Skeletons of Some Relatives of the n-cube

Antoine DEZA*
Michel DEZA

Laboratoire d'Informatique, URA 1327 du CNRS
Département de Mathématiques et d'Informatique
Ecole Normale Supérieure
*Tokyo Institute of Technology, Japan

LIENS - 94 - 16

September 1994

Skeletons of some relatives of the n-cube

Antoine DEZA [* *] Michel DEZA [⋆]

[*] Tokyo Institute of Technology, Dept. Inf. Sci., Japan
[⋆] LIENS, Ecole Normale Supérieure, Paris, France

October 1994

Abstract

We study the skeleton of several polytopes related to the n-cube, the halved n-cube, and the folded n-cube. In particular, the Gale polytope of the n-cube, its dual and the duals of the halved n-cube and the complete bipartite subgraphs polytope.

1 Introduction

The general references are [**?**, **?**, **?**] for polytopes, [**?**] for graphs and [**?**] for lattices. We first recall some basic properties of the cube and the halved cube.

The vertices of the n-cube $\gamma_n = [0,1]^n$ are all the 2^n characteristic vectors χ^S for $S \subset N = \{1, 2, \ldots, n\}$, that is, $\chi_i^S = 1$ for $i \in S$ and 0 otherwise. With $|S \Delta S'|$ denoting the size of the symmetric difference of the subsets S and S', two vertices χ^S and $\chi^{S'}$ are adjacent if and only if $|S \Delta S'| = 1$. The skeleton of γ_n is denoted by $H(n, 2)$ and the skeleton of its dual, the cross-polytope $\beta_n = \gamma_n^*$, is $K_{2\times n}$, which is also called the Cocktail-Party graph. The diameter of the n-cube and its dual are, respectively, n and 2.

The *halved n-cube* $h\gamma_n$ (see Section 8.6 of [**?**]) is obtained from the n-cube γ_n by selecting the vertex of even cardinality on each edge, that is, $h\gamma_n$ is the convex hull of all the 2^{n-1} characteristic vectors χ^S for $S \subset N = \{1, 2, \ldots, n\}$ and $|S|$ even. Two vertices χ^S and $\chi^{S'}$ are adjacent if and only if $|S \Delta S'| = 2$. The skeleton of the halved n-cube is denoted by $\frac{1}{2}H(n, 2)$; its diameter is $\lfloor \frac{n}{2} \rfloor$.

2 Skeleton of the dual halved n-cube

The halved 3-cube is a regular tetrahedron α_3. The halved 4-cube is the simplicial polytope $h\gamma_4 = \beta_4$. For $n > 4$, the facets of $h\gamma_n$-cube are partitioned into the following two orbits of its symmetry group $2^{n-1}Sym(n)$. The orbit O_1^n consists of the $2n$ facets belonging to the facets of the n-cube and defined by the inequalities:

$$x_i \leq 1 \qquad \text{for } i \in N, \qquad (1)$$
$$x_i \geq 0 \qquad \text{for } i \in N. \qquad (2)$$

The orbit O_2^n consists of the 2^{n-1} facets cutting off the vertices of odd cardinality from the n-cube and defined by the inequalities:

$$\sum_{i=1}^{n} x_i(1 - 2\chi_i^A) \leq |A| - 1 \qquad \text{for } A \subset N \text{ and } |A| \text{ odd.} \qquad (3)$$

The facets defined by the inequalities (1), (2) and (3) are respectively denoted by F_1^i, F_0^i and F^A. Since the symmetries of a polytope preserve adjacency and linear independence, we can describe the properties of its facets by simply considering a representative facet of each orbit. The facets $F_1^i \simeq F_0^i \simeq h\gamma_{n-1}$ (here and in the following " $\simeq$ " denotes the affine equivalency) and each facet F^A is the simplex containing the n vertices: $\chi^{A\cup\{i\}}$ for $i \in \bar{A}$ and $\chi^{A\setminus\{i\}}$ for $i \in A$.

The skeleton of the dual halved n-cube, denoted by $h\gamma_n^*$, is the graph whose nodes are the facets of $h\gamma_n$, two facets being adjacent if and only if their intersection is a face of codimension 2. This skeleton is given below.

