Hypermetrics in Geometry of Numbers

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Abstract.

A finite semimetric d on a set X is hypermetric if it satisfies the inequality $\sum_{i,j \in X} b_i b_j d_{ij} \leq 0$ for all $b \in \mathbb{Z}^X$ with $\sum_{i \in X} b_i = 1$.

Hypermetricity turns out to be the appropriate notion for describing the metric structure of holes in lattices.

We survey hypermetrics, their connections with lattices and applications.

Contents

1 Introduction

The central concept studied in this paper is hypermetricity. This is a natural strenghtening of the notion of metric, which has many applications and connections. The main topics to which hypermetrics relate include ℓ_1- and ℓ_2-metrics in analysis, the cut cone and the cut polytope in combinatorial optimization, graphs with high regularity, and, most importantly for our treatment, quadratic forms, Delaunay polytopes and holes in lattices.

The notion of hypermetrics sheds a new light and gives a more ordered view on some well studied questions; for example, on equiangular sets of lines, on graphs with minimum eigenvalue -2, on the metric properties of regular graphs. For instance, the parameter characterizing the three layers composing the famous list of 187 graphs with minimum eigenvalue -2 from [20] has now a more clear meaning: it comes from the radius of the L-polytope associated with the graph metrics in each layer (see Section 6.2).

Our central objects are hypermetric inequalities and hypermetric spaces. Given $b \in \mathbb{Z}^n$ with $\sum_{1\le i\le n} b_i = 1$, the inequality

$$\sum_{1\le i<j\le n} b_i b_j x_{ij} \le 0 \tag{1}$$

is called a **hypermetric inequality**. When $b = (1, 1, -1, 0, \ldots, 0)$, the inequality (1) is simply the metric condition or triangle inequality. A distance space (X, d) is said to be **hypermetric** if d satisfies all hypermetric inequalities. The **hypermetric cone** HYP_n is the cone defined by the inequalities (1) for all $b \in \mathbb{Z}^n$ with $\sum_{1\le i\le n} b_i = 1$.

Many important metrics are hypermetric. In particular, all ℓ_1-metrics are hypermetric. More precisely, given a distance d, we have the following chain of implications:

d is isometrically ℓ_2-embeddable
$\Longrightarrow$ d is isometrically ℓ_1-embeddable
$\Longrightarrow$ d is hypermetric
$\Longrightarrow$ $\sqrt{d}$ is isometrically ℓ_2-embeddable

Moreover, if d is hypermetric, then $\sqrt{d}$ has an ℓ_2-embedding on a sphere and, as we see below, this sphere corresponds to a hole in a lattice.

The last property in the above chain of implications is well characterized. Namely, $\sqrt{d}$ is isometrically ℓ_2-embeddable if and only if d satisfies the inequalities (1) for all $b \in \mathbb{Z}^n$ with $\sum_{1\leq i\leq n} b_i = 0$ (see Proposition 2.3) or, equivalently, if and only if the image $\xi_0(d)$ of d under a linear bijective mapping ξ_0 (the covariance mapping, defined in relation (9)) corresponds to a positive semidefinite quadratic form. Therefore, our object, the hypermetric cone, is (via the covariance mapping) contained in the cone of positive semidefinite quadratic forms. On the other hand, the polar of the image of the hypermetric cone under ξ_0 contains the cone of nonpositive quadratic forms and is contained in the cone of the quadratic forms that are nonpositive on binary variables.

A distance that plays a fundamental role in our treatment is the square of the euclidian distance, namely the distance $d^{(2)}$ defined by $d^{(2)}(x,y) = (x-y)^T(x-y)$ for $x, y \in \mathbb{R}^n$.

In fact, the study of hypermetrics amouts to the study of holes in lattices.

Let L be a lattice. Blow up a sphere S in one of the interstices of L until it is held rigidly by lattices points. Then, there are no lattice points in the interior and sufficiently many lie on the boundary of the sphere so that their convex hull is a full dimensional polytope P. The sphere S is called an **empty sphere** in L, its center is called a **hole** of L and the polytope P is called a **Delaunay polytope**, or L**-polytope**. So the vertices of P are the lattices points lying on the boundary of the empty sphere S. Let $V(P)$ denote the set of vertices of P. Then, the distance space $(V(P), d^{(2)})$ (with the square of the euclidian distance between vertices) is called an L**-polytope space**; such spaces are fundamental in our treatment.

Usually, empty spheres in lattices are studied from the point of view of their centers (i.e. the holes of L). But, hypermetrics provide a new way of studying empty spheres, namely from the point of view of the lattice points lying on their boundary, i.e. from the point of view of L-polytope spaces.

Indeed, L-polytopes have the remarkable property (discovered in [6]) that their L-polytope spaces are hypermetric and, conversely, every hypermetric space can be realized as a subspace of an L-polytope space (see Theorem 3.3). To each hypermetric space (X, d) corresponds an, essentially unique, L-polytope P_d whose dimension is less or equal to $|X| - 1$.

Using this connection and Voronoi's result stating that the number of distinct (up to affine equivalence) L-polytopes in fixed dimension is finite, we showed that the hypermetric cone is polyhedral ([32]).

So we have a connection between the hypermetric cone HYP_n and L-polytopes of dimension $k \leq n-1$. These two objects (hypermetric cone and L-polytopes) have been studied for their own sake. For instance, the hypermetric cone HYP_n was mostly studied from a polyhedral point of view, in particular, in connection with ℓ_1-metrics and the cut cone for which it forms a linear relaxation. On the other hand, L-polytopes were studied from the classical point of view of geometry of numbers: holes, L-decomposition of the space, dual tiling by Voronoi polytopes, etc. Our new approach is to study the metric structure of their sets of vertices. Moreover, taking advantage of the interplay with hypermetrics, we can transport and exploit some of the notions defined for the hypermetric cone to L-polytopes and vice-versa.

For instance, there is a natural notion of rank for hypermetrics (namely, the dimension of the smallest face of the hypermetric cone that contains a given hypermetric distance). We introduce the cooresponding notion of rank for L-polytopes. This notion of rank permits, in particular, to shed a new light on a classical notion studied by Voronoi, namely, the repartitioning polytopes which, indeed, correspond to facets of the hypermetric cone. The other extreme case for the rank, namely the case of rank 1 for the extreme rays of the hypermetric cone, corresponds to the class of **extreme** L-polytopes. An L-polytope P is extreme if and only if the only affine transformations T for which $T(P)$ is still an L-polytope are the homotheties. We present several examples of extreme L-polytopes: in root lattices, in sections of the Leech lattice Λ_{24} and of the Barnes-Wall lattice Λ_{16}. We also touch some other topics as perfect lattices and perfect quadratic forms (see Section 5.5).

Historically, L-polytopes and the corresponding L-partitions of the space were introduced by Voronoi at the beginning of this century. They have been studied extensively mainly by the Russian school, especially by B.N. Delaunay, E.P. Baranovskii, S.S. Ryshkov, also by R.M. Erdahl from Canada. In dimension 2 and 3, L-decompositions are used in computational geometry, under the name of Delaunay triangulation; actually, non lattice triangulations are also studied there. L-polytopes have been also used for the study of coverings in lattices (see [23], [56]); for instance, the covering radius of a lattice L is the maximum radius of an empty sphere in L, i.e. a deep hole in L. There is the following connection between Voronoi polytopes and L-polytopes: The vertices of the Voronoi polytope at a lattice point u are the centers of the L-polytopes that contain u as a vertex. Moreover, the two partitions of the space by L-polytopes and by Voronoi polytopes are in combinatorial duality.

Within the list of references, the more relevant and fundamental ones are Voronoi's Deuxième mémoire [66], the survey [58] and the collection [23] of surveys on lattices and applications.

Our treatment uses mainly technics from linear algebra, polyhedral theory and euclidian geometry.

We now briefly describe the main results presented in the paper. Actually, a good overview of the topics treated in the paper is provided by the list of Contents.

Section 2 contains all preliminaries on distance spaces, lattices and L-polytopes. Section 2.3 gives a short proof of Voronoi's result stating the finiteness of the number (up to affine equivalence) of L-polytopes in fixed dimension.

We present in Section 3 the basic connection existing between hypermetric spaces and L-polytopes; namely, every L-polytope space is hypermetric and to each hypermetric space (X, d) is associated an L-polytope P_d (see Theorem 3.3). In Section 3.1, this connection is described together with some first results showing how the polytope P_d inherits some of the properties of the hypermetric space (X, d), in particular, about subspaces (see Corollary 3.6) and ℓ_1-embeddability (see Proposition 3.7). In Section 3.2, we prove that the hypermetric cone is polyhedral.

Section 3.3 describes all the L-polytopes arising in root lattices; see, in particular, Figure 1 which shows the L-polytopes in the irreducible root lattices together with their 1-skeleton and radius. If P is an L-polytope in a root lattice, then its edges are the pairs of vertices at squared distance 2, i.e. its 1-skeleton is determined by the metric structure of its L-polytope space (see Proposition 3.9). As application, we give a characterization of the connected strongly even distance spaces that are hypermetric or ℓ_1-embeddable (see Theorems 3.12 and 3.13).

In Section 3.4, we group several results dealing with the radius of the sphere circumscribing L-polytopes. We consider, in particular, the spherical t-extension operation which consists of adding a new element to a distance space at distance t from the other elements.

The notion of rank for L-polytopes is considered in detail in Section 4. If (X, d) is a hypermetric space with $|X| = n$, then $d \in \mathrm{HYP}_n$ and the rank of (X, d) is defined as the dimension of the smallest (by inclusion) face of HYP_n that contains d. If P is an L-polytope, then the L-polytope space $(V(P), d^{(2)})$ is hypermetric and the rank of P is defined as the rank of the space $(V(P), d^{(2)})$. P is said to be extreme if its rank is equal to 1. In Section 4.1, we consider several properties for this notion of rank, in particular, its invariance (see Theorem 4.5) and its additivity (see Proposition 4.6). We describe in Section 4.2 how faces of the hypermetric cone relate to L-polytopes; see, in particular, Figure 2. In particular, hypermetrics lying on the interior of the same face of the hypermetric cone correspond to affinely equivalent L-polytopes (see Corollary 4.8), a geometric characterization for extreme L-polytopes is given in Corollary 4.9, L-polytopes associated with facets of the hypermetric cone are described in Proposition 4.10, and an upper bound on the number of facets of the hypermetric cone is derived (see Theorem 4.11).

We present in Section 4.3 some bounds on the number of vertices of an L-polytope which is basic, i.e. whose set of vertices contains a base of the lattice it spans (see

Proposition 4.15).

Section 5 is devoted to the study of extreme L-polytopes, which correspond to extreme rays of the hypermetric cone. The extreme L-polytopes in root lattices are characterized in Theorem 5.1. In Section 5.1, we derive bounds on the number of vertices of an extreme basic L-polytope which turn out to be closely related with known bounds on the cardinality of equiangular sets of lines. We also present a general construction for equiangular sets of lines from integral lattices (see Proposition 5.3). In the next Sections 5.2, 5.3 and 5.4, we describe examples of extreme L-polytopes arising in sections of the root lattice E_8, the Leech lattice Λ_{24} and the Barnes-Wall lattice Λ_{16}. In Section 5.5, we present results on the construction of perfect lattices from extreme L-polytopes.

Section 6 applies the notion of hypermetricity to graphs. Given a graph G, we consider two distances: its path metric d_G or its truncated distance d^*_G (with distance 1 on an edge and distance 2 on a nonedge). G is called hypermetric if its path metric is hypermetric. In Section 6.1, a characterization of the hypermetric graphs and of the ℓ_1-graphs is given in Theorem 6.1; see also Theorems 6.7 and 6.8 for a refined result for the class of suspension graphs.

In Section 6.2, we study the connected regular graphs whose truncated distance is hypermetric; see Proposition 6.10 for several equivalent characterizations, one of them is that their minimum eigenvalue is greater or equal to -2. The graphs with minimum eigenvalue -2 are well studied. Those that are not line graphs or cocktail-party graphs belong to the well known list of 187 graphs from [20]. This list is partitioned into three layers, each of them being characterized by a parameter which is directly related to the radius of the L-polytopes associated with the graphs in the layer.

We consider in Section 6.3 extreme hypermetric graphs, i.e. the graphs whose path metric lies on an extreme ray of the hypermetric cone. In fact, all of them are isometric subgraphs of the Gosset graph or of the Schläfli graph (which are the 1-skeletons of the Gosset polytope 3_{21} and of the Schläfli polytope 2_{21}, respectively). See Proposition 6.18 for their characterization.

In the last Section 7, we consider hypermetric inequalities. They are valid for the cut cone and, in fact, this was the initial motivation for introducing them ([28]). We describe in Section 7.1 some classes of hypermetric inequalities that define facets of the cut cone. We consider in Section 7.2 some analogues of hypermetric inequalities that are valid for other cut families as even T-cuts, t-ary cuts, multicuts. We consider in Section 7.3 a non homogeneous version of hypermetric inequalities, obtained by switching and valid for the cut polytope.

2 Preliminaries

2.1 Distance spaces

Metric notions

A **distance space** (X, d) consists of a finite set X and a symmetric function $d : X \times X \mapsto \mathbb{R}_+$ with $d(i,i) = 0$ for all $i \in X$. Let $d_{\min}$ denote the minimum non zero value taken by d.

(X, d) is said to be **connected** if the graph with vertex set X and whose edges are the pairs (i, j) with $d(i, j) = d_{\min}$, is connected. (X, d) is said to be **strongly even** if d is integer valued with even values and $d_{\min} = 2$.

A **representation** of (X, d) is a mapping $i \in X \mapsto v_i \in \mathbb{R}^n$ $(n \geq 1)$ such that

$$d(i,j) = (v_i - v_j)^2 \quad \text{for } i, j \in X \tag{2}$$

or, equivalently,

$$2v_i^T v_j = v_i^2 + v_j^2 - d(i,j) \quad \text{for } i, j \in X. \tag{3}$$

Clearly, every translation of a representation of (X, d) is again a representation of (X, d). The representation is said to be **spherical** if all v_i's lie on a sphere.

For $x, y \in \mathbb{R}^n$, $x^T y = \sum_{1 \leq i \leq n} x_i y_i$ denotes their scalar product and $\| x \|_2 = \sqrt{x^T x}$ is the euclidian norm of x. Given $x, y \in \mathbb{R}^n$, we let $d_{\ell_1}(x,y) = \| x - y \|_1 = \sum_{1 \leq i \leq n} |x_i - y_i|$ and $d_{\ell_2}(x,y) = \| x - y \|_2 = \sqrt{\sum_{1 \leq i \leq n} (x_i - y_i)^2}$ denote, respectively, the distance associated with the ℓ_1-norm and with the ℓ_2-norm on $\mathbb{R}^n$. We also set $d^{(2)}(x,y) = (d_{\ell_2}(x,y))^2 = (x-y)^T(x-y)$; so $d^{(2)}$ is the square of the euclidian distance d_{ℓ_2}. This distance will play a fundamental role in the paper.

Given two distance spaces (X, d) and (X', d'), we say that (X, d) is an **isometric subspace** of (X', d') if there exists a mapping $f : X \longrightarrow X'$ such that $d(i,j) = d'(f(i), f(j))$ for all $i, j \in X$.

A distance space (X, d) is said to be **isometrically ℓ_1-embeddable** (resp. **hypercube embeddable**, **ℓ_2-embeddable**) if it is an isometric subspace of an ℓ_1-space $(\mathbb{R}^n, d_{\ell_1})$ (resp. of a Hamming space $(\{0,1\}^n, d_{\ell_1})$, of an ℓ_2-space $(\mathbb{R}^n, d_{\ell_2})$). Therefore, (X, d) has a representation if and only if $(X, \sqrt{d})$ is isometrically ℓ_2-embeddable, i.e. (X, d) is an isometric subspace of a space $(\mathbb{R}^n, d^{(2)})$.

A **graphic space** $(V(G), d_G)$ is a distance space where $V(G)$ is the set of vertices of a graph G and d_G is its path metric. A graph G is said to be an **hypermetric graph** (resp.

an ℓ_1**-graph**) if its path metric d_G is hypermetric (resp. isometrically ℓ_1-embeddable). We will use extensively the following graphs: the complete graph K_n, the cocktail-party graph $K_{n\times 2}$ (i.e. K_{2n} with a perfect matching deleted), the hypercube $H(n,2)$ (i.e. the graph whose nodes are the vectors $x \in \{0,1\}^n$ with two nodes x, y adjacent if $d_{\ell_1}(x,y) = 1$), the half-cube graph $\frac{1}{2}H(n,2)$ (i.e. the graph whose nodes are the vectors $x \in \mathbb{R}^n$ with $\sum_{1\leq i\leq n} x_i$ even and two nodes x, y are adjacent if $d_{\ell_1}(x,y) = 2$).

Given a graph G, its **suspension** ∇G is the graph obtained from G by adding a new node adjacent to all nodes of G.

Let (X,d) be a distance space. (X,d) is a **semimetric** space if d satisfies the triangle inequality $d(i,j) \leq d(i,k) + d(j,k)$ for all $i,j,k \in X$, and (X,d) is a **metric** space if, moreover, $d(i,j) = 0$ only if $i = j$. The set of all semimetrics on X is the cone $\mathrm{MET}(X)$, or MET_n if $|X| = n$.

(X,d) is a **hypermetric space** if d satisfies the inequality

$$\sum_{i,j\in X} b_i b_j d(i,j) \leq 0 \tag{4}$$

for all $b \in \mathbb{Z}^X$ with $\sum_{i\in X} b_i = 1$. (X,d) is of **negative type** if d satisfies the inequality (4) for all $b \in \mathbb{Z}^X$ with $\sum_{i\in X} b_i = 0$. The inequality (4) is called m**-gonal** if $\sum_{i\in X} |b_i| = m$; when $\sum_{i\in X} b_i = 1$ (resp. $= 0$), the inequality (4) is called **hypermetric** (resp. **of negative type**). Note that the 3-gonal inequality (4), obtained for $b_i = b_j = 1$, $b_k = -1$ and $b_h = 0$ otherwise, coincides with the triangle inequality. Note also that the $2m+2$-gonal inequalities are implied by the $2m+1$-gonal inequalities ([28]). But, for instance, the path metric of $K_{2,3}$ is not 5-gonal (i.e. does not satisfy the 5-gonal inequalities) and the path metric of $K_7 - C_4$ is 5-gonal but not 7-gonal.

Let $X = \{0, 1, \ldots, n\}$, $|X| = n+1$. The **hypermetric cone** HYP_{n+1}, or $\mathrm{HYP}(X)$, (resp. the **negative type** cone NEG_{n+1}, or $\mathrm{NEG}(X)$) is the cone in $\mathbb{R}^{\binom{n+1}{2}}$ consisting of the vectors $d = (d_{ij})_{0\leq i<j\leq n}$ satisfying all hypermetric inequalities (resp. all negative type inequalities). Note that, for a symmetric function d on X with zero value on the diagonal pairs, we can alternatively view d as a vector indexed by the pairs (i,j), $i<j$, of X.

We denote by PSD_n the set of all $p = (p_{ij})_{1\leq i\leq j\leq n}$ for which the symmetric matrix $(p_{ij})_{1\leq i,j\leq n}$ (setting $p_{ji} = p_{ij}$) is **positive semidefinite**, i.e. satisfies $\sum_{1\leq i,j\leq n} p_{ij}x_i x_j \geq 0$ for all $x \in \mathbb{R}^n$. So, PSD_n is the cone of the positive semidefinite quadratic forms on n variables. For $p \in \mathrm{PSD}_n$, if $\sum_{1\leq i,j\leq n} p_{ij}x_i x_j = 0$ holds only for $x = 0$, then p is said to be **positive definite**.

Given a subset S of X, the **cut semimetric** $\delta(S)$ is defined by $\delta(S)_{ij} = 1$ if $|S \cap \{i,j\}| = 1$ and $\delta(S)_{ij} = 0$ otherwise, for $0 \leq i < j \leq n$. There are $2^{|X|-1}$ distinct cut semimetrics on X, since $\delta(S) = \delta(X - S)$. The **cut cone** CUT_{n+1}, or $\mathrm{CUT}(X)$, is the cone generated by the cut semimetrics $\delta(S)$ for $S \subseteq X$.

Operations on distance spaces

We consider the following operations on distance spaces: direct product, 1-sum, spherical t-extension and 0-lifting.

Let (X_1, d_1) and (X_2, d_2) be two distance spaces. Their **direct product** is the distance space $(X_1 \times X_2, d)$ where d is defined by

$$d((i_1, i_2), (j_1, j_2)) = d_1(i_1, j_1) + d_2(i_2, j_2) \text{ for } i_1, j_1 \in X_1, i_2, j_2 \in X_2. \tag{5}$$

Let (X_1, d_1) and (X_2, d_2) be two distance spaces with $|X_1 \cap X_2| = 1$, $X_1 \cap X_2 = \{i_0\}$. Their **1-sum** is the distance space $(X_1 \cup X_2, d)$ where d is defined by

$$\begin{cases} d(i,j) = d_h(i,j) & \text{for } i, j \in X_h,\ h = 1, 2 \\ d(i,j) = d(i, i_0) + d(j, i_0) & \text{for } i \in X_1,\ j \in X_2. \end{cases} \tag{6}$$

Let (X, d) be a distance space, let i_0 be an element that does not belong to X and let $t \in \mathbb{R}_+$. The **spherical t-extension** of (X, d) is the distance space $(X \cup \{i_0\}, d')$ where d' is defined by

$$\begin{cases} d'(i,j) = d(i,j) & \text{for } i, j \in X \\ d'(i, i_0) = t & \text{for } i \in X. \end{cases} \tag{7}$$

We denote the spherical t-extension d' of d as $\mathrm{sph}_t(d)$ and we set $\mathrm{sph}_t^m(d) = \mathrm{sph}_t(\mathrm{sph}_t^{m-1}(d))$ for any integer $m \geq 2$ (so $\mathrm{sph}_t^1(d) = \mathrm{sph}_t(d)$).

Let (X, d) be a distance space and let $i_0 \in X$, $j_0 \notin X$. Its **0-lifting** is the space $(X \cup \{j_0\}, d')$ defined by

$$\begin{cases} d'(i,j) = d(i,j) & \text{for } i, j \in X \\ d'(i, j_0) = d(i_0, i) & \text{for } i \in X. \end{cases} \tag{8}$$

In particular, $d(i_0, j_0) = 0$. So, every semimetric space is the 0-lifting of a metric subspace.

The direct product, the 1-sum and the 0-lifting operations preserve metricity, ℓ_1-, hypercube embeddability, hypermetricity and the properties of having a spherical representation or of being of negative type. See Lemma 3.17 and Proposition 3.18 for some conditions on t ensuring that the spherical t-extension operation also preserves these metric notions.

Note that, for graphic spaces, the direct product and the 1-sum operations on the path metric correspond, respectively, to the cartesian product and the 1-sum operations on graphs. For $t = 1$, the spherical 1-extension of the path metric of a graph G with diameter 2 is the path metric of the suspension ∇G of G.

The tensor product operation was considered in [3].

Preliminary results on distance spaces

We now group several preliminary results on distance spaces, linking the notions introduced above.

Proposition 2.1 *([4], [10]) Let (X,d) be a distance space. Then, (X,d) is isometrically ℓ_1-embeddable (resp. hypercube embeddable) if and only if $d \in CUT(X)$, i.e. $d = \sum_{S \subseteq X} \lambda_S \delta(S)$ for some scalars $\lambda_S \geq 0$ (resp. $d \in \mathbb{Z}_+(X)$, i.e. $d = \sum_S \lambda_S \delta(S)$ for some integers $\lambda_S \geq 0$).*

As an immediate consequense, we have the following result.

Proposition 2.2 *[9] Let (X,d) be a distance space with rational values. Then, (X,d) is isometrically ℓ_1-embeddable if and only if λd is isometrically hypercube embeddable for some scalar λ. The smallest such λ is called the* **scale** *of d.*

Proposition 2.3 *[59] Let (X,d) be a distance space. Then, (X,d) is of negative type if and only if (X,d) has a representation, i.e. $(X,\sqrt{d})$ is isometrically ℓ_2-embeddable. Moreover, the representation is unique, up to translation and orthogonal transformation.*

We give a proof of Proposition 2.3 which relies on the following Lemma 2.4 and on a fundamental tool, namely, the covariance mapping.

The **covariance mapping** ξ_0 is the mapping on $\mathbb{R}^{\binom{n+1}{2}}$ defined by $p = \xi_0(d)$, for $d = (d_{ij})_{0 \leq i < j \leq n}$, $p = (p_{ij})_{1 \leq i \leq j \leq n}$, with

$$\begin{cases} p_{ii} = d_{0i} & \text{for } 1 \leq i \leq n \\ p_{ij} = \frac{d_{0i}+d_{0j}-d_{ij}}{2} & \text{for } 1 \leq i < j \leq n. \end{cases} \tag{9}$$

It is easy to verify that

$$d \in \mathrm{HYP}_{n+1} \text{ if and only if } \quad \begin{array}{l} p = \xi_0(d) \text{ satisfies the inequalities} \\ \sum_{1 \leq i,j \leq n} b_i b_j p_{ij} - \sum_{1 \leq i \leq n} b_i p_{ii} \geq 0 \\ \text{for all } b \in \mathbb{Z}^n \end{array} \tag{10}$$

$$d \in \mathrm{NEG}_{n+1} \text{ if and only if } \quad \begin{array}{l} p = \xi_0(d) \text{ satisfies the inequalities} \\ \sum_{1 \le i,j \le n} b_i b_j p_{ij} \ge 0 \\ \text{for all } b \in \mathbb{Z}^n \end{array} \tag{11}$$

Therefore, $p \in \xi_0(\mathrm{NEG}_{n+1})$ if and only if the symmetric matrix $(p_{ij})_{1 \le i,j \le n}$ (setting $p_{ji} = p_{ij}$) is positive semidefinite. In other words,

$$\xi_0(\mathrm{NEG}_{n+1}) = \mathrm{PSD}_n. \tag{12}$$

It follows immediately from relations (10) and (11) that every hypermetric space is of negative type, i.e. $\mathrm{HYP}_{n+1} \subseteq \mathrm{NEG}_{n+1}$. Therefore,

$$\xi_0(\mathrm{HYP}_{n+1}) \subseteq \mathrm{PSD}_n. \tag{13}$$

Note that $\xi_0(\mathrm{CUT}_{n+1})$ is the cone generated by the vectors $(x_i x_j)_{1 \le i \le j \le n}$, for $x \in \{0,1\}^n$. Hence, its polar consists of the quadratic forms that are nonpositive on binary variables.

It is easy to see that the polar $(\mathrm{PSD}_n)^\circ$ consists of the nonpositive quadratic forms, i.e. $(\mathrm{PSD}_n)^\circ = -\mathrm{PSD}_n$. So the following chain of inclusions

$$\xi_0(\mathrm{CUT}_{n+1}) \subseteq \xi_0(\mathrm{HYP}_{n+1}) \subseteq \mathrm{PSD}_n = -(\mathrm{PSD}_n)^\circ \subseteq -(\xi_0(\mathrm{HYP}_{n+1}))^\circ \subseteq -(\xi_0(\mathrm{CUT}_{n+1}))^\circ$$

shows that our central object, namely the hypermetric cone, is (up to polar and minus sign) a subcone of the cone of quadratic forms that are nonnegative on binary variables and contains all quadratic forms that are nonnegative on integer (or real) variables.

LEMMA 2.4 *Let $A = (a_{ij})_{1 \le i,j \le n}$ be a symmetric matrix which is positive semidefinite and let $k \le n$ be its rank. There exist vectors $v_1, \ldots, v_n \in \mathbb{R}^k$ such that $a_{ij} = v_i^T v_j$ for $1 \le i,j \le n$. Moreover, if $v'_1, \ldots, v'_n$ are other vectors of $\mathbb{R}^k$ such that $a_{ij} = v'^T_i v'_j$ for $1 \le i,j \le n$, then $v'_i = T(v_i)$, $1 \le i \le n$, for some orthogonal transformation T of $\mathbb{R}^k$. The system $(v_1, \ldots, v_n)$ has rank k.*

PROOF. By assumption, A has k non zero eigenvalues which are positive. Hence, there exists an $n \times n$ matrix Q_0 such that $A = Q_0 D Q_0^T$, where D is an $n \times n$ matrix whose entries are all zero except k diagonal entries, say with indices $(1,1), \ldots, (k,k)$, equal to 1. Denote by Q the $n \times k$ submatrix of Q_0 consisting of its first k columns. Then, $A = QQ^T$ holds, i.e. $a_{ij} = v_i^T v_j$ for $1 \le i,j \le n$, where $v_1, \ldots, v_n$ denote the rows of Q. It is easy to see that $(v_1, \ldots, v_n)$ has the same rank k as A.

Let Q' be another $n \times k$ matrix such that $A = Q'Q'^T$. Both matrices Q, Q' have rank k, hence there exists a $k \times k$ non singular matrix B such that $Q' = QB$. Let Q_1 be a non singular $k \times k$ submatrix of Q formed, say, by its first k rows, and let Q_1' denote the $k \times k$ submatrix of Q' formed by its first k rows. Then, $Q_1' = Q_1 B$. From the equality $Q_1 Q_1^T = Q_1'(Q_1')^T$, we obtain that BB^T is the identity matrix, i.e. B is an orthogonal transformation of $\mathbb{R}^k$. $\square$

Proposition 2.3 now follows easily from relation (11), Lemma 2.4 and the following observation: If $p = \xi_0(d)$, then $p_{ij} = v_i^T v_j$ holds for $1 \le i \le j \le n$ if and only if $d_{ij} = (v_i - v_j)^2$ holds for $0 \le i < j \le n$, after setting $v_0 = 0$.

PROPOSITION 2.5 *Let (X, d) be a distance space. Consider the assertions:*
(i) (X, d) is isometrically hypercube embeddable.
(ii) (X, d) is isometrically ℓ_1-embeddable.
(iii) (X, d) is hypermetric.
(iv) (X, d) has a spherical representation.
(v) (X, d) is of negative type.
(vi) The distance matrix $(d_{ij})_{i,j \in X}$ has exactly one positive eigenvalue.
The implications $(i) \Longrightarrow (ii) \Longrightarrow (iii) \Longrightarrow (iv) \Longrightarrow (v) \Longrightarrow (vi)$ hold.

PROOF. $(i) \Longrightarrow (ii)$ is obvious.
$(ii) \Longrightarrow (iii)$ Using Proposition 2.1, it suffices to check that each cut semimetric $\delta(S)$ satisfies all hypermetric inequalities. Indeed, if $b \in \mathbb{Z}^X$ with $\sum_{i \in X} b_i = 1$, then $\sum b_i b_j \delta(S)_{ij} = \sum_{i \in S, j \notin S} b_i b_j = (\sum_{i \in S} b_i)(1 - \sum_{i \in S} b_i) \le 0$.
$(iii) \Longrightarrow (iv)$ will be shown in Proposition 3.2.
$(iv) \Longrightarrow (v)$ Let $(v_i, i \in X)$ be a spherical representation of (X, d), on a sphere of radius r and center c. We can suppose that c is in the origin (up to translation). Hence, $v_i^2 = r^2$ for all $i \in X$ and $d(i,j) = (v_i - v_j)^2$ for $i, j \in X$. Take $b \in \mathbb{Z}^X$ with $\sum_{i \in X} b_i = 0$. Then, $\sum_{i,j \in X} b_i b_j d(i,j) = \sum_{i,j \in X} b_i b_j (2r^2 - 2v_i^T v_j) = -2(\sum_{i \in X} b_i v_i)^2 \le 0$.
$(v) \Longrightarrow (vi)$ If d is of negative type, then there exist some vectors v_i $(i \in X)$ such that $d_{ij} = (v_i - v_j)^2$ for all $i, j \in X$. Therefore, the quadratic form $\sum_{i,j \in X} d_{ij} x_i x_j$ can be expressed as $2 \sum_{i,j \in X} v_i^2 x_i x_j - (\sum_{i \in X} x_i v_i)^2$. The first term is a quadratic form whose eigenvalues are 0 (with multiplicity $|X| - 1$) and $\sum_{i \in X} v_i^2$. Therefore, $\sum_{i,j \in X} d_{ij} x_i x_j$ can be decomposed as a sum and difference of squares involving only one positive square. This implies that the distance matrix $(d_{ij})_{i,j \in X}$ has at most one positive eigenvalue and, thus, it has exactly one since its diagonal terms are all zero. $\square$

Two partial converses to the implications $(iv) \Longrightarrow (v)$ and $(iii) \Longrightarrow (iv)$ will be given later in Propositions 3.14 and 3.16, respectively. There are many examples of distance spaces for which (ii) and (iii) are equivalent; see, in particular, Section 6. For instance, for a graphic space $(V(G), d_G)$, if G is bipartite, then all (i) – (vi) are equivalent; see Proposition 6.3.

2.2 Lattices and L-polytopes

Lattices

A subset L of $\mathbb{R}^k$ is called a **lattice** (or **point lattice**) if L is a discrete subgroup of $\mathbb{R}^k$, i.e. there exists a ball of radius $\beta > 0$ centered at each lattice point which contains no other lattice point. A subset $V = \{v_1, \ldots, v_m\}$ of L is said to be **generating** (resp. a **base**) for L if, for every $v \in L$, there exist some integers (resp. a unique system of integers) $b_1, \ldots, b_m$ such that $v = \sum_{1 \leq i \leq m} b_i v_i$. The **dimension** of L is the cardinality of a base.

Any two bases B_1, B_2 are **integral unimodular equivalent**, i.e. $M_{B_1} = AM_{B_2}$, where A is an integer matrix with determinant $|det(A)| = 1$ and M_{B_1} (resp. M_{B_2}) is the $k \times k$ matrix whose rows are the members of B_1 (resp. B_2) and k is the dimension of L. The common value $|det(B)|$ for any base B of L is called the **determinant** of L and denoted as $det(L)$.

Given $a \in \mathbb{R}^k$, the translate $L' = L+a = \{v+a : v \in L\}$ of a lattice L is called an **affine lattice**. A subset $V' = \{v_0, v_1, \ldots, v_m\}$ of L' is called an **affine generating set** for L' (resp. an **affine base** of L') if, for every $v \in L'$, there exist some integers (resp. a unique system of integers) $b_0, b_1, \ldots, b_m$ such that $v = \sum_{0 \leq i \leq m} b_i v_i$ and $\sum_{0 \leq i \leq m} b_i = 1$. Clearly, V' is an affine generating set (resp. affine base) of L' if and only if $V = \{v_1 - v_0, \ldots, v_m - v_0\}$ is a (linear) generating set (resp. base) of the lattice L.

For simplicity, we use the same word "lattice" for denoting both a usual lattice (i.e. containing 0) and an affine lattice (i.e. translate of a lattice). We also often omit to precise whether we consider linear or affine bases or generating sets.

