On a Positive Semidefinite Relaxation of the Cut Polytope

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Abstract

We study the convex body $\widetilde{\mathcal{L}}_n$ defined by

$$\widetilde{\mathcal{L}}_n := \{X \mid X = (x_{ij}) \text{ positive semidefinite } n \times n \text{ matrix }, x_{ii} = 1 \text{ for all } i\}.$$

Our main motivation for investigating this body comes from combinatorial optimization, namely from approximating the max-cut problem. An important property of $\widetilde{\mathcal{L}}_n$ is that, due to the positive semidefinite constraints, one can optimize over it in polynomial time. On the other hand, $\widetilde{\mathcal{L}}_n$ still inherits the difficult structure of the underlying combinatorial problem. In particular, it is NP-hard to decide whether the optimum of the problem

$$\min \mathrm{Tr}(CX), X \in \widetilde{\mathcal{L}}_n$$

is reached in a vertex. This result follows from the complete characterization of the matrices C of the form $C = bb^t$ for some vector b, for which the optimum of the above program is reached in a vertex.

We describe several geometric properties of $\widetilde{\mathcal{L}}_n$. Among other facts, we show that $\widetilde{\mathcal{L}}_n$ has 2^{n-1} vertices corresponding to all bipartitions of the set $\{1, 2, \ldots, n\}$.

1 Introduction

This paper is motivated by a 'hard' combinatorial optimization problem, the *maximum cut* problem (abbreviated as *max-cut*)

$$\max_{x \in \{0,1\}^n} \sum_{1 \le i < j \le n} c_{ij} |x_i - x_j|. \tag{1}$$

The max-cut problem is well known to be equivalent with the *discrete 01-quadratic programming* problem

$$\max_{x \in \{0,1\}^n} x^t Q x \tag{2}$$

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where Q is an $n \times n$ symmetric matrix. Since the exact optimum of the max-cut problem (or of the discrete quadratic programming problem) cannot be found efficiently unless $NP = P$, various approximating procedures have been proposed in the literature. An approximation of the max-cut based on the minimization of the maximum eigenvalue of the Laplacian matrix with respect to diagonal changes, has been introduced and studied in [3, 4, 5]. The computational experiments of [18] show that the eigenvalue bound provides a good approximation of the max-cut, since the relative error typically ranges between $1\% - 5\%$. It has been shown in [17] that the dual formulation of the eigenvalue bound is the optimization problem

$$\begin{array}{ll} \max & c^t x \\ x \in \mathcal{J}_n & \end{array} \tag{3}$$

where $\mathcal{J}_n$ is a convex body which is non-polyhedral and non-smooth. Actually, $\mathcal{J}_n$ is a relaxation of the well studied *cut polytope* (see Section 3 for details). Recently, it has been proved in [10] that the optimization problem (3) provides a 0.878 approximation for the max-cut problem. The goal of our paper is to study the geometrical properties of the body $\mathcal{J}_n$, in order to understand better the structure of the optimization problem (3). It appears more convenient to work with a geometric translate $\mathcal{L}_n$ instead of $\mathcal{J}_n$.

Let $\widetilde{\mathcal{L}}_n$ denote the set of $n \times n$ (symmetric) positive semidefinite matrices $X = (x_{ij})$ which satisfy $x_{ii} = 1$ for all $i = 1, \ldots, n$. Thus,

$$\widetilde{\mathcal{L}}_n := \{X \mid X \succeq 0, x_{ii} = 1, i = 1, \ldots, n\}.$$

The matrices belonging to $\widetilde{\mathcal{L}}_n$ are called correlation matrices, see [11] and references there. For a symmetric matrix $X = (x_{ij})$, let $\tau(X) := (x_{ij})_{1 \leq i < j \leq n}$ denote the $\binom{n}{2}$-vector which is the upper triangular part of X. We set

$$\mathcal{L}_n := \{\tau(X) \mid X \in \widetilde{\mathcal{L}}_n\}$$

and observe that $\mathcal{L}_n$ is the projection of $\widetilde{\mathcal{L}}_n$ on the $\binom{n}{2}$-dimensional subspace. As an example, see the body $\mathcal{L}_3$ depicted in Fig. 1. We call the body $\mathcal{L}_n$ an *elliptope* (coming from *ellip*soid and poly*tope*).

The elliptope $\mathcal{L}_n$ is the central object studied in this paper. By definition, $\widetilde{\mathcal{L}}_n$ is nothing but a section of the cone PSD_n by the hyperplanes $x_{ii} = 1$ for all i. The cone PSD_n has been extensively studied in the literature; see e.g.[2, 8, 15] for results on its faces. As a matter of fact, $\widetilde{\mathcal{L}}_n$ inherits some of the good properties of PSD_n but, however, its structure is much more complicated than that of PSD_n. For instance, the description of the faces of $\widetilde{\mathcal{L}}_n$ follows from that of the faces of PSD_n (see Proposition 2.6) but, unlike the case of PSD_n, there is no direct link between the dimension of a face and the rank of a matrix of $\widetilde{\mathcal{L}}_n$ lying in its relative interior (see Proposition 2.9). This leads to the interesting question of characterizing the subspaces that can be realized as kernels of matrices from $\widetilde{\mathcal{L}}_n$.

Extreme points of $\widetilde{\mathcal{L}}_n$ have been studied in several papers, most recently, in [11]. It is shown there that $\widetilde{\mathcal{L}}_n$ has extreme points of rank k if and only if $k(k+1) \leq 2n$.

The study of the elliptope $\mathcal{L}_n$ is also closely related to that of Euclidian distance matrices, which has an extensive literature (see, e.g., [15, 14]). A symmetric matrix $X = (x_{ij})$ is called a *Euclidian distance matrix* if $x_{ij} = \| v_i - v_j \|^2$ for some vectors $v_1, \ldots, v_n \in \mathbb{R}^k$ $(k \geq 1)$. A well known result by Schoenberg [20] asserts that X is a Euclidian distance matrix if and only if the $(n-1) \times (n-1)$ matrix $P = (p_{ij})$ defined by

$$p_{ij} = \frac{1}{2}(x_{in} + x_{jn} - x_{ij})$$

for $1 \leq i, j \leq n-1$, is positive semidefinite. Equivalently, X is a Euclidian distance matrix if and only if $x = \tau(X)$ belongs to the cone NEG_n, called *negative type cone*, and defined by

$$\mathrm{NEG}_n = \{x \in \mathbb{R}_+^{\binom{n}{2}} \mid \sum_{1 \leq i < j \leq n} b_i b_j x_{ij} \leq 0 \text{ for all } b \in \mathbb{R}^n \text{ with } \sum_{1 \leq i \leq n} b_i = 0\}.$$

The supporting cone of $\widetilde{\mathcal{L}}_n$ at each of its vertices is, up to symmetry, the cone NEG_n, i.e., the cone PSD_{n-1} (up to linear bijection) (see Remark 2.5).

