ time (and linear storage) algorithm of [19], that computes the visibility graph (of size k) of a collection of n disjoint segments; this seems especially interesting (for implementation issues) in view of the comparison of this algorithm with a straight sweep technique (see [22, 23]). An optimal time algorithm to compute the visibility complex of n disjoint convex objects is described in [20].

3 Equivalence of arrangements of pseudolines and configurations of pseudotriangles

Every arrangement of pseudolines is realizable by a configuration of pseudotriangles.

Theorem 11 *1. Every arrangement of straight lines is isomorphic to the dual image of a configuration of disjoint pseudotriangles.*
2. Every arrangement of pseudolines is isomorphic to the dual image of a collection of pseudotriangles.

Remark 12 It follows from our results in section 4, see corollary 18, that the class of arrangements (of n pseudolines) that is realizable by *disjoint* pseudotriangles is rather large, viz. of size at least 2^{cn^2} for some constant $c > 0$. We conjecture that in fact any arrangement of pseudolines belongs to this class, in which case part 1 of theorem 11 becomes obsolete.

Proof. 1. If the arrangement of lines is simple, it corresponds, under duality, to a simple configuration of points. Put small pseudotriangles at each of the points. The dual image of this configuration of pseudotriangles is isomorphic to the arrangement of pseudolines. The same procedure, carried out a little more carefully, also works if the arrangement of lines is not simple.

2. For convenience we use coordinates x_1, x_2 in the primal plane, and ξ_1, ξ_2 in the dual plane. The map $\mathcal{D}$ from the set of non–vertical lines in the primal plane to the set of points in the dual plane maps the line with equation $x_2 = \xi_1 x_1 + \xi_2$ to the point (ξ_1, ξ_2). By definition, the dual image of a pseudotriangle is the image under $\mathcal{D}$ of its set of tangent lines.

Given an arrangement $\mathcal{A}$ of pseudolines, the idea is to construct in the strip $-1 \leq \xi_1 \leq 1$ in the dual plane an arrangement $\mathcal{C}$ of *convex curves*, that is equivalent to $\mathcal{A}$. We extend each curve in directions $\xi_1 = \pm\infty$ by semi–infinite segments contained in the line through the endpoints of the curve. It is not hard to see that the pre–image of such an extended curve under $\mathcal{D}$ is a pseudotriangle. This way we obtain a collection of pseudotriangles, whose dual image is isomorphic to $\mathcal{A}$.

Let us now describe the construction of the arrangement $\mathcal{C}$. The idea is related to the construction of *wiring diagrams*, see e.g. [13]. For convenience we assume that the arrangement $\mathcal{A}$ is simple. Consider an embedding of $\mathcal{A}$ in the (dual) plane, such that no two vertices have the same ξ_1 coordinate, and such that all pseudolines are monotone with respect to the ξ_1–direction. Let the pseudolines of $\mathcal{A}$ be $l_1, \cdots, l_n$, in the order in which they intersect the line $\xi_1 = -\infty$. The vertices of $\mathcal{A}$ are $v_1, \cdots, v_{n(n-1)/2}$, ordered according to increasing ξ_1–coordinate. The curves of $\mathcal{C}$ will be denoted by $c_1, \cdots, c_n$, such that c_i corresponds to l_i. Subdivide the strip $-1 \leq \xi_1 \leq 1$ into $n(n-1)+1$ vertical strips $S_1, \cdots, S_{n(n-1)+1}$. Strip S_{2i} will contain exactly one vertex of $\mathcal{C}$, corresponding to vertex v_i of $\mathcal{A}$. The intersection of the curves of $\mathcal{C}$ with strip S_{2i-1}, $1 \leq i \leq n(n-1)$, will consist of a sequence of n *parallel* line segments on curves $c_{i_1}, \cdots, c_{i_n}$, where $i_1, \cdots, i_n$ is the permutation of the indices $1, \cdots, n$ corresponding to the order in which $l_1, \cdots, l_n$ intersect any vertical line directly to the left of v_i.