We use the following notation. Given a subset V of $\mathbb{R}^k$, we define its **integer hull** $\mathbb{Z}(V)$ by

$$\mathbb{Z}(V) = \{\sum_{v \in V} b_v v : b \in \mathbb{Z}^V\}$$

and its **affine integer hull** $\mathbb{Z}_{af}(V)$ by

$$\mathbb{Z}_{af}(V) = \{\sum_{v \in V} b_v v : b \in \mathbb{Z}^V \text{ and } \sum_{v \in V} b_v = 1\}.$$

The **minimum norm** t of a lattice L is defined as

$$t = \min((u-v)^2 : u, v \in L, u \neq v).$$

This terminology of minimal "norm" is classical in the theory of lattices, although it actually denotes the square of the euclidian norm. In particular, if $0 \in L$, then $t = \min(u^2 : u \in L, u \neq 0)$. The **minimal vectors** of L are the vectors $v \in L$ with $v^2 = t$. Their set is denoted as $L_{\min}$. Then, the polytope $\mathrm{conv}(L_{\min})$ is known as the **contact polytope** of L. Note that $2t$ coincides with the packing radius of L (see e.g. [23]).

L is **integral** if $u^T v \in \mathbb{Z}$ for all $u, v \in L$. An **even** lattice is an integral lattice with $u^2 \in 2\mathbb{Z}$ for each lattice vector u. L is a **root lattice** if L is integral and L is generated by a set of vectors v with $v^2 = 2$; then, each $v \in L$ with $v^2 = 2$ is called a **root** of L. Observe that, in a root lattice L,

$$u^T v \in \{0, -1, 1\} \text{ for } u, v \text{ roots of } L, u \neq \pm v. \tag{14}$$

This follows from the fact that $(u - v)^2 = 4 - 2u^T v > 0$ and $(u + v)^2 = 4 + 2u^T v > 0$. The **dual** L^* of L is defined as $L^* = \{x \in \mathbb{R}^k : x^T u \in \mathbb{Z} \text{ for all } u \in L\}$. If L is an integral lattice, then $L \subseteq L^*$ holds. L is called **self-dual** if $L = L^*$ holds. L is called **unimodular** if $det(L) = 1$. Hence, an integral unimodular lattice is self-dual. For example, the root lattice E_8, the Leech lattice Λ_{24} are even and unimodular.

The **direct sum** of two lattices L_1 and L_2 is defined if L_1 and L_2 are orthogonal, i.e. $u_1^T u_2 = 0$ for all $u_1 \in L_1$, $u_2 \in L_2$, as

$$L_1 \oplus L_2 = \{u_1 + u_2 : u_1 \in L_1, u_2 \in L_2\}.$$

L is called **irreducible** if $L = L_1 \oplus L_2$ implies $L_1 = \{0\}$ or $L_2 = \{0\}$ and **reducible** otherwise. A well known result by Witt states that the only irreducible root lattices are A_n $(n \geq 0)$, D_n $(n \geq 4)$, E_n $(n = 6, 7, 8)$; we describe them in Section 3.3.

L-polytopes

Let $L \subseteq \mathbb{R}^k$ be a k-dimensional lattice and let $S = S(c, r)$ be a sphere with center c and radius r in $\mathbb{R}^k$. Then, S is called an **empty sphere** (in Russian literature) in L if the following two conditions hold:

- $(v - c)^2 \geq r$ for all $v \in L$,
- $S \cap L$ has affine rank $k + 1$.

Then, the center of S is called a **hole** (in English literature).

Then, the polytope P defined as the convex hull of $S \cap L$ is called an L-**polytope**, or **Delaunay polytope**. Equivalently, a k-dimensional polytope P in $\mathbb{R}^k$ with set of vertices $V(P)$ is an L-polytope if the following conditions hold:

- P is inscribed on a sphere $S(c, r)$, i.e. $(v - c)^2 = r^2$ for all $v \in V(P)$,
- $L(P) = \mathbb{Z}_{af}(V(P)) = \{\sum_{v \in V(P)} b_v v : b \in \mathbb{Z}^{V(P)} \text{ and } \sum_{v \in V(P)} b_v = 1\}$ is a lattice,

• $(v-c)^2 \geq r^2$ for all $v \in L(P)$.
If P is an L-polytope, then the distance space $(V(P), d^{(2)})$ is called an **L-polytope space**.

Let P be an L-polytope in a lattice L. P is said to be **generating** in L if $V(P)$ generates L, i.e. $L = L(P)$. There are examples of lattices for which none of their L-polytopes is generating; this is the case for the root lattice E_8, the Leech lattice Λ_{24} and, more generally, for all even unimodular lattices (see Lemma 2.6). However, when we say that P is an L-polytope in L, we always assume that P is generating, i.e. we suppose that $L = L(P)$.

A subset $B \subseteq V(P)$ is said to be **basic** if it is an affine base of L. P is called **basic** if $V(P)$ contains a basic set. Actually, we do not know any example of a non basic L-polytope.

Two L-polytopes have the same **type** if they are affinely equivalent, i.e. $P' = T(P)$ for some affine bijection T.

Given a lattice point $v \in L$, the set of all the L-polytopes in L that admit v as a vertex is called the **star** of L at v. Clearly, the stars at distinct lattice points are identical (up to translation). The lattice L is called **general** if all the L-polytopes of its star are simplices (which, in general, cannot be obtained from one another by translation or orthogonal transformation), and L is called **special** otherwise.

Two k-dimensional lattices L, L' are said to be **z-equivalent** if there exits an affine bijection T such that $L' = T(L)$ and T brings the star of L on the star of L'; one also says that L and L' have the same **type**.

L-polytopes and Voronoi polytopes

Let us recall the connection between L-polytopes and Voronoi polytopes ([66]).
If L is a lattice in $\mathbb{R}^k$ and $u_0 \in L$, the **Voronoi polytope** at u_0 is the set $P_v(u_0)$ consisting of all points $x \in \mathbb{R}^k$ that are at least as close to u_0 than to any other lattice point, i.e. $P_v(u_0) = \{x \in \mathbb{R}^k : \| x - u_0 \| \leq \| x - u \| \text{ for all } u \in L\}$. The vertices of the Voronoi polytope $P_v(u_0)$ are precisely the centers of the L-polytopes in L that contain u_0 as a vertex, i.e. of the L-polytopes of the star of L at v_0.

The Voronoi polytopes $P_v(u)$, $u \in L$, form a normal (i.e. face-to-face) tiling of the space $\mathbb{R}^k$; this tiling is sometimes called the Voronoi-Dirichlet tiling. Another normal tiling is provided by the elementary cells $\{u + \sum_{1 \leq i \leq k} b_i v_i \text{ for } 0 \leq b_i \leq 1, 1 \leq i \leq k\}$ for $u \in L$, where $(v_1, \ldots, v_k)$ is a base of L. Hence, the Voronoi polytopes and the elementary cells have the same volume, equal to $det(L)$. Another normal partition of the space, called **L-decomposition**, is provided by the L-polytopes in L. However, different types of L-polytopes may occur in this partition; in particular, if L is special, then some of them are not simplices. For instance, if L is a general lattice of dimension 2, then the normal partition of $\mathbb{R}^2$ by the L-polytopes in L is a Delaunay triangulation of the plane. For the use of L-decompositions in computational geometry, see, for instance, Chapter 13 in [41].

Given a k-dimensional lattice L, the two normal partitions of the space by the Voronoi polytopes and by the L-polytopes in L are in combinatorial duality. Namely, there is a one-to-one correspondence $F \mapsto F^*$ between the faces F of one partition and the faces F^* of the other partition in such a way that:
• F and F^* are orthogonal,
• if F has dimension h, then F^* has dimension $k - h$ and
• if $F_1 \subseteq F_2$, then $F_2^* \subseteq F_1^*$.

Lattices and positive quadratic forms

Let $p \in \mathrm{PSD}_n$ be a positive semidefinite quadratic form. Then, by Lemma 2.4, there exist n vectors $v_1, \ldots, v_n \in \mathbb{R}^k$ $(1 \leq k \leq n)$ such that $p_{ij} = v_i^T v_j$ for all $i, j = 1, \ldots, n$ where k is the rank of the system $(v_1, \ldots, v_n)$ and of the matrix $(p_{ij})_{1 \leq i,j \leq n}$. So, $k = n$ if p is positive definite, i.e. p lies in the interior of PSD_n, and $k < n$ otherwise, i.e. if p lies on the boundary of PSD_n. Set $L = \mathbb{Z}(v_1, \ldots, v_n)$. Sometimes, L is a lattice. This is the case, in particular, if p is positive definite. Recall from relation (13) that $\xi_0(\mathrm{HYP}_{n+1}) \subseteq \mathrm{PSD}_n$. In fact, if $p \in \xi_0(\mathrm{HYP}_{n+1})$, then L is also a lattice. Actually, for $p \in \mathrm{PSD}_n$, we have that $p \in \xi_0(\mathrm{HYP}_{n+1})$ if and only if L is a lattice and $v_1, \ldots, v_n$ are all vertices of the same L-polytope in the star of L at the origin (see Section 3.1).

There is a many-to-one correspondence between the positive definite quadratic forms p of PSD_n and the lattices in $\mathbb{R}^n$. Indeed, the action on p of the group $GL(n, \mathbb{Z})$ of integral unimodular transformations produces distinct bases of the same lattice L. Voronoi ([66]; we follow [46] for the exposition) showed that the action of $GL(n, \mathbb{Z})$ induces a partition of the cone PSD_n into disjoint relatively open convex subcones, called the **L-type domains**, of dimension $1, 2, \ldots, \binom{n+1}{2}$, such that:
• On each of these subcones the affine structure of the L-decompositions of corresponding lattices is constant, i.e. the lattices corresponding to the points of a given subcone are all z-equivalent.
• Subcones of dimension $\binom{n+1}{2}$ correspond to general lattices, i.e. having simplicial L-decompositions. These L-type domains are polyhedral.
• A subcone of dimension less than $\binom{n+1}{2}$ is a relatively open face of two or more L-type domains. If such a cone makes contact with the boundary of an L-type domain, then it is necessarily a face of that domain. The lattice corresponding to a quadratic form on such a face is special, i.e. it has among its L-polytopes some that are not simplices.

Voronoi ([66]) showed that the number of distinct, up to z-equivalence, lattices in any dimension k is finite. Therefore, many of the L-type domains correspond to z-equivalent lattices.

Note that, if $p \in \xi_0(\mathrm{HYP}_{n+1})$, then the whole L-type domain containing p is also contained in $\xi_0(\mathrm{HYP}_{n+1})$. Therefore, $\xi_0(\mathrm{HYP}_{n+1})$ is a union of L-type domains. In fact, this union is infinite; however, we know that $\xi_0(\mathrm{HYP}_{n+1})$ is a polyhedral cone (see Section

3.2).

L-polytopes and empty ellipsoids

As we will see in Section 3.1, the study of the hypermetric spaces on n points amounts to the study of the L-polytopes of dimension $k \leq n$. We would like to mention another connection between L-polytopes and the quadratic functions that are nonnegative on integer variables.

There is a sequence of papers ([42], [43], [44], [45], [46]) studying the set of integer solutions of equations of the form

$$f(x) = a_0 + \sum_{1 \leq i \leq n} a_i x_i + \sum_{1 \leq i,j \leq n} a_{ij} x_i x_j \tag{15}$$

where $a_0, a_1, \ldots, a_n \in \mathbb{R}$, $a_{ij} = a_{ji} \in \mathbb{R}$, and f satisfies the condition

$$f(x) \geq 0 \text{ for all } x \in \mathbb{Z}^n. \tag{16}$$

The set of integer solutions of $f(x) = 0$ is called the **root figure** of f and is denoted by R_f. From relation (16), the matrix $a_f = (a_{ij})_{1 \leq i,j \leq n}$ is positive semidefinite and the region $\{x \in \mathbb{R}^n : f(x) < 0\}$ is free of integral points.

Suppose that a_f is positive definite. Then, the surface defined by the equation $f(x) = 0$ is an ellipsoid E_f whose interior is free of integral points; E_f is said to be an **empty ellipsoid**. The root figure R_f consists of the integral points lying on E_f and, thus, is finite. In fact, the root figure R_f is affinely equivalent to the set of vertices V of an L-polytope, with $dim(V) = dim(R_f) \leq n$. Moreover, every finite root figure arises in this way. (See [45].)

On the other hand, if a_f is not positive definite, then the root figure R_f may be infinite. However, every infinite root figure arises from the finite ones by a simple construction (essentially, every infinite root figure is of the form $R + \Gamma$ where R is a finite root figure and Γ is a sublattice of $\mathbb{Z}^n$). (See Theorem 2.1 in [45].)

Therefore, the study of the root figures amounts to the classification of the L-polytopes of dimension $k \leq n$.

Consider the cone $Q^+(\mathbb{Z}^n)$ defined by

$$\begin{aligned} Q^+(\mathbb{Z}^n) = \ & \{a = (a_0, a_1, \ldots, a_n, a_{ij}, 1 \leq i \leq j \leq n) : \\ & a_0 + \textstyle\sum_{1 \leq i \leq n} a_i x_i + \sum_{1 \leq i,j \leq n} a_{ij} x_i x_j \geq 0 \text{ for all } x \in \mathbb{Z}^n\}, \end{aligned}$$

i.e. each member $a \in Q^+(\mathbb{Z}^n)$ corresponds to a function f_a satisfying (15) and (16). Erdahl [44] shows that every $a \in Q^+(\mathbb{Z}^n)$ lying on an extreme ray of $Q^+(\mathbb{Z}^n)$ satisfies one of the following:

- f_a is constant (i.e. $a_1 = \ldots = a_n = a_{ij} = 0$),
- $f_a(x) = (\sum_{1 \le i \le n} \alpha_i x_i + \eta)^2$ where $(\alpha_1, \ldots, \alpha_n)$ is not proportional to an integer vector,
- f_a is **perfect**, i.e. the dimension of the set $\{b \in Q^+(\mathbb{Z}^n) : R_{f_a} \subseteq R_{f_b}\}$ is equal to 1.

We note the following connection between the hypermetric cone HYP_{n+1} and the cone $Q^+(\mathbb{Z}^n)$ (it follows from (10)):

$$\xi_0(\mathrm{HYP}_{n+1}) = \{a \in Q^+(\mathbb{Z}^n) : a_0 = 0 \text{ and } a_i = -a_{ii} \text{ for } i = 1, \ldots, n\}.$$

Therefore, HYP_{n+1} is, via the covariance mapping, a section of the cone $Q^+(\mathbb{Z}^n)$. Note that the notion of root figure corresponds to that of annullator that we use in Section 4.1. Moreover, there is the following link between the perfect elements of $Q^+(\mathbb{Z}^n)$ and the extreme rays of HYP_{n+1}. For $d \in \mathrm{HYP}_{n+1}$, d lies on an extreme ray of HYP_{n+1} if and only if $\xi_0(d)$ is a perfect element of $Q^+(\mathbb{Z}^n)$.

Basic facts on L-polytopes

We group several basic properties on the symmetry, the number of vertices, the volume of L-polytopes.

We start with an observation on generating L-polytopes in even lattices.

LEMMA 2.6 *[44] Let P be a generating L-polytope in an even lattice L. Then, the center of the sphere circumscribing P belongs to the dual lattice L^*. Therefore, an even unimodular lattice contains no generating L-polytope.*

PROOF. We can suppose that the origin is a vertex of P. Let c denote the center of the sphere S circumscribing P. Since L is generated by the set of vertices $V(P)$ of P, it suffices to check that $c^T v \in \mathbb{Z}$ for each $v \in V(P)$, for showing that $c \in L^*$. For $v \in V(P)$, we have that $(c - v)^2 = c^2$, i.e. $2c^T v = v^2$, implying that $c^T v \in \mathbb{Z}$ since v^2 is even. If L is even unimodular, then $c \in L^* = L$, contradicting the fact that S is an empty sphere in L. $\square$

Let S be a sphere with center c. For $x \in S$, its **antipode** on S is the point $x^* = 2c - x$. It is immediate to see that:

LEMMA 2.7 *Every L-polytope P is*
- *either* **centrally symmetric**, *i.e. $v^* \in V(P)$ for all $v \in V(P)$,*
- *or* **asymmetric**, *i.e. $v^* \notin V(P)$ for all $v \in V(P)$.*

PROPOSITION 2.8 *Let P be an L-polytope in $\mathbb{R}^k$. Then, P has at most 2^k vertices.*

PROOF. Without loss of generality, we can suppose that the origin is a vertex of P. Let $\{v_1, \ldots, v_k\}$ be a base of L. We consider the following equivalence relation on L: $u \sim v$

if $u+v \in 2L$ for $u, v \in L$. Clearly, every vertex of P is in relation by $\sim$ with one of the elements $\sum_{i \in I} v_i$ for $I \subseteq \{1, \ldots, k\}$. On the other hand, no two vertices of P are in relation by $\sim$. Indeed, if, for $u, v \in V(P)$, $u \sim v$, then $\frac{u+v}{2} \in L$, contradicting the fact that the sphere circumscribing P is empty in L. This shows that P has at most 2^k vertices. □

Let u, v, w be vertices of an L-polytope P. One can check directly that the following inequality holds:

$$(u-w)^2 \leq (u-v)^2 + (v-w)^2.$$

This is the triangle inequality expressing the fact that the L-polytope space $(V(P), d^{(2)})$ is a metric space. Actually, we will see in Proposition 3.2 that every L-polytope space is hypermetric, which is a much stronger property. The above relation means that any three vertices of an L-polytope form a triangle with no obtuse angles. In fact, the latter property is already sufficient for proving the upper bound 2^k on the number of vertices ([26], see also [32]).

The upper bounds from Proposition 2.9 below are shown in [13] as a refinement of the upper bound $vol(P) \leq 2^k det(L)$ given in [32].

Proposition 2.9 *Let P be an L-polytope in a lattice $L \subseteq \mathbb{R}^k$ with N vertices and let $vol(P)$ denote its volume. Then, $vol(P) \leq \frac{2^k det(L)}{N}$ if P is centrally symmetric and $vol(P) \leq \frac{2^k det(L)}{2N}$ if P is asymmetric.*

Proof. Let $P_v(0)$ denote the Voronoi polytope at the origin. We can assume without loss of generality that the origin is a vertex of P. Let h_0 denote the $\frac{1}{2}$-fold homothety with center 0, i.e. $h_0(x) = \frac{x}{2}$. We show that $h_0(P) \subseteq P_v(0)$.
Take a vertex v of P and suppose for contradiction that $\frac{v}{2} \notin P_v(0)$. Take a hyperplane H supporting a facet of $P_v(0)$ which separates $P_v(0)$ and $\frac{v}{2}$; H is the hyperplane going through $\frac{w}{2}$ and orthogonal to the segment $[0, w]$ for some $w \in L$. Let $R(0, w)$ denote the region comprised between the two hyperplanes perpendicular to the segment $[0, w]$ and going, respectively, through the points 0 and w. Clearly, $v \notin R(0, w)$, which implies that $v^T w > w^2$. Moreover, w is not a vertex of P, since there is an obtuse angle in the triangle $(0, v, w)$. Therefore, $\| w - c \| > \| c \|$ (where c is the center of the sphere circumscribing P) and, thus, $w^2 > 2w^T c$. On the other hand, $\| v - c \| = \| c \|$, i.e. $v^2 = 2v^T c$. We now show that $\| v - w - c \| < \| c \|$, which contradicts the fact that $v - w$ is a lattice point. Indeed, $(v-w-c)^2 - c^2 = (v-w)^2 - 2c^T(v-w) = v^2 + w^2 - 2v^T w - 2c^T(v-w)$
$= w^2 - 2v^T w + 2c^T w < 2(w^2 - v^T w) < 0$.

Consider the star of the L-polytopes in L with the origin as a vertex and let P_{star} denote their union. Hence, P is contained in P_{star} as well as any translate $P - v$ of P, for v vertex of P, and any translate of the symmetric $-P$ of P with respect to the origin. Note that the translates of $-P$ coincide with those of P if P is centrally symmetric. Therefore, P_{star} contains ϵN copies of P, with $\epsilon = 1$ if P is centrally symmetric and $\epsilon = 2$ if P is asymmetric. We deduce that $\epsilon N\ vol(P) \leq vol(P_{star})$. We have shown above that $h_0(P_{star}) \subseteq P_v(0)$. This implies that $\frac{1}{2^k} vol(P_{star}) \leq vol(P_v(0)) = det(L)$. Therefore, $vol(P) \leq \frac{2^k\ det(L)}{\epsilon N}$. □

Construction of L-polytopes

We now present some methods for constructing L-polytopes: by taking suitable sections of the sphere of minimal vectors in a lattice, by direct product and by pyramid or bipyramid extension. Note that every face of an L-polytope is again an L-polytope.

Construction by sectioning the sphere of minimal vectors in a lattice

Let L be a lattice in $\mathbb{R}^k$ and let V be the set of minimal vectors (i.e. of minimum norm) of L. Given non collinear vectors $a, b \in \mathbb{R}^k$ and some non zero scalars α, β, we set

$$V_a = \{v \in V : \ v^T a = \alpha\} \text{ and } V_b = \{v \in V : \ v^T b = \beta\}.$$

Lemma 2.10 *If the sets V_a and $V_a \cap V_b$ are not empty, then the polytopes $P = conv(V_a)$ and $P' = conv(V_a \cap V_b)$ are $L-$polytopes.*

Note that this is precisely how the Schläfli polytope 2_{21} and the Gosset polytope 3_{21} are constructed from the root lattice E_8, and how the polytopes P_{22} and P_{23} are constructed from the Leech lattice Λ_{24} (see Sections 5.2 and 5.3).

Direct product

Let L_i be a lattice in $\mathbb{R}^{k_i}$, let P_i be an L-polytope in L_i whose circumscribed sphere has radius r_i and is centered in the origin, for $i = 1, 2$. Then, $L = L_1 \times L_2 = \{(v_1, v_2) : v_1 \in L_1, v_2 \in L_2\}$ is a lattice in $\mathbb{R}^k$, $k = k_1 + k_2$ and $P = P_1 \times P_2$ is an L-polytope in L whose circumscribed sphere is centered in the origin and has radius $r = \sqrt{r_1^2 + r_2^2}$. Therefore, the **direct product** of two L-polytopes is an L-polytope. The direct product of P and a segment α_1 is called the **prism with base** P.

Call an L-polytope **reducible** if it is the direct product of two other non trivial L-polytopes (i.e. not reduced to a point) and **irreducible** otherwise. Note that irreducible L-polytopes arise in irreducible lattices. Indeed, if L is a reducible lattice, i.e. L is the direct sum $L_1 \oplus L_2$ of two orthogonal lattices L_1 and L_2 and if P is an L-polytope in L, then the projection P_i of P on the subspace spanned by L_i is an L-polytope in L_i, for $i = 1, 2$, and $P = P_1 \times P_2$ (up to affine equivalence).

Pyramid and bipyramid

If P is a polytope and v is a point that does not lie in the affine space spanned by P, then $Pyr_v(P) = \text{conv}(P \cup \{v\})$ is called the **pyramid with base** P and **apex** v. Under some conditions, the pyramid of an L-polytope is still an L-polytope.

Namely, let P be an L-polytope with radius r and suppose v is at squared distance t from all vertices of P with $t > 2r^2$. Then, the pyramid $Pyr_v(P)$ is an L-polytope with radius $R = \frac{t}{2\sqrt{t-r^2}}$ (see Proposition 3.18).

Moreover, if P is centrally symmetric and if $t = 2r^2$, then the **bipyramid** $Bipyr_v(P) = \text{conv}(P \cup \{v, v^*\})$ is an L-polytope with radius r, where v^* is the antipode of v on the sphere circumscribing $Pyr_v(P)$ (see Proposition 3.18).

The layerwise construction

The following **layerwise construction** for L-polytopes is described in [64]. Actually, rather than a construction, it is a way of visualizing a given k-dimensional L-polytope in a lattice L as the convex hull of its sections by the $k-1$-dimensional layers composing L.

Let L be a k-dimensional lattice and let $(v_1, \ldots, v_k)$ be a base of L. Then, $L_0 = \mathbb{Z}(v_1, \ldots, v_{k-1})$ is a $k-1$-dimensional sublattice of L and $L = \bigcup_{a \in \mathbb{Z}}(L_0 + av_k)$. The layers $L_0 + av_k$ $(a \in \mathbb{Z})$ are affine translates of L_0 lying in parallel hyperplanes.

Let P be a k-dimensional L-polytope, let L denote the lattice generated by $V(P)$ and let S be the sphere circumscribing P. Let F be a facet of P and let H denote the hyperplane spanned by F. Then, $L_0 = L \cap H$ is a $k-1$-dimensional sublattice of L and L is composed by the layers $L_0 + av$ $(a \in \mathbb{Z})$ for some $v \in L - L_0$. Therefore, $P = \text{conv}(\bigcup_{a \in \mathbb{Z}}(S \cap (L_0 + av)))$, where $S \cap L_0$ is the set of vertices of F and, for $a \in \mathbb{Z}$, $S \cap (L_0 + av)$ is empty or is the set of vertices of a face of an L-polytope in L_0. So, we have the following result:

PROPOSITION 2.11 *[64] For each k-dimensional L-polytope P, there exists a $k-1$-dimensional lattice L_0, an integer $p \geq 1$, and a sequence $F_0, F_1, \ldots, F_p$ of polytopes that are faces of L-polytopes in L_0 (where $dim(F_0) = k-1$, but $F_1, \ldots, F_p$ may be empty) such that $P = conv(\bigcup_{0 \leq a \leq p}(F_a + av))$, where v is a vector not lying in the space spanned by L_0.*

For instance, the pyramid construction can be viewed as the above layerwise construction with $p = 1$, a facet on the layer L_0 and a single point on the layer $L_0 + v$.

Let $p(k)$ denote the smallest number p of polytopes $F_1, \ldots, F_p$ in Proposition 2.11 needed for constructing any k-dimensional L-polytope.

Given a lattice L, if P is an L-polytope in L which is a simplex, then its volume is an integer multiple of $\frac{det(L)}{k!}$ (this can be checked by induction on the dimension). This integer is called the **relative volume** of the simplex P. The maximum relative volume of all simplices that are L-polytopes in any k-dimensional lattice is denoted by $p_0(k)$.

It is shown in [64] that $p(k) = p_0(k)$ holds. In particular, $p(2) = p(3) = p(4) = 1$, $p(5) = 2$ and $\lfloor \frac{k-1}{2} \rfloor \leq p(k) \leq k!$.

There is an L-polytope of dimension 6, namely the Schläfli polytope 2_{21}, for which the integer p (from Proposition 2.11) satisfies $p > 1$. In fact, for 2_{21}, $p = 2$, i.e. three layers are needed to obtain 2_{21} from its 5-dimensional sections. We mention two ways of visualizing 2_{21} via the layerwise construction. In the first construction, L_0 is the root lattice D_5 and the layers L_0, $L_0 + v$, $L_0 + 2v$ carry, respectively, $F_0 = \beta_5$, $F_1 = h\gamma_5$ and F_2 which is a single point. In the second construction, L_0 is the root lattice A_5 and the layers carry, respectively, $F_0 = \alpha_5$, $F_1 = J(6,2)$ and $F_3 = \alpha_5$. We refer to [25] for a description of all faces of 2_{21}.

L-polytopes in dimension $k \leq 4$

Examples of L-polytopes include the k-dimensional simplex α_k, cross-polytope β_k and hypercube γ_k. Note that $\alpha_k = Pyr(\alpha_{k-1})$, $\beta_k = Bipyr(\beta_{k-1})$ and $\gamma_k = \gamma_{k-1} \times \gamma_1$ for $k \geq 2$. We remind that every k-dimensional simplex with no obtuse angles is an L-polytope which is affinely equivalent to α_k; similarly, every k-dimensional parallepiped (with square angles) is an L-polytope which is affinely equivalent to γ_k.

All types of L-polytopes in dimension $k \leq 4$ are classified in [45].

• There is only one type of L-polytope of dimension $k = 1$, namely, the segment $\alpha_1 = \beta_1 = \gamma_1$.

• There are two types of L-polytopes of dimension $k = 2$, namely, the triangle (with no obtuse angles) α_2 and the rectangle $\beta_2 = \gamma_2$.

• There are five types of L-polytopes of dimension $k = 3$. They are the tetrahedron α_3, the octahedron β_3, the cube γ_3, the prism with triangular base (i.e. $\alpha_2 \times \alpha_1$) and the pyramid with square base (i.e. $Pyr(\gamma_2)$).

• There are 19 types of L-polytopes of dimension $k = 4$. They are described in Tables V and VII from [45]. Among them, 13 can be obtained from the L-polytopes of dimension 1, 2 or 3 by applying the direct product, pyramid and bipyramid constructions.

- Using the pyramid construction, we obtain the pyramids with base α_3 (this gives α_4), with base β_3, with base γ_3, with base the triangular prism and with base the squared base pyramid.

- Using the bipyramid construction, we obtain the bipyramids with base β_3 (this gives β_4) and with base γ_3.

- By taking the direct product of the 3-dimensional L-polytopes with α_1, we obtain the prisms with base α_3, with base β_3, with base γ_3 (this gives γ_4), with base the triangular prism and with base the squared base pyramid.

- By taking the direct product of two 2-dimensional L-polytopes, we obtain $\alpha_2 \times \alpha_2$ (indeed, $\alpha_2 \times \gamma_2$ and $\gamma_2 \times \gamma_2$ have already been mentioned).

We have also the repartitioning polytope $P^0_{2,2}$ (associated with the pentagonal facet; see Section 4.2) which is one more L-polytope of dimension 4; it is the polytope A in the Table VI from [45]. The remaining 5 L-polytopes are those numbered 4, 5, 6, 9 and 13 in

Table V from [45].

2.3 Finiteness of the number of types of L-polytopes in given dimension

Recall that two lattices L, L' are called z-**equivalent** if there exist an affine bijection T such that $L' = T(L)$ and T brings the star of L on the star of L'. (Note that any k-dimensional lattice is affinely equivalent to $\mathbb{Z}^k$.) Voronoi ([66]) proved that the number of distinct, up to z-equivalence, k-dimensional lattices is finite. This implies that the number of distinct, up to affine equivalence, k-dimensional L-polytopes is finite. Recall that two L-polytopes are said to have the same **type** if they are affinely equivalent.

We give here a direct proof of the finiteness of the number of types of L-polytopes in $\mathbb{R}^k$ since Voronoi's original proof is very involved; it is taken from [32].

We first observe that:

Fact 2.12 *Every type γ of L-polytopes is characterized by some integer matrix Y_γ (once a representative base for the type has been fixed).*

Indeed, let P be an L-polytope of type γ. Say, $P \subseteq \mathbb{R}^k$ has dimension k and P has N vertices. Let L be a lattice in $\mathbb{R}^k$ containing the set of vertices $V(P)$ of P, but L may be larger than the lattice $L(P)$ generated by $V(P)$. Let $B = \{b_1, \ldots, b_k\}$ be a base of L.

For each $v \in V(P)$, let $y_v \in \mathbb{Z}^k$ such that $v = \sum_{1\le i\le k}(y_v)_i b_i$ holds. Denote by Q_P the $N \times k$ matrix whose rows are the vectors $v \in V(P)$, by M_B the $k \times k$ matrix whose rows are the members of B and by $Y_{P,B}$ the $N \times k$ matrix whose rows are the vectors $y_v, v \in V(P)$. Then, the following relation holds:

$$Q_P = Y_{P,B} M_B \tag{17}$$

Clearly, if B' is another base of L, then $Y_{P,B'}$ is unimodular equivalent to $Y_{P,B}$. On the other hand, let P' be another L-polytope of the same type γ as P, i.e. $P' = T(P)$ for some affine bijective transformation T. Then, $T(L)$ is a lattice containing the set of vertices $V(P') = T(V(P))$ of P', $T(B)$ is a base of $T(L)$ and, thus, $Y_{P',T(B)} = Y_{P,B}$ holds. Therefore, the matrix $Y_\gamma = Y_{P,B}$ characterizes the type γ of L-polytopes. It is uniquely determined once the starting base B, called **representative base** of γ, has been fixed.

Proposition 2.13 *Let γ be a type of L-polytopes of dimension k. One can choose the representative base B in such a way that the matrix Y_γ satisfies the following conditions:*
(i) There exists a $k \times k$ submatrix $D = (\alpha_{ij})_{1\le i,j\le n}$ of Y_γ which is lower triangular and satisfies: $0 \le \alpha_{ij} < \alpha_{ii}$ for all $1 \le j < i \le k$.

(ii) $p = |det(D)|$ *is the maximum possible value of the absolute value of the determinant of any* $k \times k$ *submatrix of* Y_γ.
(iii) $p \leq k!2^k$.

For the proof, we need the following classical result about lattices.

Proposition 2.14 *[21] Let* L, L' *be two* k*-dimensional lattices in* $\mathbb{R}^k$ *such that* $L' \subseteq L$. *For every base* $\{a_1, \dots, a_k\}$ *of* L', *there exists a base* $\{b_1, \dots, b_k\}$ *of* L *such that*
(i) $a_i = \alpha_{i1} b_1 + \ldots + \alpha_{ii} b_i$, *for* $i = 1, \dots, k$, *where* $(\alpha_{ij})_{1 \leq i,j \leq k}$ *are integers satisfying*
(ii) $0 \leq \alpha_{ij} < \alpha_{ii}$, *for all* $1 \leq j < i \leq k$.

Proof of Proposition 2.13. Let P be an L-polytope of type γ with N vertices and let L be a lattice in $\mathbb{R}^k$ containing the set of vertices $V(P)$ of P. Let V_0 be a subset $V(P)$ of size k and let Q_0 denote the $k \times k$ submatrix of Q_P whose rows are the members of V_0. We choose V_0 in such a way that $|det(Q_0)|$ is largest possible. We can suppose that Q_0 is the submatrix of Q_P formed by its first k rows. The lattice $L' = \mathbb{Z}(V_0)$ is a sublattice of L and admits V_0 as a base. Applying Proposition 2.14, we deduce the existence of a base B of L such that

$$Q_0 = DM_B$$

where D is a lower triangular integer matrix satisfying Proposition 2.14 (ii). Since $Q_P = (Q_P M_B^{-1}) M_B$, by comparing with relation (17), we obtain that $Q_P M_B^{-1}$ is the integer matrix $Y_{P,B}$. Set $Y = Y_{P,B}$; it is an $N \times k$ matrix whose first k rows form the matrix D. Note that $p = |det(D)| = \frac{|det(Q_0)|}{|det(M_B)|} = \frac{|det(Q_0)|}{det(L)}$. Hence, by the choice of Q_0, the absolute value of the determinant of any $k \times k$ submatrix of Y is less or equal to p. Therefore, if B is chosen as representative matrix of the type γ, then $Y_\gamma = Y$ satisfies the conditions (i),(ii) of Proposition 2.13.

Finally, we check (iii). Let Δ denote the simplex whose vertices are the members of V_0, i.e. the rows or Q_0. Then, Δ is contained in P and, thus, $vol(\Delta) \leq vol(P)$. But, $vol(\Delta) = \frac{|det(Q_0)|}{k!} = \frac{p\ det(L)}{k!}$ and $vol(P) \leq 2^k det(L)$ from Proposition 2.9. Therefore, we obtain that $p \leq k!2^k$. □

We can now show the finiteness of the number of types of L-polytopes in $\mathbb{R}^k$.