Our main motivation for the study of the elliptope $\mathcal{L}_n$ comes from its role in approximating the max-cut problem. Actually, the elliptope $\mathcal{L}_n$ displays an example of an interesting complexity phenomenon. Namely, the weak optimization problem over $\mathcal{L}_n$ is polynomial (by the theory of [12] since checking whether a matrix belongs to $\widetilde{\mathcal{L}}_n$ can be done efficiently), but testing whether the optimum is reached in a vertex of $\mathcal{L}_n$ is NP-hard.

In Section 1, we describe some basic geometric properties of $\widetilde{\mathcal{L}}_n$ and $\mathcal{L}_n$. Namely, we provide the formulas for polars, normal cones, and faces of $\widetilde{\mathcal{L}}_n$ and $\mathcal{L}_n$. As a consequence, we conclude that $\widetilde{\mathcal{L}}_n$ has 2^{n-1} vertices corresponding to all bipartitions of the set $\{1, 2, \ldots, n\}$. In Section 2, we study the optimization problem

$$\begin{array}{c} \min \mathrm{Tr}(CX) \\ X \in \widetilde{\mathcal{L}}_n \end{array} \tag{4}$$

and its equivalent formulation

$$\begin{array}{c} \min c^t x \\ x \in \mathcal{L}_n. \end{array} \tag{5}$$

Since $\widetilde{\mathcal{L}}_n$ is not polyhedral, the optimum need not be attained in a vertex of $\widetilde{\mathcal{L}}_n$. We call a symmetric matrix C *exact* if the optimum of (4) is attained in a vertex of $\widetilde{\mathcal{L}}_n$. Our main result is the complete characterization of the exact matrices of the form $C = bb^t$ for some vector b. In Section 3, we explain in more detail the connection with the approximation of the max-cut problem.

We now give some preliminaries.

Given two $n \times n$ matrices $A = (a_{ij})$, $B = (b_{ij})$, we set

$$\langle A, B\rangle = \sum_{i,j=1}^{n} a_{ij} b_{ij}.$$

If A, B are symmetric, then we have the identity $\langle A, B\rangle = \mathrm{Tr}(AB)$. We write $A \succeq 0$ if A is a symmetric positive semidefinite matrix, i.e., $x^t A x \geq 0$ for all $x \in \mathbb{R}^n$. Let SYM_n (resp. PSD_n, DIAG_n) denote the set of all $n \times n$ symmetric (resp. symmetric positive semidefinite, diagonal) matrices. For $x \in \mathbb{R}^n$, $diag(x)$ denotes the diagonal matrix with diagonal entries $x_1, \ldots, x_n$ and, for a matrix A, $diag(A)$ denotes the vector consisting of the diagonal entries of A.

Let K be a convex body in $\mathbb{R}^n$, i.e., K is a compact convex subset of $\mathbb{R}^n$. The *polar* K^* of K is defined by

$$K^* = \{x \in \mathbb{R}^n \mid x^t y \leq 1 \text{ for all } y \in K\}.$$

If K is a convex cone, then its polar coincides with the set $\{x \in \mathbb{R}^n \mid x^t y \leq 0\}$. Given a boundary point x_0 of K, its *normal cone* $N(K, x_0)$ is defined by

$$N(K, x_0) = \{c \in \mathbb{R}^n \mid c^t x \leq c^t x_0 \text{ for all } x \in K\}.$$

The dimension of the normal cone permits to classify the boundary points of K. Namely, a boundary point x_0 is a *vertex* of K if its normal cone is full dimensional, and x_0 is a *regular* (or *smooth*) point of K if $N(K, x_0)$ has dimension 1, i.e., there is only one supporting hyperplane for K passing through x_0. The *supporting cone* $C(K, x_0)$ at x_0 is then defined by

$$C(K, x_0) = \{x \in \mathbb{R}^n \mid c^t x \leq 0 \text{ for all } c \in N(K, x_0)\}.$$

A subset F of K is called a *face* (or *extreme set*) of K if, for all $x \in F, y, z \in K$, $0 \leq \alpha \leq 1$, $x = \alpha y + (1-\alpha) z$ implies that $y, z \in F$. The set F is called an *exposed set* if $S = K \cap H$ for some supporting hyperplane H for K. Clearly, each exposed set is a face.

The convex bodies considered in this paper are $\widetilde{\mathcal{J}}_n$, $\widetilde{\mathcal{L}}_n$ and their projections $\mathcal{J}_n$, $\mathcal{L}_n$; we recall the precise definitions.

$$\widetilde{\mathcal{J}}_n = \{Y = (y_{ij}) \in \mathrm{SYM}_n \mid \frac{1}{2}J - Y \succeq 0,\ y_{ii} = 0 \text{ for all } i = 1, \ldots, n\},$$

$$\widetilde{\mathcal{L}}_n = \{X = (x_{ij}) \in \mathrm{SYM}_n \mid X \succeq 0,\ x_{ii} = 1 \text{ for all } i = 1, \ldots, n\},$$

$$\mathcal{J}_n = \tau(\widetilde{\mathcal{J}}_n), \text{ and } \mathcal{L}_n = \tau(\widetilde{\mathcal{L}}_n).$$

Hence, $\widetilde{\mathcal{L}}_n$ is the image of $\widetilde{\mathcal{J}}_n$ under the linear bijection $X = J - 2Y$. Clearly, $\mathcal{J}_n$ and $\mathcal{L}_n$ can be alternatively descibed by

$$\mathcal{J}_n = \{y \in \mathbb{R}^{\binom{n}{2}} \mid \sum_{1 \le i < j \le n} b_i b_j y_{ij} \le \frac{1}{4}(\sum_{1 \le i \le n} b_i)^2, \text{ for all } b \in \mathbb{R}^n\},$$

$$\mathcal{L}_n = \{x \in \mathbb{R}^{\binom{n}{2}} \mid \sum_{1 \le i < j \le n} b_i b_j x_{ij} \ge -\frac{1}{2} \sum_{1 \le i \le n} b_i^2, \text{ for all } b \in \mathbb{R}^n\}.$$