For the intersection of $c_1, \cdots, c_n$ with strip S_1 we take any set of parallel non–vertical line segments. Suppose we have constructed the intersection with strips $S_1, \cdots, S_{2i-1}$, and suppose vertex v_i of $\mathcal{A}$ corresponds to the intersection of l_k and l_h, where l_h lies below l_k. Note that $c_h \cap S_{2i-1}$ and $c_k \cap S_{2i-1}$ are adjacent line segments, such that the former lies below the latter. For $j \neq h$ extend the line segment $c_j \cap S_{2i-1}$ until it intersects the right boundary of S_{2i}. The line segment $c_h \cap S_{2i}$ connects the intersection of c_h and the right boundary of S_{2i-1} with a point on the right boundary of S_{2i} lying just above $c_k \cap S_{2i}$. This way we introduce exactly one vertex in strip S_{2i}, viz. $c_k \cap c_h$. Now extend c_h until it hits the right boundary of S_{2i+1}, and, for $j \neq h$, extend c_j across S_{2i} by a line segment parallel to the latter line segment. The curves $c_1, \cdots, c_n$ are convex, which concludes the proof of 2. □

4 Enumeration of Configurations of Convex Sets

We derive upper and lower bounds for the number of order types and visibility types of configurations of n disjoint convex objects, both with and without restrictions on their complexity. If the convex objects are of bounded complexity it is likely that the the number of distinct configurations of n objects is smaller than $2^{\Theta(n^2)}$. In the special case where all objects are points the upper bound is $2^{\Theta(n \log n)}$, see [12] and [15], Chapter 9.

We shall say that an object is of degree d if its boundary is a connected component of an algebraic curve of degree at most d, i.e. a curve defined by an equation of the form $P(x, y) = 0$, where P is a polynomial in x and y of degree at most d. For a similar context, and some remarks on the computational model, see [1]. For simplicity we merely consider *simple* order types, corresponding to configurations in which no three objects share a common tangent.

Theorem 13 *The number of simple order types of n convex objects is*
1. $2^{O(n^2)}$*;*
2. $2^{O(d^2 n \log(dn))}$*, if the objects are of degree d.*

Proof. 1. In view of the optimal $O(n^2)$ algorithm that constructs an arrangement of n pseudo-

dolines an information–theoretic upper bound for the number of order types of configurations of n disjoint convex obstacles is $2^{O(n^2)}$. This holds under the assumption that the common tangents of two objects can be computed in $O(1)$ time, which is obviously not true if we don't bound the complexity of the objects. However, in the algebraic decision tree model, bitangent computations take place in computation nodes. If we contract the computation tree, so that it contains merely branch nodes, the number of leaves doesn't change, but its depth reduces to $O(n^2)$.
2. Let $P_i(x,y) = 0$ define the boundary of the i-th object, $1 \leq i \leq n$, where P_i is a polynomial of degree at most d in x, y. We shall express the geometric condition that three objects don't share a common tangent as a single polynomial equation in the coefficients of P_i, P_j, P_k. Multiplying the polynomials we get for each triple of indices together we obtain a single polynomial, R say, in the coefficients of $P_1, \cdots, P_n$. It is not hard to see that a simple order type is the union of a number of connected components of the complement of the zero locus of R. As in [12], an upper bound for the number of connected components of $R^{-1}(0)$ is derived using a theorem of Milnor. There is a small subtlety in this argument: in fact we are counting the number of configurations of curves defined by $P_i(x,y) = 0$, $1 \leq i \leq n$, of which no triple shares a common tangent. Since O_i is *merely a connected component* of the curve $P_i(x,y) = 0$, we are in fact over–counting the number of configurations (which doesn't hurt, since we are dealing with upper bounds).