Theorem 2.15 *[32] The number of types of* L*-polytopes in* $\mathbb{R}^k$ *is finite.*

Proof. Every type γ of L-polytopes in $\mathbb{R}^k$ with N vertices is characterized by an $N \times k$ integer matrix Y_γ satisfying Proposition 2.13 (i)-(iii). It suffices now to show that there is only a finite number of such matrices. For this, we show that, for fixed p, there is only a finite number of matrices satisfying Proposition 2.13 (i)-(ii).

Let Y be an $N \times k$ integer matrix satisfying (i),(ii). Suppose that D is the upper $k \times k$ submatrix of Y. Then, the upper $k \times k$ submatrix of YD^{-1} is the identity matrix. Let r_{ih} be a non zero entry of YD^{-1}, where $k+1 \leq i \leq N$ and $1 \leq h \leq k$. Let C denote the matrix obtained from D by replacing its h-th row by the i-th row of Y. By Proposition 2.13 (ii), $|det(C)| \leq p$. On the other hand, $|det(CD^{-1})| = |r_{ih}|$, implying that $|r_{ih}| = \frac{|det(C)|}{p}$ belongs to $\{0, \frac{1}{p}, \ldots, \frac{p-1}{p}, 1\}$. Since YD^{-1} is an $N \times k$ matrix with $N \leq 2^k$ (from Proposition 2.8), we deduce that, for fixed p and k, there is only a finite number of such matrices YD^{-1}. Now, D is a $k \times k$ integer matrix with $p = \alpha_{11} \ldots \alpha_{kk}$ and satisfying Proposition 2.13 (i); therefore, there is only a finite number of such matrices D. Consequently, there is a finite number of possibilities for Y. □

3 Hypermetrics and L-polytopes

3.1 The connection between hypermetrics and L-polytopes

In this section, we establish the fundamental connection existing between hypermetrics spaces and L-polytopes.

The following Lemmas 3.1 and 3.2 are crucial for Theorem 3.3, which establishes this connection.

LEMMA 3.1 *[6] Let $c, v_0 = 0, v_1, \ldots, v_n \in \mathbb{R}^k$ be vectors satisfying*
(i) $\| v_i - c \| = \| c \|$, for $1 \leq i \leq n$,
(ii) $\| \sum_{1 \leq i \leq n} b_i v_i - c \| \geq \| c \|$ for all $b \in \mathbb{Z}^n$.
Then, $L = \mathbb{Z}(v_1, \ldots, v_n)$ is a lattice.

PROOF. For $b \in \mathbb{Z}^n$, set $v(b) = \sum_{1 \leq i \leq n} b_i v_i$; then, $v(b) \pm v_i \in L$. Hence, (ii) yields $(v_i \pm v(b) - c)^2 \geq c^2$, i.e. $(v_i - c)^2 + (v(b))^2 \pm 2(v_i - c)^T v(b) \geq c^2$ and, using (i), we obtain that:

$$(*) \qquad (v(b))^2 \geq 2|(v_i - c)^T v(b)| \text{ for } 1 \leq i \leq n.$$

Consider the units vectors $e_i = \frac{v_i - c}{\|c\|}$ for $i = 0, 1, \ldots, n$ and $e(b) = \frac{v(b)}{\|v(b)\|}$. Set $\beta = \min\{\max(e_i^T u : 1 \leq i \leq n) : u \in \mathbb{R}^k, \| u \| = 1\}$. In order to conclude the proof, it is enough to show that $\beta \neq 0$, since we obtain from (*) that $\| v(b) \| \geq 2\beta \| c \|$ for all $b \in \mathbb{Z}^n$ such that $v(b) \neq 0$. Suppose, for contradiction, that $\beta = 0$. Then, we can find a sequence $(u_p)_{p \geq 1}$ of unit vectors of $\mathbb{R}^k$ such that $|e_i^T u_p| \leq \frac{1}{p}$ for any $1 \leq i \leq n, p \geq 1$. By compacity of the unit sphere, we can suppose that the sequence $(u_p)_{p \geq 1}$ admits a limit u when p goes to infinity (else, replace $(u_p)_{p \geq 1}$ by a subsequence). Therefore, $\| u \| = 1$, while $e_i^T u = 0$

for $i = 1, \ldots, n$, implying that $u = 0$ since the vectors $v_1, \ldots, v_n$ span $\mathbb{R}^k$. We have a contradiction. $\square$

Lemma 3.2 *[6] Let (X, d) be a distance space, $X = \{0, 1, \ldots, n\}$. The following assertions are equivalent.*
(i) (X, d) is hypermetric,
(ii) (X, d) has a representation $i \in X \mapsto v_i \in \mathbb{R}^k$ $(k \leq n)$ on a sphere S, and S is empty in $L_{af}(X, d) = \{\sum_{i \in X} b_i v_i : b \in \mathbb{Z}^X$ and $\sum_{i \in X} b_i = 1\}$, i.e. $\| \sum_{i \in X} b_i v_i - c \| \geq r$ for all $b \in \mathbb{Z}^X$ with $\sum_{i \in X} b_i = 1$, where c and r denote the center and radius of S.

Proof. $(i) \Longrightarrow (ii)$ Since (X, d) is of negative type, (X, d) has a representation $v_0, v_1, \ldots, v_n \in \mathbb{R}^k$, the system $(v_0, \ldots, v_k)$ has rank k and we can suppose without loss of generality that $v_0 = 0$. We first show that the vectors $v_0, v_1, \ldots, v_n$ lie on a sphere S, i.e. that there exists $c \in \mathbb{R}^k$ such that

$$(*) \qquad 2c^T v_i = v_i^2 \text{ for } 1 \leq i \leq n.$$

If $k = n$, i.e. the vectors $v_1, \ldots, v_n$ are linearly independent, then the above equation (*) admits a unique solution c. Let $k \leq n - 1$. Let M denote the $n \times k$ matrix whose rows are the vectors $v_1, \ldots, v_n$, let U denote the subspace of $\mathbb{R}^n$ spanned by the columns of M and set $f = (v_1^2, \ldots, v_n^2)$. The above equation (*) has a solution if and only if $f \in U$, or equivalently, $f^T g = 0$ for each $g \in U^\perp$ (the orthogonal complement of U in $\mathbb{R}^k$). Take $g \in U^\perp$, let $b \in \mathbb{Z}^n$ such that $|g_i - b_i| < 1$ for $i = 1, \ldots, n$ and set $\delta = g - b$; so δ belongs to the unit cube.
Consider $p = \xi_0(d)$; then, $p_{ij} = v_i^T v_j$ for $1 \leq i < j \leq n$. Using relation (10), we deduce that p satisfies the inequality $\sum_{1 \leq i,j \leq n} b_i b_j p_{ij} - \sum_{1 \leq i \leq n} b_i p_{ii} \geq 0$, i.e.

$$(**) \qquad (\sum_{1 \leq i \leq n} b_i v_i)^2 - \sum_{1 \leq i \leq n} b_i v_i^2 \geq 0.$$

Hence, from relation (**), $f^T b = \sum_{1 \leq i \leq n} b_i v_i^2 \leq (b^T M)^2 = (g^T M - \delta^T M)^2 = (\delta^T M)^2$, since $g \in U^\perp$. Therefore, $f^T b \leq (\delta^T M)^2$, implying that $f^T g \leq f^T \delta + (\delta^T M)^2$. This implies that $f^T g = 0$; otherwise, the left hand side of the latter inequality could be made arbitrarily large while its right hand side is bounded. Note that the solution c to the equation (*) is unique since $(v_1, \ldots, v_n)$ has full rank k.

The fact that S is empty in $L_{af}(X, d)$ follows from relations (*) and (**).
$(ii) \Longrightarrow (i)$. Let $b \in \mathbb{Z}^X$ with $\sum_{i \in X} b_i = 1$. Then,

$$\begin{aligned}
\sum_{i,j \in X} b_i b_j d(i,j) &= \sum_{i,j \in X} b_i b_j (v_i - v_j)^2 = \sum_{i,j \in X} b_i b_j (v_i - c + c - v_j)^2 \\
&= \sum_{i,j \in X} b_i b_j (2r^2 - 2(v_i - c)^T (v_j - c)) \\
&= 2r^2 - 2(\sum_{i \in X} b_i (v_i - c))^2 \\
&= 2(r^2 - (\sum_{i \in X} b_i v_i - c)^2) \leq 0
\end{aligned}$$

since the sphere S is empty in $L_{af}(X,d)$. Hence d satisfies the hypermetric inequalities. □

In particular, every L-polytope space $(V(P), d^{(2)})$ is hypermetric, where $V(P)$ is the set of vertices of an L-polytope P. Conversely, from Lemmas 3.1 and 3.2, every hypermetric space can be realized as isometric subspace of an L-polytope space. We summarize in the next theorem this fundamental connection.

THEOREM 3.3 *[6] Let (X,d) be a hypermetric space, $|X| = n+1$. There exist a k-dimensional L-polytope P_d in $\mathbb{R}^k$, for some $k \leq n$, and a mapping $f_d : i \in X \mapsto v_i \in V(P_d)$ which is* **generating**, *i.e. $\{v_i, i \in X\}$ generates the set of vertices $V(P_d)$ of P_d, and such that*

$$d(i,j) = (v_i - v_j)^2 \text{ for } i,j \in X.$$

Moreover, the pair (P_d, f_d) is unique, up to translation and orthogonal transformation.

We refer to P_d as the **L-polytope associated with the hypermetric space** (X,d), the lattice $\mathbb{Z}_{af}(V(P_d)) = \{\sum_{v \in V(P_d)} b_v v : b \in \mathbb{Z}^{V(P_d)}$ and $\sum_{v \in V(P_d)} b_v = 1\}$ is denoted as L_d and the sphere circumscribing P_d as S_d.

Note that, if $(X' = X \cup \{j_0\}, d')$ is a 0-lifting of the hypermetric space (X,d) with $d(i_0, j_0) = 0$ for $i_0 \in X$, as in relation (8), then both (X,d) and (X', d') have the same associated L-polytope P_d (simply representing j_0 by the same vertex of P_d as i_0).

We would also like to emphasize the following fact, since it will be very useful in the sequel.

PROPOSITION 3.4 *Let (X,d) be a hypermetric space with representation $(v_i, i \in X)$ in the set of vertices of its associated L-polytope P_d. For $b \in \mathbb{Z}^X$ with $\sum_{i \in X} b_i = 1$, the equality $\sum_{i,j \in X} b_i b_j d(i,j) = 0$ holds if and only if the vector $\sum_{i \in X} b_i v_i$ is a vertex of P_d.*

PROOF. It follows from the equality $\sum_{i,j \in X} b_i b_j d(i,j) = 2(r^2 - (\sum_{i \in X} b_i v_i - c)^2)$, stated in the proof of Lemma 3.2 $(ii) \Longrightarrow (i)$. □

We give two examples illustrating this connection between hypermetric spaces and L-polytopes.

Example 1. Consider the cut semimetric $d = \delta(S)$ for some subset $S \subseteq X$. It is obviously hypermetric. Its associated L-polytope is the segment $\alpha_1 = [0,1]$ and the representation of the hypermetric space $(X, \delta(S))$ is $i \in S \mapsto v_i = 1$, $i \in X - S \mapsto v_i = 0$.

Example 2. Let (X,d) be a semimetric space. Then, d lies in the interior of the hypermetric cone HYP(X) if and only if its associated L-polytope is the simplex $\alpha_{|X|-1}$ of dimension $|X|-1$ (since, by Proposition 3.4, d satisfies no nontrivial hypermetric equality if and only if $|V(P_d)| = dim(P_d) - 1$).

We conclude this section with two additional properties concerning the connection between hypermetrics and L-polytopes.

A first observation is that, if (Y, d) is a subspace of the hypermetric space (X, d), then its associated L-polytope is embedded in the L-polytope associated to (X, d).

LEMMA 3.5 *Let P be an L-polytope with set of vertices $V(P)$ and let X be a subset of $V(P)$. Let P_X denote the L-polytope associated with the hypermetric space $(X, d^{(2)})$. Then, $V(P_X) \subseteq V(P)$ with equality if and only if X is a generating subset of $V(P)$.*

PROOF. Let L_X denote the sublattice of L generated by X and let A_X denote the affine space generated by X. Let S be the circumscribed sphere to P, so S is an empty sphere in L. The sphere $S_X = S \cap A_X$ is empty in L_X. Hence, $P_X = \text{conv}(S_X \cap L_X)$ is an L-polytope and it is the L-polytope associated with the hypermetric space $(X, d^{(2)})$. Therefore, $V(P_X) = S_X \cap L_X$ is indeed contained in $V(P) = S \cap L$. It is easy to see that $V(P_X) = V(P)$ if and only if X generates the lattice L. □

In particular, every face of an L-polytope is an $L-$polytope. For instance, every 2-dimensional face of an L-polytope is a rectangle or a triangle with no obtuse angles.

COROLLARY 3.6 *Let (X, d) be a hypermetric space and (Y, d) be a subspace of (X, d), i.e. $Y \subseteq X$. Let P_X, P_Y denote the L-polytopes associated, respectively, with (X, d), (Y, d). Then, $V(P_Y) \subseteq V(P_X)$ holds.*

There are some properties of the hypermetric space (X, d) which are inherited by its associated L-polytope. This is the case for hypercube or ℓ_1-embeddability as shown in the next result. Another such property is the notion of rank and extremality as we will see later in Section 4.1.

PROPOSITION 3.7 *(i) [5] Let (X, d) be a hypermetric space and let P_d be its associated L-polytope with set of vertices $V(P_d)$. Then, (X, d) is isometrically ℓ_1-embeddable if and only if P_d is embedded in a parallepiped, i.e. $(V(P_d), d^{(2)})$ is isometrically ℓ_1-embeddable. (ii) [31] Moreover, if d is rational valued, then (X, d) is ℓ_1-embeddable if and only if P_d is embedded in a hypercube with side length λ; the smallest such λ is the scale of both spaces (X, d) and $(V(P_d), d^{(2)})$.*

PROOF. (i) From Proposition 2.1, d is ℓ_1-embeddable if and only if $d = \sum_{1 \le h \le m} \lambda_h \delta(S_h)$ for some scalars $\lambda_h > 0$, where the S_h's are subsets of X, for $1 \le h \le m$.

Set $e'_h = \sqrt{\lambda_h} e_h$, where e_h is the h-th unit vector in $\mathbb{R}^m$, for $1 \leq h \leq m$. Let L denote the lattice in $\mathbb{R}^m$ generated by the system of orthogonal vectors $(e'_1, \ldots, e'_m)$. It is easy to check that the sphere S with center $c = \frac{1}{2}\sum_{1\leq h\leq m} e'_h$ and radius $\| c \|$ is empty in L. For $i = 1, \ldots, n$, set $I_i = \{h \in \{1, \ldots, m\} : i \in S_h\}$. So, $h \in I_i$ if and only if $i \in S_h$. Therefore, $d(i,j) = \sum_{1\leq h\leq m,\ |S_h\cap\{i,j\}|=1} \lambda_h = \sum_{1\leq h\leq m,\ h\in I_i\triangle I_j} \lambda_h$. From this, we deduce that the mapping $i \in X \mapsto v_i = \sum_{h\in I_i} e'_h \in \mathbb{R}^m$ is a representation of (X,d). Indeed, $d^{(2)}(v_i, v_j) = (\sum_{h\in I_i\triangle I_j} e'_h)^2 = \sum_{h\in I_i\triangle I_j} \lambda_h = d(i,j)$ holds. This shows that the L-polytope P_d associated with (X,d) is embedded in the parallepiped spanned by e'_h, $1 \leq h \leq m$.
(ii) is based on Corollary 2.2. Indeed, if d is rational valued, then, (X,d) is ℓ_1-embeddable if and only if $(X, \lambda d)$ is hypercube embeddable for some scalar λ. Let λ denote the scale of (X,d), i.e. the smallest such scalar. Let $i \in X \mapsto w_i \in \{0,1\}^n$ be an isometric embedding of λd into the hypercube, i.e. $\lambda d(i,j) = d_1(w_i, w_j)$ for $i,j \in X$. But $d_1(w_i, w_j) = d^{(2)}(w_i, w_j)$ because the w_i's are binary. Hence, they lie on the sphere S circumscribing the hypercube $[0,1]^n$ and S is empty in $\mathbb{Z}^n$. Therefore, $(w_i, i \in X)$ is also the hypermetric representation of λd, i.e. the L-polytope $P_{\lambda d}$ associated with $(X, \lambda d)$ is embedded in the hypercube $[0,1]^n$. Hence, $\frac{1}{\lambda} P_{\lambda d}$ is clearly the L-polytope P_d associated with (X,d) and, thus, $v \in V(P_d) \mapsto \sqrt{\lambda} v \in V(P_{\lambda d}) \subseteq \{0,1\}^n$ is an h-embedding of $(V(P_d), \lambda d^{(2)})$. λ is clearly the scale of $(V(P_d), d^{(2)})$ since every h-embedding of $(V(P_d), \alpha d^{(2)})$ yields an h-embedding of $(X, \alpha d)$. □

3.2 Polyhedrality of the hypermetric cone

An important application of the connection between hypermetric spaces and L-polytopes is for proving that the hypermetric cone is polyhedral. Indeed, the hypermetric cone HYP_{n+1} is defined by infinitely many inequalities and it is therefore natural to ask whether a finite subset of them suffices for describing HYP_{n+1}. The answer is yes, as shown in the next Theorem 3.8. Based on the fact that facets of HYP_{n+1} correspond to a very special class of L-polytopes (the repartitioning L-polytopes, see Section 4.2), an upper bound on the coefficients of the hypermetric inequalities that define facets of HYP_{n+1} is given in Theorem 4.11.

THEOREM 3.8 *[32] For any $n \geq 2$, the hypermetric cone HYP_{n+1} is polyhedral.*

PROOF. Set $X = \{0, 1, \ldots, n\}$ and let d be a distance on X. Recall, from relation (10), that, for $b_0, b_1, \ldots, b_n \in \mathbb{Z}$ with $\sum_{0\leq i\leq n} b_i = 1$, d satisfies $\sum_{0\leq i<j\leq n+1} b_i b_j d(i,j) \leq 0$ (resp. $= 0$) if and only if its image $p = \xi_0(d)$ under the covariance mapping ξ_0 satisfies $\sum_{1\leq i,j\leq n} b_i b_j p_{ij} - \sum_{1\leq i\leq n} b_i p_{ii} \geq 0$ (resp. $= 0$).

Since the mapping ξ_0 is bijective linear, the cone HYP_{n+1} is polyhedral if and only if the cone $\xi_0(\mathrm{HYP}_{n+1})$ polyhedral.
For $p \in \xi_0(\mathrm{HYP}_{n+1})$, we define its **annullator** $Ann(p)$ by

$$Ann(p) = \{b \in \mathbb{Z}^n : b \neq 0, e_1, \ldots, e_n \text{ and } \sum_{1\le i,j\le n} b_i b_j p_{ij} - \sum_{1\le i\le n} b_i p_{ii} = 0\}$$

where $e_1, \ldots, e_n$ denote the unit vectors in $\mathbb{R}^n$. Let $F(p)$ denote the smallest face of $\xi_0(\mathrm{HYP}_{n+1})$ containing p, i.e.

$$F(p) = \xi_0(\mathrm{HYP}_{n+1}) \cap \bigcap_{b\in Ann(p)} H_b$$

where H_b denotes the hyperplane in $\mathbb{R}^{\binom{n+1}{2}}$ defined by the equation $\sum_{1\le i,j\le n} b_i b_j p_{ij} - \sum_{1\le i\le n} b_i p_{ii} = 0$. Clearly, showing that $\xi_0(\mathrm{HYP}_{n+1})$ is polyhedral amounts to showing that the number of its distinct faces is finite or, equivalently, that the number of distinct annullators $Ann(p)$, for $p \in \xi_0(\mathrm{HYP}_{n+1})$, is finite.

Let P_d denote the L-polytope associated with d, let L_d be the associated lattice and let $i \in X \mapsto v_i \in V(P_d)$ be the representation of (X, d) on the sphere S_d with center c circumscribing P_d. We can assume that $v_0 = 0$; then, $L_d = \mathbb{Z}(v_1, \ldots, v_n)$. For $v \in L_d$, set

$$Z(v) = \{b \in \mathbb{Z}^n : v = \sum_{1\le i\le n} b_i v_i\}.$$

Then, from Proposition 3.4, we obtain that

$$(*) \qquad Ann(p) \cup \{0, e_1, \ldots, e_n\} = \bigcup_{v\in V(P_d)} Z(v).$$

Suppose that the polytope P_d is of type γ and let Y_γ be the integer matrix characterizing the type γ, as in Proposition 2.13. From Fact 2.12, there exists a base B in $\mathbb{R}^k$ such that, if Q_{P_d} denotes the matrix whose rows are the vectors $v \in V(P_d)$ and if M_B denotes the matrix whose rows are the vectors of B, then relation (17) reads

$$Q_{P_d} = Y_\gamma M_B.$$

Let Q denote the $n \times k$ matrix whose rows are the vectors v_i, $1 \le i \le n$. So, Q may have repeated rows and every row of Q is a row of Q_{P_d}. We have $Q = Y M_B$, for some integer matrix Y. If we denote by y_v, $v \in V(P_d)$, the rows of Y_γ, then the rows of Y are the vectors y_{v_i} for $1 \le i \le n$. Note that the equality $v = \sum_{1\le i\le n} b_i v_i$ is equivalent to the equality $y_v = \sum_{1\le i\le n} b_i y_{v_i}$. Therefore, $Z(v) = \{b \in \mathbb{Z}^n : y_v = \sum_{1\le i\le n} b_i y_{v_i}\}$ for each $v \in V(P_d)$. Hence, for $v \in V(P_d)$, $Z(v)$ depends only on $(y_v, y_{v_1}, \ldots, y_{v_n})$. Using (*), we deduce that

$Ann(p)$ is entirely determined by the matrix Y_γ and the subsystem $(y_{v_1}, \ldots, y_{v_n})$ of its rows. In other words, for each $d \in \mathrm{HYP}_{n+1}$, the annullator $Ann(\xi_0(d))$ is completely determined by a pair (γ, θ) where γ is a type of L-polytopes in $\mathbb{R}^k$ with $k \leq n$, and θ is a mapping from $\{1, \ldots, n\}$ to the set of rows of Y_γ. Therefore, since the number of such mappings θ is obviously finite and since the number of types of L-polytopes in given dimension is finite (from Theorem 3.8), we deduce that the number of distinct annullators $Ann(\xi_0(d))$, for $d \in \mathrm{HYP}_{n+1}$, is finite. □

3.3 L-polytopes in root lattices

In this section, we group several results on L-polytopes in root lattices.

First, we recall the description of the irreducible root lattices and of their L-polytopes. We also show that, if P is an L-polytope in a root lattice, then its 1-skeleton is completely determined by the metric structure of P; namely, two vertices form an edge of P if and only if their squared euclidian distance is equal to 2 (see Proposition 3.9).

Then, we see that L-polytopes in root lattices arise in a natural way from hypermetric spaces that are connected and strongly even (see Proposition 3.10). As a consequense, we obtain a characterization of the connected strongly even distance spaces that are hypermetric, or isometrically ℓ_1-embeddable (see Theorems 3.12 and 3.13).

Let P be an L-polytope which is generating in a root lattice L. If L is reducible, then $L = L_1 \oplus L_2$ where L_1 and L_2 are root lattices. Hence, $P = P_1 \times P_2$ where P_i is an L-polytope in L_i, for $i = 1, 2$. Therefore, it suffices to describe the L-polytopes that are generating in some irreducible root lattice.

The irreducible root lattices have been classified by Witt (see, for instance, [19]). They are A_n $(n \geq 0)$, D_n $(n \geq 4)$, and E_n $(n = 6, 7, 8)$. We recall their description below; we will consider in more detail the lattices E_6, E_7 and E_8 in Section 5.2.

For each of the lattices A_n, D_n and E_n, we present some information about its roots (i.e. its minimal vectors) and about its empty spheres (i.e. its holes). For details, we refer, for instance, to [19], [23] or [24].

Case of A_n, $n \geq 0$

- $A_n = \{x \in \mathbb{Z}^{n+1} : \sum_{0 \leq i \leq n} x_i = 0\}$.
- The roots of A_n are the $n(n+1)$ vectors $e_i - e_j$, $0 \leq i \neq j \leq n$, where e_i denote the i-th unit vector in $\mathbb{R}^{n+1}$.
- There are $\lfloor \frac{n+1}{2} \rfloor$ types of empty spheres in A_n. Their centers are

$$c_a = \left(\frac{a}{n+1}, \ldots, \frac{a}{n+1}; -\frac{n+1-a}{n+1}, \ldots, -\frac{n+1-a}{n+1}\right),$$

where $\frac{a}{n+1}$ is repeated $n+1-a$ times and $-\frac{n+1-a}{n+1}$ is repeated a times, with corresponding radius $r_a = \sqrt{\frac{a(n+1-a)}{n+1}}$, for $1 \leq a \leq \lfloor\frac{n+1}{2}\rfloor$. The case $a = \lfloor\frac{n+1}{2}\rfloor$ corresponds to a deep hole, i.e. a hole with maximum radius.

• The L-polytope circumscribed by the empty sphere with center c_a and radius r_a has for vertices the following $\binom{n+1-a}{b} + \binom{a}{b}$ vectors $(1^b, 0^{n+1-a-b}; 1^b, 0^{a-b})$ for $1 \leq b \leq a$, where the first b ones are chosen among the $n+1-a$ positions of the entries $\frac{a}{n+1}$ of c_a and the last b ones are chosen among the a positions of the entries $-\frac{n+1-a}{n+1}$ of c_a.
Its 1-skeleton is the Johnson graph $J(n+1, a)$.

Case of D_n, $n \geq 4$

• $D_n = \{x \in \mathbb{Z}^n : \sum_{1 \leq i \leq n} x_i \in 2\mathbb{Z}\}$.

• The roots of D_n are the $2n(n-1)$ vectors $\pm e_i \pm e_j$ for $1 \leq i \neq j \leq n$.

• There are two types of empty spheres in D_n, namely, an empty sphere S_1 with center $c_1 = (0, \ldots, 0, 1)$ and radius $r_1 = 1$, and an empty sphere S_2 with center $c_2 = (\frac{1}{2}, \ldots, \frac{1}{2})$ and radius $r_2 = \frac{\sqrt{n}}{2}$.

• The L-polytope circumscribed by the sphere S_1 has for vertices the $2n$ vectors $(0, \ldots, 0)$, $(0, \ldots, 0, 2)$ and $(0, \ldots, 0, \pm 1, 0, \ldots, 0, 1)$ where the ± 1 is in one of the first $n-1$ positions. This is the cross-polytope β_n whose 1-skeleton is the cocktail-party graph $K_{n \times 2}$.

• The L-polytope circumscribed by the second sphere S_2 has for vertices the 2^{n-1} vectors $x \in \{0,1\}^n$ with $\sum_{1 \leq i \leq n} x_i \in 2\mathbb{Z}$. This is the half-cube $h\gamma_n$ whose 1-skeleton is the half-cube graph $\frac{1}{2}H(n, 2)$. It corresponds to a deep hole in D_n.
Note that, for $n = 4$, β_4 and $h\gamma_4$ are affinely equivalent.

Case of E_8

• $E_8 = \{x \in \mathbb{R}^8 : x \in \mathbb{Z}^8$, or $x \in (\frac{1}{2} + \mathbb{Z})^8$ and $\sum_{1 \leq i \leq 8} x_i \in 2\mathbb{Z}\}$, i.e. E_8 is the lattice generated by D_8 and $\frac{1}{2}\sum_{1 \leq i \leq 8} e_i$. E_8 is unimodular, so $E_8^* = E_8$.

• The roots of E_8 are the 240 vectors $\pm e_i \pm e_j$ and $\frac{1}{2}(\pm e_1 \pm \ldots \pm e_n)$, where there is an even number of minus signs in a root of the second kind.

• There are two types of empty spheres in E_8, namely, the sphere S_1 with center $c_1 = (1, 0^7)$ and radius $r_1 = 1$, and the sphere S_2 with center $c_2 = (\frac{5}{6}, \frac{1}{6}^7)$ and radius $r_2 = \sqrt{\frac{8}{9}}$.

• The L-polytope circumscribed by the sphere S_1 has for vertices the following 16 vectors (0^8), $(2, 0^7)$, $(1, 0, \ldots, 0, \pm 1, 0, \ldots, 0)$, where ± 1 is in one of the last seven positions. This is the cross-polytope β_8 whose 1-skeleton is $K_{8 \times 2}$. It corresponds to a deep hole in E_8.

• The L-polytope circumscribed by the sphere S_2 has for vertices the following 9 vectors (0^8), $(\frac{1}{2}, \ldots, \frac{1}{2})$ and $(1, 0, \ldots, 0, 1, 0, \ldots, 0)$, where the second 1 is in one of the last seven positions. This is the simplex α_8 with 1-skeleton K_9.

Case of E_7

• The root lattice E_7 consists of the vectors of E_8 that are orthogonal to a given minimal vector v_0 of E_8. If we choose $v_0 = (\frac{1}{2}, \ldots, \frac{1}{2})$, then $E_7 = \{x \in E_8 : \sum_{1 \le i \le 8} x_i = 0\}$. Another choice for v_0 could be $v_0 = (1, 1, 0^6)$; we will work with this definition of E_7 in Section 5.2 (actually, we shall use there for E_7 the following affine translate $\{x \in E_8 : x^T v_0 = x_1 + x_2 = 1\}$).

• There are two types of empty spheres in E_7, namely, the sphere S_1 with center $c_1 = (\frac{3}{4}^2, -\frac{1}{4}^6)$ and radius $r_1 = \sqrt{\frac{3}{2}}$, and the sphere S_2 with center $c_2 = (\frac{7}{8}, -\frac{1}{8}^7)$ and radius $r_2 = \sqrt{\frac{7}{8}}$.

• The L-polytope circumscribed by the sphere S_1 has for vertices the 56 vectors $c_1 \pm (\frac{3}{4}^2, -\frac{1}{4}^6)$. This is the Gosset polytope 3_{21} whose 1-skeleton is the Gosset graph G_{56}. It corresponds to a deep hole in E_7.

• The L-polytope circumscribed by the sphere S_2 has for vertices the 8 following vectors (0^8) and $(1, 0, \ldots, 0, -1, 0, \ldots, 0)$, where -1 is in one of the last seven positions. This is the 7-dimensional simplex α_7 with 1-skeleton K_8.

Case of E_6

• The root lattice E_6 consists of the vectors of E_7 that are orthogonal to two nonorthogonal given minimal vectors v_0 and w_0 of E_8. If we choose $v_0 = (1, 1, 0^6)$ and $w_0 = (-\frac{1}{2}^8)$, then $E_6 = \{x \in E_8 : x_1 + x_2 = x_3 + \ldots + x_8 = 0\}$. (In Section 5.2, we select differently v_0 and w_0 and we consider an affine translate as E_6.)

• There is only one type of empty sphere in E_6. Its radius is $\sqrt{\frac{4}{3}}$ and it circumscribes the L-polytope whose vertices are the following 27 vectors $(\frac{1}{2}, -\frac{1}{2}, \frac{5}{6}, -\frac{1}{6}^5)$, $(-\frac{1}{2}, \frac{1}{2}, \frac{5}{6}, -\frac{1}{6}^5)$ where $\frac{5}{6}$ is in one of the last six positions, and $(0, 0, -\frac{2}{3}^2, \frac{1}{3}^4)$ where the two $-\frac{2}{3}$'s are in the last six positions. This is the Schläfli polytope 2_{21} whose 1-skeleton is the Schläfli graph G_{27}. So the star of E_6 contains only copies of 2_{21} and of its image under central symmetry.

We summarize in Figure 1 below some information about the L-polytopes P arising in the irreducible root lattices. For each of them, we indicate the square r^2 of its radius and its 1-skeleton, denoted by $H(P)$ and called an L-**polytope graph**.

lattice L	L-polytope P	L-polytope graph $H(P)$	squared radius r^2
A_n $(n \geq 0)$		$J(n+1,t)$ for $1 \leq t \leq \lfloor \frac{n+1}{2} \rfloor$	$\frac{t(n+1-t)}{n+1}$
D_n $(n \geq 4)$	β_n $h\gamma_n$	$K_{n\times 2}$ $\frac{1}{2}H(n,2)$	1 $n/4$
E_8	α_8 β_8	K_9 $K_{8\times 2}$	$8/9$ 1
E_7	α_7 3_{21}	K_8 G_{56}	$7/8$ $3/2$
E_6	2_{21}	G_{27}	$4/3$

Figure 1

We group several observations about the graphs $J(n,t)$, $\frac{1}{2}H(n,2)$, $K_{n\times 2}$, K_n, the Schläfli graph G_{27} and the Gosset graph G_{56} occurring in Figure 1.

- There are some isomorphisms among them, namely, $J(n,1) = K_n$, $\frac{1}{2}H(2,2) = K_2$, $\frac{1}{2}H(3,2) = K_4$, $K_{3\times 2} = J(4,2)$, $K_{4\times 2} = \frac{1}{2}H(4,2)$. Note that $J(n,2)$ is the line graph $L(K_n)$ of K_n and $J(n,2)$ is also called the triangular graph denoted by $T(n)$. The half-cube graph $\frac{1}{2}H(5,2)$ is also called the Clebsch graph.
- $J(n,t)$ is an isometric subgraph of $\frac{1}{2}H(n,2)$ and of $J(n+1,t)$, $\frac{1}{2}H(n,2)$ is an isometric subgraph of $\frac{1}{2}H(n+1,2)$, $K_{n\times 2}$ is an isometric subgraph of $K_{(n+1)\times 2}$. Also, $\nabla\frac{1}{2}H(5,2)$ is an isometric subgraph of G_{27}; $\frac{1}{2}H(6,2)$, $K_{6\times 2}$, $J(8,2)$ and ∇G_{27} are isometric subgraphs of G_{56}. In fact, $J(5,2)$ (resp. $\frac{1}{2}H(5,2)$, G_{27}) is the subgraph of $\frac{1}{2}H(5,2)$ (resp. of G_{27}, G_{56}) induced by the neighbourhood of one of its nodes.
- $J(n,t)$, $K_{n\times 2}$ $(n \geq 2)$, $\frac{1}{2}H(n,2)$ are ℓ_1-graphs, but G_{27}, G_{56} are not ℓ_1-graphs.

We consider in the next result an interesting property for an L-polytope P in a root lattice. Namely, a geometric feature of P is entirely determined by the metric structure of P: its 1-skeleton consists of the pairs of vertices at squared distance 2.