Given a subset S of $\{1, \ldots, n\}$, let χ^S denote its characteristic vector, defined by $\chi_i^S = 1$ if $i \in S$ and $\chi_i^S = 0$ otherwise, and set $\overline{S} = \{1, \ldots, n\} \setminus S$. Let J denote the all ones matrix. We set

$$J_S = J - \chi^S(\chi^S)^t - \chi^{\overline{S}}(\chi^{\overline{S}})^t = \left(\begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \end{array}\right),$$

$$L_S = 2\chi^S(\chi^S)^t + 2\chi^{\overline{S}}(\chi^{\overline{S}})^t - J = \left(\begin{array}{c|c} 1 & -1 \\ \hline -1 & 1 \end{array}\right).$$

Hence, $J_S \in \widetilde{\mathcal{J}}_n$, $L_S = J - 2J_S \in \widetilde{\mathcal{L}}_n$, $L_S = L_{\overline{S}}$, and $L_\emptyset = J_\emptyset = J$. We call the matrices J_S,L_S a 01-cut matrix and a ± 1-cut matrix, respectively; we refer to J_S,L_S as to the *cut matrices*.

2 Geometry of the elliptope $\mathcal{L}_n$

The main result of this section is the characterization of the vertices of the elliptope $\mathcal{L}_n$. As tools for this result, we describe the polar of $\mathcal{L}_n$ and the normal cone at any point of $\mathcal{L}_n$. We also present results on the faces of $\mathcal{L}_n$ and full treatment in the case of $\mathcal{L}_3$.

Let us first observe that the bodies $\widetilde{\mathcal{L}}_n$ and $\mathcal{L}_n$ have some symmetries. Given a subset A of $\{1, \ldots, n\}$, consider the mapping Sw_A on SYM_n, called *switching*, which is defined by $Y = Sw_A(X)$ with

$$y_{ij} = \begin{cases} x_{ij} & \text{if } i, j \in S \text{ or } i, j \notin S \\ -x_{ij} & \text{otherwise.} \end{cases}$$

Then, $Sw_A(L_S) = L_{S \triangle A}$ for each subset S and $Sw_A(\widetilde{\mathcal{L}}_n) = \widetilde{\mathcal{L}}_n$. There is an obvious analogue of switching for the body $\mathcal{L}_n$.

Let W denote the set of all $n \times n$ matrices X with diagonal entries equal to 1. Then, we have the equality

$$\mathcal{L}_n = \mathrm{PSD}_n \cap W. \tag{6}$$

Hence, the inclusion $\mathrm{PSD}_n^* \oplus W^* \subseteq \widetilde{\mathcal{L}}_n^*$ holds trivially. In fact, equality holds as shown in the next result. Note that $\mathrm{PSD}_n^* = -\mathrm{PSD}_n$ and W^* is the set of diagonal matrices with trace less than or equal to 1.

Proposition 2.1 *(i)* $\widetilde{\mathcal{L}}_n^* = \{D - M \mid M \succeq 0,\ D \in \mathrm{DIAG}_n, \mathrm{Tr}(D) = 1\}$.
(ii) $\mathcal{L}_n^* = \mathrm{Conv}(-2\tau(bb^t) \mid b \in \mathbb{R}^n, \| b \| = 1)$.

Proof. (i) Let $Y \in \widetilde{\mathcal{L}}_n^*$ and assume, for contradiction, that $Y \notin \{D - M \mid M \in \mathrm{PSD}_n,\ D \in \mathrm{DIAG}_n, \mathrm{Tr}(D) = 1\}$. Then, for each $D \in \mathrm{DIAG}_n$ with $\mathrm{Tr}(D) = 1$, the matrix $D - Y$ is not positive semidefinite, i.e., $\lambda_{\min}(D - Y) < 0$. Therefore, we have that $\max(\lambda_{\min}(D - Y) \mid D \in \mathrm{DIAG}_n, \mathrm{Tr}(D) = 1) < 0$. The following result is shown in [3]. Let D_0 be the diagonal matrix with trace one for which the above maximization problem attains its optimum and set $\lambda_0 = \lambda_{\min}(D_0 - Y) < 0$. Then, there exists a set of vectors $v_1, \ldots, v_k$ which are eigenvectors of $D_0 - Y$ for the eigenvalue λ_0 and such that all diagonal entries of the matrix $X := \sum_{h=1}^{k} v_h v_h^t$ are equal to 1. Hence, the matrix X belongs to $\widetilde{\mathcal{L}}_n$ and

$$\mathrm{Tr}((Y - D_0)X) = \sum_{h=1}^{k} \mathrm{Tr}((Y - D_0)v_h v_h^t) = -\sum_{h=1}^{k} \lambda_0 \mathrm{Tr}(v_h v_h^t) = -\lambda_0 \mathrm{Tr}(X) = -n\lambda_0.$$

Therefore, $\langle Y, X\rangle = \langle D_0, X\rangle + \langle Y - D_0, X\rangle = 1 - n\lambda_0 > 1$, contradicting the fact that $Y \in \widetilde{\mathcal{L}}_n^*$.
(ii) Given $y \in \mathbb{R}^{\binom{n}{2}}$, let Y denote the $n \times n$ symmetric matrix whose upper triangular part is y and with diagonal entries equal to $-\frac{1}{n}$. Then, $y \in \mathcal{L}_n^*$ if and only if $Y \in \widetilde{\mathcal{L}}_n^*$ since $\langle X, Y\rangle = 2x^t y - 1$ for $X \in \mathcal{L}_n$ and $x = \tau(X)$. By (i), we know that $Y \in \widetilde{\mathcal{L}}_n^*$ if and only if $Y = D - \sum_{h=1}^{k} \lambda_h b_h b_h^t$ for some diagonal matrix D with trace 1, $b_1, \ldots, b_k$ unit vectors, and $\lambda_1, \ldots, \lambda_k \geq 0$ with $\sum_{h=1}^{k} \lambda_h = 2$ (since $-1 = \mathrm{Tr}(Y) = \mathrm{Tr}(D) - \sum_h \lambda_h$). Therefore, we deduce that $y \in \mathcal{L}_n^*$ if and only if $y = -\sum_h \lambda_h \tau(b_h b_h^t)$ for some unit vectors b_h and scalars $\lambda_h \geq 0$ with $\sum_h \lambda_h = 2$, i.e., $y \in \mathrm{Conv}(-2\tau(bb^t) \mid b \in \mathbb{R}^n, \| b \| = 1)$. ∎

Proposition 2.2 *Let $A \in \widetilde{\mathcal{L}}_n$ and $a = \tau(A) \in \mathcal{L}_n$. Then*
(i) $N(\widetilde{\mathcal{L}}_n, A) = \{D - M \mid D \in DIAG_n, M \succeq 0, \langle M, A\rangle = 0\}$.
(ii) $N(\mathcal{L}_n, a) = \mathrm{Cone}(-\tau(bb^t) \mid b \in \mathrm{Ker}(A))$.