The line $y = \xi x + \eta$ is tangent to the curve $P_i(x,y) = 0$ if the polynomial equation $p_i(x_1, \xi, \eta) := P_i(x_1, \xi x_1 + \eta)$ in x_1 has two coinciding real roots. In other words: (ξ, η) must be such that the system of equations $p_i(x_i, \xi, \eta) = 0$, $p_i'(x_i, \xi, \eta) = 0$ has a (real) solution (x_i, ξ, η). Similarly the line $y = \xi x + \eta$ is a common tangent of the curves $P_\mu(x,y) = 0$, $\mu = i, j, k$, if the system of 6 polynomial equations $p_\mu(x_\mu, \xi, \eta) = 0$, $p_\mu'(x_\mu, \xi, \eta) = 0$, $\mu = i, j, k$ (in 5 variables, viz. x_i, x_j, x_k, ξ, η), has a solution. By introducing a variable of homogeneity, this system is transformed into a system of 6 homogeneneous polynomial equations, of degree $d, d-1, d, d-1, d, d-1$, respectively, in 6 variables. This system has a non–trivial solution iff. the *multivariate resultant* R_{ijk} of the 6 polynomials vanishes, see [5] and [16], chapter 1. R_{ijk} is a polynomial of degree $C(d) := (d^3(d-1)^3)^5 = O(d^{30})$ in the coefficients of P_μ, $\mu = i, j, k$. Now we multiply all R_{ijk} together, where (i,j,k) ranges over all $\Theta(n^3)$ triples of indices in $[1,n]$. This yields a polynomial R of degree $\Theta(C(d)n^3)$ in the $\Theta(d^2 n)$ variables $a^l_{\mu\nu}$, $l = 1, \cdots, n$, $0 \leq \mu, \nu$ and $\mu + \nu \leq d$. According to a theorem of Milnor, see [18], the complement of the zero locus of R has at most $(2 + \Theta(C(d)n^3))(1 + \Theta(C(d)n^3))^{\Theta(d^2 n)}$ $= 2^{\Theta(d^2 n \log(dn))}$ connected components. This clearly is an upper bound for the number of simple order types of configurations of objects of degree d. □

The order type uniquely determines the visibility type. Therefore:

Corollary 14 *The number of visibility types of simple configurations of n convex objects is*
1. $2^{O(n^2)}$*;*
2. $2^{O(d^2 n \log(dn))}$*, if the objects are of degree d.*

Starting from an example in [11], Proposition 6.2, we derive a lower bound for the number of order types. More precisely we shall prove

Theorem 15 *The number of order types of configurations of n disjoint convex objects in the plane is at least*
1. $2^{n^2/8}$*;*
2. $2^{\Omega(dn \log n)}$*, if the objects are of degree $d = O(n^\alpha)$, for some fixed α with $0 \leq \alpha < 1$.*

In the proof we need the following lemma, whose (not very difficult) proof we omit from this version.

Lemma 16 *The number of labeled graphs with n vertices and maximal degree at most d is $2^{\Omega(dn \log n)}$, provided $d = O(n^\alpha)$, for some fixed α with $0 \leq \alpha < 1$.*

Remark For an asymptotically sharp result, under stricter conditions on d, we refer to [17].

Proof. We shall prove both parts simultaneously. Let $k = \lceil n/2 \rceil$. Consider a regular k–gon with vertices $p_1, \cdots, p_k$. Put a small circle C_i of radius ϱ centered at p_i. Here ϱ is small enough to guarantee that no line in the plane intersects more than 2 of the circles. Draw all common tangent lines of any pair of circles parallel to the sides and the diagonals of the k–gon (so for any pair of distinct circles we draw exactly 2 out of their 4 common tangent lines), see Figure 5. This set of lines is partitioned into k classes of parallel lines, denoted by $L_{k+1}, \cdots, L_{2k}$. All lines in class L_h are given the same, arbitrarily chosen, direction. Let $\mathcal{T}$ be the set of triples (i, j, h) such that $L(C_i, C_j) \in L_{|h|}$. So for $(i, j, h) \in \mathcal{T}$ we have $1 \leq i, j \leq k <| h |\leq 2k$, $i \neq j$, and $(i, j, h) \in \mathcal{T}$ iff. $(i, j, -h) \in \mathcal{T}$. Obviously $|\mathcal{T}| = \Theta(k^2)$. Let $V = \{1, \cdots, k\}$, and let $\mathcal{G}$ be the set of labeled undirected graphs, in the the first case, and the set of labeled undirected graphs of maximal degree not exceeding $d-2$ in the second case. The restriction on the degree will become clear from the construction below. For a pair (E, σ), such that $(V, E) \in \mathcal{G}$ and $\sigma : E \to \{-1, +1\}$, define $\tau_{(E,\sigma)} : \mathcal{T} \to \{-1, +1\}$ by

$$\tau_{(E,\sigma)}(i, j, h) = \begin{cases} \sigma(\{i, j\}), \text{if } \{i, j\} \in E, \\ -1, \text{if } \{i, j\} \notin E \text{ and } h > 0, \\ +1, \text{if } \{i, j\} \notin E \text{ and } h < 0. \end{cases}$$

