Bounds on the Covering Radius of a Lattice

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Abstract

This paper leans on results of Baranovskii [1], [2]. The covering radius $R(L)$ of a lattice L is the radius of smallest balls with centers in points of L which cover all the space spanned by L. $R(L)$ is tightly related to minimal vectors of classes of the quotient $\frac{1}{2}L/L$. The convex hull of all minimal vectors of a class Q is a Delaunay polytope $P(Q)$ of dimension $\leq n$,dimension of L. Let $\frac{1}{4}v^2_{max}$ ($\frac{1}{4}u^2_{max}$) be a maximal squared radius of $P(Q)$ of dimension n (of dimension less than n, respectively). If $\frac{1}{3}u^2_{max} \leq \frac{1}{4}v^2_{max}$, then $R^2(L) = \frac{1}{4}v^2_{max}$. This is a case of the well-known Barnes-Wall and Leech lattices. Otherwise, $\frac{1}{4}v^2_{max} \leq R^2(L) \leq \frac{1}{3}u^2_{max}$. This is a refinement of a result of Norton ([3], ch.22).

Let L be an n-dimensional lattice. There are two important normal (face-to-face) partitions of the space $\mathbf{R}^n$ related to the lattice L, namely the well-known Voronoi partition and an L-partition. These partitions are combinatorially dual each to other: a k-dimensional face of one partition is orthogonal to a $(n-k)$-dimensional face of the other partition. A Voronoi partition consists of mutually congruent Voronoi polytopes. An L-partition consists of Delaunay polytopes and, in general, contains noncongruent polytopes.

A Delaunay polytope of a lattice L of dimension n is the convex hull of all points of L lying on an empty sphere and spanning n-dimensional space. A sphere is called *empty* if there is no lattice points in its interior. Here and below, we use only $(n-1)$-dimensional spheres. Since the L-partition is normal, each face of a Delaunay polytope of the L-partition is a face of this L-partition.

There are only two types of Delaunay polytopes: symmetric and asymmetric. A Delaunay polytope is called *symmetric* if the antipode of each its vertex is a vertex of it. A Delaunay polytope is called *asymmetric* if it has no pair of antipodal vertices. (Two vertices of a polytope inscribed in a sphere are antipodal if they are endpoints of a diameter of this sphere.)

We distinguish lattice points and lattice vectors. If origin is in general position with respect to a lattice, then we say that a vector with a lattice point as its endpoint *represents* this lattice point. The difference of two vectors representing two lattice points is called *a lattice vector.* If origin coincides with a lattice point, then any vector representing a lattice point is a lattice vector.

If origin is a lattice point, then the lattice $\frac{1}{2}L$ is well defined, and we can consider the quotient $\frac{1}{2}L/L$. Minimal vectors of a class of $\frac{1}{2}L/L$ are tightly related to symmetric Delaunay polytopes. (A vector a of a set Q is called minimal if it has minimal *norm* a^2 among all vectors of Q.)

Proposition 1 *Let $\frac{1}{2}v$ be a minimal vector of a class Q of the quotient $\frac{1}{2}L/L$. Let $S(Q)$ be a sphere with the center in the point $\frac{1}{2}v$ and squared radius $r^2=\frac{1}{4}v^2$. Then the sphere $S(Q)$ is empty.*

Proof. If we take the point $\frac{1}{2}v$ as origin, then each point of L is represented by a vector $a-\frac{1}{2}v$, where a is a lattice vector. Hence vectors $a-\frac{1}{2}v$ for all $a\in L$ belong to the class Q. Since $\frac{1}{2}v$ is a minimal vector of Q, the sphere $S(Q)$ does not contain lattice points of L in its interior, i.e. it is empty. □

Let Q_{min} be the set of minimal vectors of the class Q. Let $P(Q)$ be the convex hull of endpoints of vectors of Q_{min}. Baranovskii [1], [2] calls the polytope $P(Q)$ by "primary element" of the L-partition. Let $H(Q)$ be the space spanned by Q_{min}. Clearly that all lattice points of L lying on the empty sphere $S(Q)$ lie in the space $H(Q)$. Using definition of a Delaunay polytope, we obtain

Corollary 1 *$P(Q)$ is a symmetric Delaunay polytope of the lattice $L\cap H(Q)$.* □