Proof. (i) First, if $D \in \mathrm{DIAG}_n$, $M \succeq 0$ with $\langle M, A\rangle = 0$, then $D - M \in N(\widetilde{\mathcal{L}}_n, A)$ since, for all $X \in \widetilde{\mathcal{L}}_n$, $\langle D - M, X\rangle = \mathrm{Tr}(D) - \langle M, X\rangle \leq \mathrm{Tr}(D) = \langle D - M, A\rangle$. Conversely, let $Y \in N(\widetilde{\mathcal{L}}_n, A)$, i.e., $\langle Y, X\rangle \leq \langle Y, A\rangle$ holds for all $X \in \widetilde{\mathcal{L}}_n$. We can suppose that the diagonal entries of Y are equal to 0 (since $\langle D, X\rangle = \mathrm{Tr}(D)$ holds

for all $X \in \widetilde{\mathcal{L}}_n$ and $D \in \mathrm{DIAG}_n$). Suppose first that $\langle Y, A\rangle = 0$. We show that $-Y$ is positive semidefinite, i.e., that $\langle Y, X\rangle \leq 0$ for all $X \in \mathrm{PSD}_n$. If $X \in \widetilde{\mathcal{L}}_n$, then $\langle Y, X\rangle \leq 0$ holds by the assumption that $Y \in N(\widetilde{\mathcal{L}}_n, A)$. If $X \succeq 0$ with $x_{ii} \leq 1$ for all $1 \leq i \leq n$, then $X' := X + \mathrm{diag}(1 - x_{11}, \ldots, 1 - x_{nn}) \in \widetilde{\mathcal{L}}_n$. Hence, $\langle Y, X'\rangle \leq 0$, i.e., $\langle Y, X\rangle \leq 0$. Finally, if $X \succeq 0$, let α be a positive scalar such that the diagonal entries of αX are less than or equal to 1. By the previous case, $\langle Y, \alpha X\rangle \leq 0$ which implies that $\langle Y, X\rangle \leq 0$. We now suppose that $a := \langle Y, A\rangle \neq 0$. Then, $a > 0$ since $0 = \langle Y, I\rangle \leq \langle Y, A\rangle$. So, $a^{-1}Y \in \widetilde{\mathcal{L}}_n^*$ and, therefore, by Proposition 2.1, $Y = D - M$ for some diagonal matrix D with trace a and $M \succeq 0$ with $\langle M, A\rangle = \langle D, A\rangle - \langle Y, A\rangle = \mathrm{Tr}(D) - a = 0$. This concludes the proof of (i).
(ii) Applying (i), we obtain that $N(\mathcal{L}_n, a) = \{-\tau(M) \mid M \succeq 0,\ \langle M, A\rangle = 0\}$. The result follows since, for a decomposition $M = \sum_{1\leq h\leq k} b_h b_h^t$ of M as a sum of rank one matrices, $\langle M, A\rangle = 0$ holds if and only if $Ab_h = 0$, i.e., $b_h \in \mathrm{Ker}(A)$, for all h.
∎

Remark 2.3 Let us remark that, for $n = 3$, the normal cone at each point $\tau(L_S)$ of $\mathcal{L}_n$ is a circular cone. By symmetry, it suffices to check this fact for the cut matrix $L_\emptyset = J$. Let us consider the section of the normal cone $N(\mathcal{L}_n, \tau(J))$ by the hyperplane with equation $x_{12} + x_{13} + x_{13} = 3$. Note that the point $c = (1, 1, 1)$ belongs to $N(\mathcal{L}_n, \tau(J)) \cap H$. One can easily check that each extreme ray $-\tau(bb^t)$ of $N(\mathcal{L}_n, \tau(J))$ intersects H in a point which is at constant distance $\sqrt{6}$ from c. This shows, therefore, that $N(\mathcal{L}_n, \tau(J))$ is a circular cone. We show in Fig. 2 the normal cone at a vertex.

We can now characterize the vertices of $\mathcal{L}_n$.

Theorem 2.4 *$\mathcal{L}_n$ has 2^{n-1} vertices, namely, the vectors $\tau(L_S)$, for $S \subseteq \{1, \ldots, n\}$.*

Proof. We first check that each vector $\tau(L_S)$ is a vertex of $\mathcal{L}_n$. Indeed, for $1 \leq i < j \leq n$, the hyperplane $x_{ij} = 1$ (resp. $-x_{ij} = 1$) is supporting for $\mathcal{L}_n$ at $\tau(L_S)$ if $i, j \in S^2 \cup (\{1, \ldots, n\} \setminus S)^2$ (resp. if $i \in S, j \notin S$ or vice versa). This shows that the normal cone of $\mathcal{L}_n$ at $\tau(L_S)$ is full dimensional, i.e., that $\tau(L_S)$ is a vertex of $\mathcal{L}_n$. Conversely, let $A \in \widetilde{\mathcal{L}}_n$ and suppose that $a = \tau(A)$ is a vertex of $\mathcal{L}_n$. Then, there exist $\binom{n}{2}$ vectors $b_1, \ldots, b_{\binom{n}{2}}$ such that the system $(b_i b_i^t \mid 1 \leq i \leq \binom{n}{2})$ is linearly independent. Consider the $\binom{n}{2} \times \binom{n}{2}$ matrix M whose rows are the vectors $b_i b_i^t$ and the submatrix M_1 formed by its first $n-1$ columns, indexed by the pairs $(1, j)$ for $2 \leq j \leq n$. Then, M_1 has rank $n-1$ and, thus, contains an $(n-1)\times(n-1)$ nonsingular submatrix which is indexed, say, by the vectors $b_1, \ldots, b_{n-1}$. It is easy to check that the vectors $b_1, \ldots, b_{n-1}$ are linearly independent. This shows that the matrix A has rank one and, thus, $A = aa^t$ for some $a \in \mathbb{R}^n$. But, $a \in \{-1, 1\}^n$

since the diagonal entries $(a_i)^2$ of A are all equal to 1. Therefore, A is one of the cut matrices L_S. ∎

In particular, the vectors $\tau(L_S)$ are the only ± 1-valued members of $\mathcal{L}_n$ (indeed, every ± 1-member of $\mathcal{L}_n$ has a full dimensional normal cone, i.e., is a vertex of $\mathcal{L}_n$).