Note that $\tau_{(E,\sigma)} \neq \tau_{(E',\sigma')}$ for $(E, \sigma) \neq (E', \sigma')$. Hence there are at least $\sum_{E:(V,E)\in\mathcal{G}} 2^{|E|} \geq |\mathcal{G}|$ such mappings $\mathcal{T} \to \{-1, +1\}$. Since the number of graphs in $\mathcal{G}$ is $2^{\binom{k}{2}}$, in the first case, and $2^{\Omega(dk \log k)}$ in the second case, the proof is complete, provided we show that every $\tau_{(E,\sigma)}$ is *realizable*. By this we mean that there is a collection of $2k$ disjoint convex objects $O_1, \cdots, O_{2k}$ such that $\tau_{(E,\sigma)(i,j,\pm h)}$ satisfies, for all triples $(i, j, \pm h) \in \mathcal{T}$ with $h > 0$:

condition ($\star$): $\tau_{(E,\sigma)}(i, j, \pm h) = 1(-1)$ if the support line of $\pm O_h$, parallel to $L(O_i, O_j)$, lies to the left (right) of $\pm O_h$.

(Recall that a support line of O_h $(-O_h)$ contains O_h in its left (right) half plane.

So let us describe the construction of the convex objects $O_1, \cdots, O_{2k}$ for some fixed $\sigma : E \to \{-1, +1\}$. These objects are obtained by

1. slightly perturbing the objects bounded by the circles C_i; this yields objects O_i, $1 \leq i \leq k$;
2. adding a convex object O_h, $k < h \leq 2k$, that intersects all lines in L_h ahead of the regular k–gon.

On each circle C_i, $1 \leq i \leq k$, we introduce a collection T_i of $2k$ disjoint small chords, centered at the points whose tangent lines are parallel to the sides and diagonals of the k–gon, see Figure 6.

Consider a triple $(i, j, h) \in \mathcal{T}$ with $h > 0$. Note that $L(C_i, C_j) \in L_h$. If $\{i, j\} \in E$ we perturb the line $L(C_i, C_j)$ into a line $L_{(E,\sigma)}(i, j)$, such that

(i) the tilt of $L_{(E,\sigma)}(i, j)$ with respect to $L(C_i, C_j)$ is $\pm\varphi$ if $\sigma(i, j) = \mp 1$;

(ii) $L_{(E,\sigma)}(i, j)$ intersects C_i (C_j) in the same chord of T_i (T_j) as $L(C_i, C_j)$.

It is not hard to see that there is a small $\varphi > 0$ satisfying these conditions, see also figure 6. The sign of the tilt is determined by our intention to insert a convex object O_h that intersects $L(C_i, C_j)$ ahead of C_i and C_j, and that lies to the right (left) of $L_{(E,\sigma)}(i, j)$ iff. $\tau_{(E,\sigma)}(i, j, h) = -1$ $(+1)$. If $\{i, j\} \notin E$ we take $L_{(E,\sigma)}(i, j) = L(C_i, C_j)$. Note that this way we perturb exactly d_i of the lines tangent at C_i, where d_i is the degree of $i \in V$ in the graph (V, E).

We first put, for $1 \leq i \leq k$, a convex object O_i at vertex p_i of the regular k–gon, that is tangent to all lines of the form $L_{(E,\sigma)}(i, j)$ or $L_{(E,\sigma)}(j, i)$. To this end consider the convex object O'_i bounded by the circle C_i, and the d_i lines $L_{(E,\sigma)}(i, j)$ or $L_{(E,\sigma)}(j, i)$, that intersect the interior of this circle. Note that we still have to perturb object O'_i so that its boundary becomes algebraic (and of degree at most d in case 2). However, all of the k lines of the form $L_{(E,\sigma)}(i, j)$ or $L_{(E,\sigma)}(j, i)$ are tangent to it.