Example. Consider the polytopes $P(Q)$ of the root lattice A_n. A vector $a\in A_n$ can be written in the form $a=\sum_{i=1}^{n+1}z_ie_i$, where $\{e_i:1\le i\le n+1\}$ is an orthonormal basis of $\mathbf{R}^{n+1}$, $z_i\in\mathbf{Z}$ and $\sum_{i=1}^{n+1}z_i=0$. A class of the quotient $\frac{1}{2}A_n/A_n$ is uniquely determined by a set $T\subseteq\{1,2,...,n+1\}$ of even cardinality $|T|$. Denote this class by Q_T. Since there are 2^n even subsets T of a $(n+1)$-set, the classes Q_T exhaust all classes of $\frac{1}{2}A_n/A_n$. One can easily to verify that minimal vectors of Q_T are vectors $\frac{1}{2}\sum_{i\in T}\varepsilon_ie_i$, where $\varepsilon_i\in\{\pm1\}$ and $\sum_{i\in T}\varepsilon_i=0$. The polytope $P(Q_T)$ is a middle section of a $|T|$-dimensional unite cube. Let $|T|=2k$, then $0\le k\le\lfloor\frac{n+1}{2}\rfloor$. The one-dimensional skeleton of $P(Q_T)$ is the Johnson graph $J(2k,k)$. The squared radius of $P(Q_T)$ is equal to $(\frac{1}{2}\sum_{i\in T}\varepsilon_ie_i)^2=\frac{1}{4}|T|=\frac{1}{2}k$ (see also [3], ch.4, section 6).

Proposition 2 *Let P be a symmetric Delaunay polytope of a lattice L with the center in origin. Then vectors representing vertices of P are all minimal vectors of a class of $\frac{1}{2}L/L$.*

Proof. Since the vector connecting a pair of antipodal vertices of P is a lattice vector, i.e. it belongs to L, the center of P belongs to $\frac{1}{2}L$. Hence, when the center of P is origin, vectors representing points of L belong to a class Q of $\frac{1}{2}L/L$. Clearly, the vectors representing vertices of P are all minimal vectors of Q. □

Proposition 3 *Let P be a Delaunay polytope of a lattice L with a vertex in origin. Let v be the lattice vector representing a vertex of P. Let Q be the class of $\frac{1}{2}L/L$ containing $\frac{1}{2}v$, and let $P(Q)$ be convex hull of all minimal vectors of Q. Then*

(i) $\frac{1}{2}v$ is minimal vector of Q,

(ii) $P(Q)$ is a symmetric face of P.

Remark. Proposition 3 is a reformulation of a result of Baranovskii ([1], Lemma 1).

Proof. Let n be dimension of L (and P). Let S_v be the sphere of squared radius $\frac{1}{4}v^2$ with the center in the point $\frac{1}{2}v$. If we take the point $\frac{1}{2}v$ as a new origin, then the vectors representing points of L belong to the class Q. Suppose that $\frac{1}{2}v$ is not minimal vector of Q. Then the sphere S_v contains in its interior points of L. Let u be such a point. Then $(u-\frac{1}{2}v)^2 < \frac{1}{4}v^2$, i.e.

$$u^2 < uv. \tag{1}$$

Let x be a vector representing the center of P. Then $(w-x)^2 \geq x^2$ for all points w of L, i.e. $w^2 \geq 2wx$, with equality if w is a vertex of P. In particular, $u^2 \geq 2ux$. Since v is a vertex of P, the equality

$$v^2 = 2vx \tag{2}$$

holds. We show that the point $v-u \in L$ belongs to the interior of the sphere S circumscribing P. We have

$$(v-u)^2 - 2(v-u)x = v^2 - 2vu + u^2 - 2vx + 2ux.$$

Using the equality (2) and the inequality $2ux \leq u^2$, we have

$$(v-u)^2 - 2(v-u)x \leq 2(u^2 - vu).$$

Using the inequality (1), we obtain that the point $v-u$ lies in the interior of the sphere S. This contradiction shows that $\frac{1}{2}v$ is a minimal vector of Q.

So, the sphere S_v is empty, and the convex hull of all points lying on S_v is $P(Q)$.