Remark 2.5 As a consequence of Proposition 2.2, we have the following assertions.
(i) The regular points of $\mathcal{L}_n$, i.e., having a normal cone of dimension one, are of the form $\tau(A)$ for $A \in \widetilde{\mathcal{L}}_n$ whose kernel $\mathrm{Ker}(A)$ has dimension 1.
(ii) Given $A \in \widetilde{\mathcal{L}}_n$, the supporting cone of $\mathcal{L}_n$ at the point $a = \tau(A)$ is given by $C(\mathcal{L}_n, a) = \{x \in \mathbb{R}^{\binom{n}{2}} \mid \sum_{1 \leq i < j \leq n} b_i b_j x_{ij} \geq 0 \text{ for all } b \in \mathrm{Ker}(A)\}$.
In particular, the supporting cone at the vertex $\tau(L_\emptyset)$ coincides with the cone $-\mathrm{NEG}_n$ (i.e., is a linear bijective image of PSD_{n-1}) and the suporting cone at any other vertex $\tau(L_S)$ is a symmetric image of it, namely, $C(\mathcal{L}_n, \tau(L_S)) = Sw_S(-\mathrm{NEG}_n)$.

We now turn to the description of the faces of $\mathcal{L}_n$. Let us first recall some well known facts about the faces of the cone PSD_n of positive semidefinite matrices (see [15]). Let $A, B \in \mathrm{PSD}_n$ and let $\Phi(A)$ denote the smallest face of PSD_n containing A. Then, $B \in \Phi(A)$ if and only if $\mathrm{Ker}(A) \subseteq \mathrm{Ker}(B)$. In particular, every face of PSD_n is of the form

$$\Phi_V := \{X \in \mathrm{PSD}_n \mid V \subseteq \mathrm{Ker}(X)\}$$

for some subspace V of $\mathbb{R}^n$ and, conversely, Φ_V is a face of PSD_n for each subspace V. Moreover, each face of PSD_n is isomorphic to PSD_r and, thus, has dimension $\binom{r+1}{2}$, for some $0 \leq r \leq n$.

It is well known that, if K_1, K_2 are two convex bodies and F_1, F_2 are faces of K_1, K_2, respectively, then $F_1 \cap F_2$ is a face of $K_1 \cap K_2$ and, moreover, each face of $K_1 \cap K_2$ arises in this way.

As noted in relation (6), the body $\widetilde{\mathcal{L}}_n$ is the intersection of PSD_n and W. Clearly, W is the only face of W. Hence, we have the following result.

Proposition 2.6 *Let $A, B \in \widetilde{\mathcal{L}}_n$ and let $F(A)$ denote the smallest face of $\widetilde{\mathcal{L}}_n$ containing A. Then, $B \in F(A)$ if and only if Ker $(A) \subseteq$ Ker(B). In particular, every face of $\widetilde{\mathcal{L}}_n$ is of the form*

$$F_V := \{X \in \widetilde{\mathcal{L}}_n \mid V \subseteq Ker(X)\}$$

for some subspace V of $\mathbb{R}^n$ and $F(A) = F_{Ker(A)}$ for $A \in \widetilde{\mathcal{L}}_n$.

Corollary 2.7 *Every face of $\widetilde{\mathcal{L}}_n$ is exposed.*

Proof. Consider a face F_V where V is a subspace of $\mathbb{R}^n$. Let $b_1, \ldots, b_k$ be an orthogonal base of V. Then, for $X \in \widetilde{\mathcal{L}}_n$, $X \in F_V$ if and only if $b_i^t X b_i = 0$ for all $i = 1, \ldots, k$ or, equivalently, $\sum_{1 \le i \le k} b_i^t X b_i = 0$. Hence, the face F_V arises as the intersection of $\widetilde{\mathcal{L}}_n$ by the supporting hyperplane $\sum_{1 \le i \le k} b_i^t X b_i = 0$. This shows that F_V is exposed. ∎

Note that, given a subspace V, there always exists $X \succeq 0$ such that $V \subseteq \mathrm{Ker}(X)$, but there may exist no such $X \in \widetilde{\mathcal{L}}_n$. This is the case, for instance, if $V \subseteq \mathbb{R}^2$ is generated by the vector $(2,1)$. For this reason, we call a subspace V of $\mathbb{R}^n$ *realizable* if there exists $X \in \widetilde{\mathcal{L}}_n$ such that $V \subseteq \mathrm{Ker}(X)$. Clearly, the only realizable subspaces of dimension $n-1$ are the kernels of the cut matrices L_S. We give in the next section the characterization of the 1-dimensional realizable subspaces.

Unlike the case of the cone PSD_n, a more precise description of the faces of the convex body $\widetilde{\mathcal{L}}_n$ (or $\mathcal{L}_n$), as e.g. their dimension, seems a hard problem. We give some partial results. In particular, we show that the convex segment joining any two vertices of $\widetilde{\mathcal{L}}_n$ is a face of $\widetilde{\mathcal{L}}_n$. We also indicate how every face of $\widetilde{\mathcal{L}}_n$ can be "lifted" to a face of $\widetilde{\mathcal{L}}_{n+1}$.

Proposition 2.8 *Let A, B be distinct subsets of $\{1, \ldots, n\}$. Then, the convex segment $[L_A, L_B] = \{\alpha L_A + (1-\alpha) L_B \mid 0 \le \alpha \le 1\}$ is a face of $\widetilde{\mathcal{L}}_n$.*

Proof. Using the switching symmetry, we can suppose that $B = \emptyset$. We show that the segment $[L_\emptyset, L_A]$ is a face of $\widetilde{\mathcal{L}}_n$. Set $Y = \frac{1}{2}(L_\emptyset + L_A)$. Then, $\mathrm{Ker}(Y) = \{b \in \mathbb{R}^n \mid \sum_{i \in A} b_i = \sum_{i \notin A} b_i = 0\}$. One can easily check that a symmetric $n \times n$ matrix matrix X belongs to $F(Y)$ if and only if there exists a scalar a such that $|a| \le 1$ and $X = (x_{ij})$ with $x_{ij} = 1$ for $i, j \in A$ or $i, j \notin A$ and $x_{ij} = a$ for $i \in A, j \notin A$. In other words, $X \in F(Y)$ if and only if X is the convex combination $\frac{a+1}{2} L_\emptyset + \frac{1-a}{2} L_A$ of $L_\emptyset$ and L_A. This shows that $[L_\emptyset, L_A] = F(Y)$ is, thus, a face of $\widetilde{\mathcal{L}}_n$. ∎

Note that there exist faces of $\widetilde{\mathcal{L}}_n$ of dimension 2 that are not polyhedral. We describe such a face for $\mathcal{L}_4$ in Example 2.10 below. We now present a full description of the faces of the body $\mathcal{L}_3$.