We now introduce a convex object O_h (of degree $O(1)$) that

(i) intersects all lines $L(C_i, C_j) \in L_h$ ahead of C_i and C_j;

(ii) lies to the right (left) of $L_{(E,\sigma)}(i, j)$ iff. $\tau_{(E,\sigma)}(i, j, h) = -1$ $(+1)$.

Condition (i) implies that $\tau_{(E,\sigma)}(i, j, \pm h)$ satisfies condition $(\star)$, if $L_{(E,\sigma)}(i, j) \neq L(C_i, C_j)$, viz. if $\{i, j\} \notin E$. Condition (ii) implies that $\tau_{(E,\sigma)}(i, j, \pm h)$ satisfies condition $(\star)$, if $L_{(E,\sigma)}(i, j) = L(C_i, C_j)$, viz. if $\{i, j\} \in E$. Therefore the order type of the collection $\{O'_1, \cdots, O'_k, O_{k+1}, \cdots, O_{2k}\}$ is a realization of $\tau_{(E,\sigma)}$. It remains to smoothen O'_i, for $1 \leq i \leq k$.

Note that O_i' is a connected component of the set with equation $P_i(x,y) := c_i(x,y)\prod_{j=1}^{d_i} l_j^i(x,y) = 0$, where c_i is a quadratic form whose zero locus is the circle C_i, and l_j^i are linear forms defining the d_i lines $L_{(E,\sigma)}(i,j)$ or $L_{(E,\sigma)}(j,i)$. Assuming that P_i is positive at p_i, we see that the convex object O_i' can be approximated by the interior of a connected component of the curve $P_i(x,y) = \varepsilon$, for some sufficiently small positive ε. It is not hard to prove that any line intersects this component in at most two points. Therefore it is a convex curve, of degree $d_i + 2 \leq d$, in the second case. We take O_i to be its interior. □

In the full version of the paper we show that all configurations in the proof of theorem 15 have distinct visibility types. This shows:

Corollary 17 *The number of visibility types of configurations of n disjoint convex objects in the plane is at least*
1. $2^{n^2/8}$*;*
2. $2^{\Omega(dn\log n)}$ *if the objects are of degree* $d = O(n^\alpha)$*, for some fixed* α *with* $0 \leq \alpha < 1$.

Consider a pseudo–triangulation of any of the configurations constructed in the proof of theorem 15. Its dual image is an arrangement of pseudolines, that is obviously realizable by *disjoint* pseudotriangles. Since there are only $2^{O(n\log n)}$ different pseudo–triangulations for each configuration, we have, cf. remark 12:

Corollary 18 *The number of arrangements of n pseudolines that is realizable by disjoint pseudotriangles is* $2^{\Theta(n^2)}$.

Minimal visibility types Using simple properties of pseudo–triangulations we are able to caracterize the minimal visibility types on n disjoints convex objects. In the companion paper [21] we prove the following results.

Theorem 19 *The number of free bitangents shared by n disjoint convex objects in general position is at least $6n-6-h$ where h is the number of bitangents on the convex hull of the objects.* □

In particular the minimal number of free bitangents shared by n disjoint convex objects (in general poistion) is at least $4n-4$ since h is at most $2n-2$. For the same lower bound in the case of line segments: see [24], and also [4, 25].

Theorem 20 *There is a 1-1 correspondence between*
1. the set of minimal visibility types of n disjoint convex objects;
2. the set of maximal convex hulls of n disjoints convex objects;
3. the set of plane trees on n vertices.
Furthermore the set of minimal weak (labeled) visibility graphs on n disjoint objects is in 1-1 correspondance with the set of (labeled) trees on n nodes; their number is therefore n^{n-2}. □

5 Conclusion

We have shown that configurations of plane disjoint convex sets are strongly related (via the notion of pseudo–triangulation) to arrangements of pseudolines. This enables us to extend the classical algorithms, that compute order types of points and visibility graphs of line segments, to deal with collections of convex sets with the same (time and space) complexities. Our approach gives also new insights in the problem of the caracterization of visibility graphs.