Lemma 2.1 *The facets of O_1^n and O_2^n form, respectively, the coclique $\bar{K}_{2n}$, and the coclique $\bar{K}_{2^{n-1}}$; each facet F^A is adjacent, either to F_1^i if $i \in A$, or to F_0^i if $i \in \bar{A}$ for each $i \in N$.*

Corollary 2.2 *For $n \geq 4$, the skeleton of the dual halved n-cube is a bipartite graph of diameter 4.*

PROOF. Since the valency of a facet belonging to O_1^n, respectively to O_2^n, is half the size of O_2^n, respectively of O^1, we have $\delta(h\gamma_n^*) \leq 4$. On the other hand, the facets F_1^i and F_0^i, having no common neighbour, we get $\delta(h\gamma_n^*) > 3$. □

Corollary 2.3 *The halved n-cube has $n\,2^{n-2}$ faces of codimension 2 which are all simplices, that is $h\gamma_n$ is quasi-simplicial. For $n \to \infty$, $h\gamma_n$ is asymptotically simplicial.*

PROOF. Since the number of faces of codimension 2 of a polytope is half of the total valency of the skeleton of its dual, the result is a straightforward calculation. All faces of codimension 2 being incident to the simplex facets of $h\gamma_n$, the halved n-cube is a quasi-simplicial. □

3 Gale transform of the n-cube

Let A be a $(2^n - n - 1) \times 2^n$ matrix which rows form a basis for the space of all the affine dependencies on the vertices of the n-cube. A Gale transform of γ_n is the collection of the 2^n points in $\mathbb{R}^{2^n-n-1}$ which are the columns of A.

We consider the matrix A induced by the following $2^n - n - 1$ affine dependencies on the vertices of γ_n:

$$(1 - |T|)\chi^{\emptyset} + \sum_{i \in T} \chi^{\{i\}} - \chi^{T} = 0 \qquad \text{for } T \subset N \text{ and } |T| \geq 2. \tag{4}$$

Since each column of A corresponds to a vertex χ^S of γ_n for $S \subset N$, we simply denote by v^S the vector formed by this column of A. For example, the first column of A corresponds to $\chi^{\emptyset}$ and forms the vector $v^{\emptyset}$ which $2^n - n - 1$ coordinates are $v^{\emptyset}_T = (1 - |T|)$, where $\mathbb{R}^{2^n-n-1}$ is naturally indexed by $T \subset N$, $|T| \geq 2$.

A *Gale polytope*, $Gale(P)$, of a polytope P is the convex hull of a Gale transform of P. In the following we consider $Gale(\gamma_n)$ associated to the affine dependencies (4). The polytope $Gale(\gamma_3)$ is a prism over a tetrahedron; see also Example 5.6 in [?] for relation with Lawrence polytopes. For $n \geq 4$, we introduce some edges and facets of $Gale(\gamma_n)$ in order to compute its diameter and the one of its dual.

Consider the following inequalities, where x_T for $T \subset N$ and $|T| \geq 2$ are the coordinates of a point x in $\mathbb{R}^{2^n-n-1}$ indexed by $T \subset N$, $|T| \geq 2$.

$$-x_A \leq 1 \qquad \text{for } |A| = 2, \tag{e_1}$$

$$x_{A\setminus\{i\}} - x_A \leq 1 \qquad \text{for } |A| \geq 3 \text{ and } i \in A, \tag{e_2}$$

$$x_A \leq 1 \qquad \text{for } |A| = 2, \tag{e_3}$$

$$x_{A\cup\{i\}} - x_A \leq 1 \qquad \text{for } |A| \geq 2 \text{ and } i \notin A, \tag{e_4}$$

$$2\sum_{j \in N} x_{\{j\}} - 2x_{\{i\}} + (n-1)(x_N - 1) \leq 0 \qquad \text{for } i \in N, \tag{e_5}$$

$$\sum_{|T|\geq 2} x_T - 2^n(x_A+x_B) \leq 2^n-1 \qquad \text{for } |A|,|B|\geq 2 \text{ and } 2(|A|+|B|)\leq n+3. \quad (e_6)$$