PROPOSITION 3.9 *[31] Let P be a generating L-polytope in a root lattice. Let $G(P)$ denote the graph with set of vertices $V(P)$ and with edges the pairs (u,v) for which $d^{(2)}(u,v)=2$, for $u,v \in V(P)$, and let $d_{G(P)}$ denote its path metric. Then, $d^{(2)}(u,v) = 2d_{G(P)}(u,v)$ holds for all $u,v \in V(P)$, i.e. the L-polytope space $(V(P), d^{(2)})$ coincides with the space $(V(P), 2d_{G(P)})$. Moreover, $G(P)$ coincides with the 1-skeleton $H(P)$ of P, i.e. two vertices u,v form an edge of P if and only if $d^{(2)}(u,v)=2$.*

PROOF. Let $u,v \in V(P)$ such that $d_{G(P)}(u,v) = 2$. Let (u,u_1,v) be a path in $G(P)$ from u to v, i.e. $(u-u_1)^2 = (u_1-v)^2 = 2$ and $(u-v)^2 > 2$. Observe that

$$(*) \qquad (u_1-u)^T(u_1-v) = 0.$$

Indeed, $(u_1-u)^T(u_1-v) \geq 0$ since any three vertices of P form a triangle with no obtuse angles. Using relation (14), we obtain that $(u_1-u)^T(u_1-v) = 0, 1$. Moreover, $(u-v)^2 = 4 - 2(u_1-u)^T(u_1-v) > 2$, implying that $(u_1-u)^T(u_1-v) = 0$ and, thus, $(u-v)^2 = 4$.

Consider now $u,v \in V(P)$ such that $d_{G(P)}(u,v) = k \geq 2$. Let $(u_0 = u, u_1, \ldots, u_k = v)$ be a shortest path from u to v in $G(P)$. Then, $u - v = \sum_{1 \leq i \leq k} r_i$, where $r_i = u_i - u_{i-1}$ is a root, i.e. $r_i^2 = 2$, for $1 \leq i \leq k$. So this path corresponds to the sequence of roots $(r_1, \ldots, r_k)$. Consider the subpath (u_{i-1}, u_i, u_{i+1}). Applying relation (*), we deduce that $r_i^T r_{i+1} = 0$ holds. So, any two consecutive roots are orthogonal.

Note that $w = u_{i-1} + u_{i+1} - u_i$ is also a vertex of P since $w \in L$ and w also lies on the sphere. Hence, $(u_0, u_1, \ldots, u_{i-1}, w, u_{i+1}, \ldots, u_k)$ is another shortest path from u to v; it corresponds to the sequence of roots $(r_1, \ldots, r_{i-1}, r_{i+1}, r_i, r_{i+2}, \ldots, r_k)$. By the above argument, $r_i^T r_{i+2} = (r_{i-1})^T r_{i+1} = 0$. After iteration, we obtain that any two roots r_i, r_j, $i \neq j$, are orthogonal. Therefore, $(u-v)^2 = \sum_{1 \leq i \leq k} r_i^2 = 2k = 2d_{G(P)}(u,v)$ holds. Moreover, $u - v$ is the diagonal of the k-cube spanned by $r_1, \ldots, r_k$, whose vertices all are vertices of P. Therefore, u, v do not form an edge of P.

It is easy to see that, conversely, any two vertices u,v of P with $(u-v)^2 = 2$ form an edge of P. □

We now see that L-polytopes in root lattices arise in a natural way from connected strongly even hypermetric spaces.

PROPOSITION 3.10 *Let (X,d) be a connected distance space with minimum distance $d_{\min} = 2$. The following assertions are equivalent.*

(i) (X,d) *is of negative type and* d *takes only even values.*
(ii) (X,d) *has a representation* $i \mapsto v_i \in \mathbb{R}^m$ *(*$m \geq 1$*) and* $L = \{\sum_{i\in X} b_i v_i : b \in \mathbb{Z}^X$ *and* $\sum_{i\in X} b_i = 0\}$ *is a root lattice.*

Proof. For $(i) \Longrightarrow (ii)$, observe that, since (X,d) is connected, then L is generated by the set $\{v_i - v_j : i,j \in X \text{ with } d(i,j) = d_{\min}\}$. Hence, L is generated by a set of vectors v with $v^2 = 2$, implying that L is a root lattice.
$(ii) \Longrightarrow (i)$ is obvious. □

Corollary 3.11 *Let* (X,d) *be a connected strongly even distance space. If* (X,d) *is hypermetric with associated* L*-polytope* P_d *generating the lattice* L_d*, then* L_d *is a root lattice.*

As an application, we have the following two Theorems 3.12 and 3.13 which give a characterization of the connected strongly even distance spaces which are hypermetric, or ℓ_1-embeddable. The application to graphs is formulated in Section 6.1.

Theorem 3.12 *[65] Let* (X,d) *be a connected strongly even distance space. The following assertions are equivalent.*
(i) (X,d) *is hypermetric*
(ii) $(X, \frac{1}{2}d)$ *is an isometric subspace of a direct product of half-cube graphs* $\frac{1}{2}H(n,2)$ *(*$n \geq 7$*), cocktail-party graphs* $K_{n\times 2}$ *(*$n \geq 7$*), and copies of the Gosset graph* G_{56}*.*

Proof. $(i) \Longrightarrow (ii)$ From Corollary 3.11, the L-polytope P_d associated with (X,d) is generating in a root lattice. Therefore, from Proposition 3.9, The L-polytope space $(V(P_d), \frac{d^{(2)}}{2})$ coincides with the graphic space $(V(P_d), d_{G(P_d)})$ which, using Figure 1, is a direct product of Johnson graphs, cocktail-party graphs, half-cube graphs, copies of G_{27} and G_{56}. The result now follows using the remarks formulated after Figure 1.
The converse implication is obvious. □

Theorem 3.13 *([31]) Let* (X,d) *be a connected strongly even distance space. The following assertions are equivalent.*
(i) (X,d) *is an* ℓ_1*-space*
(ii) $(X, \frac{1}{2}d)$ *is an isometric subspace of a product of half-cube graphs and cocktail-party graphs.*

The proof given in [31] is identical to that of Theorem 3.12, using Proposition 3.7 and the fact that the graphs G_{27} and G_{56} are not ℓ_1-graphs.
In the (main) subcase of graphic spaces, another proof was given earlier in [61]; it is elementary (it does not use L-polytopes), but longer.

3.4 On the radius of L-polytopes

We now present several results which give, in some cases, a more precise information on the radius of L-polytopes.

The first result is a partial converse to the implication $(iv) \Longrightarrow (v)$ from Proposition 2.5; it gives explicitely the value of the radius of the spherical representation of a distance space (X, d) of negative type when $\sum_{i \in X} d(i,j)$ is a constant.

Proposition 3.14 *Assume that (X, d) is of negative type and that the sum $\sum_{i \in X} d(i,j)$ does not depend on j. Then, (X, d) has a spherical representation, on a sphere whose center is the center of mass of the representation and whose radius r is given by the following relation*

$$r^2 = \frac{1}{2|X|} \sum_{j \in X} d(i,j) \tag{18}$$

Proof. By Proposition 2.3, (X, d) has a representation $i \in X \mapsto v_i \in \mathbb{R}^n$ and, up to translation, we can suppose that $\sum_{i \in X} v_i = 0$. Then, using relation (3), we obtain that

$$\begin{aligned} 0 = 2v_j^T(\textstyle\sum_{i \in X} v_i) &= \textstyle\sum_{i \in X}(v_i^2 + v_j^2 - d(i,j)) \\ &= \textstyle\sum_{i \in X} v_i^2 + |X| v_j^2 - \sum_{i \in X} d(i,j) \end{aligned}$$

which implies that v_i^2 is a constant r^2 determined by $r^2 = \frac{1}{2|X|} \sum_{i \in X} d(i,j)$. □

Note that (18) can be reformulated as $r^2 = \frac{\sum_{i,j \in X} d_{ij}}{2|X|^2}$, i.e. $\sqrt{2}r$ is the quadratic mean of the $|X|^2$ values $\sqrt{d_{ij}}$ for $i, j \in X$.

An example of distance space with constant sum $\sum_{i \in X} d(i,j)$ is $(V(G), d_G)$, where d_G is the path metric of a distance regular graph G or of a regular graph G of diameter 2.

The next result is a specification of Proposition 3.14 to hypermetric spaces.

Proposition 3.15 *[31] Let (X, d) be a hypermetric space, let P_d be its associated L-polytope and let r denote the radius of its circumscribed sphere S_d. If $\sum_{i \in X} d(i,j)$ does not depend on j, then the radius r is given by relation (18), namely $r^2 = \frac{1}{2|X|} \sum_{i \in X} d(i,j)$.*

Proof. From Proposition 3.14, we can suppose that X lies on a sphere S with center the center of mass of X and with radius r given by (18). On the other hand, S_d is a minimal dimension sphere containing X. Hence, $S_d \subseteq S$ holds. The affine space spanned by S_d contains X and, thus, its center of mass, i.e. the center of S. Therefore, S and S_d have the same radius. □

The following result is a partial converse to the implication $(iii) \Longrightarrow (iv)$ from Proposition 2.5.

PROPOSITION 3.16 *[31] Let (X, d) be a connected strongly even distance space. Suppose that (X, d) has a representation on a sphere with radius r such that $r^2 < 2$. Then, (X, d) is hypermetric.*

PROOF. Let $(v_i, i \in X)$ be a representation of (X, d) on a sphere S. Up to translation, we can suppose that $v_i = 0$ for some index $i \in X$. From Proposition 3.10, $L(X, d) = \mathbb{Z}(v_i, i \in X)$ is a root lattice. We show that the sphere S is empty in $L(X, d)$ which, by Proposition 3.2, implies that (X, d) is hypermetric. Let H be the affine space spanned by $\{v_i : i \in X\}$. We can suppose that S lies in H (else replace S by $S \cap H$). Let ℓ be the line in $\mathbb{R}^{n+1}$ orthogonal to H going through the center of S, and let q be a point on ℓ such that $(q - v_i)^2 = 2$ for all $i \in X$. Note that $(q - v)^2 < 2$ for each point v lying in the interior of the ball delimited by S. Note also that $(q - v_i)^T(q - v_j) \in \{0, -1, 1\}$ for all $i \neq j \in X$. Indeed, $(v_i - v_j)^2$ is even since (X, d) is strongly even and $(v_i - v_j)^2 = 4 - 2(q - v_i)^T(q - v_j) \leq 4r^2 < 8$, implying that $(v_i - v_j)^2 \in \{2, 4, 6\}$. Therefore, $L' = \mathbb{Z}(q - v_i : i \in X)$ is a root lattice and, in particular, $a^2 \geq 2$ for each $a \in L'$, $a \neq 0$. Suppose now that some point $v \in L(X, d)$ lies in the interior of the ball delimited by S. Then, $(v - q)^2 < 2$, yielding a contradiction with the fact that $v - q \in L'$. □

We present now some results on spherical t-extensions of hypermetric spaces. Recall from relation (7) that $(X' = X \cup \{i_0\}, d')$ is the spherical t-extension of (X, d) if $d'(i, j) = d(i, j)$ for $i, j \in X$ and $d'(i, i_0) = t$ for $i \in X$. Denote d' by $\mathrm{sph}_t(d)$ and the iterated spherical t-extensions of d by $\mathrm{sph}_t^m(d)$.

LEMMA 3.17 *[48] Let (X, d) be a distance space. Then, $sph_t(d) \in NEG_{n+1}$ if and only if (X, d) has a spherical representation with radius r and $r^2 \leq t$. Moreover,*
(i) If $r^2 < t$, then $sph_t(d)$ has a spherical representation with radius $R = \frac{t}{2\sqrt{t-r^2}}$.
(ii) If $t = r^2$, then $sph_t(d)$ has no spherical representation.

PROOF. If $\mathrm{sph}_t(d) \in \mathrm{NEG}_{n+1}$, then $\mathrm{sph}_t(d)$ has a representation $i \in X' \mapsto v_i$ with $(v_i - v_{i_0})^2 = t$; hence, the v_i's $(i \in X)$ lie on a sphere S_0 with center v_{i_0} and radius $\sqrt{t}$. Let H denote the affine space spanned by $(v_i, i \in X)$. Therefore, the v_i's $(i \in X)$ lie on the sphere $S_d = S_0 \cap H$ whose radius r is less or equal to $\sqrt{t}$.
Conversely, consider a representation $(v_i, i \in X)$ of (X, d) on a sphere S_d with radius r, $r^2 \leq t$. Choose a point v_{i_0} on the line orthogonal to the affine space H spanned by $(v_i, i \in X)$ and going through the center of S_d such that v_{i_0} is at squared distance $t - r^2$ from H. Then, $(v_i, i \in X')$ is a representation of $\mathrm{sph}_t(d)$, i.e. $\mathrm{sph}_t(d) \in \mathrm{NEG}_{n+1}$.
If $r^2 < t$, then let S denote the sphere of dimension one higher than that of S_d, which contains S_d and v_{i_0}. So, $S_d = S \cap H$ and the radius R of S is $R = \frac{t}{2\sqrt{t-r^2}}$.

On the other hand, if $r^2 = t$, then v_{i_0} is the center of $S_0 = S_d$ and the representation of $\mathrm{sph}_t(d)$ is not spherical. $\square$

Next, we study the values of t for which the spherical t-extension of a hypermetric space remains hypermetric.

PROPOSITION 3.18 *[48] Let (X, d) be a hypermetric space and let r denote the radius of the sphere S_d circumscribing P_d.*
(i) If $t \geq 2r^2$, then $sph_t(d)$ is hypermetric, its radius is $R = \frac{t}{2\sqrt{t-r^2}}$ with $t \geq 2R^2$ (and $R \geq r$ with equality if and only if $t = 2r^2$). Therefore, $sph_t^m(d)$ is hypermetric for any integer $m \geq 1$.
Moreover, the L-polytope P associated with $sph_t(d)$ is a pyramid with base P_d if $t > 2r^2$, or if $t = 2r^2$ and P_d is asymmetric, and P is a bipyramid with base P_d if P_d is centrally symmetric.
(ii) If $r^2 < t < 2r^2$ and if P_d is centrally symmetric, then $sph_t(d)$ is not hypermetric.

PROOF. We use the notation from the proof of Lemma 3.17.
(i) Let L_d be the lattice spanned by $V(P_d)$ and let L denote the lattice generated by L_d and v_{i_0}. So, L consists of layers which are translates of L_d, the distance between consecutive layers being $h = \sqrt{t-r^2}$. By assumption, $t \geq 2r^2$, implying that $h \geq R = \frac{t}{2\sqrt{t-r^2}}$. This shows that the sphere S is empty in L. Therefore, $\mathrm{sph}_t(d)$ is hypermetric and its associated L-polytope P has radius R. If $t > 2r^2$, then P is the pyramid with base P_d and apex v_{i_0}. If $t = 2r^2$, then one checks easily that the antipode $v_{i_0}^*$ of v_{i_0} on the sphere S belongs to L if and only if P_d is centrally symmetric. Therefore, if P_d is centrally symmetric, then P is the bipyramid with base P_d and apex v_{i_0} and, if P_d is asymmetric, then P is the pyramid with base P_d and apex v_{i_0}. Note that $t > 2R^2$ follows from $R = \frac{t}{2h}$ and $h > R$.
(ii) Let $v \in V(P_d)$ and let v^* be its antipode on the sphere S_d. Then, $w = v + v^* - v_{i_0}$ belongs to L and we show that w lies inside S, which implies that $\mathrm{sph}_t(d)$ is not hypermetric. We have that $(v-v^*)^2 = 4r^2$, $(v-v_{i_0})^2 = (v^*-v_{i_0})^2 = t$, $(v-c)^2 = (v^*-c)^2 = (v_{i_0}-c)^2 = R^2$, from which we deduce that $(w-c)^2 = R^2 + 2t - 4r^2 < R^2$. $\square$

We present some examples of applications (see [14], [47]).
Example 3. Consider the complete bipartite graph $K_{1,n}$. Its path metric is $\mathrm{sph}_1(d)$, where $d = 2d(K_n)$ takes value 2 on all pairs of $\{1, \ldots, n\}$. So, d is hypermetric with radius $r = \sqrt{\frac{n-1}{n}}$, since it can be represented by the $n-1$-simplex with side length 2. Therefore, by Lemma 3.17, $d(K_{n,1})$ has a spherical representation with radius $R = \frac{1}{2\sqrt{1-r^2}} = \sqrt{\frac{n}{4}}$. If $n = 2, 3$, then $\nabla K_{n,1}$ has a spherical representation; if $n = 4$, $\nabla K_{4,1}$ is of negative type but not spherical.

Example 4. Consider the graph $K_n - P_3$, for $n \geq 4$. Let d denote the distance on 3 points with two values equal to 2 and one equal to 1. Clearly, $d(K_4 - P_3) = \mathrm{sph}_1(d)$ and $K_{n+1} - P_3 = \nabla(K_n - P_3)$, yielding that $d(K_n - P_3) = \mathrm{sph}_1^{n-3}(d)$. One checks easily that d is hypermetric with radius r_3, $r_3^2 = \frac{4}{7}$, and with associated L-polytope α_2. Set $(r_{n+1})^2 = \frac{1}{4(1-r_n^2)}$ for $n \geq 3$. Then, $r_4^2 = \frac{7}{12}$, $r_5^2 = \frac{3}{5}$, $r_6^2 = \frac{5}{8}$, $r_7^2 = \frac{2}{3}$, $r_8^2 = \frac{3}{4}$, $r_9^2 = 1$. From Lemma 3.17, $K_n - P_3$ is spherical with radius r_n^2, for $n \leq 9$, but not for $n = 10$. From Proposition 3.16 applied to $2d(K_n - P_3)$, we obtain that $K_n - P_3$ is hypermetric for $n \leq 8$. It is known that the L-polytope associated with $2d(K_7 - P_3)$ is the Schläfli polytope 2_{21} and that the L-poytope associated with $2d(K_8 - P_3)$ is the Gosset polytope 3_{21} (see Section 5.2). From Proposition 3.18 (ii), $K_9 - P_3$ is not hypermetric since 3_{21} is centrally symmetric and $r_8^2 < t = 1 < 2r_8^2$. Note that $K_n - P_3$ for $n \leq 6$ is an ℓ_1-graph.

We conclude this section with an additional observation on the spherical t-extension operation. Let (X, d) be a distance space with $|X| = n$ and consider the spherical t-extension $\mathrm{sph}_t(d)$ of d. Proposition 3.18 (i) can be rephrased as follows.
(i) Suppose $d \in \mathrm{HYP}_n$. Then, $\mathrm{sph}_t^m(d) \in \mathrm{HYP}_{n+m}$ for all $m \geq 1$ if $t \geq \frac{1}{2}(diam(P_d))^2$, where $diam(P_d)$ denotes the diameter of the sphere circumscribing the L-polytope P_d associated with d; moreover, we have an "if and only if" statement if P_d is centrally symmetric.

Compare (i) above with the following two assertions (ii) and (iii) that deal, respectively, with the case when the spherical t-extension is a semimetric, or is isometrically ℓ_1-embeddable.

(ii) Suppose $d \in \mathrm{MET}_n$. Then, $\mathrm{sph}_t^m(d) \in \mathrm{MET}_{n+m}$ for all $m \geq 1$ if and only if $t \geq \frac{1}{2}\max(d_{ij} : i, j \in X)$.

(iii) Suppose $d \in \mathrm{CUT}_n$. Then, $\mathrm{sph}_t^m(d) \in \mathrm{CUT}_{n+m}$ if $t \geq \frac{1}{2}s(d)$, where $s(d)$ denotes the minimum **size** $\sum_S \lambda_S$ of a realization of d as $d = \sum_S \lambda_S \delta(S)$ with $\lambda_S \geq 0$.
Indeed, if $d = \sum_S \lambda_S \delta(S)$ with $\sum_S \lambda_S = s(d) \leq 2t$, then $\mathrm{sph}_t(d) = \frac{1}{2}(\sum_S \lambda_S \delta(S) + \sum_S \lambda_S \delta(S \cup \{i_0\}) + (2t - s(d))\delta(\{i_0\}))$ has a realization of size $t + \frac{s(d)}{2}$ (i_0 denoting the extension point).

Observe also that, in (i), the limit value when m goes to infinity of $(diam(P_{\mathrm{sph}_t^m(d)}))^2$ is equal to $2t$. Similarly, in (iii), the limit value when m goes to infinity of $s(\mathrm{sph}_t^m(d))$ is equal to $2t$.

Example 5. Let $d = d(K_n)$, i.e. $d_{ij} = 1$ for all distinct i, j; d is hypermetric with radius $\sqrt{\frac{n-1}{2n}}$ (see Example 3). Then, $\mathrm{sph}_t(d) \in \mathrm{NEG}_{n+1}$ if and only if $t \geq \frac{n-1}{2n}$ (by Lemma 3.17 (i)) and it is easy to see that $\mathrm{sph}_t(d) \in \mathrm{MET}_{n+1}$ (or CUT_{n+1}, or HYP_{n+1}) if and only if $t \geq \frac{1}{2}$.

Example 6. Let $d = d(K_{n\times 2})$. Since $K_{n\times 2}$ is the 1-skeleton of the cross-polytope β_n, d is hypermetric with radius $\frac{1}{\sqrt{2}}$ (see Figure 1 and Proposition 3.9), i.e. the squared diameter of the L-polytope P_d is equal to 2. The minimum size $s(d)$ of a realization of d in CUT_{2n} is also equal to 2. It is easy to see that, for each integer $m \geq 1$, $\mathrm{sph}_t^m(d) \in \mathrm{MET}_{n+m}$ (or CUT_{n+m}, or HYP_{n+m}) if and only if $t \geq 1$.

4 L-polytopes: rank and hypermetric faces

There is a natural notion of rank for hypermetric spaces. Namely, if (X, d) is a hypermetric space, then its rank $rk(X, d)$ is the dimension of the smallest face of the cone $\mathrm{HYP}(X)$ that contains d. The extremal cases when the rank is equal to 1, or the corank is equal to 1, correspond, respectively, to extreme rays and facets of the hypermetric cone.

Correspondingly, we define the rank $rk(P)$ of an L-polytope P as the rank of its L-polytope space $(V(P), d^{(2)})$. L-polytopes of rank 1 are called extreme; they are associated to hypermetrics lying on an extreme ray of the hypermetric cone.

Extreme L-polytopes have a highly rigid geometric structure; indeed, their only affine transforms which are still L-polytopes are their homothetic transforms (see Corollary 4.9). The first example of an extreme L-polytope is the segment α_1, associated with the cut semimetrics. Other examples are known, as the Schläfli polytope 2_{21}, the Gosset polytope 3_{21} constructed from the root lattice E_8 and some others constructed from the Leech lattice Λ_{24} and the Barnes-Wall lattice Λ_{16}; they are described in the next Section 5.

L-polytopes of corank 1, which correspond to facets of the hypermetric cone, are well understood; they are the repartitioning polytopes considered by Voronoi, described in Section 4.2. Based on this connection, an upper bound on the coefficients of hypermetric facets is given in Theorem 4.11.

We study several properties for the notion of rank of an L-polytope, in particular, in Section 4.1, including

- its invariance; namely, for each generating subset V of the set of vertices of P, the rank of the space $(V, d^{(2)})$ is equal to the rank of P (see Theorem 4.5),
- its additivity; namely, the rank of the direct product of two L-polytopes is equal to the sum of their ranks (see Proposition 4.6).

We present in Section 4.3 some bounds for the rank of an L-polytope in terms of its number of vertices (see Proposition 4.15). We also investigate in detail in Section 4.2 the links between faces of the hypermetric cone and their associated L-polytopes.

4.1 Rank of an L-polytope

Let (X, d) be a hypermetric space, i.e. $d \in \mathrm{HYP}(X)$. We define its **annullator** $Ann(X, d)$ by

$$Ann(X, d) = \{b \in \mathbb{Z}^X : b \neq e_i, i \in X, \sum_{i \in X} b_i = 1, \sum_{i,j \in X} b_i b_j d(i, j) = 0\}.$$

(This notion was already used in the proof of Theorem 3.8.) The system $\mathcal{S}(X, d)$ consists of the equations $\sum_{i,j \in X} b_i b_j x(i, j) = 0$ for $b \in Ann(X, d)$, i.e. $\mathcal{S}(X, d)$ consists of the hypermetric equalities satisfied by d. Let $F(X, d)$ (or $F(d)$) denote the smallest (by inclusion) face of the hypermetric cone $\mathrm{HYP}(X)$ that contains d. Hence,

$$F(X, d) = \mathrm{HYP}(X) \cap \bigcap_{b \in Ann(X,d)} H_b,$$

where H_b denotes the hyperplane in $\mathbb{R}^{\binom{|X|}{2}}$ defined by the equation $\sum_{i,j \in X} b_i b_j d(i, j) = 0$. The dimension of $F(X, d)$ is equal to the rank of the solution set to the system $\mathcal{S}(X, d)$.

Definition 4.1 *(i)The* **rank** *$rk(X, d)$ of a hypermetric space (X, d) is defined as the dimension of the smallest face $F(X, d)$ of HYP(X) that contains d. Its* **corank** *is defined as $\binom{|X|}{2} - rk(X, d)$.*
(ii) The **rank** *$rk(P)$ of an L-polytope P is defined as the rank of the L-polytope space $(V(P), d^{(2)})$, i.e. $rk(P) = rk(V(P), d^{(2)})$. An L-polytope of rank 1 is called* **extreme**.

Hence, $rk(X, d) = 1$ if d lies on an extreme ray of the hypermetric cone; $rk(X, d) = \binom{|X|}{2}$ if d lies in the interior of $\mathrm{HYP}(X)$, i.e. $F(X, d) = \mathrm{HYP}(X)$, and $rk(X, d) = \binom{|X|}{2} - 1$ if $F(X, d)$ is a facet of $\mathrm{HYP}(X)$.

In fact, the rank of a hypermetric space is an invariant of its associated L-polytope, namely, $rk(X, d) = rk(P_d)$ holds.

We first observe that a hypermatric space and any 0-lifting of it have the same rank. This means that we may consider only metrics rather than semimetrics.

Lemma 4.2 *Let (X', d') be a 0-lifting of the hypermetric space (X, d). Then, $rk(X', d') = rk(X, d)$.*

Proof. We take the notation of relation (8). We have to show that the solution sets of the systems $\mathcal{S}(X, d)$ and $\mathcal{S}(X', d')$ have the same rank. Since $\mathcal{S}(X, d)$ is a subsystem of $\mathcal{S}(X', d')$, it suffices to check that, in the system $\mathcal{S}(X', d')$, each additional variable $x(i, j_0)$, for $i \in X$, can be expressed in terms of the variables $x(i, j)$, for $i, j \in X$. Indeed, since $d(i_0, j_0) = 0$, the triangle equalities $x(i_0, i) - x(j_0, i) - x(i_0, j_0) = 0$ and

$x(j_0,i) - x(i_0,i) - x(i_0,j_0) = 0$ belong to $\mathcal{S}(X',d')$. This implies that the equality $x(j_0,i) = x(i_0,i)$, for $i \in X$, follows from $\mathcal{S}(X',d')$. □

An immediate consequence of Lemma 4.2 is that $rk(X,d) = rk(V_X,d)$ holds, if V_X is the set of vertices of P_d representing (X,d). In fact, $rk(V,d^{(2)}) = rk(P)$ holds for each generating subset V of $V(P)$, as shown in Theorem 4.5, implying that $rk(X,d) = rk(P_d)$.

Let $P \subseteq \mathbb{R}^k$ be a k-dimensional L-polytope with set of vertices $V(P)$ and let $V \subseteq V(P)$ be a generating subset of $V(P)$.

For $w \in V(P)$, every $a \in \mathbb{Z}^V$ such that $w = \sum_{v\in V} a_v v$ and $\sum_{v\in V} a_v = 1$ is called an **affine realization** of w **in the set** V.

From Proposition 3.4, we have that, for $b \in \mathbb{Z}^V$ with $\sum_{v\in V} b_v = 1$,

$$b \in Ann(V,d^{(2)}) \text{ if and only if } \sum_{v\in V} b_v v \in V(P).$$

In other words, there is a one-to-one correspondence between
- the equations of $\mathcal{S}(V,d^{(2)})$ and
- the affine realizations of the vertices of P in the set V.

In particular, if B is a basic set in $V(P)$, then each vertex has a unique affine realization in the set B and, therefore, $\mathcal{S}(B,d^{(2)})$ is a system of $|V(P)-B| = |V(P)|-k-1$ equations in $\binom{k+1}{2}$ variables. We deduce that

$$\binom{k+2}{2} - |V(P)| \leq rk(B,d^{(2)}) \leq \binom{k+1}{2}. \tag{19}$$

LEMMA 4.3 *Let V be a generating subset of $V(P)$ and let $c \in \mathbb{Z}^V$ such that $\sum_{v\in V} c_v = 0$ and $\sum_{v\in V} c_v v = 0$. then, the folllowing equations*

$$\sum_{v\in V} c_v x(u,v) = 0 \quad \text{for } u \in V \tag{20}$$

$$\sum_{u,v\in V} c_u c_v x(u,v) = 0 \tag{21}$$

are implied by the system $\mathcal{S}(V,d^{(2)})$.

PROOF. Take $u \in V$. Set $c'_u = c_u + 1$, $c'_v = c_v$ for $v \in V$, $v \neq u$. Then, c' is an affine realization of u in V, implying that the equation $\sum_{v,w\in V} c'_u c'_w x(v,w) = 0$ belongs to $\mathcal{S}(V,d^{(2)})$. It can be rewritten as

$$(*) \qquad \sum_{v,w\in V} c_v c_w x(v,w) + 2\sum_{v\in V} c_v x(u,v) = 0.$$

By multiplying (*) by c_u and summing over $u \in V$, we obtain the equation from (21). Then, (20) follows. $\square$

Let $\mathcal{S}'(V, d^{(2)})$ denote the system consisting of the equations (20) and (21) together with the hypermetric equations $\sum_{u,v \in V} a_u^w a_v^w x(u,v) = 0$ where, for $w \in V(P)$, a^w is a given affine realization of w in the set V.

Lemma 4.4 *The systems $\mathcal{S}(V, d^{(2)})$ and $\mathcal{S}'(V, d^{(2)})$ have the same solutions.*

Proof. It remains only to show that each equation of $\mathcal{S}(V, d^{(2)})$ follows from the system $\mathcal{S}'(V, d^{(2)})$. For $w \in V(P)$, let b be another affine realization of w in V. Then, we can apply (21) to $a^w - b$, yielding that $\sum_{u,v \in V}(a_u^w - b_u)(a_v^w - b_v)x(u,v) = 0$, i.e.

$$(*) \qquad \sum_{u,v \in V} a_u^w a_v^w x(u,v) - 2 \sum_{u,v \in V} a_u^w b_v x(u,v) + \sum_{u,v \in V} b_u b_v x(u,v) = 0.$$

The first term is equal to 0 since it is corresponds to an equation of $\mathcal{S}'(V, d^{(2)})$. From (20), we have that, for $u \in V$, $\sum_{v \in V} a_v^w x(u,v) = \sum_{v \in V} b_v x(u,v)$ and, thus, the second term of (*) is equal to $-2\sum_{u,v \in V} a_u^w a_v^w x(u,v) = 0$. Hence, the equation $\sum_{u,v \in V} b_u b_v x(u,v) = 0$ follows from $\mathcal{S}'(V, d^{(2)})$. $\square$

Theorem 4.5 *[33] Let V be a generating subset of $V(P)$. Then, $rk(V, d^{(2)}) = rk(V(P), d^{(2)})$ holds.*

Proof. We show that the solution sets to the systems $\mathcal{S}(V, d^{(2)})$ and $\mathcal{S}(V(P), d^{(2)})$ have the same rank. Since $\mathcal{S}(V, d^{(2)})$ is a subsystem of $\mathcal{S}(V(P), d^{(2)})$, it suffices to check that each variable $x(u,w)$, for $u \in V, w \in V(P) - V$, or $u, w \in V(P) - V$, can be expressed in terms of the variables $x(u,v)$, for $u, v \in V$.

Let $w, w' \in V(P) - V$ and let a, a' denote affine realizations of w, w' in V, respectively. We show that the following equations (22) and (23) are implied by $\mathcal{S}(V(P), d^{(2)})$.

$$x(w,u) = \sum_{v \in V} a_v x(u,v) \quad \text{for} \quad u \in V \tag{22}$$

$$x(w,w') = \sum_{u,v \in V} a_u a'_v x(u,v) \tag{23}$$

For this, set $b_w = -1$, $b_v = a_v$ for $v \in V$ and $b_v = 0$ for $v \in V(P) - (V \cup \{w\})$, $b'_{w'} = -1$, $b'_v = a'_v$ for $v \in V$ and $b'_v = 0$ for $v \in V(P) - (V \cup \{w'\})$. We can apply Lemma 4.3. From

(20), we obtain that, for $u \in V$, $\sum_{v\in V(P)} b_v x(u,v) = 0$, i.e.
$-x(u,w) + \sum_{v\in V} a_v x(u,v) = 0$, thus stating (22). We now apply (21) for b, b' and $b+b'$. We obtain that $\sum_{u,v\in V(P)} b_u b_v x(u,v) = 0$, $\sum_{u,v\in V(P)} b'_u b'_v x(u,v) = 0$ and $\sum_{u,v\in V(P)} (b_u + b'_u)(b_v + b'_v) x(u,v) = 0$, from which we deduce that $\sum_{u,v\in V(P)} b_u b'_v x(u,v) = 0$. Expressing b' in terms of a', we obtain that $-\sum_{u\in V(P)} b_u x(u,w) + \sum_{u\in V(P),\ v\in V} b_u a'_v x(u,v) = 0$, where the first term is 0 by (20). Then, expressing b in terms of a, we obtain that $-\sum_{v\in V} a'_v x(w,v) + \sum_{u,v\in V} a_u a'_v x(u,v) = 0$, where the first term is equal to $x(w,w')$ by (20). This concludes the proof. □

We conclude this section with an additivity property of the rank of an L-polytope.

Proposition 4.6 *[33] Let P_1 and P_2 be L-polytopes; then, their direct product $P_1 \times P_2$ is an L-polytope with rank $rk(P_1 \times P_2) = rk(P_1) + rk(P_2)$.*

Proof. Let V_i denote the set of vertices of P_i, $i = 1, 2$; so, $V = V_1 \times V_2$ is the set of vertices of $P_1 \times P_2$. Fix $b_1 \in V_1$ and $b_2 \in V_2$. Let $\mathcal{S}_1$ denote the subsystem of $\mathcal{S}(V, d^{(2)})$ consisting of the equations involving only the variables $x((u_1, b_2), (v_1, b_2))$ for $u_1, v_1 \in V_1$. Similarly, $\mathcal{S}_2$ is the subsystem of $\mathcal{S}(V, d^{(2)})$ involving only the variables $x((b_1, u_2), (b_1, v_2))$ for $u_2, v_2 \in V_2$. Clearly, the rank of the solution set to the system $\mathcal{S}_i$ is equal to $rk(P_i)$, for $i = 1, 2$. In order to conclude that $rk(P) = rk(P_1) + rk(P_2)$, it suffices to show that the variables $x((u_1, u_2), (v_1, v_2))$, for $(u_1, u_2), (v_1, v_2) \in V - ((V_1 \times \{b_2\}) \cup (\{b_1\} \times V_2))$, can be expressed in terms of the variables $x((b_1, u_2), (b_1, v_2))$ and $x((u_1, b_2), (v_1, b_2))$, for $u_1, v_1 \in V_1$, $u_2, v_2 \in V_2$. We show that the following relation (*) holds.

$$(*) \qquad x((u_1,u_2),(v_1,v_2)) = x((b_1,u_2),(b_1,v_2)) + x((u_1,b_2),(v_1,b_2))$$

for $u_1, v_1 \in V_1, u_2, v_2 \in V_2$.