Several natural questions arise from our study. Is every pseudoline arrangement realizable by a configuration of disjoint convex plane sets? We have shown that this is the case for a very large class of pseudoline arrangements. What is the complexity of the problem of deciding if a pseudoline arrangement is realizable by convex plane sets? An other challenge is to extend the ideas of this paper to the study of configurations of disjoint convex sets in more than 2 dimensions. It is to be expected that results in this direction also yield new points of view in the theory of oriented matroids.

References

[1] C. Bajaj and Myung-Soo Kim. Convex Hulls of Objects Bounded by Algebraic Curves. *Algorithmica*, 6 (4), 1991.

[2] M. Bern, D. Eppstein, P. Glassman, and F. Yao. Horizon theorms for lines and polygons. In J.E. Goodman, R. Pollack, and W. Steiger, editors, *Discrete and Computational Geometry: Papers from the DIMACS Special Year*, volume 6 of *DIMACS Series in Discrete Mathematics and Theoretical Computer Science*, pages 45–66. AMS-ACM, 1991.

[3] A. Björner, M. Las Vergnas, B. Sturmfels, N. White, and G. Ziegler. *Oriented Matroids*. Encyclopedia of Mathematics and Applications. Cambridge University Press, 1993.

[4] D. Campbell and J. Higgins. Minimal visibility graphs. *Information Processing Letter*, 37:49–53, 1991.

[5] J.F. Canny. *The Complexity of Robot Motion Planning*. The MIT Press, 1987.

[6] B. Chazelle, L. J. Guibas, and D. T. Lee. The power of geometric duality. *BIT*, pages 76–90, 1985.

[7] H. Edelsbrunner and L. J. Guibas. Topologically sweeping an arrangement. *J. Comput. System Sci.*, 38:165–194, 1989.

[8] H. Edelsbrunner and L. J. Guibas. Corrigendum. *J. Comput. System Sci.*, 42:249–251, 1991.

[9] H. Edelsbrunner, J. O'Rourke, and R. Seidel. Constructing arrangements of lines and hyperplanes with applications. *Siam J. Comput.*, 15:341–363, 1986.

[10] L. Fejes-Tóth. Illumination of convex discs. *Acta Math. Acad. Sci. Hungar.*, 29:355–360, 1977.

[11] J.E. Goodman and R. Pollack. Multidimensional Sorting. *Siam J. Comput.*, 12:484–507, 1983.

[12] J.E. Goodman and R. Pollack. Upper Bounds for Configurations and Polytopes in $\mathbb{R}^d$. *Discrete Comput. Geom.*, 1:219–227, 1986.

[13] J.E. Goodman and R. Pollack. *Allowable Sequences and Order Types in Discrete and Computational Geometry*, volume 10 of *Algorithms and Combinatorics*, chapter V. Springer Verlag, Berlin, Heidelberg, New York, 1993.

[14] L. Guibas and M. Sharir. *Combinatorics and Algorithms of Arrangements*, volume 10 of *Algorithms and Combinatorics*, chapter I. Springer Verlag, Berlin, Heidelberg, New York, 1993.

[15] D.E. Knuth. *Axioms and Hulls*, volume 606 of *Springer Lect. Notes in Comp. Science*. Springer Verlag, 1992.

[16] F.S. Macaulay. *Algebraic Theory of Modular Systems*. Cambridge University Press, London, 1916.

[17] B. D. McKay and N. C. Wormald. The asymptotic enumeration by degree sequence of graphs with degree $o(n^{1/2})$. Technical report, 1989.

[18] J. Milnor. On the Betti Number of Real Varieties. *Proc. Amer. Math. Soc.*, 15:275–280, 1964.

[19] M. H. Overmars and E. Welzl. New methods for computing visibility graphs. In *Proceedings of the 4th ACM Symposium on Computational Geometry*, pages 164–171, 1988.

[20] M. Pocchiola and G. Vegter. The Visibility Complex. In *Proceedings 9th ACM Symposium on Computational Geometry*, pages 328–337, San Diego, 1993.