One can easily check that each of those inequalities induces an edge of $Gale(\gamma_n)$. More precisely, (e_1) and (e_2) induce the edges $[v^\emptyset, v^A]$ for $|A| \geq 2$, $(e_3), (e_4)$ and (e_5) induce the edges $[v^i, v^A]$ for $|A| \geq 1$ and $i \notin A$ or $A = N$ and (e_6) induce the edges $[v^A, v^B]$ for $|A|, |B| \geq 2$ and $2(|A| + |B|) \leq n + 3$.

Property 3.1 *The diameter of $Gale(\gamma_n)$ is at most 2. Moreover, $\delta(Gale(\gamma_3)) = 2$ and $\delta(Gale(\gamma_4)) = 1$.*

PROOF. The vertices $v^\emptyset$ and v^A are respectively linked by the edges $[v^\emptyset, v^N]$ and $[v^N, v^A]$ for $|A| = 1$ and by the edge $[v^\emptyset, v^A]$ for $|A| \geq 2$. The vertices v^i and v^j always form an edge, v^i and v^A are linked by $[v^i, v^j]$ and $[v^j, v^A]$ with $j \notin A$, for $2 \leq |A| \leq n-1$, and $[v^i, v^N]$ form an edge. Finally, the vertices v^A and v^B are linked by the edges $[v^A, v^\emptyset]$ and $[v^\emptyset, v^B]$ for $|A|, |B| \geq 2$. □

We then consider the following 2^{n-1} inequalities.

$$2^{n-1} x_{\bar{A}} - \sum_{|T|\geq 2} x_T \leq 1 \qquad \text{for } A \subset N \text{ and } |A| \leq 1,$$

$$2^{n-1}(x_A + x_{\bar{A}}) - \sum_{|T|\geq 2} x_T \leq 1 \qquad \text{for } A \subset N \text{ and } 2 \leq |A| \leq n-1.$$

One can easily check that each of those inequalities induces a facet G^A of $Gale(\gamma_n)$ for $A \subset N$ and $|A| \leq n-1$. Since each facet G^A contains all vertices except the pair $\{v^S, v^{\bar{S}}\}$, we call them the *huge facets*.

Lemma 3.2 *The huge facets form the clique $K_{2^{n-1}}$ in the skeleton of $Gale^*(\gamma_n)$.*

PROOF. Let us first consider $g = G^A \cap G^B$ with $A, B \subset N$ and $2 \leq |A|, |B| \leq n-1$. The face g contains all the vertices of $Gale(\gamma_n)$ except $\{v^A, v^{\bar{A}}, v^B, v^{\bar{B}}\}$. We show that g is of codimension 2 by exhibiting a family V of $2^n - n - 2$ affinely independent vertices belonging to g, this will imply that G^A and G^B are adjacent. Namely, V is formed by the vertices v^S with $S \notin \{A, \bar{A}, B, \bar{B}\}$ and $|S| \geq 2$ and the vertices $\{v^i, v^j\}$ with $1 \leq i < j \leq n$ such that $v^i_A = v^j_B = 1$ and $v^i_B = v^j_A = 0$. In the case $0 \leq |A|, |B| \leq 1$, V is formed by the vertices v^S with $S \notin \{\bar{A}, \bar{B}\}$ and $|S| \geq 2$. Finally, in the case $0 \leq |A| \leq 1$ and $2 \leq |B| \leq n-1$, V is formed by the vertices v^S with $S \notin \{\bar{A}, B, \bar{B}\}$ and $|S| \geq 2$ and the vertex $v^\emptyset$. □

Property 3.3 *The huge facets form a dominating clique in the skeleton of* $Gale^*(\gamma_n)$.

PROOF. Since the pairs $\{v^S, v^{\bar{S}}\}$ form a partition of all the vertices of $Gale(\gamma_n)$, for any facet F, at least one huge facet G^A satisfies $|G^A \cap F| = |F| - 1$. This implies that G^A is adjacent to F; in other words, the huge facets form a dominating clique. □

Corollary 3.4 *The diameter of* $Gale^*(\gamma_n)$ *is at most* 3. *Moreover, it is* 2 *for* $n = 3, 4$.