From the identity $(u_1, u_2) = (v_1, v_2) + (b_1, u_2) + (u_1, b_2) - (v_1, b_2) - (b_1, v_2)$, we have the following equation of $\mathcal{S}(V(P), d^{(2)})$

$$(a) \qquad 0 = x((b_1,v_2),(v_1,b_2)) + x((v_1,v_2),(b_1,u_2)) + x((v_1,v_2),(u_1,b_2)) + x((b_1,u_2),(u_1,b_2)) - \sum_{\substack{(s_1,s_2)=(v_1,v_2),(b_1,u_2),(u_1,b_2)\\(t_1,t_2)=(b_1,v_2),(v_1,b_2)}} x((s_1,s_2),(t_1,t_2)).$$

Moreover, the affine dependency $(v_1, v_2) + (b_1, u_2) + (u_1, b_2) - (v_1, b_2) - (b_1, v_2) - (u_1, u_2) = 0$ yields the equation

$$(b) \qquad 0 = x((v_1,v_2),(b_1,u_2)) + x((v_1,v_2),(u_1,b_2)) + x((b_1,u_2),(u_1,b_2)) + x((b_1,v_2),(v_1,b_2)) + x((u_1,u_2),(b_1,v_2)) + x((u_1,u_2),(v_1,b_2)) - \sum_{\substack{(s_1,s_2)=(v_1,v_2),(b_1,u_2),(u_1,b_2)\\(t_1,t_2)=(u_1,u_2),(b_1,v_2),(v_1,b_2)}} x((s_1,s_2),(t_1,t_2)).$$

Substracting (a) from (b) yields

$$(c) \qquad 0 = x((u_1,u_2),(b_1,v_2)) + x((u_1,u_2),(v_1,b_2)) - x((u_1,u_2),(b_1,u_2)) \\ -x((u_1,u_2),(u_1,b_2)) - x((u_1,u_2),(v_1,v_2)).$$

The next two hypermetric equations follow, respectively, from the identities: $(v_1,u_2) = (u_1,u_2)+(v_1,b_2)-(u_1,b_2)$ and $(u_1,v_2) = (u_1,u_2)+(b_1,v_2)-(b_1,u_2)$.

$$(d) \qquad x((u_1,u_2),(u_1,b_2)) + x((u_1,b_2),(v_1,b_2)) - x((u_1,u_2),(v_1,b_2)) = 0$$

$$(e) \qquad x((u_1,u_2),(b_1,u_2)) + x((b_1,v_2),(b_1,u_2)) - x((u_1,u_2),(b_1,v_2)) = 0.$$

Using (c), (d) and (e), we deduce the relation (*). □

For instance, $rk(\gamma_k) = k$ since γ_k is the direct product $(\gamma_1)^k$ and $rk(\gamma_1) = 1$.

4.2 L-polytopes related to faces

We show that hypermetrics that lie in the interior of the same face of the hypermetric cone are associated with affinely equivalent L-polytopes. Therefore, one can speak of the L-polytope associated with a face of the hypermetric cone.

Let T be an affine bijection of $\mathbb{R}^k$. We set

$$d_T(u,v) = (T(u)-T(v))^2 \quad \text{for } u,v \in \mathbb{R}^k.$$

Proposition 4.7 *[33] Let $P \subseteq \mathbb{R}^k$ be an L-polytope and let V be a generating subset of $V(P)$. Let T be an affine bijection of $\mathbb{R}^k$. Let F denote the smallest face of the hypermetric cone HYP(V) that contains $(V, d^{(2)})$.*
(i) If $T(P)$ is an L-polytope, then d_T lies in the interior of F, i.e. $F(d_T) = F$.
(ii) If d lies in the interior of F, then the L-polytope P_d associated with d is affinely equivalent to P.

Proof. (i) Let r denote the radius of the L-polytope $T(P)$. We suppose that $T(P)$ is centered at the origin. We show that d_T satisfies the system $\mathcal{S}(V, d^{(2)})$. Let $a \in \mathbb{Z}^V$ be an affine realization of $w \in V(P)$. Then, $T(w) = \sum_{v\in V} a_v T(v)$ is a vertex of $T(P)$. Therefore, $\sum_{u,v\in V} a_u a_v d_T(u,v) = \sum_{u,v\in V} a_u a_v (T(u)-T(v))^2 = \sum_{u,v\in V} a_u a_v (2r^2 - 2T(u)^T T(v)) = 2r^2 - 2(\sum_{v\in V} a_v T(v))^2 = 0$.
(ii) Let P_d be the L-polytope associated with d and let $T : V \longrightarrow V(P_d)$ be a generating mapping such that $d(u,v) = (T(u)-T(v))^2$ for $u,v \in V$. The mapping T is one-to-one since $d(u,v) \neq 0$ for $u \neq v \in V$. We show that T can be extended to an affine bijective mapping of the space spanned by V, mapping $V(P)$ to $V(P_d)$.

First, we verify that T preserves the affine dependencies on V, i.e., for $c \in \mathbb{Z}^V$ with $\sum_{v\in V} c_v = 0$, $\sum_{v\in V} c_v v = 0$ holds if and only if $\sum_{v\in V} c_v T(v) = 0$ holds. Since the vectors

$v, v \in V$, lie on a sphere, we have that

$$(*) \quad \sum_{u,v \in V} c_u c_v d^{(2)}(u,v) = \sum_{u,v \in V} c_u c_v (u-v)^2 = -2\left(\sum_{v \in V} c_v v)^2\right).$$

For the same reason,

$$(**) \quad \sum_{u,v \in V} c_u c_v d(u,v) = \sum_{u,v \in V} (T(u)-T(v))^2 = -2\left(\sum_{v \in V} c_v T(v)\right)^2.$$

By assumption, $F(d) = F$, i.e. the systems $\mathcal{S}'(V,d)$ and $\mathcal{S}'(V,d^{(2)})$ have the same sets of solutions (using Lemma 4.4). This implies that the quantities in (*) and (**) are simultaneously equal to zero, i.e. $\sum_{v \in V} c_v v = 0$ if and only if $\sum_{v \in V} c_v T(v) = 0$.

We now check that, for $b \in \mathbb{Z}^V$ with $\sum_{v \in V} b_v = 1$, $\sum_{v \in V} b_v v$ is a vertex of P if and only if $\sum_{v \in V} b_v T(v)$ is a vertex of P_d. But, by Proposition 3.4, $\sum_{v \in V} b_v v \in V(P)$ if and only if $d^{(2)}$ satisfies the equation $\sum_{u,v \in V} b_u b_v x(u,v) = 0$ and $\sum_{v \in V} b_v T(v) \in V(P_d)$ if and only if d satisfies the same equation.

Therefore, we can extend T to the space spanned by V by setting $T(\sum_{v \in V} b_v v) = \sum_{v \in V} b_v T(v)$; T is affine bijective and maps P on P_d. □

COROLLARY 4.8 *Let (X,d) and (X,d') be two hypermetric spaces with associated L-polytopes P_d and $P_{d'}$. Let $F(d)$, $F(d')$ denote the smallest face of $HYP(X)$ that contains d, d', respectively. Then, $F(d) = F(d')$ if and only if P_d and $P_{d'}$ are affinely equivalent.*

COROLLARY 4.9 *Let P be an L-polytope in $\mathbb{R}^k$. Then, P is extreme if and only if the only affine bijective transformations T of $\mathbb{R}^k$ for which $T(P)$ is an L-polytope are the homotheties.*

PROOF. Suppose first that $rk(P) = 1$, i.e. $(V(P), d^{(2)})$ lies on an extreme ray of HYP($V(P)$). Assume that $T(P)$ is an L-polytope. By Proposition 4.7 (i), $d_T = \lambda^2 d^{(2)}$ for some scalar λ. hence, $(T(u)-T(v))^2 = \lambda^2 (u-v)^2$ for all $u, v \in V(P)$. It is not difficult to see that, up to translation, $\lambda^{-1}T$ is an orthogonal transformation.
(ii) Let $d \in$ HYP($V(P)$) with $F(d) = F(d^{(2)})$. By Proposition 4.7 (ii), the L-polytope P_d associated to d is of the form λP, where $\lambda > 0$, implying that $d = \lambda^2 d^{(2)}$. This shows that $(V(P), d^{(2)})$ lies on an extreme ray of HYP($V(P)$), i.e. $rk(P) = 1$. □

We now describe the L-polytopes which are associated with facets of the hypermetric cone.

Let Sim_1, Sim_2 be two simplices lying in affine spaces that intersect in one point which belongs to Sim_1 and Sim_2. Then, their convex hull $P = \text{conv}(Sim_1 \cup Sim_2)$ is called a **repartitioning polytope**. This polytope was studied by Voronoi ([66]). There is only one affine dependency among the vertices of Sim_1 and Sim_2:

$$\sum_{v \in V_1} b_v v = \sum_{v \in V_2} b_v v,$$

where $\sum_{v \in V_1} b_v = \sum_{v \in V_2} b_v = 1$, $b_v \geq 0$ for $v \in V_1 \cup V_2$ and V_i denotes the set of vertices of Sim_i, $i = 1, 2$. Set $V_0 = \{v \in V_1 \cup V_2 : b_v = 0\}$. Then, $P_1 = \text{conv}(V_1 \cup V_2 - V_0)$ is also a repartitioning polytope, with the same affine dependency between its vertices as P and $P = \Pi_{v \in V_0} Pyr_v(P_1)$. We denote the repartitioning polytope P by $P^m_{p,q}$, where $m = |V_0|$, $p + 1 = |V_1 - V_0|$ and $q + 1 = |V_2 - V_0|$. Hence, $P^m_{p,q}$ has $m + p + q + 2$ vertices (if $p, q \geq 1$) and its dimension is $m + p + q$. Note that $P^m_{p,q}$ does not denote a concrete polytope, but a class of affinely equivalent repartitioning polytopes.

We now show that the L-polytope associated with a facet is a repartitioning polytope. Let $d \in \text{HYP}(X)$ and suppose that d lies in the interior of the facet defined by the equation

$$\sum_{i,j \in X} b_i b_j x(i,j) = 0, \tag{24}$$

where $b \in \mathbb{Z}^X$ and $\sum_{i \in X} b_i = 1$. Hence, (24) is the only hypermetric equality satisfied by d. In particular, $d(i,j) > 0$ for distinct i, j; else, d would satisfy the $2(|X| - 2)$ triangle equalities $d(i,k) - d(i,j) - d(j,k) = 0$ and $d(j,k) - d(i,j) - d(i,k) = 0$ for $k \in X - \{i,j\}$.

PROPOSITION 4.10 *[33] Let P_d be the L-polytope associated with d lying in the interior of the facet defined by (24). Then, P_d is basic and P_d is a repartitioning polytope $P^m_{p,q}$ where $m = |\{i : b_i = 0\}|$, $p + 1 = |\{i : b_i > 0\}|$ and $q = |\{i : b_i < 0\}|$.*

PROOF. Let $(v_i, i \in X)$ denote the representation of d on $V(P_d)$. From Proposition 3.4, the equality (24) is equivalent to the point

$$v_0 = \sum_{i \in X} b_i v_i \tag{25}$$

being a vertex of P_d. From Proposition 3.4 and the fact that (24) is the only hypermetric equality satisfied by d, we deduce that $v_0 \notin \{v_i : i \in X\}$, $V(P_d) = \{v_i : i \in X\} \cup \{v_0\}$ and the set $\{v_i : i \in X\}$ is affinely independent. Hence, P_d has $|X| + 1$ vertices and $\sum_{v \in V(P_d)} b_v v = 0$ is the only affine dependency between the vertices of P_d, after setting

$b_v = b_i$ if $v = v_i$ for $i \in X$ and $b_v = -1$ if $v = v_0$. Set $V_0 = \{v \in V(P_d) : b_v = 0\}$, $V_+ = \{v \in V(P_d) : b_v > 0\}$, $V_- = \{v \in V(P_d) : b_v < 0\}$, $m = |V_0|$, $p + 1 = |V_+|$ and $q + 1 = |V_-|$. Then, $P_1 = \text{conv}(V_+ \cup V_-)$ is a repartitioning polytope $P^0_{p,q}$ and $P_d = \Pi_{v \in V_0} Pyr_v(P_1)$ is a repartitioning polytope $P^m_{p,q}$. □

As we see in the next Example 7, there exist distinct hypermetric facets for with the b_i's have the same numbers of positive and negative components; hence, they correspond to repartitioning polytopes with the same parameters p and q. For this reason, we denote the repartitioning polytope associated with the hypermetric facet (24) by $P^m_{p,q}(b)$.

Note that the matrix Y_γ characterizing the type of the repartitioning polytope $P^m_{p,q}(b)$ is of the form $\left[\frac{I_n}{b_1 \ldots b_n}\right]$ (recall Fact 2.12).

Example 7. Let (24) be a triangle equality, i.e. $b_1 = b_2 = 1, b_3 = -1$ and $b_i = 0$ otherwise. Then (25) reads $v_0 = v_1 + v_2 - v_3$ and $V_+ = \{v_1, v_2\}$, $V_- = \{v_3, v_0\}$. Therefore, the L-polytope associated with a triangle facet is $P^0_{1,1}$ or, more precisely, $P^0_{1,1}(1, 1, -1)$, a rectangle whose diagonals are the segments $[v_1, v_2]$ and $[v_0, v_3]$.
Let (24) be a pentagonal facet, i.e. $b_1 = b_2 = b_3 = 1$, $b_4 = b_5 = -1$ and $b_i = 0$ otherwise. Then, (25) reads $v_0 = v_1 + v_2 + v_3 - v_4 - v_5$. Therefore, the L-polytope associated with the pentagonal facet is $P^0_{2,2}$ or, more precisely, $P^0_{2,2}(1, 1, 1, -1, -1)$, the convex hull of two intersecting triangles.
We give two examples of distinct hypermetric facets for which the associated repartitioning polytopes have the same parameters p, q. Set $b_1 = (2, 2, 2, 1, 1, 1, -2, -2, -2, -1, -1)$ and $b_2 = (1, 1, 1, 1, 1, 1, -1, -1, -1, -1, -1)$. Then, (24) defines a facet for both b_1 and b_2; both are associated with a repartitioning polytope with parameters $p = q = 5$ (with, of course, distinct affine dependency (25) between their vertices).

We can now derive an upper bound for the largest coefficients of hypermetric facets. Set

$$b^n_{\max} = \max_{1 \le i \le n}(|b_i| : \; b \in \mathbb{Z}^n, \textstyle\sum_{1 \le i \le n} b_i = 1, \sum_{1 \le i < j \le n} b_i b_j x(i, j) \le 0 \text{ defines a facet of } \text{HYP}_n).$$

THEOREM 4.11 *[13] For $n \ge 4$, $b^n_{\max} < \frac{2^{n-2}\,(n-1)!}{n+1}$.*

PROOF. Let P denote the L-polytope associated with a hypermetric facet $\sum_{1 \le i < j \le n} b_i b_j x(i, j) = 0$. Let L denote the lattice generated by $V(P)$. From Proposition 4.10, P is a repartitioning polytope of dimension $n - 1$, with $n + 1$ vertices $v_0, v_1, \ldots, v_n$ where $v_0 = \sum_{1 \le i \le n} b_i v_i$ is the unique affine dependency among them.
We consider the $(n+1) \times n$ matrix M whose rows are the vectors $(1, v_i)$ for $i = 0, 1, \ldots, n$. Let M_i denote the $n \times n$ matrix obtained from M by deleting its i-th row. We have that

$|det(M_0)| = det(L)$ since $(v_1, \ldots, v_n)$ is an affine base of L, and $|det(M_i)| = |b_i| det(L)$ for $i = 1, \ldots, n$. Let Sim_i denote the $(n-1)$-simplex whose vertices are $(1, v_0)$ and $(1, v_j)$ for $1 \le j \le n$, $j \neq i$, and let Q_i denote the parallepiped spanned by these vectors. Then, $vol(Sim_i) = \frac{vol(Q_i)}{(n-1)!}$. But, $vol(Q_i) = |det(M_i)|$ and $vol(Sim_i) \le vol(P)$, since Sim_i is contained in an affine translate of P. Therefore, $\frac{|b_i| det(L)}{(n-1)!} \le vol(P)$ with $vol(P) \le \frac{2^{n-1} det(L)}{2(n+1)}$ by Proposition 2.9. This implies that $|b_i| \le \frac{2^{n-2}(n-1)!}{n+1}$. □

As a consequence of Theorem 4.11, we obtain that the hypermetric cone HYP_n has at most $2^{(n-1)^2} \left(\frac{(n-1)!}{n+1}\right)^{n-1}$ facets ([13]). This shows again that the hypermetric cone is polyhedral. Actually, the proof of Theorem 4.11 is a refinement of that of Theorem 3.8. It takes advantage of the fact that L-polytopes associated with facets, namely repartitioning polytopes, have a much simpler structure than the L-polytopes of arbitrary faces.

REMARK 4.12 *[13] As a consequence of Theorem 4.11, we obtain that testing whether a given distance d is hypermetric is in co-NP. It is not known whether testing hypermetricity is NP-hard. But the following complexity results are known.*
(i) Given an integral distance d and an integer m, does d satisfy all $(2m+1)$-gonal hypermetric inequalities ? This problem is in co-NP.
(ii) Given an integral distance d. Is d hypermetric ? If not, give the smallest k such that d violates a $(2k+1)$-gonal inequality. This problem is NP-hard.

We conclude this section with an observation on L-polytopes with small corank. We recall that we do not know any example of a non basic L-polytope. We conjecture that every L-polytope is basic. This is indeed the case for simplices and repartitioning polytopes, i.e. for L-polytopes associated with hypermetrics with corank 0 and 1. We extend this fact to the case of hypermetrics with corank 2 and 3.

PROPOSITION 4.13 *Let P be a k-dimensional L-polytope and let V be a generating subset of $V(P)$. If the hypermetric space $(V, d^{(2)})$ has corank $\binom{|V|}{2} - rk(V, d^{(2)}) \le 3$, then P is basic.*

PROOF. We show that V is affinely independent, which implies that P is basic. Suppose, for contradiction, that $\sum_{v \in C} b_v v = 0$ is an affine dependency with $C \subseteq V$ and $b_v \neq 0$ for $v \in C$. By Lemma 4.3, the equations $\sum_{v \in C} b_v x(u, v) = 0$, for $u \in V$, follow from the system $\mathcal{S}(V, d^{(2)})$. One can check that the matrix of the subsystem $\sum_{v \in C} b_v x(u, v) = 0$, for $u \in C$, has full rank $|C|$. Since the corank of $(V, d^{(2)})$ is equal to the rank of the matrix of the system $\mathcal{S}(V, d^{(2)})$, we deduce that $corank(V, d^{(2)}) \ge |C|$, implying that $|C| \le 3$. Hence, $C = \{v_1, v_2, v_3\}$ and, for instance, v_3 belongs to the segment $[v_1, v_2]$. So we have a triangle with an obtuse angle, yielding a contradiction. □

We summarize in Figure 2 below some of the main facts we know about the connections between faces of the hypermetric cone and their associated L-polytopes.

hypermetric d		associated L-polytope P_d
d is a cut semimetric	$\Longleftrightarrow$	$P = \alpha_1$
$F(d) = \mathrm{HYP}_{n+1}$	$\Longleftrightarrow$	$P_d = \alpha_n$
$d \in \mathrm{CUT}_{n+1}$	$\Longleftrightarrow$	$V(P)$ is contained in the set of vertices of a parallepiped
$F(d)$ is a facet	$\Longleftrightarrow$	P_d is a repartitioning polytope
$F(d)$ is an extreme ray	$\Longleftrightarrow$	P_d is extreme
$F(d) = F(d')$	$\Longleftrightarrow$	$P_d, P_{d'}$ are affinely equivalent

Figure 2

For the first two equivalences, see Examples 1 and 2 and, for the last four equivalences, see, respectively, Propositions 3.7, 4.10, Theorem 4.5 and Corollary 4.8.

4.3 Bounds on the rank of basic L-polytopes

In this section, we present some bounds for the rank of a basic L-polytope. Recall that an L-polytope P is basic if its set of vertices $V(P)$ contains a base of the lattice generated by $V(P)$.

LEMMA 4.14 *Let P be a basic k-dimensional L-polytope. Then, the following relations hold.*

$$rk(P) \geq \binom{k+2}{2} - |V(P)| \tag{26}$$

$$rk(P) \leq \binom{k+1}{2} \tag{27}$$

Proof. It follows immediately from relation (19) and Theorem 4.5. □

For centrally symmetric L-polytopes, we can improve the bound (26).

Proposition 4.15 *[33] Let P be a centrally symmetric k-dimensional L-polytope. Then,*

$$rk(P) \geq \binom{k+1}{2} - \frac{|V(P)|}{2} + 1. \tag{28}$$

Proof. Let B be a basic set in $V(P)$. For each $w \in V(P)$, a^w denotes the affine realization of w in B and $h(w)$ denotes the corresponding hypermetric equality of the system $\mathcal{S}(B, d^{(2)})$, i.e. we set $h(w) := \sum_{u,v \in B} a_u^w a_v^w x(u,v)$. Let $v \in B$. Since $w^* = v + v^* - w$, the affine realization a^{w^*} of w^* in B is given by $a^{w^*} = e_v + a^{v^*} - a^w$, where e_v is the v-th unit vector in $\mathbb{R}^B$. Hence, we have that

$$\begin{aligned} h(w^*) &= h(v^*) + h(w) + 2\textstyle\sum_{u' \in B} a_{u'}^{v^*} x(v,u') \\ &\quad -2\textstyle\sum_{u' \in B} a_{u'}^{w} x(v,u') - 2\sum_{u,u' \in B} a_{u'}^{v^*} a_u^w x(u,u'), \end{aligned}$$

i.e.

$$(a) \qquad h(w^*) = h(w) + \sum_{u \in B} a_u^w \left(h(v^*) - 2x(v,u) + 2 \sum_{u' \in B} a_{u'}^{v^*} (x(v,u') - x(u,u')) \right).$$

If $w \in B$, then $h(w)$ is zero and, thus, (a) implies that

$$(b) \qquad h(w^*) = h(v^*) - 2x(v,w) + 2 \sum_{u' \in B} a_{u'}^{v^*} (x(v,u') - x(w,u')).$$

Now, we deduce from (a) and (b) that, for each $w \in V(P)$,

$$(c) \qquad h(w^*) = h(w) + \sum_{u \in B} a_u^w h(u^*).$$

Using (c) for $w = v^*$, we deduce that

$$(d) \qquad 0 = h(v^*) + \sum_{u \in B} a_u^{v^*} h(u^*).$$

We now show that the system $\mathcal{S}(B, d^{(2)})$ can be reduced to a system of $\frac{|V(P)|}{2} - 1$ equations, which implies that the rank of its solution set is greater or equal to $\binom{k+1}{2} -$

$\frac{|V(P)|}{2}+1$. Clearly, the base B contains at most one pair of antipodal points. For a set A, we set $A^* = \{a^* : a \in A\}$.
Suppose first that B contains no pair of antipodal points. Then, $V(P) = B \cup B^* \cup A \cup A^*$, for some $A \subseteq V(P) - B$. By (c), each equation $h(a^*) = 0$, for $a \in A$, follows from the equations $h(a) = 0$, for $a \in A \cup B^*$. From (d), one of the equations $h(b^*) = 0$, for $b \in B$, follows from the others. Therefore, the system $\mathcal{S}(B, d^{(2)})$ reduces to $|A| + |B^*| - 1 = \frac{|V(P)|}{2} - 1$ equations.
Suppose now that B contains one antipodal pair, i.e. $B = B' \cup \{v, v^*\}$ with $|B'| = k - 1$. Then, $V(P) = B \cup (B')^* \cup A \cup A^*$ for some $A \subseteq V(P) - B$. Hence, $\mathcal{S}(B)$ reduces again to $|A| + |(B')^*| = \frac{|V(P)}{2} - 1$ equations. □

For example, the k-dimensional simplex α_k has $k+1$ vertices; relations (26) and (27) hold at equality. It is easy to check that the rank of the k-dimensional cross-polytope β_k is $rk(\beta_k) = \binom{k+1}{2} - k + 1$. Hence, β_k realizes equality in the bound (28).

The following Lemma 4.16 may be useful for computing the rank of L-polytopes.

LEMMA 4.16 *[33] Let P be a basic k-dimensional centrally symmetric L-polytope and let $B = \{v_0, v_1, \ldots, v_k\}$ be a basic set in $V(P)$. Let H denote the affine space spanned by $B_1 = \{v_1, \ldots, v_n\}$ and set $P_1 = P \cap H$. If P_1 is an asymmetric L-polytope and if there exists $w \in V(P) - H$ such that $w \notin \{v_1^*, \ldots, v_k^*\}$ and $w - v_0 \notin H$, then $rk(P_1) = rk(P)$ holds.*

PROOF. The set B_1 is basic in $V(P_1) = V(P) \cap H$. Hence, $rk(P_1)$ is equal to the rank of the solution set to the system $\mathcal{S}(B_1, d^{(2)})$. In order to show that $rk(P) = rk(P_1)$, it suffices to check that each variable $x(v_0, v_i)$, for $1 \le i \le k$, can be expressed in terms of the variables $x(v_i, v_j)$, for $1 \le i, j \le k$, in the system $\mathcal{S}(B, d^{(2)})$. Let $a, b \in \mathbb{Z}^{k+1}$ denote the affine realizations of w, v_0^* in B. We have $a_0 \ne 0, 1$ since $w \notin H$ and $w - v_0 \notin H$; also, $b_0 \ne -1$, else the center $\frac{v_0 + v_0^*}{2}$ of P would lie in H contradicting the fact that P_1 is asymmetric. Using relation (b) from the proof of Proposition 4.15 (applied to $v = v_0$ and $w = v_i$), we deduce that
$h(v_i^*) = h(v_0^*) - 2x(v_0, v_i) + 2\sum_{0 \le j \le k} b_j(x(v_0, v_j) - x(v_i, v_j))$.
Set $h_i = -2\sum_{1 \le j \le k} b_j x(v_i, v_j)$ for $1 \le i \le k$. Then,
$h(v_i^*) = h(v_0^*) - 2x(v_0, v_i)(b_0 + 1) + h_i + 2\sum_{0 \le j \le k} b_j x(v_0, v_j)$.
Substracting the above relations with indices i and 1, we obtain that the equation

$$(*) \qquad x(v_0, v_i) = x(v_0, v_1) + \frac{h_i - h_1}{2(b_0 + 1)}$$

follows from $\mathcal{S}(B, d^{(2)})$. Consider now the equation $h(w) = 0$, i.e.
$0 = \sum_{1 \le i < j \le k} b_i b_j x(v_i, v_j) + \sum_{1 \le i \le k} b_i b_0 x(v_0, v_i)$. Using (*), it can be rewritten as

$0 = \sum_{1\le i<j\le k} b_ib_jx(v_i,v_j) + b_0(1-b_0)x(v_0,v_1) + \frac{b_0}{2(b_0+1)}\sum_{1\le i\le k} b_i(h_i-1)$. Therefore, $x(v_0,v_1)$ and, thus, each $x(v_i,v_0)$, can be expressed in terms of $x(v_i,v_j)$, $1\le i<j\le k$. □

5 Extreme L-polytopes

In this section, we consider extreme L-polytopes, i.e. L-polytopes with rank 1. If P is an L-polytope, then P is extreme if and only if the only affine bijective mappings T for which $T(P)$ is still an L-polytope are the homotheties (see Corollary 4.9). Extreme L-polytopes are of particular interest since they correspond to the extreme rays of the hypermetric cone.

More precisely, if $d \in \mathrm{HYP}_n$ lies on an extreme ray of HYP_n, then its associated L-polytope P_d is an extreme L-polytope of dimension $k\le n-1$. Conversely, if P is a k-dimensional extreme L-polytope, then, for each generating subset V of its set of vertices, the hypermetric space $(V,d^{(2)})$ lies on an extreme ray of the hypermetric cone $\mathrm{HYP}(V)$. Moreover, by taking 0-liftings of $(V,d^{(2)})$, we obtain extreme rays of the cone HYP_n, for any $n\ge|V|$. In particular, if P is basic, then each basic subset of $V(P)$ yields an extreme ray of the hypermetric cone HYP_{k+1} and, thus, of HYP_n, for $n\ge k+1$. Therefore, finding all extreme rays of the hypermetric cone HYP_n yields the question of finding all extreme L-polytopes of dimension $k\le n-1$.

The only basic extreme L-polytope of dimension $k\le 5$ is the segment α_1, of dimension 1. Indeed, it is known that the only extreme rays of the hypermetric cone HYP_n, for $n\le 6$, are the cut semimetrics with associated L-polytope α_1 (see [28] for $n\le 5$ and [15] for $n=6$). Actually, it is announced in [44] that α_1 is the only extreme L-polytope of dimension $k\le 5$, i.e. the assumption about "basic" can be dropped.

For $n\ge 7$, the hypermetric cone has extreme rays which are not generated by cut semimetrics. Indeed, there exists a basic extreme L-polytope of dimension 6, namely, the Schläfli polytope 2_{21}; it is asymmetric and has 27 vertices. The Gosset polytope 3_{21} is a basic centrally symmetric extreme L-polytope of dimension 7 with 56 vertices. We describe the polytopes 2_{21} and 3_{21} in Section 5.2. Other examples of extreme L-polytopes are presented in Sections 5.3 and 5.4. We refer to [33] for a detailed treatment of the topics treated in this section.

Actually, α_1, 2_{21} and 3_{21} are the only extreme L-polytopes occurring in root lattices.

Theorem 5.1 *Let P be a generating L-polytope in a root lattice. Then, P is extreme if and only if P is α_1, 2_{21} or 3_{21}.*

PROOF. Let L denote the lattice generated by $V(P)$. By assumption, L is a root lattice and, by Proposition 4.6, L is irreducible. Hence, P is one of the L-polytopes from Figure 1. This implies that P is α_1, 2_{21} or 3_{21} since the other polytopes give ℓ_1-spaces. □

5.1 Extreme L-polytopes and equiangular sets of lines

In this section, we present bounds on the number of vertices of a basic extreme L-polytope and we compare them with some known bounds for the cardinality of equiangular sets of lines. We also present some constructions of equiangular sets of lines by taking sections of the sphere of minimal vectors in a lattice.

As an immediate consequence of Lemma 4.14 and Proposition 4.15, we have the following lower bounds for the number of vertices of an extreme basic L-polytope.

THEOREM 5.2 *Let P be a k-dimensional basic L-polytope. If P is extreme, then,*

$$|V(P)| \geq \frac{k(k+3)}{2} \quad \textit{if } P \textit{ is asymmetric} \tag{29}$$

$$|V(P)| \geq k(k+1) \quad \textit{if } P \textit{ is centrally symmetric.} \tag{30}$$

There is a striking analogy between the lower bounds (29), (30) and the following known upper bounds (31), (32) for the number N_p of points in a spherical two-distance set of dimension k and the number N_ℓ of lines is an equiangular set of lines of dimension k (see [53]).

$$N_p \leq \frac{k(k+3)}{2} \tag{31}$$

$$N_\ell \leq \frac{k(k+1)}{2} \tag{32}$$

Recall that equiangular sets of lines and spherical two-distance sets are in one-to-one correspondence. Namely, let $\mathcal{L}$ be a set of equiangular lines of dimension $k+1$ and let $\ell_0 \in \mathcal{L}$. Choose a unit vector e_0 along ℓ_0 and, for each $\ell \in \mathcal{L}$, $\ell \neq \ell_0$, choose a unit vector e_ℓ along ℓ which forms an acute angle with e_0. Then, the set $\mathcal{P} = \{e_\ell : \ \ell \in \mathcal{L} - \{\ell_0\}\}$ is a spherical two-distance set in dimension k; indeed, if ϕ denotes the common acute angle between the lines of $\mathcal{L}$, then $\mathcal{P}$ lies on the sphere of center $(\cos\phi)e_0$, radius $\sin\phi$, in the hyperplane $x^T e_0 = \cos\phi$. The construction can be reversed. Also, $|\mathcal{P}| = |\mathcal{L}| - 1$ and thus the two bounds (31), (32) can be deduced from one another.

The bound (32) was given by Gerzon who proved, furthermore, that, if equality holds in (32), then $k+2=4,5$ or $k+2=q^2$ for some odd integer $q, q \geq 3$ (see [53]). The first case of equality in (32) is $N_\ell = 28$ for $q=3, k=7$; it is well-known that an equiangular set of 28 lines can be constructed from the Gosset polytope 3_{21} (see Section 5.2). Also, the set of vertices of the Schläfli polytope 2_{21} is a spherical two-distance set in $\mathbb{R}^6$, realizing equality in (31). The next case of equality is $N_\ell = 276$ for $q=5, k=23$. Neumaier ([54]) has shown how to construct a set of 276 equiangular lines using the Leech lattice Λ_{24}. In Section 5.3, we shall see that an extreme centrally symmetric L-polytope of dimension 23 and with 552 vertices can be constructed from this set of lines, also that a suitable section of it is an extreme asymmetric L-polytope of dimension 22 and with 275 vertices. The next cases of equality in (32) are $N_\ell = 1128$ for $q=7, k=47$, and $N_\ell = 3160$ for $q=9, k=79$; but it is not known whether such sets of equiangular lines exist in these two cases.

On the other hand, we shall see in Section 5.4 some examples of extreme $L-$polytopes realizing equality in the bound (29) or (30), but not arising from some spherical 2 - distance set or from some equiangular set of lines. Also, we shall have examples of extreme $L-$polytopes that do not realize equality in the bound (29) or (30).

We now present a general construction for equiangular sets of lines by taking a suitable section of the sphere of minimal vectors in an integral lattice.

Let L be a lattice with minimal norm t and let $L_{\min}$ be its set of minimal vectors. Given $a \in L$, $a \neq 0$, set $V = \{u \in L_{\min} : 2u^T a = a^2\}$. Hence, all $u \in V$ lie on a sphere with center $\frac{a}{2}$. By Lemma 2.10, if $V \neq \emptyset$, then the polytope $P = \mathrm{conv}(V)$ is an L-polytope. Moreover, P is centrally symmetric.

The following properties can be easily checked: $V \neq \emptyset$ if and only if $a = a_1 + a_2$ for some $a_1, a_2 \in L_{\min}$ and, then, $a_1, a_2 \in V$. If $V \neq \emptyset$, then $|V| = 1$ if and only if $a^2 = 4t$. If $|V| \geq 2$, then, for all $u, v \in V$ such that $v \neq u, a-u$, we have that

$$\frac{a^2 - t}{2} \leq u^T v \leq \frac{t}{2}. \tag{33}$$

(This follows from the fact that $(u-v)^2 \geq t$ and $(u+v-a)^2 \geq t$.) This implies that $t \leq a^2 \leq 2t$ if $|V| \geq 3$.