At first we show that all vertices of $P(Q)$ are vertices of P. This assertion is obvious if there are only 2 minimal vectors $\pm\frac{1}{2}v$ in the class Q. Let $P(Q)$ has a vertex u distinct from v and 0. Since $(u-\frac{1}{2}v)^2 = \frac{1}{4}v^2$, we have $u^2 = uv$. Recall that the point $v-u$ is also a vertex of $P(Q)$, since $P(Q)$ is symmetric. Hence the equality $(v-u)^2 = (v-u)v$ holds. Using (2) and the equality $u^2 = uv$, we obtain

$$(v-u)^2 + u^2 = v^2 - 2vu + 2u^2 = 2vx = 2(v-u)x + 2ux.$$

Since $u^2 \geq 2ux$ and $(v-u)^2 \geq 2(v-u)x$, the above equality shows that these last inequalities hold as equalities. This means that u and $v-u$ lie on the sphere S, and, by definition of a Delaunay polytope, u and $v-u$ are vertices of P.

Now we show that $P(Q)$ is a face of P. Let H be a hyperplane spanning the intersection $S \cap S_v$. Clearly, $P(Q) \subset H$. Note that if $x = \frac{1}{2}v$, then $S = S_v$ and H coincides with the whole space spanned by L (and P). In this case P is symmetric and, by Proposition 2, $P = P(Q)$.

Let $S \neq S_v$. Then H is a hyperplane which partitions the space spanned by L into two open halfspaces. Let U be that of these halfspaces that does not contain the center x of S. Then $B_v \cap U \supset B \cap U$, where B_v (B) is the ball with the surface S_v (S, respectively). Since B_v is empty and all points of L on S_v are contained in $H \cap S_v$, $B \cap U$ does not contain vertices of P. This means that H is a hyperplane supporting $P(Q)$, and $P(Q)$ is a face of P. □

We say that two faces of the L-partition are *equivalent* if one face can be obtained from another by a translation or by central reflection.

Corollary 2 *There is one-to-one correspondence between classes of equivalent symmetric faces of the L-partition of a lattice L and polytopes $P(Q)$ of classes of $\frac{1}{2}L/L$.*

Proof. By Proposition 3, each symmetric face of the L-partition (as a face of a Delaunay polytope) is equivalent to $P(Q)$ for some class Q of $\frac{1}{2}L/L$. Proposition 1, Corollary 1 and analysis of the proof of Proposition 3 show that converse is also true. □

Since any edge of a Delaunay polytope is a symmetric one-dimensional Delaunay polytope, we obtain

Corollary 3 *1-dimensional $P(Q)$'s describe all types of edges (i.e. classes of equivalent edges) of Delaunay polytopes of a lattice.*

Now we can deduce our main result. Recall that covering radius $R(L)$ of a lattice L is the greatest radius of spheres circumscribing Delaunay polytopes of L. Call a Delaunay polytope *deep* (*s-deep, a-deep, u-deep*, respectively) if it has maximal radius among radii of all (symmetric, asymmetric, symmetric of dimension less than n, respectively) Delaunay polytopes of the L-partition of L. If L has symmetric Delaunay polytopes, then they are $P(Q)$ for some classes of $\frac{1}{2}L/L$. The squared radius of $P(Q)$ is equal to $\frac{1}{4}v^2$, where $\frac{1}{2}v$ is a minimal vector of the class Q. Let $\frac{1}{4}v^2_{max}$ be squared radius of a s-deep (symmetric) Delaunay polytope of L of dimension n. Then clearly

$$R^2(L) \geq \frac{1}{4}v^2_{max}.$$

This bound is attained if L has a symmetric deep Delaunay polytope.

Let all deep Delaunay polytopes of L be asymmetric, and let $\frac{1}{4}u^2_{max}$ be squared radius of a u-deep (symmetric) face of the L-partition of L. So, $\frac{1}{4}u^2_{max}$ is squared radius of a u-deep polytope $P(Q)$ (of dimension less than dimension of L).