Proposition 2.9 *Every proper face of $\mathcal{L}_3$ is, either reduced to a single point of $\mathcal{L}_3$, or is an edge (1-dimensional face) joining two vertices of $\mathcal{L}_3$ (there are six such faces).*

Proof. Let F_V be a face of $\mathcal{L}_3$, where V is a (realizable) subspace of $\mathbb{R}^3$. If $\dim(V) = 2$, then F_V is reduced to a vertex of $\mathcal{L}_3$. Suppose now that V has dimension 1. Let $b \in V$. Then, by Lemma 3.1, b is balanced. We can suppose that $|b_1|, |b_2|, |b_3| \leq 1$ and, for instance, $b_1 = 1$. Then, $b = (1, \alpha, \beta)$ with $1 \leq |\alpha| + |\beta|$. Let $X \in \widetilde{\mathcal{L}}_3$ be of the form $\begin{pmatrix} 1 & x & y \\ x & 1 & z \\ y & z & 1 \end{pmatrix}$, where $x, y, z \in \mathbb{R}$. Then, $X \in F_V$ if and only if $Xb = 0$, i.e., x, y, z satisfy the system

$$\begin{cases} \alpha x + \beta y & = -1 \\ x + \beta z & = -\alpha \\ y + \alpha z & = -\beta. \end{cases} \tag{7}$$

The determinant of the system (7) is equal to $-2\alpha\beta$. If $\alpha\beta \neq 0$, then the system (7) has a unique solution (x, y, z), i.e., F_V consists of a single point of $\widetilde{\mathcal{L}}_3$. If, say, $\alpha = 0$, then $\beta = \pm 1$. The solutions of the sytem (7) are of the form $(-\beta z, -\beta, z)$ for $z \in \mathbb{R}$. For $\beta = 1$, we obtain that $X \in F_V$ if and only if $X = \frac{z+1}{2} L_{\{1\}} + \frac{1-z}{2} L_{\{3\}}$ with $|z| \leq 1$ and, thus, $F_V = [L_{\{1\}}, L_{\{3\}}]$. Similarly, for $\beta = -1$, $X \in F_V$ if and only if $X = \frac{z+1}{2} L_{\emptyset} + \frac{1-z}{2} L_{\{2\}}$, i.e., $F_V = [L_{\emptyset}, L_{\{2\}}]$. ∎

Example 2.10 Let V denote the 1-dimensional subspace of $\mathbb{R}^4$ spanned by the vector $b = (1, 1, 1, 0)$. One can check easily that a symmetric 4×4 matrix X belongs to F_V if and only if X is of the form $X = \begin{pmatrix} 1 & -1/2 & -1/2 & x \\ -1/2 & 1 & -1/2 & y \\ -1/2 & -1/2 & 1 & z \\ x & y & z & 1 \end{pmatrix}$, where $x, y, z \in \mathbb{R}$ satisfy $x + y + z = 0$ and $x^2 + xy + y^2 \leq \frac{3}{4}$ (the first condition ensures that $Xb = 0$ and the second one that $X \succeq 0$). Hence, F_4 is a 2-dimensional face of $\widetilde{\mathcal{L}}_4$ with the shape of an ellipse.

Finally, we present an operation which permits to lift each face of $\widetilde{\mathcal{L}}_n$ to a face of $\widetilde{\mathcal{L}}_{n+1}$. Let X be a symmetric $n \times n$ matrix with diagonal entries equal to 1 and let $c \in \mathbb{R}^n$ denote its last n-th column. Consider the $(n+1) \times (n+1)$ symmetric matrix $L_n(X)$ defined by $L_n(X) = \left(\begin{array}{c|c} X & c \\ \hline c^t & 1 \end{array}\right)$.

Lemma 2.11 *$X \in \widetilde{\mathcal{L}}_n$ if and only if $L_n(X) \in \widetilde{\mathcal{L}}_{n+1}$.*

Proof. Let $y \in \mathbb{R}^{n-1}, x_n, x_{n+1} \in \mathbb{R}$ and set $x = (y, x_n)$, $x' = (y, x_n + x_{n+1}) \in \mathbb{R}^n$, $z = (y, x_n, x_{n+1})$ and $z' = (y, x_n, 0) \in \mathbb{R}^{n+1}$. Then, we have that $z^t L_n(X) z = x'^t X x'$ and $x^t X x = z'^t L_n(X) z'$. This shows that $X \succeq 0$ if and only if $L_n(X) \succeq 0$

and, thus, $X \in \widetilde{\mathcal{L}}_n$ if and only if $L_n(X) \in \widetilde{\mathcal{L}}_{n+1}$. ∎

Corollary 2.12 *Let F be a face of $\widetilde{\mathcal{L}}_n$. Then, $F' := \{L_n(X) \mid X \in F\}$ is a face of $\widetilde{\mathcal{L}}_{n+1}$.*

Proof. Suppose $F = F(Y)$ is the smallest face of $\widetilde{\mathcal{L}}_n$ containing some $Y \in \widetilde{\mathcal{L}}_n$. We show that F' coincides with $F(L_n(Y))$, the smallest face of $\widetilde{\mathcal{L}}_{n+1}$ containing $L_n(Y)$. The kernel of $L_n(Y)$ is spanned by the vectors $(b, 0)$ for $b \in \mathrm{Ker}(Y)$ and $(0, \ldots, 0, 1, -1)$. Hence, if $Z \in F(L_n(Y))$, then the n-th and $(n+1)$-th columns of Z coincide because $(0, \ldots, 0, 1, -1) \in \mathrm{Ker}(Z)$, and the submatrix of Z formed by its first n columns and rows belongs to $F(Y)$ because $(b, 0) \in \mathrm{Ker}(Z)$ for all $b \in \mathrm{Ker}(Y)$. Therefore, $Z = L_n(X)$ for some $X \in F(Y)$. This shows that $F' = F(L_n(Y))$ is, thus, a face of $\widetilde{\mathcal{L}}_{n+1}$. ∎

3 Optimizing over $\mathcal{L}_n$

Let us recall that a symmetric matrix C is called exact if the optimum of (4) is attained in a vertex of $\widetilde{\mathcal{L}}_n$. The motivation to study this question comes from the application to the max-cut problem, which will be discussed in the next section. The main result of this section is the characterization of the exact matrices C which are of the form $C = bb^t$ for a vector b.