Since P is centrally symmetric, we can arrange its vertices into pairs of antipodal vertices. Each such pair determines a line going through $\frac{a}{2}$ and with direction $2u - a$, for $u \in V$. Let $\mathcal{L}$ denote this set of lines and let $V' = \{\sqrt{2}(u - \frac{a}{2}) : u \in V\}$ denote the set of their directions. Note that $u'^2 = 2t - \frac{a^2}{2}$ for $u' \in V'$, and $u'^T v' = 2u^T v - \frac{a^2}{2}$ for $u', v' \in V'$. Therefore, if L is an integral lattice, then u'^2, $u'^T v'$ are integers with the same parity as

$\frac{a^2}{2}$. Note also that, from relation (33), we have that $-(t-\frac{a^2}{2}) \leq u'^T v' \leq (t-\frac{a^2}{2})$ for $u', v' \in V'$, $v' \neq u', -u'$. Using the above observations, we obtain the following result.

Proposition 5.3 *[30] Let $\mathcal{L}$ denote the set lines determined by the diagonals of the polytope $P = conv(V)$. The following assertions hold.*
(i) If $a^2 = 2t$, then the lines in $\mathcal{L}$ are pairwise orthogonal.
(ii) Suppose $a^2 = 2t-2$, $t \geq 2$ and L is an integral lattice. Then, $\mathcal{L}$ is an equiangular set of lines with common angle $\arccos(\frac{1}{t+1})$ (resp. $\arccos(0) = \frac{\pi}{2}$) if t is even (resp. odd).
(iii) Suppose $a^2 = 2t-4$, $t \geq 4$ and L is an integral lattice. If t is odd, then $\mathcal{L}$ is equiangular with common angle $\arccos(\frac{1}{t+2})$ and, if t is even, then there are two possible angles between the lines of $\mathcal{L}$, namely $\arccos(\frac{2}{t+2})$ and $\arccos(0) = \frac{\pi}{2}$.

We give an illustration of the above construction in the case (ii) when $a^2 = 2t-2$, $t = 2$ and L is a root lattice (see [30] for details). For each irreducible root lattice, we indicate what is the L-polytope P produced by the construction, the number of lines in the equiangular set $\mathcal{L}$ of its diagonals and the dimension in which $\mathcal{L}$ occurs.
- for $L = A_{n-1}$, $P = \beta_{n-1}$, $|\mathcal{L}| = n-1$, in dimension $n-1$,
- for $L = D_n$, $P = \alpha_1 \times \beta_{n-2}$, $|\mathcal{L}| = 2(n-2)$, in dimension $n-1$,
- for $L = E_6$, the 1-skeleton of P is $J(6,3)$, $|\mathcal{L}| = 10$, in dimension 5,
- for $L = E_7$, $P = \frac{1}{2}H(6,2)$, $|\mathcal{L}| = 16$, in dimension 6,
- for $L = E_8$, $P = 3_{21}$, $|\mathcal{L}| = 28$, in dimension 7.
Note that, in dimensions 5 and 6, the maximum cardinality of an equiangular set of lines is equal to 10 and 16, respectively; so the two examples above from E_6 and E_7 are maximum.

5.2 The Schläfli polytope 2_{21} and the Gosset polytope 3_{21} are extreme

In this section, we show that the Schläfli polytope 2_{21} and the Gosset polytope 3_{21} are extreme. The proof uses the treatment for the notion of rank developed in Section 4.1. The main steps of the proof are:
• find an affine base B, so $|B| = 7$ for 2_{21} and $|B| = 8$ for 3_{21} (therefore, showing that both $2_{21}, 3_{21}$ are basic $L-$polytopes),
• using the affine decomposition of each non basic vertex in B, find the explicit description of the system $\mathcal{S}(B, d^{(2)})$ (it consists of 27-7=20 equations for 2_{21} and $\frac{56}{2} - 1 = 27$ for 3_{21}),
• show that the solution set to the system $\mathcal{S}(B, d^{(2)})$ has rank 1.

We need an explicit description of the polytopes $2_{21}, 3_{21}$. We refer, for instance, to [19], [23], [24] for a detailed account of the facts about E_6, E_7, E_8 mentioned below.

The lattice E_8 is defined by

$$E_8 = \{x \in \mathbb{R}^8 : \ x \in \mathbb{Z}^8 \text{ or } x \in (\frac{1}{2} + \mathbb{Z})^8 \ \text{ and } \sum_{1 \leq i \leq 8} x_i \in 2\mathbb{Z}\}.$$

Let V_8 denote the set of minimal vectors of E_8; V_8 consists of
- the 112 vectors $(\pm 1^2, 0^6)$ and
- the 128 vectors $\left(\pm\frac{1}{2}^8\right)$ that have an even number of minus signs.

So, $|V_8| = 240$ and $v^T v = 2$ for $v \in V_8$. The set V_8 lies on the sphere S_8 with center 0 and radius $\sqrt{2}$.

Let $v_0 = (1, 1, 0^6)$ be a given minimal vector. One can check that $v^T v_0 = 0, \pm 1$ for all $v \in V_8, v \neq \pm v_0$. The lattice E_7 is defined by

$$E_7 = \{x \in E_8 : \ x^T v_0 = 1\}.$$

Let H_7 denote the hyperplane defined by the equation $x^T v_0 = 1$; then, $S_7 = S_8 \cap H_7$ is the 7-dimensional sphere with center $\frac{v_0}{2}$ and radius $\sqrt{\frac{3}{2}}$. Set

$$V_7 = \{x \in V_8 : \ x^T v_0 = 1\}.$$

Then, V_7 consists of
- the 12 vectors $(1, 0, \pm 1, 0^5)$,
- the 12 vectors $(0, 1, \pm 1, 0^5)$ and
- the 32 vectors $\left(\frac{1}{2}, \frac{1}{2}, \pm\frac{1}{2}^6\right)$ with an even number of minus signs.

So, $|V_7| = 56$ and V_7 lies on the sphere S_7. By Lemma 2.10, the polytope $\mathrm{conv}(V_7)$ is an L-polytope; it is the so-called Gosset polytope 3_{21}. Observe that the 56 points of V_7 are partitioned in 28 pairs of antipodal points (with respect to the sphere S_7, i.e. the antipode of v is $v^* = v_0 - v$). So, the polytope 3_{21} is centrally symmetric.

Let $w_0 = \left(\frac{1}{2}\right)^8$ be a given minimal vector of V_7; so, $w_0^* = \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}^6\right)$. One can check that $v^T w_0 = 0, 1$ for all $v \in V_7, v \neq w_0$ and $v \neq w_0^*$. Then, the lattice E_6 is defined by

$$E_6 = \{x \in E_7 : \ x^T w_0 = 1\}.$$

Note that, if v^* is the antipode of $v \in V_7$, then $v^T w_0 + (v^*)^T w_0 = v_0^T w_0 = 1$ and, thus, $v^T w_0 = 1$ if and only if $(v^*)^T w_0 = 0$. Let H_6 denote the hyperplane defined by the equation $x^T w_0 = 1$; then, $S_6 = S_7 \cap H_6 = S_8 \cap H_7 \cap H_6$ is the 6-dimensional sphere with center $\frac{v_0 + w_0}{3}$ and radius $\sqrt{\frac{4}{3}}$. Set

$$V_6 = \{x \in V_7 : \ x^T w_0 = 1\}$$

and $V_6^* = \{v^* : \ v \in V_6\}$. Hence, $V_7 = V_6 \cup V_6^* \cup \{w_0, w_0^*\}$. The set V_6 consists of
- the 6 vectors $(1, 0, 1, 0^5)$,
- the 6 vectors $(0, 1, 1, 0^5)$ and
- the 15 vectors $\left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}^2, \frac{1}{2}^4\right)$.

Hence, $|V_6| = 27$ and V_6 lies on the sphere S_6. The polytope conv(V_6) is the so-called Schläfli polytope 2_{21}; from Lemma 2.10, it is indeed an L-polytope and it is clearly asymmetric.

Remark 5.4 *(i) The 28 distinct lines determined by the diagonals of 3_{21} form a 7-dimensional set of equiangular lines with common angle* $\arccos(\frac{1}{3})$*; this can be seen directly or as an application of Proposition 5.3 (ii).*
(ii)For $u, v \in V_6, v \neq u$, $u^T v = 0, 1$ and, thus, $d^{(2)}(u,v) = (u-v)^2 = 4$ (if $u^T v = 0$) or 2 (if $u^T v = 1$). Therefore, the 27 vertices of 2_{21} form a 6-dimensional spherical two-distance set of points.
(iii)The graph whose nodes are the vertices of 2_{21} and with edges the pairs (u, v) of vertices at the smallest distance $d^{(2)}(u, v) = 2$, is called the **Schläfli graph** *and is denoted by G_{27}. The graph whose nodes are the vertices of 3_{21} and with edges the pairs (u, v) of vertices with $d^{(2)}(u, v) = 2$ is called the* **Gosset graph** *and is denoted by G_{56}. From Proposition 3.9, G_{27} (resp. G_{56}) is the 1-skeleton of 2_{21} (resp. of 3_{21}).*

We now show that the polytopes 2_{21} and 3_{21} are extreme. This result was proved in [33]; another proof was given in [44].

Theorem 5.5 *The Schläfli polytope 2_{21} and the Gosset polytope 3_{21} are basic extreme L-polytopes.*

Proof. We denote the vectors of V_6 by $u_i = (1, 0, 1_i, 0^5)$, $v_i = (0, 1, 1_i, 0^5)$, where the first two coordinates are fixed and the second 1 stays in the $(2+i)-$th position, for $1 \le i \le 6$, and $u_{ij} = \left(\frac{1}{2}, \frac{1}{2}, \left(-\frac{1}{2}\right)_i, \left(-\frac{1}{2}\right)_j, \frac{1}{2}^4\right)$ where the two $-\frac{1}{2}$'s stay in the $(2+i)-$th and $(2+j)-$th positions for $1 \le i < j \le 6$. One can verify that the distances between the points of V_6 are as follows, where we set $t = 2$.

$$(*)\begin{cases} d(u_i,u_j) & = d(v_i,v_j) = t \text{ for } i \neq j \\ d(u_i,v_j) & = \begin{cases} t & \text{if } i = j \\ 2t & \text{if } i \neq j \end{cases} \\ d(u_i,u_{kl}) & = d(v_i,u_{kl}) = \begin{cases} t & \text{if } i \notin \{k,l\} \\ 2t & \text{if } i \in \{k,l\} \end{cases} \\ d(u_{ij},u_{kl}) & = \begin{cases} t & \text{if } |\{i,j\}\cap\{k,l\}| = 1 \\ 2t & \text{if } |\{i,j\}\cap\{k,l\}| = 0 \end{cases} \end{cases}$$

Consider the following subset of V_6

$$B_6 = \{u_{12}, u_{24}, u_{34}, u_{35}, u_{15}, u_6, v_6\}.$$

One can check that B_6 is an affine basis of E_6, i.e. that B_6 generates the set V_6. The affine decompositions of the non basic points of $V_6 - B_6$ in B_6 give the following system of 20 hypermetric equalities in the 21 variables $d(i,j)$ for $1 \leq i < j \leq 7$ (the indices are modulo 5).

$$\begin{cases} d(i,6)+d(i+1,6)-d(i,i+1) & =0 \quad \text{for } 1\leq i\leq 5 \\ d(i,7)+d(i+1,7)-d(i,i+1) & =0 \quad \text{for } 1\leq i\leq 5 \\ d(i,i+2)+d(i,i+3)-d(i+2,i+3) & =0 \quad \text{for } 1\leq i\leq 5 \\ d(6,7)+\sum\limits_{\substack{i<j \\ i,j\in\{k,k+1,k+2\}}} d(i,j)-\sum\limits_{i\in\{k,k+1,k+2\}}(d(i,6)+d(i,7)) & =0 \quad \text{for } 1\leq k\leq 5 \end{cases}$$

In fact, the equalities of the first, second and fourth lines corrspond to the representations of v_i, u_i and u_{k6}, respectively. The equalities of the third line correspond to the representations of $u_{45}, u_{25}, u_{23}, u_{13}$ and u_{14}.
For example, the equality $d(1,6)+d(2,6)-d(1,2)=0$ comes from the affine decomposition $v_5 = u_{12}+u_{34}-u_6$ of v_5 in B_6.

One can verify that the solution set to the system $\mathcal{S}(B_6,d_0)$ described above, is precisely given by (*) and, thus, has rank 1. Therefore, $rk(2_{21}) = rk(B_6,d_0) = 1$, showing that 2_{21} is extreme.

We now turn to the case of 3_{21}. Consider the set $B_7 = B_6 \cup \{w_0\}$. It is clear that B_7 is an affine base of E_7, i.e. that B_7 generates the set V_7. Indeed, $V_7 = V_6 \cup V_6^* \cup \{w_0, w_0^*\}$, $v_0 = u_{12}+u_{34}+u_{56}-w_0$ and, for $v \in V_6$, $v^* = v_0 - v = u_{12}+u_{34}+u_{56}-w_0-v$ is, thus, affinely decomposable in B_7. Since $w_0^T v = 1$ for all $v \in B_6$, we have that $d^{(2)}(w_0,v) = 2$ for $v \in B_6$.

From Lemma 4.16 (applied to $P = 3_{21}$, $P_1 = 2_{21}$, $H = H_6$ and $w = u_{13}^*$), we deduce that $rk(2_{21}) = rk(3_{21})$, implying that 3_{21} is extreme.
Note that the system $\mathcal{S}(B_7,d^{(2)})$ consists of the system $\mathcal{S}(B_6,d^{(2)})$ together with the 7 equations corresponding to the decomposition of v^* in B_7, for $v \in B_6$, and shown below.

$$\begin{cases} d(i,8)+d(i+1,8)-d(i,i+1) & =0 \quad \text{for } 1\leq i\leq 5 \\ d(1,2)+d(1,3)+d(2,3)+d(k,8)-\sum\limits_{i=1,2,3}(d(i,k)+d(i,8)) & =0 \quad \text{for } k=6,7 \end{cases}$$

□

Since 2_{21} is extreme and basic, each basic set $B \subseteq V(2_{21})$ yields an extreme ray of the hypermetric cone Hyp_7. We have constructed in the proof of Theorem 5.5 the basic set B_6. It is interesting to know how many distinct (up to permutation) extreme rays of Hyp_7 arise in this way from 2_{21}. Actually, we believe that all the extreme rays of Hyp_7, other than those generated by the cut semimetrics, arise from 2_{21}.

For each basic subset $B \subseteq V(2_{21}) = V_6$, we define the graph $G_{27}[B]$ with set of nodes B and with edges the pairs of points of B at the smallest distance 2. So, $G_{27}[B]$ is the subgraph of the Schläfli graph G_{27} induced by B; $G_{27}[B]$ is called a **basic** subgraph of G_{27}. For instance, for the basic set B_6 defined above, $G_{27}[B_6]$ is $K_7 - C_5$ (where C_5 is the cycle on the nodes $(u_{12}, u_{34}, u_{15}, u_{24}, u_{35})$).

By direct inspection of the 7-vertices subgraphs of the Schläfli graph, we found that there are in total 26 distinct basic subsets in 2_{21}.

Eight of them are connected with Theorem 6.8; namely, they are the graphs G_i, $1 \leq i \leq 8$, where $G_1 = \nabla B_9$ (so, $G_1 = G_{27}[B_6]$), $G_2 = \nabla H_2$, $G_3 = \nabla H_1$, $G_4 = \nabla B_8$, $G_5 = \nabla B_7$, $G_6 = \nabla H_4$, $G_7 = \nabla H_3$ and $G_8 = \nabla B_5$. The graphs B_i $(1 \leq i \leq 8)$ and H_i $(1 \leq i \leq 4)$ are shown in Figures 9 and 10, respectively.

We show in Figures 3, 4, 5 and 6 the 26 basic subgraphs of G_{27}. Actually, we depict there the complements $\bar{G}_i$ of the graphs G_i since they appear to be simpler to draw. Hence, in Figures 3, 4, 5 and 6, an edge means a pair of points at the largest distance 4. The 26 basic graphs G_i $(1 \leq i \leq 26)$ are partitioned in five classes indexed by some integer $q, q = 8, 11, 12, 14, 15$. In fact, all basic graphs of the same class are switching equivalent and the invariant q of each switching class is the number of odd tuples, i.e. triples of nodes carrying an odd number of edges. We refer to [33] for more details about the occurrence of switching here.

Finally, note that one obtains at least 26 distinct extreme rays for Hyp_8 from the Gosset polytope 3_{21}. Indeed, each basic set of 2_{21} can be augmented to a basic set of 3_{21}. We do not know about the classification of all other basic sets of 3_{21}.

Figure 3

Figure 4

Figure 5

Figure 6

5.3 Extreme L-polytopes in the Leech lattice Λ_{24}

In this section, we describe two extreme L-polytopes coming from the Leech lattice Λ_{24}. They have dimension 22, 23 and they are constructed by taking two consecutive suitable sections of the sphere of minimal vectors of Λ_{24}, precisely in the same way as the Gosset polytopes $3_{21}, 2_{21}$ were constructed from the lattice E_8 in Section 5.2.

We refer to [23] for a precise description of the Leech lattice Λ_{24}; we only recall some facts that we need for our treatment.

The Leech lattice Λ_{24} is a 24-dimensional lattice in $\mathbb{R}^{24}$. For convenience, the coordinates of the vectors $x \in \mathbb{R}^{24}$ are indexed by the elements of $I = \{\infty, 0, 1, \ldots, 22\}$. For $i \in I$, let e_i denote the i-th unit vector whose coordinates are all equal to zero except the i-th one equal to 1. For a subset S of I, set $e_S = \sum_{i \in S} e_i$.

Let $\mathcal{B}_{24}$ denote the family of blocks of the Steiner system $S(5,8,24)$ defined on the set I; hence, $|\mathcal{B}_{24}| = 759$. Set $\mathcal{B}_{23} = \{B - \{\infty\} : B \in \mathcal{B}_{24} \text{ with } \infty \in B\}$; so $\mathcal{B}_{23}$ is the family of blocks of the Steiner system $S(4,7,23)$ defined on the set $\{0,1,\ldots,22\}$ and $|\mathcal{B}_{23}| = 253$. In $\mathcal{B}_{23}$, there are exactly 176 blocks that do not contain a given point and there are exactly 77 blocks that do contain a given point.

The Leech lattice Λ_{24} is generated by the vectors $e_I - 4e_\infty$ and $2e_B$ for all blocks $B \in \mathcal{B}_{24}$. Let V denote the set of minimal vectors of Λ_{24}; so, $x^T x = 32$ for $x \in V$. (Note

that, in the usual definition, all vectors are scaled by a factor of $\frac{1}{\sqrt{8}}$ and the minimal norm is 4; we choose to omit this factor in order to make the notation easier.) The set V consists of the following vectors:
(I) $(\pm 4^2, 0^{22})$ ($1104 = 2 \times 24 \times 23$ such vectors),
(II) $(\pm 2^8, 0^{16})$, where the positions of the nonzero components form a block of $\mathcal{B}_{24}$ and there is an even number of minus signs ($2^7 \times 759$ such vectors),
(III) $(\pm 3, \pm 1^{23})$, where the ± 3 may be in any position, but the lower signs are taken on a codeword of the Golay code $\mathcal{C}_{24}$.
Recall that the codewords of $\mathcal{C}_{24}$ which have exactly 8 nonzero coordinates are precisely the blocks of $\mathcal{B}_{23}$.

Set $c = (5, 1^{23})$ and $a_0 = (4, 4, 0^{22})$; so $c, a_0 \in \Lambda_{24}$, $c^T c = 48$ and $a_0 \in V$. Set

$$V_{23} = \{v \in V : v^T c = 24\} \text{ and } V_{22} = \{v \in V : v^T c = 24 \text{ and } v^T a_0 = 16\}.$$

Then, by Lemma 2.10, the polytopes $P_{23} = \text{conv}(V_{23}), P_{22} = \text{conv}(V_{22})$ are L-polytopes; they have dimension 23, 22, respectively.

The center of the sphere circumscribing P_{23} is the vector $\frac{c}{2}$. Clearly, $a_0 \in V_{23}$ and its antipode $a_0^* = c - a_0 = (1, -3, 1^{22})$ also belongs to V_{23}; therefore, P_{23} is centrally symmetric. The set V_{23} consists of the vectors a_0, a_0^* together with the following vectors:
(aI) $a_i := (4, 0, 0, \cdots, 4_i, 0, \cdots, 0)$, where the second 4 is in the i-th position, for $1 \leq i \leq 22$, and their antipodes $a_i^* = c - a_i = (1, 1, 1, \cdots, -3_i, 1, \cdots, 1)$ where the -3 is in the i-th position, for $1 \leq i \leq 22$,
(aII) $b(S) := (2, 2^7, 0^{16})$, where the first 2 is in the first position (∞) and the positions of the seven other 2's form the block S of $\mathcal{B}_{23}$,
(aIII) $c(T) := (3, -1^7, 1^{16})$, where the 3 is in the first position and the positions of the seven -1's form the block T of $\mathcal{B}_{23}$.
Therefore, $|V_{23}| = 2 + 2 \times 22 + 2 \times 253 = 552$; the polytope P_{23} is centrally symmetric and realizes equality in the bound (30).

The set V_{22} consists of the following vectors:
(bI) a_i for $1 \leq i \leq 22$,
(bII) $b(S)$ for all blocks S of $\mathcal{B}_{23}$ containing 0,
(bIII) $c(T)$ for all blocks T of $\mathcal{B}_{23}$ not containing 0.
Therefore, $|V_{22}| = 22+77+176 = 275$; the polytope P_{22} is asymmetric and realizes equality in the bound (29). Note that $V_{23} = V_{22} \cup V_{22}^* \cup \{a_0, a_0^*\}$, where $V_{22}^* = \{v^* : v \in V_{22}\}$.

In fact, both polytopes P_{22}, P_{23} are basic and extreme. We indicate how to construct an affine base. We first recall a property of the Steiner system $\mathcal{B}_{23}$: The set $\{0, 1, \ldots, 22\}$ can be partitioned into two sets A, B such that $0 \in A$, $|A| = 11$, $|B| = 12$ and for any $i \in A$, there exist two blocks T_i, T_i' of $\mathcal{B}_{23}$ satisfying $T_i \cap T_i' = \{i\}$ and $T_i \cup T_i' = B \cup \{i\}$.

Namely, we can take $A = \{0,1,3,4,5,8,10,11,12,17,21\}$,
$B = \{2,6,7,9,13,14,15,16,18,19,20,22\}$ and

$$T_0 = \{0,7,15,16,19,20,22\}\ ,\ T_1 = \{1,6,7,9,13,15,22\},$$

$$T_3 = \{2,3,9,14,15,16,22\}\ ,\ T_4 = \{2,4,6,9,19,20,22\},$$

$$T_5 = \{5,9,13,16,18,19,22\}\ ,\ T_8 = \{6,8,13,14,16,20,22\},$$

$$T_{10} = \{7,9,10,14,18,20,22\}\ ,\ T_{11} = \{2,6,7,11,16,18,22\},$$

$$T_{12} = \{2,12,13,15,18,20,22\}\ ,\ T_{17} = \{2,7,13,14,17,19,22\},$$

$$T_{21} = \{6,14,15,18,19,21,22\} \text{ and } T'_{21} = \{2,7,9,13,16,20,21\}.$$

We consider the following set of 23 vectors of V_{22}:

$$\mathcal{B} = \{c(T_i) : i \in A - \{0\}\} \cup \{a_i : i \in B - \{22\}\} \cup \{a_{21}, c(T'_{21})\}.$$

Then, $\mathcal{B}$ is a basic set for the polytope P_{22}. One can check (using computer) that the rank of the solution set to the system $\mathcal{S}(\mathcal{B}, d^{(2)})$, which consists of $252 = 275 - 23$ equations in $\binom{23}{2} = 253$ variables, is equal to 1. Therefore, the polytope P_{22} is extreme.

One can extend $\mathcal{B}$ to an affine basis for P_{23}. Namely, the set $\mathcal{B} \cup \{b(T_0)^*\}$ is an affine basis for P_{23}. Indeed, one can check that

$$a_0 = b(T'_0) + c(T_1) + c(T'_1) + a_1 - b(T_0) - 2b(T_0)^*$$

and, thus, a_0 is spanned by $\mathcal{B} \cup \{b(T_0)^*\}$. Then, $a_0^* = b(T_0) + b(T_0)^* - a_0$ is also spanned by $\mathcal{B} \cup \{b(T_0)^*\}$, as well as any v^* for $v \in V_{22}$. Now the extremality of P_{23} follows from that of P_{22}, using Lemma 6.5 (taking P_{23} for P, P_{22} for P_1 and a_0 for v). In conclusion, we have shown:

THEOREM 5.6 *(i) The polytope P_{23} is a centrally symmetric extreme L-polytope of dimension 23 with 552 vertices, hence realizing equality in the bound (30).*
(ii) The polytope P_{22} is an asymmetric extreme L-polytope of dimension 22 with 275 vertices, hence realizing equality in the bound (29).

Observe that the set V_{22} is a spherical two-distance set; namely, the distances between the points of V_{22} take the two values 32 or 48. Also, the 276 lines defined by the 276 pairs of antipodal vertices of the polytope P_{23} are equiangular (with common angle $\arccos(\frac{1}{5})$).

5.4 Extreme L-polytopes from the Barnes-Wall lattice Λ_{16}

In this section, we describe some more examples of extreme $L-$polytopes coming from the Barnes-Wall lattice.

We refer to [23] for a precise description of the Barnes-Wall lattice Λ_{16}.

The Barnes-Wall lattice Λ_{16} is a 16-dimensional lattice in $\mathbb{R}^{16}$. Let V denote the set of minimal vectors of Λ_{16} . Then, V consists of the following vectors:
(I) 480 vectors of the form $(\pm 2^2, 0^{14})$, where there are two non zero components equal to 2 or -2,
(II) 3840 vectors of the form $(\pm 1^8, 0^8)$, where the positions of the ± 1's form one of the 30 codewords of weight 8 of the first order Reed-Muller code and there are an even number of minus signs.

We show in Figure 7 a list of 15 codewords of weight 8 of the first order Reed-Muller code; the other 15 codewords of weight 8 are obtained by complementation of the codewords shown in Figure 7.

c_{12}	0 0 1 1 1 1	1 1 1 1	0 0 0 0	0 0
c_{13}	0 1 0 1 1 1	0 0 1 0	1 0 1 0	0 1
c_{14}	0 1 1 0 1 1	0 1 0 0	0 0 1 1	1 0
c_{15}	0 1 1 1 0 1	0 0 0 1	0 1 0 0	1 1
c_{16}	0 1 1 1 1 0	1 0 0 0	1 1 0 1	0 0
c_{23}	1 0 0 1 1 1	0 0 1 0	0 1 0 1	1 0
c_{24}	1 0 1 0 1 1	0 1 0 0	1 1 0 0	0 1
c_{25}	1 0 1 1 0 1	0 0 0 1	1 0 1 1	0 0
c_{26}	1 0 1 1 1 0	1 0 0 0	0 0 1 0	1 1
c_{34}	1 1 0 0 1 1	1 0 0 1	0 1 1 0	0 0
c_{35}	1 1 0 1 0 1	1 1 0 0	0 0 0 1	0 1
c_{36}	1 1 0 1 1 0	0 1 0 1	1 0 0 0	1 0
c_{45}	1 1 1 0 0 1	1 0 1 0	1 0 0 0	1 0
c_{46}	1 1 1 0 1 0	0 0 1 1	0 0 0 1	0 1
c_{56}	1 1 1 1 0 0	0 1 1 0	0 1 1 0	0 0

Figure 7

Hence, there are 4320 minimal vectors in Λ_{16} and $v^T v = 8$ for every minimal vector. (Note that in the usual definition, the minimal norm is 4 and all vectors should be scaled by a factor $\frac{1}{\sqrt{2}}$; we omit this factor in order to make the notation easier.)

Set $a = (2^6, 0^{10})$ (the six 2's are in the first six positions which are precisely the first six positions distinguished in Figure 7). Let S denote the sphere of center $\frac{a}{2}$ and radius $\sqrt{6}$; then, S is an empty sphere in Λ_{16} corresponding to a deep hole (i.e. with maximum radius). The associated $L-$polytope P, defined by $P = \{v \in \Lambda_{16} : (v - \frac{a}{2})^2 = 6\}$, has exactly 512 vertices that we now describe. Note first that the vectors $0 = (0^{16})$ and $a = (2^6, 0^{10})$ are both vertices of P, since $a \in \Lambda_{16}$ and $(\frac{a}{2})^2 = 6$; they are, in fact, antipodal on the sphere S. Therefore, P is a centrally symmetric $L-$polytope. Let $v \in \Lambda_{16}$; v is a vertex of P if and only if $v^T a = v^2$ holds. The remaining vertices of P, apart from 0 and a, can be partitioned into the following three classes:
(a) First, those lying in the hyperplane H_a^8 defined by the equation $x^T a = 8$, i.e. those that are minimal vectors; denote their set by V^8. There are 135 such vertices and they are of the form:
(aI) $(2^2, 0^4, 0^{10})$, where the two 2's stay in the first six positions,
(aII) $(1^4, 0^2, \pm 1^4, 0^6)$, where the first four 1's stay in the first six positions, i.e. the positions of the ± 1's form one of the 15 codewords shown in Figure 7, and there is an even number of minus signs.
(b) The antipodes of the vectors of V^8; denote their set by V^{16}, so $V^{16} = \{a - v : v \in V^8\}$ and they all lie in the hyperplane H_a^{16} of equation $x^T a = 16$. There are also 135 such vertices and they are of the form:
(bI) $(2^4, 0^2, 0^{10})$, where the two 2's stay in the first six positions,
(bII) $(1^4, 2^2, \pm 1^4, 0^6)$, the $1, \pm 1$'s form one of the 15 codewords shown in Figure 7 and there is an even number of minus signs.
(c) The remaining vertices lie in the hyperplane H_a^{12} of equation $x^T a = 12$ and they are of the form $v_1 + v_2$ where v_1 is of type I and v_2 is of type II; denote their set by V^{12}. More precisely, take v_2 of the form $(1^4, 0^2, \pm 1^4, 0^6)$ (there are $15 \times 8 = 120$ such vectors) and v_1 of the form $(2, 0^5, \pm 2, 0^9)$, where the first 2 stays in the two positions of the first two zeros of v_2 and the ± 2 stays in one of the positions of the $\pm 1's$ of v_2 and has the opposite sign (there are 8 choices for v_1). For example, for $v_1 = (0,0,1,1,1,1,1,1,1,1,0,0,0,0,0,0)$, $v_2 = (2,0,0,0,0,0,-2,0,0,0,0,0,0,0,0,0)$, we obtain the vector $v = v_1 + v_2 = (2,0,1,1,1,1,-1,1,1,1,0,0,0,0,0,0)$. Note, however, that v can be obtained as the sum of three distinct pairs of vectors v_1, v_2. Namely,

$$v = (0,0,1,1,1,1,-1,-1,1,1,0^6) + (2,0,0,0,0,0,0,2,0,0,0^6),$$

$$v = (0,0,1,1,1,1,-1,1,-1,1,0^6) + (2,0,0,0,0,0,0,0,2,0,0^6) \text{ and}$$

$$v = (0,0,1,1,1,1,-1,1,1,-1,0^6) + (2,0,0,0,0,0,0,0,0,2,0^6).$$

Therefore, in total, there are $\frac{120\times 8}{4} = 240$ vectors in V^{12} and they are of the form $(2, 0, 1^4, \pm 1^4, 0^6)$, where the positions of the $1, \pm 1$'s form one of the 15 codewords of Figure 7, the 2 stays on one of the two remaining places in the first six positions and there is an odd number of minus signs. These 240 vectors are clearly divided in 120 pairs of antipodal vectors lying respectively in the hyperplanes H_b^2 (of equation $x^T b = 2$) and H_b^{-2} (of equation $x^T b = -2$), where $b = (0^6, 1^{10})$ (H_b^2 contains the vertices with exactly one minus sign and H_b^{-2} contains the vertices with three minus signs).

In summary, the set V of vertices of P is $V = V^8 \cup V^{12} \cup V^{16} \cup \{0, a\}$, $|V| = 512$. P is a centrally symmetric $L-$polytope of dimension 16 corresponding to a deep hole of Λ_{16} and having 512 vertices.

By taking some sections of the empty sphere S by some suitable hyperplanes H_a^α, one can construct some more 15-dimensional $L-$polytopes, including several examples of extreme ones.

Clearly, the sets $\Lambda_{15}^\alpha = \Lambda_{16} \cap H_a^\alpha = \{x \in \Lambda_{16} : x^T a = \alpha\}$, for $\alpha = 8, 12, 16$, are 15-dimensional lattices and they all identical up to translation; note that they are different from the laminated lattice Λ_{15} (see [23]). The sphere $S^\alpha = S \cap H_a^\alpha$ is an empty sphere in the lattice Λ_{15}^α; therefore, the polytope $P^\alpha = \mathrm{conv}(V^\alpha) = \mathrm{conv}(S \cap H_a^\alpha)$ is a 15-dimensional $L-$polytope in Λ_{15}^α, for any $\alpha = 8, 12, 16$.

Both P^8, P^{16} are asymmetric $L-$polytopes with 135 vertices. In fact, P^8 is an affine image of P^{16}. The polytope P^{12} is centrally symmetric with 240 vertices. Note however that the set of vertices of P^{16} is not a spherical two-distance set (indeed, there are three possible distances between the vertices of P^{16}, namely, 8,12,16); also, the 120 lines defined by the 120 pairs of antipodal vertices of P^{12} are not equiangular since there are two possible angles, namely, arccos(0),arccos($\frac{1}{3}$)).

Theorem 5.7 *[33] (i) The polytope P is a centrally symmetric extreme $L-$polytope of dimension 16 with 512 vertices.*
(ii) The polytopes P^8, P^{16} are asymmetric extreme L-polytopes of dimension 15, each having 135 vertices.
(iii)The polytope P^{12} is not extreme.

Proof. The set

$$B = \{v_{12}, v_{13}, v_{14}, v_{15}, v_{16}, v_{23}\} \cup$$

$$\cup \{c_{12}(13), c_{13}(24), c_{14}(24), c_{15}(\emptyset), c_{23}(24), c_{25}(12), c_{26}(14), c_{34}(34), c_{35}(24), c_{45}(23)\}$$

is a basic set in $V(P^{16})$, where we use the following notation. v_{ij} denotes the vector $(2^4, 0^2, 0^{10})$ of type (bI) with i,j denoting the positions of the first two 0's, and $c_{ij}(xy)$ denotes the vector obtained from the codeword c_{ij} (see Figure 7) by assigning a minus sign to the 1's in the x-th and y-th positions among the last four 1's of c_{ij}. For example, $c_{12}(13) = (0,0,1,1,1,1;-1,1,-1,1;0,0,0,0;0,0)$ and $c_{15}(\emptyset) = c_{15}$ (no minus sign at all).