Conjecture 3.5 *For* $n \geq 4$, *the diameters of the Gale polytope of the* n*-cube and of its dual are* 1 *and* 2, *respectively.*

4 Complete bipartite subgraphs polytope

We recall that the *folded* n*-cube* $\square_n$ is the graph whose vertices are the 2^{n-1} partitions of $N = \{1, \ldots, n\}$ into two subsets, S and $\bar{S}$; two partitions being adjacent when their common refinement contains a singleton. In particular, $\square_4 = K_{4,4}$ and $\bar{\square}_5 = \frac{1}{2}H(5, 2)$, also called the Clebsch graph.

The *complete bipartite subgraphs polytope* c_n, which is also called the cut polytope of the complete graph, is a relative of the folded n-cube. More precisely, the vertices of c_n are the 2^{n-1} incidence vectors $\delta(S)$ in $\mathbb{R}^{\binom{n}{2}}$ of the partitions of N, that is, $\delta(S)_{ij} = 1$ if exactly one of i, j is in S and 0 otherwise for $1 \leq i < j \leq n$. It is easy to check that the squared Euclidian distance between two partitions, seen as vertices of c_n, is $d(n-d)$, where d is their path distance, in the graph $\square_n$. Now, $c_3 = h\gamma_3 = \alpha_3$ and c_4 is combinatorially equivalent to the simplicial 6-dimensional cyclic polytope with 8 vertices. The symmetry group of c_n is isomorphic to the automorphism group of $\square_n$, see [?]. See [?] for a detailed treatment of c_n.

The skeleton of c_n is the clique $K_{2^{n-1}}$, see [?].The determination of all the facets of c_n for large n seems to be hopeless, but a wide range of facets has been already found (including all for $n \leq 7$). It seems that the huge majority of them are simplices for large n, that is, c_n is asymptotically simplicial, as well as $h\gamma_n$,. In [?] it was conjectured (and proved for $n \leq 7$) that $\delta(c_n^*) \leq 4$; moreover, $\delta(c_4^*) = \delta(c_5^*) = 2$ and $\delta(c_6^*) = 3$. Actually, the skeleton of c_4^* is the line graph of the folded 4-cube.

Remark 4.1 *Using the basis of the space of affine dependencies on* c_5 *given in [?], we found by computer that* $Gale(c_5) \simeq h\gamma_5$; *recall that* $\bar{\square}_5 = \frac{1}{2}H(5, 2)$. *Clearly,* $Gale(h\gamma_4) \simeq \alpha_3$ *and* $Gale(h\gamma_5) \simeq c_5$; *more generally, for* n *odd,* $Gale(h\gamma_n)$ *can be obtained from the following basis of* $2^{n-1} - n - 1$ *affine dependencies:*

$$(n-1)\sum_{i\in X} x_{N\setminus\{i\}} - |A|\sum_{i\in N} x_{N\setminus\{i\}} + (n-1)x_A = 0\, for\ |A|\ even,\ 2 \le |A| \le n-2.$$

Finally, we mention $cont_m$, the contact polytope of the lattice $\mathbb{Z}(V_m)$ in $\mathbb{R}^{\binom{m}{2}}$ studied in [?], where V_m denotes the set of vertices of c_m, that is, $cont_m$ is the convex hull of all vectors of this lattice having the minimal length $\mu = min(4, m-1)$. Clearly, it comes from the construction A given in Chapters 5, 7 of [?] with V_m seen as a linear binary code with $n = \binom{m}{2}$, $M = 2^{m-1}$ and $d = m-1$. We have,