Let $b = (b_1, \ldots, b_n)$ be a vector. The *gap* of b, denoted as $\gamma(b)$, is defined as

$$\gamma(b) := \min_{S \subset \{1, \ldots, n\}} |b(S) - b(\overline{S})| \tag{8}$$

where $b(S) := \sum_{i \in S} b_i$. In particular, we have $\gamma(b) = 0$ if $b(S) = b(\overline{S})$ for some S. We say that a vector $b = (b_1, \ldots, b_n)$ is *balanced* if

$$|b_i| \leq \sum_{j=1}^{i-1} |b_j| + \sum_{j=i+1}^{n} |b_j| \tag{9}$$

for every $i = 1, \ldots, n$. In other words, a vector b is balanced if none of its entries (in absolute value) is larger than the sum of the remaining entries (in absolute value).

Given a pair of vectors x and y, let $x \circ y$ denote the vector $z = (z_i)$ with entries $z_i := x_i y_i$. Let $V^\perp$ denote the orthogonal complement of a linear subspace V. Let $e = (1, \ldots, 1)$ denote the vector of all ones.

Lemma 3.1 *A linear subspace V is realizable if and only if $e \in$ cone$\{x \circ x \mid x \in V^\perp\}$.*

Proof. Assume that V is realizable, i.e., $V \subset \mathrm{Ker}(X)$ for some $X \in \widetilde{\mathcal{L}}_n$. Since X is positive semidefinite, $\mathrm{Ker}(X)$ is the eigenspace of the minimum eigenvalue $\lambda_{\min} = 0$, and hence X can be written as $X = \sum_{i=1}^k x_i x_i^t$ where $x_1, \ldots, x_k \in (\mathrm{Ker}(X))^\perp \subset V^\perp$. Since $diag(X) = e$, we have $e = \sum_{i=1}^k x_i \circ x_i \subset cone\{x \circ x \mid x \in V^\perp\}$. Conversely, let $e = \sum_{i=1}^k x_i \circ x_i$ for some $x_1, \ldots, x_k \in V^\perp$. Set $X := \sum_{i=1}^k x_i x_i^t$. Obviously, $V \subset \mathrm{Ker}(X)$, and $X \in \widetilde{\mathcal{L}}_n$ since X is positive semidefinite and $diag(X) = e$. ∎

The following lemma is a re-phrasing (using a different terminology) of a result of [4, Theorem 3.2].

Lemma 3.2 *([4]) Let $b \in \mathbb{R}^n$ be a vector. Then the linear space $V = \langle b \rangle$ generated by b is realizable if and only if b is balanced.*

∎

Theorem 3.3 *Let C be a matrix of the form $C = bb^t$ for some $b \in \mathbb{R}^n$. Then, C is exact if and only if one of the following holds*
(i) b is balanced and has gap $\gamma(b) = 0$, or
(ii) b is unbalanced.

Proof. We have $\langle bb^t, X \rangle = b^t X b \geq 0$ for every b and every $X \in \widetilde{\mathcal{L}}_n$. Hence $\min_{X \in \widetilde{\mathcal{L}}_n} b^t X b \geq 0$, with equality if and only if $b \in \mathrm{Ker}(X)$ for some $X \in \widetilde{\mathcal{L}}_n$, i.e., if the subspace $\langle b \rangle$ is realizable. Hence, from Lemma 3.2, the minimum is equal to zero if and only if b is balanced. Assume that b is balanced. We claim that

$$bb^t \text{ is exact if and only if } \gamma(b) = 0. \tag{10}$$

The matrix bb^t is, by definition, exact if and only if $\min_{X \in \widetilde{\mathcal{L}}_n} b^t X b$ is reached in a vertex L_S, i.e., $\langle bb^t, L_S \rangle = b^t L_S b = 0$ for some S. Hence $L_S b = 0$ since $L_S \succeq 0$. The latter is equivalent to $b(S) - b(\overline{S}) = 0$. Hence (10) is proved. Assume that b is not balanced. We claim that

$$bb^t \text{ is exact.} \tag{11}$$

Assume that all $b_i, i = 1, \ldots, n$, are nonnegative. Without loss of generality we may also assume that $b_1 > \sum_{i=2}^n b_i$. Define the vector $a = (a_i)$ by $a_1 = \sum_{i=2}^n b_i$ and $a_i = b_i$ for $i = 2, \ldots, n$. Hence, $a^t L_{\{1\}} a = 0$, which shows that the minimum of $\langle aa^t, X \rangle$ is reached in the vertex $L_{\{1\}}$ of $\widetilde{\mathcal{L}}_n$. Hence, $-aa^t$ belongs to the normal cone of $L_{\{1\}}$,

$$-aa^t \in N(\widetilde{\mathcal{L}}_n, L_{\{1\}}). \tag{12}$$

For every $i = 2, \ldots, n$, set $f_i = (1, 0, \ldots, 0, 1, 0, \ldots, 0)$ (the 1- and i-th entries are equal to 1), and observe that $f_i \in \mathrm{Ker}(L_{\{1\}})$. Hence

$$-f_i f_i^t \in N(\widetilde{\mathcal{L}}_n, L_{\{1\}}). \tag{13}$$

Now, it is easy to check that bb^t can be expressed as

$$bb^t = aa^t + \sum_{i=2}^{n} \lambda_i f_i f_i^t$$

for $\lambda_i = (b_1 - a_1)a_i \geq 0$, and hence (12) and (13) imply that $-bb^t \in N(\widetilde{\mathcal{L}}_n, L_{\{1\}})$. Thus, bb^t is exact. In case that some b_i's are negative, apply switching with the set $S = \{i \mid b_i < 0\}$. ∎

For every $S \subseteq \{1, \ldots, n\}$, let $\mathcal{O}_S \subset \mathbb{R}^{\binom{n}{2}}$ denote the orthant

$$\mathcal{O}_S := \{x = (x_{ij}) \mid x_{ij} \geq 0 \text{ for } i,j \in S, \text{ or } i,j \notin S \text{ and } x_{ij} \leq 0 \text{ otherwise}\}.$$

Since $\mathcal{L}_n$ is contained in the unit cube $[-1,1]^{\binom{n}{2}}$, we have

Lemma 3.4 *For every S, the orthant $\mathcal{O}_S$ is entirely contained in the normal cone $N(\mathcal{L}_n, \tau(L_S))$ of the vertex $\tau(L_S)$.*

∎

Let p_n denote the probability that a random vector $c \in \mathbb{R}^{\binom{n}{2}}$, with $\|c\| := \sum c_{ij}^2 = 1$, is exact.