[21] M. Pocchiola and G. Vegter. Minimal tangent visibility graphs. Technical report, Liens, 1994. in preparation.

[22] M. Pocchiola and G. Vegter. Sweep algorithms for visibility graphs of curved obstacles. Technical report, Liens, 1994. in preparation.

[23] S. Rivière. Experimental comparison of two algorithms for visibility graphs, 1993.

Diplôme d'Etude Approfondie, Ecole Norm. Sup., Paris.

[24] X. Shen and H. Edelsbrunner. A tight lower bound on the size of visibility graphs. *Information Processing Letter*, 26:61–64, 1987/88.

[25] X. Shen and Q. Hu. A note on minimal visibility graphs. *Information Processing Letter*, 46:101, 1993.

[26] P. Shor. Strechability of Pseudolines is NP-hard. In P. Gritzmann and B. Sturmfels, editors, *Applied Geometry and Discrete Mathematics: The Victor Klee Festschrift*, volume 4 of *DIMACS Series in Discrete Mathematics and Theoretical Computer Science*, pages 531–554. AMS-ACM, 1991.

[27] J. Urrutia and J. Zaks. Illuminating convex sets. Technical Report 89-31, University of Ottawa, 1989.

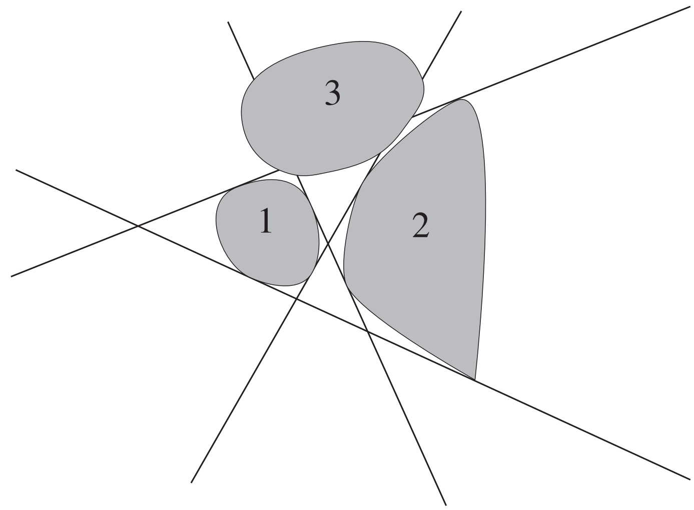

Figure 2: A configuration of 3 objects.

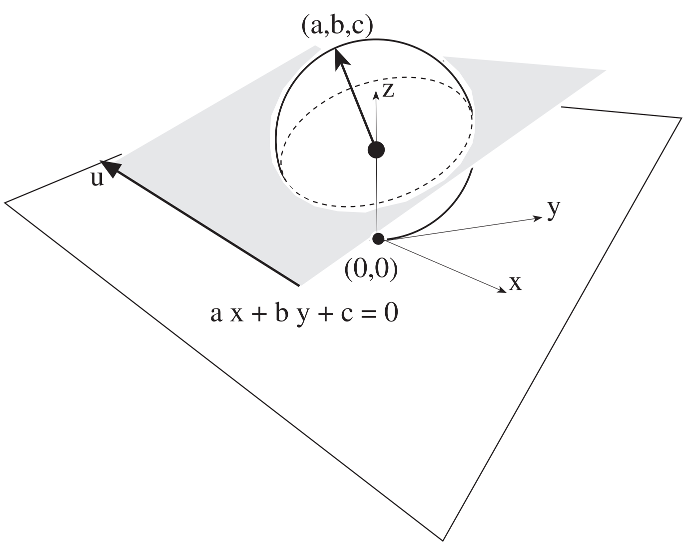

Figure 1: $\mathcal{S}^2$, the space of directed lines in the plane.