- $cont_2 = conv\{\pm e_1\} = \beta_1$ and $\mathbb{Z}(V_2) = \mathbb{Z} = A_1$,
- $cont_3 = conv\{\pm e_i \pm e_j : 1 \le i \ne j \le 3\}$ is the cubo-octahedron (the vertices of this Archimedean solid are the midpoints of the edges of γ_3) and $\mathbb{Z}(V_3)$ is the face-centered cube lattice $A_3 \cong D_3$,
- $cont_4 = conv\{\pm\delta(i), \pm\delta(i) - 2e_{ij} : 1 \le i \ne j \le 4\} \simeq h\gamma_6$,
- $cont_5$ is a 10-polytope with the following 100 vertices: $\{\pm 2e_{ij} : 1 \le i \le j \le 5\}$ $\cup\ \{\delta(i) - 2\sum_{\{jk\}\in X} e_{jk} : 1 \le i \le 5,\ X \subset E(K_{i,\{1,2,3,4,5\}-i})$. So, $cont_5$ is the union of $2\beta_{10}$ and five 4-cubes γ_4, this polytope has 4 624 facets divided into 4 orbits of its symmetry group $2^5 Sym(5)$, moreover, the orbit formed by the 384 facets equivalent to the one induced by the inequality $\sum_{\{ij\}\in C_{1,2,3,4,5}} x_{ij} \le 2$ forms a dominating set in the skeleton of $cont_5^*$,
- for $m \ge 6$, $cont_m = conv\{\pm 2e_{ij} : 1 \le i \le j \le m\} \simeq \beta_{\binom{m}{2}}$.

So, the kissing number of the lattice, that is the number of vertices of $cont_m$, is $\tau = 2, 12, 32, 100, m(m-1)$ for $m = 2, 3, 4, 5, \ge 6$.

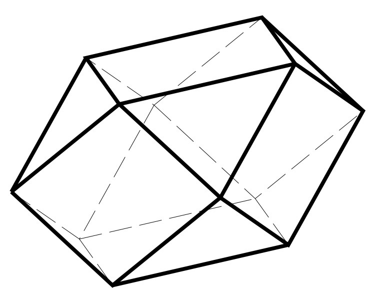

Figure 4.1: The contact polytope of $\mathbb{Z}(V_3)$ is a cubo-octahedron

References

1. F. Barahona and R. Mahjoub, "On the cut polytope", *Mathematical Programming* **36** (1986) 157-173.
2. M. Bayer and C. Lee, "Combinatorial aspects of convex polytopes ", in P. Gruber and J. Wills (eds.) *Handbook on Convex Geometry*, North Holland (1994) 485-534.
3. L. Billera, P. Filliman and B. Sturmfels, "Constructions and complexity of secondary polytopes", *Advances in Mathematics* **83** (1990) 155-179.
4. A. Brouwer, A. Cohen and A. Neumaier, *Distance-Regular Graphs*, Springer-Verlag, Berlin, 1989.
5. J. Conway and N. Sloane, *Sphere Packing, Lattices and Groups* **290** Grundlehren der Mathematischen Wissenschaften, Springer-Verlag, Berlin, 1987.
6. H. Coxeter, *Regular Polytopes*, Second edition, McMillan 1963, corrected reprint, Dover, New York, 1973.
7. A. Deza and M. Deza, "On the skeleton of the dual cut polytope", Proceedings of *Jerusalem Combinatorics 93*, to appear in H. Barcelo and G. Kalai (eds.) *Contemporary Mathematics* (1994).
8. M. Deza, "Isometries of hypergraphs", *Proceedings of the International Conference on the Theory of Graphs*, in A. Rao (ed.), McMillan, Calcutta (1979) 174-189.
9. M. Deza and V. Grishukhin, "Cut polytopes and its lattices", Rapport de Recherche du LIENS 94-18, Ecole Normale Supérieure, Paris (1994).
10. M. Deza, V. Grishukhin and M. Laurent, "The symmetries of the cut polytope and of some relatives", in P. Gritzmann and B. Sturmfels (eds.) *Applied Geometry and Discrete Mathematics, the "Victor Klee Festschrift" DIMACS Series in Discrete Mathematics and Theoretical Computer Science* **4** (1991) 205-220.
11. M. Deza, M. Grötschel and M. Laurent, *Cuts and Metrics* (book in preparation).
12. G. Ziegler, *Lectures on Polytopes*, (book to appear).