The system $\mathcal{S}(B, d^{(2)})$ consists of 135-16=119 equations in $\binom{16}{2} = 120$ variables; one can check (using computer) that its rank is 1, showing that P^{16} is extreme.

The vector $v_0 = (2,0,1,1,1,1;-1,1,1,1;0,0,0,0;0,0)$ is a vertex of P lying in H_a^{12} (having the shape of the codeword c_{12}). Then, the set $B \cup \{v_0\}$ is a basic set in $V(P)$. Using Lemma 4.16 (taking P for P, P^{16} for P_1 and the vector $v_0^* = a - v_0$ for w), we deduce that P is extreme, since P^{16} is extreme.

For (iii), consider the subset X of the vertices of P^{12} that lie in the hyperplane H_b^2; there are exactly 120 such vertices. The polytope conv(X) is a 14-dimensional asymmetric $L-$polytope in the lattice $\Lambda_{16} \cap H_a^{12} \cap H_b^2$ whose rank is equal to 35. Therefore, the $L-$polytope P^{12} is not extreme. □

Note that the hole of the lattice Λ_{15}^{16} corresponding to the extreme $L-$polytope P^{16} is not a deep hole; indeed, its radius is equal to $\frac{4}{\sqrt{3}}$, while the radius of the hole of Λ_{15}^{12} corresponding to the $L-$polytope P^{12} is equal to $\sqrt{6}$ and $6 > \frac{16}{3}$.

Another extreme $L-$polytope can be constructed from Λ_{16} as follows. Consider the polytope Q whose vertices are the vertices of P that satisfy $x^T a = 0, 8, 16$ or 24, i.e. they are the vertices of P^8, or of P^{16}, or they are 0 or a. Hence, Q has $2 \times 135 + 2 = 272$ vertices, Q is a 16-dimensional polytope and the set $B \cup \{a\}$ generates all vertices of Q (B is the set defined in the proof of Theorem 5.7). In fact, Q is an $L-$polytope in the lattice $\Lambda'_{16} = \Lambda_{16} \cap \{x : x^T a = 0 \pmod 8\}$; so, Λ'_{16} is the sublattice of Λ_{16} having points only in the layers $x^T a = 0, 8, 16, 24$,etc...

THEOREM 5.8 *[33] The polytope Q is a centrally symmetric extreme $L-$polytope of dimension 16 with 272 vertices, hence realizing equality in the bound (30).*

PROOF. Use Lemma 4.16, taking the polytope Q for P, the polytope P^{16} for P_1 and the vector $0 = a^*$ for w. □

Finally, let us look at some $L-$polytope obtained by taking a section of the sphere of minimal vectors by some hyperplane (as in the construction of Lemma 2.10). Namely, consider the section by the hyperplane H_a^4 of equation $x^T a = 4$. In this way, one obtains the $L-$polytope $Q' = \text{conv}(x \in \Lambda_{16} : x^T x = 8 \text{ and } x^T a = 4)$. Q' is a 15-dimensional $L-$polytope and it has 1080 vertices that are of the form:

(i) $(2, 0^5, \pm 2, 0^{10})$, where the first 2 stays in the first six positions (120 such vectors),

(ii) $(\pm 1^4, 0^2, \pm 1^4, 0^6)$, where the positions of the ± 1's form one of the 15 codewords of Figure 7, there is exactly one minus sign in the first four ± 1 and there is an odd number

of minus signs in the last four ± 1 (480 such vectors),
(iii)$(1^2, 0^4, \pm 1^6, 0^4)$, where the positions of the 0's form one of the 15 codewords of Figure 7 and there is an even number of minus signs (480 such vectors).

Consider the vertex $c = (2, 0, .., 0, 2)$ of Q'. Then, the distances $d^{(2)}(c, v)$ from the other vertices v to c take the values $8, 12, 16, 20, 24$; in fact, value 8 (respectively, 12,16,20,24) is taken for 119 (respectively, 336, 427,176,21) vertices of Q'. Therefore, the set of the 119 vertices that are at distance 8 from c forms a 14-dimensional asymmetric $L-$polytope which realizes equality in the bound (29). However, this polytope is not extreme. On the other hand, the polytope Q' is extreme.

We summarize in Figure 8 the results from this section about the L-polytopes constructed from the Barnes-Wall lattice Λ_{16}. Recall that $a = (2^6, 0^{10})$, $c = (2, 0^{14}, 2)$, S denotes the deep hole of Λ_{16} with center $\frac{a}{2}$ and H_a^α denotes the hyperplane $x^T a = \alpha$.

L-polytope	dimension	number of vertices	asymmetric (A) or centrally symmetric (CS)	equality in bound (29) or (30) ?	extreme ?
$P = \mathrm{conv}(S \cap \Lambda_{16})$	16	512	CS	No	Yes
$P^8 = \mathrm{conv}(S \cap \Lambda_{16} \cap H_a^8)$	15	135	A	Yes	Yes
$P^{16} = \mathrm{conv}(S \cap \Lambda_{16} \cap H_a^{16})$	15	135	A	Yes	Yes
$P^{12} = \mathrm{conv}(S \cap \Lambda_{16} \cap H_a^{12})$	15	240	CS	Yes	No
$Q = \mathrm{conv}(S \cap \Lambda_{16} \cap \{x : x.a = 0, 8, 16, 24\})$	16	272	CS	Yes	Yes
$\mathrm{conv}(x \in \Lambda_{16} : x.x = 8, a.x = 4, x.c = 8)$	14	119	A	Yes	No
$Q' = \mathrm{conv}(x \in \Lambda_{16} : x.x = 8, a.x = 4)$	15	1080	A	No	Yes

Figure 8

5.5 Extreme L-polytopes and perfect lattices

Let L be a k-dimensional lattice (containing the origin) with minimal norm t and set $L_{\min} = \{v \in L : v^2 = t\}$.

Let $(v_1, \dots, v_n)$ be a base of L and, for each $v \in L_{\min}$, let $v = \sum_{1 \le i \le k} b_i^v v_i$ denote its decomposition in the base, with $b^v \in \mathbb{Z}^k$. We consider the system $\mathcal{S}_L$ composed by the following equations

$$\sum_{1\le i\le j\le k} b_i^v b_j^v x_{ij} = t \text{ for } v \in L_{\min}$$

in $\binom{k+1}{2}$ variables. The lattice L is said to be **perfect** if the system $\mathcal{S}_L$ has full rank $\binom{k+1}{2}$, i.e. if it has a unique solution, namely, $x_{ij} = 2v_i^T v_j$ for $1 \le i < j \le k$, $x_{ii} = v_i^2$ for $1 \le i \le k$. Perfect lattices are important since they include the lattices with the locally densiest packings (see, for instance, [58]).

If L is an affine lattice, i.e. L is the translate of a lattice L_0, then we say that L is perfect if L_0 is perfect.

The notion of perfect lattice is closely related to the notion of extreme L-polytope as the following Propositions 5.9, 5.10 and 5.11 show.

PROPOSITION 5.9 *[48] Let P be an L-polytope with radius r, let L_0 denote the lattice generated by the set of vertices $V(P)$ of P and let t denote its minimal norm. Suppose that P is a basic extreme L-polytope, that there exist $u, v \in V(P)$ with $(u-v)^2 = t$ and that $t \ge \frac{4}{3}r^2$. Then, there exists w not lying on the hyperplane spanned by P such that $(w-v)^2 = t$ for all $v \in V(P)$ and the lattice L generated by $L_0 \cup \{w\}$ is perfect.*

PROOF. We can suppose without loss of generality that the origin is a vertex of P. By Lemma 3.17, the spherical t-extension of the space $(V(P), d^{(2)})$ has a spherical representation. Let w denote the vector representating the extension point. So, $(w-v)^2 \ge t$ for all $v \in L_0$ with equality if $v \in V(P)$. Let L denote the lattice generated by $L_0 \cup \{w\}$. Then, $L = \bigcup_{a\in\mathbb{Z}} L_a$, where $L_a = (L_0 + aw)$ are the layers composing L. The distance between two consecutive layers is $h = \sqrt{t - r^2}$.

We check that L has minimal norm t, i.e. $v^2 \ge t$ for all $v \in L$, $v \ne 0$. This is obvious if v lies in L_0. If v lies in a layer L_a which is not consecutive to the layer L_0, then $\| v \| \ge 2h$, i.e. $v^2 \ge 4h^2 = 4(t - r^2) \ge t$ since $t \ge \frac{4}{3}r^2$. If v lies in a layer consecutive to L_0, say $v = u - w$ where $u \in L_0$, then $v^2 \ge t$.

Since P is basic, we can find a base $(v_1, \ldots, v_k)$ of L_0 composed of vertices of P. Then, $(w, v_1, \ldots, v_k)$ is a base of L. So, the system $\mathcal{S}_L$ is composed by the equations $\sum_{0\le i\le j\le k} b_i b_j x_{ij} = t$ where $(b_0 w + \sum_{1\le i\le k} b_i v_i)^2 = t$ with $b \in \mathbb{Z}^{k+1}$. We show that $\mathcal{S}_L$ has full rank. Let x denote a solution of $\mathcal{S}_L$. Since $w, w - v_1, \ldots, w - v_k \in L_{\min}$, we deduce that the equations $x_{00} = t$, $x_{00} + x_{ii} - x_{0i} = t$ $(1 \le i \le k)$ belong to $\mathcal{S}_L$. Therefore, $x_{00} = t$ and $x_{ii} = x_{0i}$ for $i = 1, \ldots, k$.

Let $v \in V(P)$, $v = \sum_{1\le i\le k} b_i^v v_i$ with $b^v \in \mathbb{Z}^k$. Then, $v - w \in L_{\min}$, implying the equation $x_{00} - \sum_{1\le i\le k} b_i^v x_{0i} + \sum_{1\le i\le j\le k} b_i^v b_j^v x_{ij} = t$ of $\mathcal{S}_L$. Hence, x satisfies

$$(*) \qquad \sum_{1\le i\le k} ((b_i^v)^2 - b_i^v) x_{ii} + \sum_{1\le i<j\le k} b_i^v b_j^v x_{ij} = 0$$

for each $v \in V(P)$.

By assumption, P is an extreme L-polytope, i.e. the system $\mathcal{S}(V(P), d^{(2)})$, composed by the equations

$$(**) \qquad \sum_{1 \leq i \leq k} (1 - \sum_{1 \leq j \leq k} b_j^v) b_i^v d_{0i} + \sum_{1 \leq i < j \leq k} b_i^v b_j^v d_{ij} = 0$$

for all $v \in V(P)$, has rank $\binom{k+1}{2} - 1$.

Set $d_{0i} = x_{ii}$ for $1 \leq i \leq k$ and $d_{ij} = x_{ii} + x_{jj} - 2x_{ij}$ for $1 \leq i < j \leq k$, where x is a solution of $\mathcal{S}_L$. Then, since x satisfies (*), we deduce that d satisfies (**). Therefore, d and, thus, x, is uniquely determined up to multiple. The fact that there exist $u, v \in V(P)$ with $u - v \in L_{\min}$ permits to fix the multiple. Hence, $\mathcal{S}_L$ has a unique solution x, i.e. L is perfect. $\square$

Note that Proposition 5.9 still holds if we replace the assumption $t \geq \frac{4}{3}r^2$ by the assumption $t \geq r^2$ and t is the minimal norm of L.

As we saw in Lemma 2.10, every section of the contact polytope by a hyperplane not containing the origin is an L-polytope. Hence, Proposition 5.9 can be reformulated as follows.

PROPOSITION 5.10 *[48] Let L be a k-dimensional lattice with minimal norm t and let P be an L-polytope obtained by taking a section of the contact polytope of L by a hyperplane not containing the origin. If P is basic extreme and if P contains two vertices u, v with $(u - v)^2 = t$, then L is perfect.*

For example, the root lattice E_8 and the Leech lattice Λ_{24} are perfect. This can be seen by applying Proposition 5.10; for E_8, take $t = 2$, $P = 3_{21}$ with squared radius $\frac{3}{2}$ and for Λ_{24}, take $t = 32$, $P = P_{23}$ with squared radius 20 (see Sections 5.2 and 5.3). Another example of perfect lattice is the lattice Λ'_{16} (defined as $\Lambda_{16} \cap \{x : x^T a = 0 \mod (8)\}$ where Λ_{16} is the Barnes-Wall lattice and a is a minimal vector); apply Proposition 5.10 with the polytope P^{16} (see Section 5.4).

The following result can also be checked.

PROPOSITION 5.11 *[48] Let P be an extreme basic L-polytope with radius r and let L' denote the lattice generated by the set of vertices of P and the center of P (L' is known as the* **centered lattice***). If L' has minimal norm r^2, then L' is perfect.*

Note that the Schläfli polytope 2_{21} is an extreme basic L-polytope in E_6. The lattice generated by $V(2_{21})$ and its center is the dual lattice E_6^* which is indeed perfect.

We conclude this section with some remarks on perfect forms. The quadratic form $Q(x) = \sum_{1\leq i,j\leq n} a_{ij}x_ix_j$ is said to be **perfect** if the symmetric matrix $(a_{ij})_{1\leq i,j\leq n}$ is the Gram matrix of the base of a perfect lattice. Voronoi ([66]) introduced this notion and proved that the number of distinct, up to equivalence, perfect forms in any given dimension n is finite. (Two quadratic forms are **equivalent** if they coincide up to positive multiple and integral unimodular transformation.) In dimension $n = 2,3,4,5,6,7$, the number of nonequivalent perfect forms is 1,1,2,3,7,33, respectively. For details on perfect forms, see, for instance, [58] (the complete enumeration in the case $n = 7$ was done recently by Jaquet [50]).

We mention briefly some of the known perfect forms, in our terminology. If $(a_{ij})_{1\leq i,j\leq n}$ is a symmetric matrix, then $p = (a_{ij})_{1\leq i\leq j\leq n}$ belongs to $\xi_0(\mathrm{NEG}_{n+1})$ and its image $d = \xi_0^{-1}(p)$ under the inverse of the covariance map belongs to NEG_{n+1}. It turns out that several perfect forms correspond, in this way, to distances d that are related to easy graphs.

We use the following notation: K_A denotes the complete graph with set of nodes A and $K_{a_1,\ldots,a_t}$ denote the complete t-multipartite graph with a_1 nodes in the first part, ..., a_t nodes in the t-th part.

- The quadratic form $Q_0^n(x) = \sum_{1\leq i\leq j\leq n} x_ix_j$ is perfect for any $n \geq 2$. Its symmetric matrix is $a_{ii} = 1$ for $1 \leq i \leq n$ and $a_{ij} = \frac{1}{2}$ for $1 \leq i \neq j \leq n$; the corresponding distance d is the path metric of the complete graph K_{n+1}. This is the only perfect form for $n = 2,3$.
- The form $Q_1^n(x) = \sum_{1\leq i\leq j\leq n} x_ix_j - x_1x_2$ is perfect for $n \geq 4$; Q_0^4 and Q_1^4 are the only perfect forms for $n = 4$. The corresponding distance is $d = d(K_{n+1} - P_2)$ (where P_2 is the path on $(1,2)$).

The two forms Q_0^n and Q_1^n are known as the first and second principal forms, in terms of Voronoi; they are equivalent to the forms A_n, D_n (corresponding to the root lattices A_n, D_n), in terms of Coxeter.

- The last perfect form for $n = 5$, that we denote by Q_2^5, corresponds to the distance $d = \frac{1}{2}(d(K_6) + d(K_{1,2,3}))$ (which is an ℓ_1-metric).
- The perfect form $Q_3^n(x) = 2(\lfloor\frac{n-1}{2}\rfloor Q_1^n(x) + x_1x_2 - \sum_{3\leq i<j\leq n} x_ix_j)$, for $n \geq 3$, was discovered by Anzin ([2]); its corresponding distance is $d = 2\lfloor\frac{n-3}{2}\rfloor d(K_{n+1} - K_{\{1,2\}}) + 2d(K_{n+1} - K_{\{3,4,\ldots,n\}})$. (Note that $Q_3^3 = 2Q_0^3$, $Q_3^4 \simeq Q_1^4$ and $Q_3^5 = 4Q_2^5$.)
- The 7 perfect forms for $n = 6$ have as corresponding distances $d(K_7)$, $d(K_7 - P_2)$, $d(K_7 - P_3)$ (which is an extreme hypermetric), $\frac{1}{2}(d(K_7) + d(K_{1,2,2,2}))$, $\frac{1}{2}(d(K_7) + d(K_{1,2,4}))$, $\frac{1}{2}(d(K_7) + d(K_{1,1,2,3}))$ and $\frac{1}{2}(d(K_7 - P_{(2,1,6,5)}) + d(K_7 - P_{(3,1,2,5,6,4)}))$.
- Among the 33 perfect forms for $n = 7$, six of them correspond to the distances $d(K_8)$, $d(K_8 - P_2)$, $d(K_8 - P_3)$ (which is an extreme hypermetric), $\frac{1}{2}(d(K_8) + d(G))$ where G is $K_{1,2,2,3}$, $K_{2,2,2,2}$ or $K_{1,1,1,2,3}$. There is also Anzin's form $Q_3^7(x)$.

• The irredicible root lattices A_n ($n \geq 0$), D_n ($n \geq 4$) and E_n ($n = 6,7,8$) can be represented by their Coxeter-Dynkin diagrams, that are very special trees. It turns out that $\xi_0(d(K_{n+1}) + \mathrm{sph}_0(d(\bar{G})))$ is the symmetric matrix for the quadratic form corresponding to the irreducible root lattice whose Coxeter-Dynkin diagram is G.

6 Hypermetric graphs

We group in this section several results concerning hypermetricity of distance spaces arising from graphs.

There are esssentially two ways of constructing a distance space from a graph. The most classical construction of a distance space from a connected graph G is by considering the graphic space $(V(G), d_G)$, where d_G is the path metric of G. If $(V(G), d_G)$ is hypermetric (resp. isometrically ℓ_1-embeddable, of negative type), we say that G is a **hypermetric graph** (resp. an **ℓ_1-graph**, a **graph of negative type**).

Another distance space which can be constructed from a graph G is the space $(V(G), d_G^*)$, where d_G^* is the **truncated distance** of G defined by

$$\begin{cases} d_G^*(i,j) = 1 & \text{if } ij \in E(G), i \neq j, \\ d_G^*(i,j) = 2 & \text{if } ij \notin E(G),\ i \neq j, \\ d_G^*(i,i) = 0 & \text{for all } i \in V(G). \end{cases}$$

Observe that, if G has diameter ≤ 2, then the two notions of path metric and truncated distance coincide. We shall, in particular, consider the class of suspension graphs, which have diameter 2.

6.1 A characterization of hypermetric and ℓ_1-graphs

We first present a characterization of the graphs whose path metric is hypermetric, or isometrically ℓ_1-embeddable.

Theorem 6.1 *Let G be a connected graph. Then,*
(i) ([65]) G is hypermetric if and only if G is an isometric subgraph of a product of half-cube graphs, cocktail-party graphs and copies of the Gosset graph G_{56}.
(ii)([31], [61]) G is an ℓ_1-graph if and only if G is an isometric subgraph of a product of half-cube graphs and cocktail-party graphs.

Proof. This is an immediate consequense of Theorems 3.12 and 3.13 applied to the connected strongly even distance space $(V(G), 2d_G)$. □

For the sake of completeness, we recall the following result which characterizes the isometric subgraphs of a hypercube, i.e. the graphs whose path metric is isometrically hypercube embeddable. The equivalence $(i) \iff (ii)$ is from [40] and the equivalence $(i) \iff (iii)$ from [11].

THEOREM 6.2 *Let G be a connected graph. The following assertions are equivalent.*
(i) G is an isometric subgraph of a hypercube.
(ii) G is bipartite and, for all nodes $i, j \in V(G)$, the set $G(i,j) = \{k \in V(G) : d_G(i,k) < d_G(j,k)\}$ is closed under taking shortest paths.
(iii) G is bipartite and d_G is 5-gonal.

We recall that, for a bipartite graph G, the hierarchy of metric properties from Proposition 2.5 collapses. Namely,

PROPOSITION 6.3 *[57] Let G be a connected bipartite graph. Then, the following assertions are equivalent.*
(i) G is an isometric subgraph of a hypercube.
(ii) G is an ℓ_1-graph.
(iii) G is hypermetric.
(iv) G is of negative type.
(v) the distance matrix $(d_G(i,j))_{i,j \in V(G)}$ has exactly one positive eigenvalue.
Moreover, G has then an essentially unique ℓ_1-embedding.

The characterization from Theorem 6.2 is a "good" characterization, in the sense that it permits to recognize in polynomial time whether a graph is an isometric subgraph of a hypercube. The result from Theorem 6.1 (ii) does not yield, a priori, a good characterization of ℓ_1-graphs. However, the proof method developped by Shpectorov [61] permits to recognize ℓ_1-graphs in polynomial time. No good characterization is known yet for hypermetric graphs (recall Remark 4.12).

If we restrict our attention to the class of suspension graphs, then we have some refined characterizations for hypermetricity and ℓ_1-embeddability. Note that, for a graph G, its suspension ∇G is hypermetric (resp. an ℓ_1-graph) if and only if ∇H is hypermetric (resp. an ℓ_1-graph) for each connected component H of G. Indeed, the path metric of ∇G arises as the 1-sum of the path metrics of $\nabla H_1, \ldots, \nabla H_m$, if $H_1, \ldots, H_m$ are the connected components of G.

We start with a characterization of the suspension graphs that are of negative type. Given a graph G on n nodes, its **adjacency matrix** A_G is the $n \times n$ symmetric matrix with zero diagonal entries and whose (i,j)-entry is equal to 1 if i, j are adjacent in G and to 0 otherwise, for distinct $i, j \in V(G)$. Let $\lambda_{\min}(A_G)$ denote the smallest eigenvalue of A_G.

Figure 9

PROPOSITION 6.4 *[7] Let G be a graph. Then, its suspension ∇G is of negative type if and only if $\lambda_{\min}(A_G) \geq -2$ holds.*

PROOF. We use Proposition 2.3, so we show that $\lambda_{\min}(A_G) \geq -2$ if and only if the space $(V(\nabla G), d_{\nabla G})$ has a representation. Let i_0 denote the suspension node of ∇G and suppose G has n nodes. If $\lambda_{\min}(A_G) \geq -2$, then the matrix $A_G + 2I$ is positive semidefinite. Hence, there exist n vectors $u_1, \ldots, u_n \in \mathbb{R}^m$ for some m such that $A_G + 2I$ is their Gram matrix, i.e.

$$\begin{cases} (u_i)^2 = 2 & \text{for } i = 1, \ldots, n, \\ u_i^T u_j = 1 & \text{if } ij \in E(G), \\ u_i^T u_j = 0 & \text{otherwise.} \end{cases}$$

Then, the mapping $i \in V(G) \mapsto u_i$, $i_0 \mapsto u_0 = 0$, provides a representation of $(V(\nabla G), 2d_{\nabla G})$. Indeed, $(u_i - u_j)^2 = 2$ if $ij \in E(\nabla G)$ and $(u_i - u_j)^2 = 4$ otherwise. All the above arguments can be reversed, stating the converse implication: If ∇G is of negative type, then $\lambda_{\min}(A_G) \geq -2$. □

Given a graph H, its **line graph** is the graph $L(H)$ whose nodes are the edges of H with two edges adjacent in $L(H)$ if they share a common node. It is easy to see that the suspension $\nabla L(H)$ of any line graph is an ℓ_1-graph. Indeed, if we label the suspension node by 0 and each edge $e \in E(H)$, $e = ij$, by the vector $\frac{e_i+e_j}{2}$ (e_i denoting the i-th unit vector in the space $\mathbb{R}^{V(H)}$), then we obtain an ℓ_1-embedding of $\nabla L(H)$. Line graphs have been characterized by Beineke [17] by means of excluded subgraphs.

Theorem 6.5 *[17] A graph G is a line graph if and only if G does not contain as an induced subgraph any of the nine graphs B_i, $1 \leq i \leq 9$, shown in Figure 9.*

Remark 6.6 *One can verify that*
- *∇B_i is not an ℓ_1-graph for all $1 \leq i \leq 9$ except $i = 3$; in fact, ∇B_1, B_2 are not 5-gonal and ∇B_4, ∇B_6 are not 7-gonal.*
- *For each of the graphs H_i, $1 \leq i \leq 4$, shown in Figure 10, ∇H_i is not an ℓ_1-graph.*

Figure 10

Let G be a connected graph and suppose that its suspension ∇G is hypermetric. Let H denote the 1-skeleton of the L-polytope associated with the space $(V(\nabla G), d_{\nabla G})$. Then, H is one of the L-polytope graphs shown in Figure 1. Therefore, if ∇G is an ℓ_1-graph, then, by Proposition 3.7, $H \neq G_{27}$, G_{56} and H is one of $J(m,t)$, $\frac{1}{2}H(M,2)$ and $K_{m\times 2}$. More precisely, we have the following result.

THEOREM 6.7 *([7], [8]) Let G be a connected graph. Then, the following assertions are equivalent.*
(i) ∇G is an ℓ_1-graph.
(ii) G does not contain as an induced subgraph any of the graphs from the family $\mathcal{F} = \{B_1, B_2, B_4, B_5, B_6, B_7, B_8, B_9, H_1, H_2, H_3, H_4\}$.
(iii) G is a line graph or G is an induced subgraph of a cocktail-party graph.

PROOF. The implication $(i) \Longrightarrow (ii)$ follows from the fact that the suspensions of the graphs from $\mathcal{F}$ are not ℓ_1-graphs. The implication $(iii) \Longrightarrow (i)$ is clear. We show that $(ii) \Longrightarrow (iii)$ holds. Let G be a connected graph that does not contain any member of $\mathcal{F}$ as an induced subgraph. If G does not contain B_3 as an induced subgraph, then G is a line graph by Theorem 6.5. Hence, we suppose that B_3 is an induced subgraph of G; say, $B_3 = G[Y]$ is the subgraph of G induced by the subset of nodes Y, $|Y| = 5$. We show that G is an induced subgraph of a cocktail-party graph. For this, consider the following property (P).

(P) For each subset $Z \subseteq V(G)$ such that $Y \subseteq Z$ and for each $i \in V(G) - Z$, if $G[Z]$ is an induced subgraph of a cocktail-party graph and $G[Z \cup \{i\}]$ is connected, then $G[Z \cup \{i\}]$ is also an induced subgraph of a cocktail-party graph.

We show that (P) holds, by induction on $|Z| \geq 5$.

(a) We show that (P) holds for $Z = Y$. Let $i \in V(G) - Y$ such that $G[Y \cup \{i\}]$ is connected. So, $G[Y \cup \{i\}]$ is a connected graph on six nodes containing $B_3 = K_5 - P_2$ as an induced subgraph. By direct inspection, one can check that there are eleven connected graphs on six nodes containing B_3 as an induced subgraph. Among them, we find H_1, H_2, H_3, H_4; we also find two graphs containing B_2 and three graphs containing B_1; these cases are excluded since G does not contain any member of $\mathcal{F}$. The remaining two graphs are $K_6 - P_2$ and $\nabla\nabla K_{2\times 2}$ which are, respectively, induced subgraphs of $K_{5\times 2}$ and $K_{4\times 2}$. Hence, the property (P) holds for $Z = Y$.

Consider now Z such that $Y \subseteq Z \subseteq V(G)$, $|Z| \geq 6$ and $G[Z]$ is an induced subgraph of a cocktail-party graph, and consider $i \in V(G) - Z$ such that $G[Z \cup \{i\}]$ is connected. Set $Y = \{y_1, y_2, y_3, y_4, y_5\}$ where, for instance, y_1 and y_2 are not adjacent in G and, thus, every other pair of nodes of Y is adjacent in G.

(b) Let $s, t \in Z$ such that s and t are not adjacent in G. We show that i is adjacent to both s and t. Since $G[Z]$ is contained in a cocktail-party graph, every other node of Z is adjacent to both s and t. Let $u \in Z$ be a node which is adjacent to i. Then, i is adjacent to at least one of s or t (else, $G[\{u, s, t, i\}]$ would be a B_1 induced subgraph of G). Hence, for $U = \{s, t, y_3, y_4, y_5\}$, $G[U]$ is B_3 and $G[U \cup \{i\}]$ is connected. By the argument from case

(a) above, we deduce that $G[U \cup \{i\}]$ is an induced subgraph of a cocktail-party graph, which implies that i is adjacent to both s and t.

(c) Let $s, t \in Z$ such that s and t are adjacent in G. We show that i is adjacent to at least one of s or t. If there exists $r \in Z$ which is not adjacent to s, then, by the argument of case (b) above, i is adjacent to both r and s. Similarly, if there exists $r \in Z$ which is not adjacent to t, then i is adjacent to t. Else, each $r \in Z$ is adjacent to both s and t. Let $r \in Z$ which is adjacent to i. We can find a set U such that $|U| = 5$, $r, s, t \in U$ and $G[U] = B_3$. Therefore, $G[U \cup \{i\}]$ is an induced subgraph of a cocktail-party graph, which implies that i is adjacent to at least one of s or t.

We deduce from (b) and (c) that $G[Z \cup \{i\}]$ is an induced subgraph of a cocktail-party graph. So, we have shown that (P) holds. □

THEOREM 6.8 *([7], [8]) Let G be a connected graph. Then, the following assertions are equivalent.*
(i) ∇G is a hypermetric graph, but not an ℓ_1-graph.
(ii) G is an induced subgraph of the Schläfli graph G_{27} and G contains as an induced subgraph one of the graphs of the family $\mathcal{F}_0 = \mathcal{F} - \{B_1, B_2, B_4, B_6\} = \{B_5, B_7, B_8, B_9, H_1, H_2, H_3, H_4\}$.

PROOF. $(i) \Longrightarrow (ii)$ By Theorem 6.7, if ∇G is not an ℓ_1-graph, then G contains as an induced subgraph one of the members of $\mathcal{F}$ and, in fact, of $\mathcal{F}_0$ since ∇B_1, ∇B_2, ∇B_4, ∇B_6 are not hypermetric (recall Remark 6.6). Let P denote the L-polytope associated with the hypermetric space $(V(\nabla G), 2d_{\nabla G})$ and let H denote its 1-skeleton. By Corollary 3.11, P is a generating L-polytope in a root lattice. Thus, P is a direct product of L-polytopes from Figure 1 and H is a direct product of L-polytopes graphs from Figure 1. In fact, since the graph G is connected, H is not a direct product, i.e. H is one of the L-polytope graphs from Figure 1. Now, H is G_{27} or G_{56} since all the other L-polytope graphs are ℓ_1-graphs. Therefore, ∇G is an isometric subgraph of G_{56} and, thus, G is an isometric subgraph of G_{27}.
$(ii) \Longrightarrow (i)$ is clear. □

COROLLARY 6.9 *[7] Let G be a connected graph on n nodes.*
(i) If $n \geq 37$, then ∇G is an ℓ_1-graph if and only if ∇G is 5-gonal and of negative type.
(ii) If $n \geq 28$, then ∇G is an ℓ_1-graph if and only if ∇G is hypermetric.

Assouad and Delorme ([7], [8]) have studied, more generally, the graphs G whose truncated distance d^*_G is ℓ_1-embeddable. We mention their results.

Let G be a bipartite graph. Then, d_G^* is isometrically ℓ_1-embeddable if and only if d_G^* is 31-gonal, i.e. it suffices to check all induced sugraphs on at most 31 nodes.

On the other hand, arbitrary graphs whose truncated distance is isometrically ℓ_1-embeddable cannot be characterized by a finite list of forbidden subgraphs. Indeed, for each $n \geq 2$, there exists a graph on $2n+1$ nodes for which d_G^* is not $2n+1$-gonal but, for all its proper induced subgraphs, their truncated distance is isometrically ℓ_1-embeddable.

Let λ be an integer, $\lambda \geq 1$. Then, the graphs for which d_G^* is isometrically ℓ_1-embeddable with scale λ, i.e. λd_G^* is isometrically hypercube embeddable, can be characterized by finitely many forbidden subgraphs. Namely, there exists an integer $n(\lambda)$ such that, for any graph G, d_G^* is isometrically ℓ_1-embeddable with scale λ if and only if the same holds for all induced subgraphs of G on at most $n(\lambda)$ nodes. For instance, $n(2) \leq 120$, $n(\lambda) = 5$ for λ odd.

6.2 Hypermetric regular graphs

We group here several results on the hypermetricity of the truncated distance space of a regular graph. They will apply, in particular, to the usual path metric of strongly regular graphs, i.e. distance-regular graphs of diameter 2.

Given a graph G on n nodes, we denote by D_G^* the symmetric $n \times n$ matrix whose (i,j)-entry is equal to $d_G^*(i,j)$, for all $i,j \in V(G)$.

The first result gives several equivalent characterizations for the truncated distance of a regular graph to be hypermetric.

Proposition 6.10 *[31] Let G be a connected regular graph on n nodes with valency k. Then, the following assertions are equivalent.*
(i) d_G^* is of negative type.
(ii) the distance space $(V(G), 2d_G^*)$ has a spherical representation with radius r satisfying $r^2 < 2$.
(iii) d_G^* is hypermetric.
(iv) ∇G is of negative type.
(v) $\lambda_{\min}(A_G) \geq -2$.
(vi) D_G^* has exactly one positive eigenvalue.
Moreover, if d_G^* is hypermetric, then the radius r of the L-polytope associated with the space $(V(G), 2d_G^*)$ is given by

$$r^2 = 2 - \frac{k+2}{n}. \qquad (34)$$

PROOF. $(i) \Longrightarrow (ii)$ Note that $\sum_{i\in V(G)} 2d^*_G(i,j) = 2(2n-2-k)$ is a constant. Hence, by Proposition 3.14, $(V(G), 2d^*_G)$ has a spherical representation whose radius r is given by relation (18), i.e. $r^2 = 2 - \frac{k+2}{n}$ and, thus, $r^2 < 2$.
$(ii) \Longrightarrow (iii)$ follows from Proposition 3.16.
$(iii) \Longrightarrow (iv)$ By Proposition 3.15, the radius of the L-polytope associated with $(V(G), 2d^*_G)$ is given by (34). Since $(V(\nabla G), 2d_{\nabla G})$ is the spherical 2-extension of the space $(V(G), 2d^*_G)$, we deduce by Lemma 3.17 that $(V(\nabla G), d_{\nabla G})$ is of negative type.
The equivalence $(iv) \Longleftrightarrow (v)$ follows from Proposition 6.4.
$(v) \Longrightarrow (vi)$ Let $\lambda_1 = k \geq \lambda_2 \geq \ldots \geq \lambda_n \geq -2$ denote the eigenvalues of the adjacency matrix A_G of G. Note that $D^*_G = J - (A_G + 2I)$, where J is the $n \times n$ matrix of all ones. The vector of all ones is a common eigenvector of A_G and D^*_G for the eigenvalues k and $2n-2-k$, respectively. One checks easily that the other eigenvalues of D^*_G are $-\lambda_2 - 2, \ldots, -\lambda_n - 2$ with $-\lambda_2 - 2 \leq \ldots \leq -\lambda_n - 2 \leq 0$. Hence, $2n-2-k$ is the only positive eigenvalue of D^*_G.
$(vi) \Longrightarrow (v)$ follows by reversing the arguments of $(v) \Longrightarrow (vi)$.
Using the obvious implication $(iv) \Longrightarrow (i)$, we obtain the equivalence of $(i) - (vi)$. □

Proposition 6.10 applies, in particular, to regular graphs of diameter 2; then, d_G and d^*_G coincide. However, without the regularity assumption, the equivalence of $(i) - (vi)$ does not hold. For instance, $K_9 - P_3$ and $K_{10} - P_3$ have diameter 2 (are not regular), satisfy (v) but not (iii) (recall Example 2).