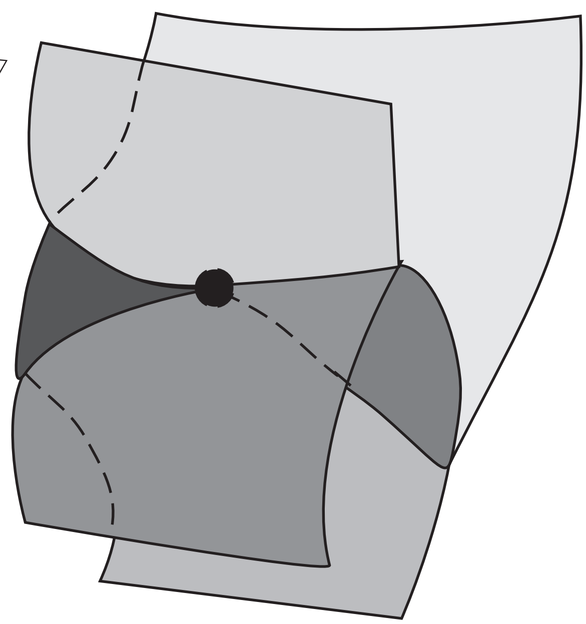

Figure 3: Neighborhood of a vertex of Γ_V.

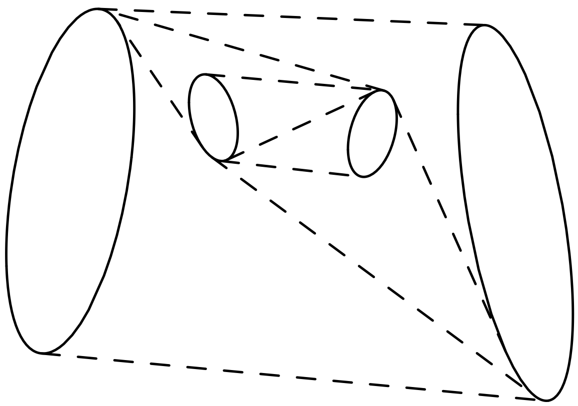

Figure 4: A pseudo–triangulation.

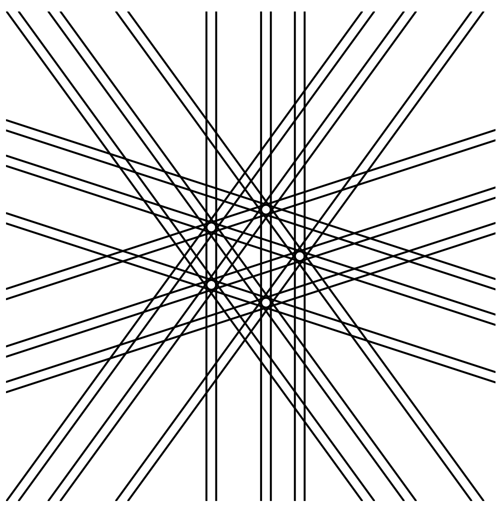

Figure 5: The regular k–gon, with small circles at its vertices.

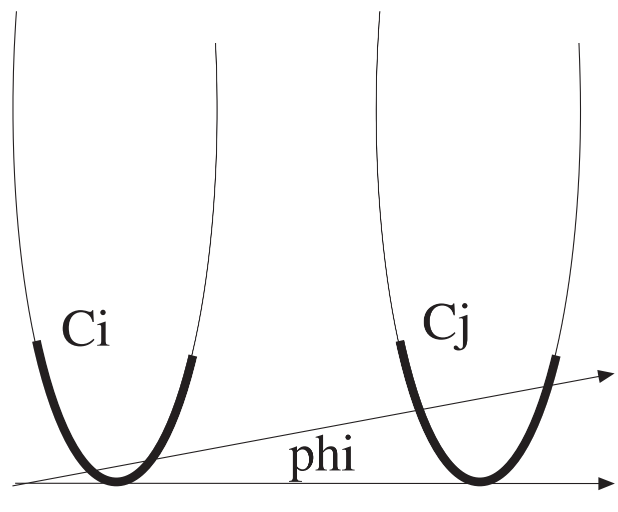

Figure 6: Perturbing $L(C_i, C_j) \in L_h$, for $\{i, j\} \in E$. Here $\tau_{(E,\sigma)}(i, j, -h) = \tau_{(E,\sigma)}(i, j, h) = \sigma(\{i, j\}) = +1$.