Let P be an a-deep asymmetric Delaunay polytope of L with center x. Obviously $\frac{1}{4}R^2(L)$ is squared covering radius of $\frac{1}{2}L$. Let $P(Q)$ be the u-deep symmetric face of P. Recall that the center y of $P(Q)$ belong to $\frac{1}{2}L$ and note that y is one of the nearest to x points of $\frac{1}{2}L$. Hence the squared distance between the centers of $P(Q)$ and P is not greater than $\frac{1}{4}R^2(L)$, since distance of any point of the space from a nearest point of $\frac{1}{2}L$ is not greater than $\frac{1}{2}R(L)$. Therefore we have

$$R^2(L) \leq \frac{1}{4}R^2(L) + \frac{1}{4}u^2_{max},$$

i.e.

$$R^2(L) \leq \frac{1}{3}u^2_{max}.$$

We have

Theorem 1 *Let $\frac{1}{4}v^2_{max}$ be the squared radius of an s-deep symmetric Delaunay polytope of a lattice L. We set $v_{max}=0$ if L has no symmetric Delaunay polytope of dimension n. Let $\frac{1}{4}u^2_{max}$ be the squared radius of a u-deep symmetric face of the L-partition of L. If $u^2_{max}\le\frac{3}{4}v^2_{max}$, then the squared covering radius of L is equal to*

$$R^2(L)=\frac{1}{4}v^2_{max}.$$

Otherwise, the following bounds on $R^2(L)$ hold:

$$\frac{1}{4}v^2_{max}\le R^2(L)\le\frac{1}{3}u^2_{max}.\square$$

The case $u^2_{max}\le\frac{3}{4}v^2_{max}$ occurs in many well-known lattices. For example, this inequality is true for Leech lattice Λ_{24}, Barnes-Wall lattice Λ_{16}, and root lattices D_{2k}, E_7, E_8. Minimal vectors of the corresponding classes of $\frac{1}{2}L/L$ are described in the book [3] (ch.6, section 5 for Barnes-Wall lattice; ch.10, Th.28 for Leech lattice; ch.6, section 3 for E_7 and E_8).

Recall that all Delaunay polytopes of the root lattice A_n are Johnson polytopes $J(n+1,k)$, $1\le k\le\lfloor\frac{n+1}{2}\rfloor$. All these polytopes are asymmetric, except $J(n+1,\frac{n+1}{2})$ for n odd. Using the given in above example description of polytopes $P(Q_T)$ of the root lattice A_n, for n odd, we obtain $\frac{1}{4}v^2_{max}=\frac{n+1}{2}$, $\frac{1}{4}u^2_{max}=\frac{n-1}{2}$, i.e. $\frac{1}{3}u^2_{max}=\frac{2}{3}(n-1)>\frac{n+1}{2}=\frac{1}{4}v^2_{max}$. Therefore, for the root lattice A_n with odd n, we have $R^2(A_n)=\frac{n+1}{2}=\frac{1}{4}v^2_{max}<\frac{1}{3}u^2_{max}$.

Corollary 4 *Let a lattice L has no symmetric Delaunay polytope. If L has an (asymmetric) Delaunay polytope of squared radius $\frac{1}{3}u^2_{max}$, then $R^2(L)=\frac{1}{3}u^2_{max}$.* $\square$

Corollary 4 is applied to the root lattice E_6, since $R^2(E_6)=\frac{4}{3}$ and $u^2_{max}=4$.

Note that the given here proof of the equality $R^2(L)=\frac{1}{4}v^2_{max}$ for the Leech and Barnes-Wall lattices is much more simple than one given in the book [3].

Recall that Baranovskii [2] discover that the 9-dimensional lattice A_9^{+5} gives the thinnest known lattice covering of a 9-dimensional space. For this lattice, both the inequalities of Theorem 1 are strict: $\frac{1}{4}v^2_{max}=\frac{9}{16}$, $\frac{1}{3}u^2_{max}=\frac{3}{4}$, and $R^2(A_9^{+5})=\frac{3}{5}$.

It is well known (see, for example, [3]) that all Delaunay polytopes of the lattice A_n^*, the dual of the root lattice A_n, are mutually congruent simplexes. Hence $P(Q)$ is a segment for all classes Q (distinct from zero class). For even n, the bound $R^2(L)\le\frac{1}{3}u^2_{max}$ gives a bound on the covering density θ of A_n^* which coincides with the following exact value:

$$\theta=\sqrt{n+1}\left(\frac{n(n+2)}{12(n+1)}\right)^{n/2}.$$

For odd n, the above upper bound on $R^2(L)$ gives the following upper bound on θ:

$$\theta\le\sqrt{n+1}\left(\frac{n(n+2)+1}{12(n+1)}\right)^{n/2}.$$

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