Corollary 3.5 *We have $p_n \geq 2^{-\frac{1}{2}(n^2-3n+2)}$.*

Proof. Since $\mathcal{O}_S \subset N(\mathcal{L}_n, \tau(L_S))$ for every S by the above lemma, we have

$$p_n = Prob(c \in \bigcup_S N(\mathcal{L}_n, \tau(L_S)) \geq Prob(c \in \bigcup_S \mathcal{O}_S).$$

Hence

$$p_n \geq \frac{2^{n-1}}{2^{\binom{n}{2}}} = 2^{-\frac{1}{2}(n^2-3n+2)}$$

since $\widetilde{\mathcal{L}}_n$ has 2^{n-1} vertices and the total number of orthants is $2^{\binom{n}{2}}$. ∎

In particular, we have $p_3 \geq 0.5$ by the corollary. Ch. Delorme (personal communication) computed the exact value $p_3 = 0.845$.

Let us conclude this section with pointing out an interesting complexity aspect of the optimization over $\mathcal{L}_n$.

(i) The weak optimization problem (WOPT) over $\mathcal{L}_n$ is polynomial time solvable;

(ii) Testing whether the optimum over $\mathcal{L}_n$ is attained at a vertex is NP-hard.

Let us recall that the *weak optimization problem* for a convex body K is defined in [12] as follows. Given a rational vector c and a rational number $\varepsilon > 0$, either (i) find a rational vector y such that $y \in S(K,\varepsilon)$ and $c^t x \leq c^t y + \varepsilon$ for all $x \in S(K,-\varepsilon)$, or (ii) assert that $S(K,-\varepsilon)$ is empty. (Here $S(K,\varepsilon)$ denote the set of points which lie in the ε-neighborhood of K; and $S(K,-\varepsilon)$ denote the set of points whose ε-neighborhood is contained in K.) The polynomial-time solvability of WOPT follows from the theory developed in [12], since one can efficiently check the (weak) membership of $X \in \widetilde{\mathcal{L}}_n$ (by computing the minimum eigenvalue of X with sufficient precision, and inspecting its diagonal entries). On the other hand, the problem (ii) is NP-hard for $\mathcal{L}_n$, as a corollary of Theorem 3.3. Given an integer vector $b = (b_1, \ldots, b_n)$, it is NP-hard to decide whether the gap $\gamma(b)$ is zero (cf. the *exact sum* problem in [9]). Thus, if we could decide whether or not the optimum of $x^t \tau(bb^t)$ is reached in a vertex of $\mathcal{L}_n$, we would be able to solve the exact sum problem.

A practically efficient algorithm which can be used for the optimization problem over the elliptope $\mathcal{L}_n$ was described in [19].

4 The max-cut problem

Let $G = (V,E)$ be a graph and $c : E \to \mathbb{R}$ be an edge-weight function. The *max-cut problem* consists of finding a subset S of vertices for which the sum of the weights on the edges between S and $\bar{S}$ is maximum. Let us denote

$$mc(G,c) := \max_{S \subset V} \sum_{i \in S, j \notin S} c_{ij}.$$

The max-cut problem is polynomial-time solvable when G is planar ([13, 16]), and it is NP-complete for G general ([9]). Barahona and Mahjoub [1] introduced a polytope associated with the max-cut problem called the cut polytope. For our purpose, it is sufficient to recall the definition only for the case when $G = K_n$ is the complete graph. For a set $S \subset V$, let $\delta(S)$ (*the cut*) denote the edge set $\delta(S) := \{ij \mid i \in S, j \notin S\}$, and let $\chi^{\delta(S)}$ denote the characteristic vector of the cut $\delta(S)$ defined by $\chi_{ij}^{\delta(S)} = 1$ for $ij \in \delta(S)$, and $\chi_{ij}^{\delta(S)} = 0$ otherwise. The *cut polytope* P_n is defined as $P_n := conv\{\chi^{\delta(S)} \mid S \subset V\}$. Hence, the max-cut problem can be alternatively defined as

$$mc(G,c) := \max_{x \in P_n} c^t x \tag{14}$$

The cut polytope P_n has been extensively studied; see, e.g., [6, 7].

Schrijver [21] introduced the convex body $\mathcal{J}_n$ as a relaxation of the cut polytope. Indeed, since $\chi^{\delta(S)} \in \mathcal{J}_n$ for every S, we have

LEMMA 4.1
(i) $P_n \subset \mathcal{J}_n$,
(ii) $mc(G,c) \leq \max_{x\in\mathcal{J}_n} c^t x$, *setting* $c_{ij} = 0$ *if the pair* ij *is not an edge of* G.

∎

Clearly, the symmetric matrix $C = (c_{ij})$ is exact if and only if the program $\max_{x\in\mathcal{J}_n} c^t x$ solves the max-cut problem.

Delorme and Poljak ([3, 4, 5]) considered earlier an eigenvalue upperbound $\varphi(G,c)$, defined as

$$\varphi(G,c) := \min_{u\in\mathbb{R}^n, \sum u_i = 0} \frac{n}{4}\lambda_{\max}(L(G,c) + diag(u))$$

where $L(G,c)$ denotes the Laplacian matrix of the weighted graph (G,c) and $\lambda_{\max}$ the maximum eigenvalue.

Actually, the two bounds coincide, i.e.,

$$\max_{x\in\mathcal{J}_n} c^t x = \varphi(G,c),$$

as was shown by Poljak and Rendl [17], using duality.

For nonnegative weigths c, the quality of the approximation can be measured by the ratio

$$\frac{\max_{x\in\mathcal{J}_n} c^t x}{mc(G,c)}.$$

It has been conjectured in [4] that this ratio is bounded by 1.131 (with C_5 as the worst case). A recent result of [10] shows that this ratio is bounded by $\frac{1}{0.878} = 1.139$. Using a result of [4], one can prove a better bound for some special classes of weights.

PROPOSITION 4.2 *Let the weights* c_{ij} *be given by* $c_{ij} = a_i a_j$ *where* $a_1, \ldots, a_n \in \mathbb{R}_+$. *Then,*

$$\frac{\max_{x\in\mathcal{J}_n} c^t x}{mc(G,c)} \leq \frac{9}{8} = 1.125.$$

PROOF. For $a_1, \ldots, a_n$ balanced, this result was shown in (Corollary 5.1,[4]). If $a_1, \ldots, a_n$ is not balanced, then C is exact by Theorem 3.3, implying that the ratio is equal to 1. ∎

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