Let G be a connected regular graph with $\lambda_{\min}(A_G) \geq -2$. Hence, its truncated distance d^*_G is hypermetric. Let P^*_G denote the L-polytope associated with the space $(V(G), 2d^*_G)$ and let H^*_G denote its 1-skeleton. By Proposition 3.9, $(V(G), d^*_G)$ is an isometric subspace of the graphic space $(V(H^*_G), d_{H^*_G})$. By Corollary 3.11, P^*_G is an L-polytope in a root lattice and, thus, H^*_G is a direct product of some of the L-polytope graphs shown in Figure 1. We show in the next result that, if H^*_G is a non trivial direct product, then it can only be the direct product of two complete graphs.

A bipartite graph B with bipartition $V_1 \cup V_2$ of its set of nodes is said to be **semiregular** if all nodes in V_1 (resp. V_2) have the same degree.

LEMMA 6.11 *[31] Let G be a connected regular graph on n nodes with valency k. Suppose that $\lambda_{\min}(A_G) \geq -2$ and let H^*_G denote the 1-skeleton of the L-polytope P^*_G associated with $(V(G), 2d^*_G)$. If H^*_G is a non trivial direct product, then $H^*_G = K_{n_1} \times K_{n_2}$ for some $n_1, n_2 \geq 1$, G is the line graph of a bipartite semiregular graph and $n = \frac{n_1+n_2}{n_1 n_2}(k+2)$.*

PROOF. Suppose that H is the non trivial direct product $H_1 \times H_2$. By assumption, $(V(G), d^*_G)$ is an isometric subspace of the graphic space $(V(H), d_H)$. Let $f : i \in V(G) \mapsto f(i) = (f_1(i), f_2(i)) = (i_1, i_2) \in V(H_1) \times V(H_2)$ denote this isometric embedding. For

$i \in V(G)$, set $V_1(i) = \{j \in V(G) : f_1(i) = f_1(j)\}$ and $V_2(i) = \{j \in V(G) : f_2(i) = f_2(j)\}$. If i, j are adjacent in G, then $j \in V_1(i) \cup V_2(i)$. Conversely, we check that, if $|V_1(i)|, |V_2(i)| > 1$, then both $V_1(i)$ and $V_2(i)$ induce a complete graph in G.

For this, let $j \in V_1(i)$ and $h \in V_2(i)$ with $j \neq i$, $h \neq i$. Then, $2 \geq d^*_G(j,h) = d_{H_1}(j_1,h_1) + d_{H_2}(j_2,h_2) = d_{H_1}(i_1,h_1) + d_{H_2}(j_2,i_2)$ (since $i_1 = j_1$ and $i_2 = h_2$) which is equal to $d^*_G(i,h) + d^*_G(i,j) \geq 2$. This implies that $d^*_G(i,h) = d^*_G(i,j) = 1$, i.e. both h and j are adjacent to i. One deduces easily that any two nodes in V_1, or in V_2 are adjacent.

Therefore, if $|V_1(i)|, |V_2(i)| > 1$, then $|V_1(i)| + |V_2(i)| = k+2$. For $j \in V_1(i)$, $V_1(i) = V_1(j)$, $k+2 \leq |V_1(j)| + |V_2(j)|$, implying that $|V_2(i)| \leq |V_2(j)|$ and, thus, $|V_1(j)|, |V_2(j)| > 1$, yielding $k+2 = |V_1(j)| + |V_2(j)|$ and, thus, $|V_2(j)| = |V_2(i)|$. Therefore, since G is connected, there exist integers $p, q \geq 1$ such that $|V_1(i)| = p$, $|V_2(i)| = q$ for all $i \in V(G)$.

Let B denote the bipartite graph with node bipartition $V_1 \cup V_2$, where $V_1 = f_1(V(G)) \subseteq V(H_1)$ and $V_2 = f_2(V(G)) \subseteq V(H_2)$, and two nodes $i_1 \in V_1$, $i_2 \in V_2$ are adjacent in B if $(i_1, i_2) = f(i)$ for some node $i \in V(G)$. So each node of V_1 (resp. of V_2) has valency p (resp. q), i.e. B is semiregular. It is immediate to see that G is the line graph of B.

We now check that H_1 and H_2 are complete graphs. Set $n_1 = |V_1|$, $n_2 = |V_2|$ and $n = |V(G)$. Let r denote the radius of the L-polytope P^*_G; r is given by relation (34). So, $r^2 = 2 - \frac{k+2}{n} = \frac{n_1-1}{n_1} + \frac{n_2-1}{n_2}$. Let r_m denote the radius of the L-polytope whose 1-skeleton is the graph H_m, for $m = 1, 2$. Then, $r^2 = r_1^2 + r_2^2$ holds. We use the following observation: For each L-polytope P in a root lattice, its radius r satisfies $r^2 \geq \frac{|V(P)|-1}{|V(P)|}$ with equality if and only if P is a simplex. Therefore, $r_m^2 \geq \frac{|V(H_m)|-1}{|V(H_m)|} \geq \frac{n_m-1}{n_m}$, since $|V(H_m)| \geq n_m$, for $m = 1, 2$. But, $r^2 = r_1^2 + r_2^2 = \frac{n_1-1}{n_1} + \frac{n_2-1}{n_2}$, from which we deduce that $r_m^2 = \frac{n_m-1}{n_m}$, $|V(H_m)| = n_m$ and, thus, H_m is the complete graph K_{n_m} for $m = 1, 2$. □

COROLLARY 6.12 *[31] Let G be a connected regular graph on n nodes with valency k and such that $\lambda_{\min}(A_G) \geq -2$. Then, one of the following assertions holds.*
(i) G is the line graph of a bipartite semiregular graph and $n = \frac{n_1 n_2}{n_1+n_2}(k+2)$, for some $n_1, n_2 \geq 1$.
(ii) G is the line graph of a regular graph and $n = \frac{m}{4}(k+2)$ for some $m \geq 3$.
(iii) $G = K_{m\times 2}$ and $n = k+2$.
(iv) G is an induced subgraph of the Gosset graph G_{56} and $n = 2(k+2)$.
(v) G is an induced subgraph of the Schläfli graph G_{27} and $n = \frac{3}{2}(k+2)$.
(vi) G is an induced subgraph of the Clebsch graph $\frac{1}{2}H(5,2)$ and $n = \frac{3}{2}(k+2)$.

PROOF. Let H^*_G denote the 1-skeleton of the L-polytope P^*_G associated with $(V(G), 2d^*_G)$. If H^*_G is a direct product, then we have (i) by Lemma 6.11. So we now suppose that H^*_G

is one of the L-polytope graphs from Figure 1. We know that the radius r of P_G^* satisfies $r^2 = 2 - \frac{k+2}{n} < 2$.

• If $H_G^* = J(m,t)$ for some $t \geq 1$, $n \geq 2t$, then $r^2 = \frac{t(m-t)}{m} < 2$ implying that $t = 1,2,3$. If $H_G^* = J(m,1) = K_m$, then $G = H_G^* = K_m$ is the line graph of the bipartite semiregular graph $K_{1,m}$; hence, $m = n$ and we have (i). If $H_G^* = J(m,2) = L(K_m)$, then G is a line graph. Since G is regular, one can check that G is the line graph of a regular graph or a bipartite semiregular graph. Since $r^2 = \frac{2(m-2)}{m}$, we deduce that $n = \frac{m}{4}(k+2)$. So, we have (i) or (ii). If $H_G^* = J(m,3)$, then $m = 6,7,8$. If $H_G^* = J(6,3)$, then G is an induced subgraph of G_{56} and $r^2 = \frac{3}{2} = 2 - \frac{k+2}{n}$, yielding $n = 2(k+2)$, i.e. we have (iv). The cases $m = 7,8$ are excluded. Indeed, one can check that every subgraph K of $J(m,3)$ $(m = 7,8)$ such that K is not contained in $J(6,3)$ or $J(n,2)$ and no pair of nodes of K are at distance 3 in $J(m,3)$ has strictly less than $\frac{m(k+2)}{9-m}$ nodes.

• If $H_G^* = K_{m\times 2}$, then we have (iii).

• If $H_G^* = \frac{1}{2}H(m,2)$ for some $m \geq 4$, then $r^2 = \frac{m}{4} < 2$, implying that $m = 4,5,6,7$. If $m = 4$, then $H_G^* = K_{4\times 2}$ and, thus, we have (iii). If $m = 5$, then $r^2 = \frac{5}{4}$ yielding $n = \frac{4}{3}(k+2)$ and, thus, we have (vi). If $m = 6$, then $r^2 = \frac{3}{2}$ yielding $n = 2(k+2)$ and, thus, we have (iv) since $\frac{1}{2}H(6,2)$ is an isometric subgraph of G_{56}. The case $m = 7$ is excluded (similarly to the exclusion above of the cases $J(7,3)$ and $(J(8,3)$; indeed, there is no k-regular subgraph of $\frac{1}{2}H(7,2)$ on $n = 4(k+2)$ nodes which is not contained in $\frac{1}{2}H(6,2)$ or $J(7,2)$ and does not contain a pair of vertices at distance 3).

• If $H_G^* = G_{56}$, then we have (iv) and, if $H_G^* = G_{27}$, then we have (v). □

REMARK 6.13 *Under the assumptions of Corollary 6.12, the only possibilities for the 1-skeleton H_G^* of the L-polytope P_G^* associated with the hypermetric space $(V(G), 2d_G^*)$ are $H_G^* = K_{n_1} \times K_{n_2}$, $J(m,1)$, $J(m,2)$, $J(6,3)$, $K_{m\times 2}$, $\frac{1}{2}H(5,2)$, $\frac{1}{2}H(6,2)$, G_{27} and G_{56}. In particular, if G is not a line graph nor a cocktail-party graph, then H_G^* is one of $J(6,3)$, $\frac{1}{2}H(5,2)$, $\frac{1}{2}H(6,2)$, G_{27} or G_{56}. Note that the radius r of the L-polytope P_G^* satisfies $r^2 = \frac{5}{4}$ for $\frac{1}{2}H(5,2)$, $r^2 = \frac{4}{3}$ for G_{27} and $r^2 = \frac{3}{2}$ for $\frac{1}{2}H(6,2)$, $J(6,3)$ and G_{56}.*

The graphs for which $\lambda_{\min}(A_G) \geq -2$ are well studied. It is easy to check that $\lambda_{\min}(A_G) \geq -2$ for every line graph G (indeed, if $G = L(H)$, then $2I + A_G = N^TN$, where N is the node-edge incidence matrix of H); moreover, $\lambda_{\min}(A_G) = -2$ if and only if G contains an even circuit or two odd circuits (see [20]). If G is a cocktail-party graph, one computes easily that $\lambda_{\min}(A_G) = -2$.

Let $\mathcal{L}_{BCS}$ denote the class of graphs which are connected, regular, not line graphs nor cocktail-party graphs and satisfy $\lambda_{\min}(A_G) = -2$. This class has been extensively studied.

In fact, it is completely classified; see [20]. $\mathcal{L}_{BCS}$ consists of 187 graphs, each of them has $n \leq 28$ nodes and valency $k \leq 16$. The graphs of $\mathcal{L}_{BCS}$ are partitioned into three layers $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$ depending on the value of the quantity $\frac{n}{k+2}$, where n is the number of nodes and k the valency of a graph in $\mathcal{L}_{BCS}$. The layer $\mathcal{L}_1$ (resp. $\mathcal{L}_2$, $\mathcal{L}_3$) consists of the graphs $G \in \mathcal{L}_{BCS}$ for which $\frac{n}{k+2} = 2$ (resp. $\frac{n}{k+2} = \frac{3}{2}$, $\frac{n}{k+2} = \frac{4}{3}$).

Our approach permits to shed a new light on the parameter $\frac{k+2}{n}$ characterizing each layer of $\mathcal{L}_{BCS}$. Namely, the parameter $\frac{k+2}{n}$ is nothing but the quantity $2 - r^2$, where r is the radius of the L-polytope associated with the hypermetric space $(V(G), 2d^*_G)$ for any graph $G \in \mathcal{L}_{BCS}$. Therefore, each layer in $\mathcal{L}_{BCS}$ is characterized by a quantity derived from the hypermetricity of its graphs.

We summarize several facts about the class $\mathcal{L}_{BCS}$ and its three layers.

• The first layer $\mathcal{L}_1$ consists of 163 graphs (the graphs NN1-163 in [20]); it is characterized by $\frac{n}{k+2} = 2$. For each graph $G \in \mathcal{L}_{BCS}$, the L-polytope P^*_G associated with the hypermetric space $(V(G), 2d^*_G)$ has radius $\frac{3}{2}$ and its 1-skeleton is $\frac{1}{2}H(6,2)$, $J(6,3)$ or G_{56}. Hence, each graph $G \in \mathcal{L}_1$ is an induced subgraph of G_{56} and, thus, has diameter 2 or 3. Therefore, the graphs of $\mathcal{L}_1$ with diameter 2 are hypermetric with L-polytope graph $\frac{1}{2}H(6,2)$, $J(6,3)$ or G_{56}.

• The second layer $\mathcal{L}_2$ consists of 21 graphs (the graphs NN164-184 in [20]) including the Schläfli graph G_{27} (which is N184 in [20]). It is characterized by the value $\frac{n}{k+2} = \frac{3}{2}$. For each $G \in \mathcal{L}_2$, the L-polytope P^*_G is 2_{21} with radius r, $r^2 = \frac{4}{3}$. Hence, each $G \in \mathcal{L}_2$ is an isometric subgraph of G_{27} and, thus, has diameter 2 and is hypermetric.

• The third layer $\mathcal{L}_3$ consists of 3 graphs; they are the Clebsch graph $\frac{1}{2}H(5,2)$ (N187 in [20]) and two of its regular subgraphs (the graphs NN185,186 in [20]). $\mathcal{L}_2$ is characterized by the value $\frac{n}{k+2} = \frac{4}{3}$. For each graph $G \in \mathcal{L}_3$, $P^*_G = h\gamma_5$ with radius r, $r^2 = \frac{5}{4}$, with 1-skeleton $\frac{1}{2}H(5,2)$. Therefore, each graph of $\mathcal{L}_3$ is an isometric subgraph of $\frac{1}{2}H(5,2)$ and, thus, has diameter 2 and is an ℓ_1-graph with L-polytope graph $\frac{1}{2}H(5,2)$.

We conclude this section with some results on hypermetric distance-regular graphs.

A graph G is **distance-regular** if there exist integers b_m, c_m $(m > 0)$ such that for any two nodes $i, j \in V(G)$ at distance $d_G(i,j) = m$ there are exactly c_m nodes at distance 1 from i and distance $m-1$ from j, and there are b_m nodes at distance 1 from i and distance $m+1$ from j. Hence, G is regular with valency b_0 and there are k_m nodes at distance m from any node $i \in V(G)$, where $k_0 = 1$, $k_1 = 1$, $k_{m+1} = \frac{k_m b_m}{c_{m+1}}$, $m \geq 0$.

If G is distance-regular, then $\sum_{i \in V(G)} d_G(i,j) = \sum_{m \geq 0} m k_m$ is a constant. Therefore, from Proposition 1.6, a distance-regular graph is of negative type if and only if the space $(V(G), d_G)$ has a spherical representation.

Let μ denote the number of common neighbours of two nodes at distance 2, i.e. $\mu = c_2$. Koolen [51] has classified the hypermetric distance-regular graphs with $\mu \geq 2$. We recall his result (see [19] or [51] for the description of the graphs not defined here).

Theorem 6.14 *(Theorem 3.15 in [51]) Let G be a distance-regular graph with $\mu \geq 2$. Then, G is a hypermetric graph if and only if one of the following holds.*
(i) $\mu = 2n-2$ and G is a cocktail-party graph $K_{n\times 2}$.
(ii) $\mu = 10$ and G is the Gosset graph G_{56}.
(iii) $\mu = 8$ and G is the Schläfli graph G_{27}.
(iv) $\mu = 6$ and G is a half-cube graph.
(v) $\mu = 4$ and G is a Chang graph.
(vi) $\mu = 4$ and G is a Johnson graph.
(vii) $\mu = 2$ and G is a Hamming graph.
(viii) $\mu = 2$ and G is a Doob graph (including the Schrikhande graph).
(ix) $\mu = 2$ and G is the icosahedron graph.

The following distance-regular graphs with $\mu = 1$ are hypermetric: the cycle C_n, the double-odd graph DO_{2n+1}, the dodecahedron graph, the Petersen graph. (In fact, they are all ℓ_1-graphs). The distance-regular graphs which are isometric subgraphs of a hypercube are precisely the double-odd graph DO_{2n+1}, the hypercube $H(n,2)$ and the even cycle C_{2n} ([51], [67]).

Proposition 6.15 *([51], [31]) Let G be a strongly regular graph. Then, G is hypermetric if and only if G is one of the following graphs: $K_n \times K_n$, $J(n,2)$, $K_{n\times 2}$, $\frac{1}{2}H(n,2)$, G_{27}, the 5-cycle C_5, the Petersen graph, the Schrikhande graph, or one of the three Chang graphs.*

6.3 Extreme hypermetric graphs

In this section, we consider extreme hypermetric graphs, i.e. the graphs G whose path metric d_G lies on an extreme ray of the hypermetric cone.

Let G be a hypermetric graph. Let P_G denote the L-polytope associated with the hypermetric space $(V(G), 2d_G)$ and let H_G denote its 1-skeleton. Hence, P_G is an L-polytope in a root lattice and G is an isometric subgraph of H_G. Moreover, G is an extreme hypermetric if and only if P is an extreme L-polytope (by Theorem 4.5). By Theorem 5.1, the only extreme L-polytopes in a root lattice are the segment α_1, the Schläfli polytope 2_{21} and the Gosset polytope 3_{21}. Therefore, if G is an extreme hypermetric graph distinct from K_2, then,

• either $H_G = G_{56}$, i.e. G is an isometric subgraph of G_{56} which is generating (i.e. $V(G)$ viewed as subset of the set of vertices $V(3_{21})$ of 3_{21} generates $V(3_{21})$); we say that G is an extreme hypergraph of **Type I**.

• or $H_G = G_{27}$, i.e. G is an isometric subgraph of G_{27} which is generating (i.e. $V(G)$ generates $V(2_{21})$); we say that G is an extreme hypermetric graph of **Type II**.

A generating subset in G_{27} has at least 7 elements. We found that there are 26 distinct (up to permutation) generating subsets in G_{27} with 7 elements (i.e. basic subsets of 2_{21}; see Section 5.2). For $B \subseteq V(G_{27})$, let $G_{27}[B]$ denote the subgraph of G_{27} induced by B. Note that $G_{27}[B]$ is an isometric subgraph of G_{27} if and only if $G_{27}[B]$ has diameter 2 and, then, $G_{27}[B]$ is a hypermetric graph. Among the 26 basic subsets B of G_{27} (whose graphs $G_{27}[B]$ are shown in Figures 3, 4, 5 and 6), the graph $G_{27}[B]$ has diameter 2 for twelve of them, namely for the graphs G_i for $1 \leq i \leq 8$, G_{16}, G_{18}, G_{24} and G_{26}. Hence, these twelve graphs are extreme hypermetric graphs on 7 nodes with L-polytope graph G_{27}. We recall that $G_1 = \nabla B_9$, $G_2 = \nabla H_2$, $G_3 = \nabla H_1$, $G_4 = \nabla B_8$, $G_5 = \nabla B_7$, $G_6 = \nabla H_4$, $G_7 = \nabla H_3$ and $G_8 = \nabla B_5$, where the graphs B_i $(1 \leq i \leq 8)$ and H_i $(1 \leq i \leq 4)$ are shown in Figures 9 and 10, respectively. We show in Figure 11 the graphs G_{16}, G_{18}, G_{24} and G_{26} (their complements are shown in Figures 4, 5 and 6).

Figure 11

For each of the above twelve graphs, their suspension ∇G_i (for $1 \leq i \leq 8$, $i = 16, 18, 24, 26$) is an extreme hypermetric graph on 8 nodes with L-polytope graph G_{56}.

Lemma 6.16 *[31] Let H be a maximal (by inclusion) L-polytope graph which is a proper isometric subgraph of G_{56}. Then, H is one of the following graphs.*
(i) $H = J(8,2)$.
(ii)$H = K_{6\times 2} \times K_2$.
(iii) $H = \frac{1}{2}H(6,2)$.
(iv) $H = G_{27}$.

Proof. We know that H is a direct product of the L-polytope graphs from Figure 1. Let r denote the radius of the L-polytope whose 1-skeleton is H. Then, $r^2 \leq \frac{3}{2}$, since H is contained in G_{56}.

• If $H = J(n,t)$, then $r^2 = \frac{t(n-t)}{n} \leq \frac{3}{2}$, implying that $t = 1, 2, 3$. Then, H is not maximal except for $J(8,2)$. Indeed, if $t = 1$, then $n \leq 7$ and $K_n \subset_{iso} J(8,2)$; if $t = 2$, then $n \leq 8$ and $J(n,2) \subset_{iso} J(8,2)$; if $t = 3$, then $n = 6$ and $J(6,3) \subset_{iso} \frac{1}{2}H(6,2)$.

• If $H = K_{n\times 2}$, then $n \le 6$ and $K_{n\times 2} \subset_{iso} K_{6\times 2} \times K_2$.

• If $H = \frac{1}{2}H(n,2)$, then $r^2 = \frac{n}{4} \le \frac{3}{2}$, implying that $n \le 6$ and, thus, $H \subset_{iso} \frac{1}{2}H(6,2)$.

Else $H = G_{27}$ or H is a direct product. Suppose that $H = H_1 \times H_2$. Denote by r_1, r_2 the radius of the L-polytope whose 1-skeleton is H_1, H_2, respectively. Then, $r^2 = r_1^2 + r_2^2 \le \frac{3}{2}$. Looking at the radii of the L-polytopes from Figure 1, it is easy to see that the only possibility is $H_1 = K_{6\times 2}$, $H_2 = K_2$ ($r_1^2 = 1$, $r_2^2 = \frac{1}{2}$) (for instance, for $H_1 = H_2 = K_4$, $r_1^2 = r_2^2 = \frac{3}{4}$ but $K_4 \times K_4 \subset_{iso} J(8,2)$). □

LEMMA 6.17 *[31] Let H be a maximal (by inclusion) L-polytope graph which is a proper isometric subgraph of G_{27}. Then, one of the following holds.*
(i) $H = J(6,2)$.
(ii) $H = K_{5\times 2}$.
(iii) $H = \frac{1}{2}H(5,2)$.
(iv) $H = K_6$.

PROOF. The proof is similar to that of Lemma 6.16. We use the fact that the radius r of the L-polytope whose 1-skeleton is H satisfies $r^2 \le \frac{4}{3}$. It is easily seen that H cannot be a direct product.
• If $H = J(n,t)$, then $r^2 = \frac{t(n-t)}{n} \le \frac{4}{3}$, implying that $t = 1, 2$ and $n \le 6$. Hence, we have (i) or (iv).
• If $H = K_{n\times 2}$, then $n \le 5$ (because $K_{6\times 2}$ is not contained in G_{27}) and, thus, $H \subset_{iso} K_{5\times 2}$.
• If $H = \frac{1}{2}H(n,2)$, then $r^2 = \frac{n}{4} \le \frac{4}{3}$, implying that $n \le 5$ and, thus, $H \subset_{iso} \frac{1}{2}H(5,2)$. □

We deduce the following characterization for extreme hypermetric graphs.

PROPOSITION 6.18 *[31] Let G be a connected graph distinct from K_2. Then, G is an extreme hypermetric graph if and only if one of the following assertions hold.*
(i) Type I: G is an isometric subgraph of G_{56} and G is not an induced subgraph of $J(8,2)$, $K_{6\times 2} \times K_2$, $\frac{1}{2}H(6,2)$ or G_{27}.
(ii) Type II: G is an isometric subgraph of G_{27} and G is not an induced subgraph of $K_{5\times 2}$, $J(6,2)$, K_6 or $\frac{1}{2}H(5,2)$.

Observe that all the excluded graphs in Proposition 6.18 are ℓ_1-graphs. In other words, every isometric subgraph of G_{56} is either an extreme hypermetric graph, or an ℓ_1-graph.

As an application of Proposition 6.18, we obtain that:
• Every isometric subgraph of G_{27} on $n \ge 17$ nodes is extreme.
• Every induced subgraph of G_{27} on $n \ge 20$ nodes is extreme (since deleting 7 nodes from

G_{27} preserves the diameter 2 because $\mu(G_{27}) = 8$).
• Every isometric subgraph of G_{56} on $n \geq 33$ nodes is extreme.
• Every induced subgraph of G_{56} on $n \geq 47$ nodes is extreme (since $\mu(G_{56}) = 10$).
• If G is a connected graph of diameter 2, then its suspension ∇G is an extreme hypermetric graph of Type I if and only if G is an extreme hypermetric graph of Type II.

We now collect some properties for extreme hypermetrics arising from the graphs $G \in \mathcal{L}_{BCS}$.

As we saw in Section 6.2, if G is a connected regular graph with $\lambda_{\min}(A_G) \geq -2$, then its truncated distance d_G^* is hypermetric. Let P_G^* denote the L-polytope associated with $(V(G), 2d_G^*)$ and let H_G^* denote its 1-skeleton.

Suppose that G belongs to the class $\mathcal{L}_{BCS}$, i.e. G is connected regular with $\lambda_{\min}(A_G) = -2$ and G is not a line graph nor a cocktail-party graph. By Remark 6.13, H_G^* is one of $J(6,3)$, $\frac{1}{2}H(5,2)$, $\frac{1}{2}H(6,2)$, G_{27} or G_{56}. Since $(V(G), d_G^*)$ is an isometric subspace of $(V(H_G^*), d_{H_G^*})$ which, in turn, is an isometric subspace of $(V(G_{56}), d_{G_{56}})$, we deduce that G does not contain any pair of nodes at distance 3 in G_{56}; in particular, if G is an induced subgraph of $\frac{1}{2}H(6,2)$, then G has at most $n \leq 16$ nodes.

This implies the following Proposition 6.19

Proposition 6.19 *[31] Let G be a graph of $\mathcal{L}_{BCS}$. Then, if G is not an induced subgraph of $\frac{1}{2}H(6,2)$, then d_G^* is extreme hypermetric. In particular, if G is on $n \geq 17$ nodes, then d_G^* is extreme hypermetric.*

Proposition 6.20 *[31] A graph $G \in \mathcal{L}_{BCS}$ is extreme hypermetric if and only if it has diameter 2 and it is not an induced subgraph of $\frac{1}{2}H(6,2)$.*
Every extreme regular hypermetric graph of diameter 2 belongs to $\mathcal{L}_{BCS}$.
Let G be an extreme hypermetric graph from $\mathcal{L}_{BCS}$; then, G is of Type I (resp. Type II) if and only if G belongs to the layer $\mathcal{L}_1$ (resp. $\mathcal{L}_2$).
Every graph from $\mathcal{L}_{BCS}$ on $n \geq 17$ and with valency $k \geq 9$ is extreme. They are the 29 graphs in layer $\mathcal{L}_1$ numbered NN135-163 in [20] and the 8 graphs in layer $\mathcal{L}_2$ numbered NN177-184 in [20].
All the 9 maximal (by inclusion) graphs of $\mathcal{L}_{BCS}$ are extreme hypermetric graphs; they are the Schläfli graph G_{27} numbered N184in [20], the three Chang graphs NN161, 162, 163, and the five graphs NN148-152 on 22 nodes.

7 Hypermetric inequalities for the cut cone

Set $X = \{1, \ldots, n\}$. We recall that the cut cone CUT_n is the cone in $\mathbb{R}^{\binom{n}{2}}$ generated by the cut semimetrics $\delta(S)$ for $S \subseteq X$. In fact, CUT_n consists of the semimetrics d on X

for which the distance space (X,d) is isometrically ℓ_1-embeddable (see Proposition 2.1). In the context of graph theory and combinatorial optimization, the set of edges ij of the complete graph K_n that have an endnode in S and the other endnode in $X-S$ is called a **cut**; so the cut semimetric $\delta(S)$ is its incidence vector. For the sake of simplicity, we call $\delta(S)$ a cut semimetric or a cut.

The cone CUT_n can be alternatively described by a system of linear inequalities, its valid inequalities. Recall that, for $v \in \mathbb{R}^{\binom{n}{2}}$, the inequality $v^T x \leq 0$ is said to be **valid** for CUT_n if it is satisfied by all cut semimetrics $\delta(S)$, $S \subseteq X$. Moreover, the valid inequality $v^T x \leq 0$ defines a **facet** of CUT_n if there exist $\binom{n}{2}-1$ linearly independent cut semimetrics satisfying the equality $v^T x = 0$. Hence, finding the valid inequalities for CUT_n amounts to characterizing ℓ_1-embeddable semimetrics by linear inequalities.

We have seen in this paper two important classes of valid inequalities for the cut cone, namely the hypermetric inequalities and the inequalities of negative type (defined in relation (4); recall Proposition 2.5). In fact, the inequalities of negative type never define facets of the cut cone since they are implied by the hypermetric inequalities (this is the implication $(iii) \Longrightarrow (v)$ from Proposition 2.5).

We describe in section 7.1 some classes of hypermetric inequalities that define facets of the cut cone; we also present some generalizations of hypermetric inequalities yielding new valid inequalities for the cut cone. In section 7.2, we describe how, by analogy with hypermetric inequalites, some inequalities can be constructed that are valid for other cut families, as even T-cuts, t-ary cuts, multicuts or even multicuts. We present in section 7.3 the inequalities that arise by "switching" the hypermetric inequalities; they are valid for the cut polytope. They are, in fact, part of the much larger class of gap inequalities.

As a curiosity, let us mention an analogue of hypermetric inequalities for a class of non necessarly symmetric distance functions. Namely, let $(\Omega, \mathcal{A}, \mu)$ be a nonnegative measure space and let A_x, $x \in X$, be members of $\mathcal{A}$ with finite measure, i.e. $\mu(A_x) < \infty$. Consider the function $d : X^2 \mapsto \mathbb{R}_+$ defined by $d(x,y) = \mu(A_x - A_y)$ for $x, y \in X$. Then, d satisfies the inequality $\sum_{1 \leq i < j \leq m} (-1)^{i+j} d(x_i, x_j) \leq 0$ for all $x_1, \ldots, x_m \in X$, $m \geq 1$ ([4]). To enable the reader to compare this inequality with hypermetric inequalities, we make the following two observations.

- First, any hypermetric inequality $\sum_{1 \leq i < j \leq n} b_i b_j x_{ij} \leq 0$, with $\sum_{1 \leq i \leq n} b_i = 1$ and $b \in \mathbb{Z}^n$, can be viewed as the inequality $\sum_{1 \leq i < j \leq P} x_{ij} + \sum_{P+1 \leq i < j \leq N} x_{ij} - \sum_{1 \leq i \leq P < j \leq N} x_{ij} \leq 0$ if we set $P = \sum_{i : b_i > 0} b_i$, $N = \sum_{1 \leq i \leq n} |b_i|$ and if we allow repetition of the points.
- Second, a semimetric d on X is isometrically ℓ_1-embeddable if and only if there exist a nonnegative measure space $(\Omega, \mathcal{A}, \mu)$ and sets $A_x \in \mathcal{A}$, $x \in X$, of finite measure such that $d(x,y) = \mu(A_x \triangle A_y)$ for all $x, y \in X$ ([4]).

7.1 Hypermetric facets for the cut cone

Hypermetric inequalities form a large class of valid inequalities for the cut cone. Therefore, we have the inclusion $\mathrm{CUT}_n \subseteq \mathrm{HYP}_n$ for all $n \geq 3$. In fact, for $n \leq 6$, the hypermetric inequalities suffice for describing the cut cone, i.e. $\mathrm{CUT}_n = \mathrm{HYP}_n$ for $n \leq 6$, but the inclusion $\mathrm{CUT}_n \subseteq \mathrm{HYP}_n$ is strict for $n \geq 7$. In other words, CUT_n ($n \geq 7$) has a facet that is not defined by a hypermetric inequality or, equivalently, HYP_n ($n \geq 7$) has an extreme ray that is not generated by a cut semimetric. Indeed, such extreme rays of HYP_n arise from the extreme L-polytope 2_{21} (see section 5.2). For examples of non hypermetric facets of CUT_n, see e.g. [36], [37]; the complete description of the facets of CUT_7 can be found there.

Hypermetric facets for the cut cone have been studied in several papers ([29], [36], [37],[38], [62]); we refer to [35] for a survey.

We now recall the description of CUT_n for $n \leq 6$. As a short notation, let us denote the hypermetric inequality $\sum_{1\leq i<j\leq n} b_ib_jx_{ij} \leq 0$, where $b \in \mathbb{Z}^n$ with $\sum_{1\leq i\leq n} b_i = 1$, by $\mathrm{Hyp}_n(b_1,\ldots,b_n)$, or $\mathrm{Hyp}_n(b)$.

For $n = 3, 4$, the cut cone CUT_n is completely determined by the triangle inequalities $x_{ij} - x_{ik} - x_{jk} \leq 0$ for $i, j, k \in X$.

For $n = 5$, the facets of CUT_5 are (up to permutation of the nodes) defined by one of the following hypermetric inequalities $\mathrm{Hyp}_5(1,1,-1,0,0)$ and $\mathrm{Hyp}_5(1,1,1,-1,-1)$ ([28], [29]).

For $n = 6$, the facets of CUT_6 are (up to permutation of the nodes) defined by one of the following hypermetric inequalities $\mathrm{Hyp}_6(1,1,-1,0,0,0)$, $\mathrm{Hyp}_6(1,1,1,-1,-1,0)$, $\mathrm{Hyp}_6(2,1,1,-1,-1,-1)$ and $\mathrm{Hyp}_6(1,1,1,1,-2,-1)$ ([15]).

The complete characterization of the hypermetric inequalities $\mathrm{Hyp}_n(b)$ that define facets of CUT_n seems to be a hard problem. Note that, if $\mathrm{Hyp}_n(b)$ defines a facet of CUT_n, then it also defines a facet of HYP_n and, thus, its associated L-polytope is a repartitioning polytope $P^m_{p,q}(b)$ ($m = |\{i : b_i = 0\}|$, $p+1 = |\{i : b_i > 0\}|$, $q = |\{i : b_i < 0\}|$) which can be embedded in a parallepiped; see Propositions 3.7 and 4.10. In particular, by Theorem 4.11, if $\mathrm{Hyp}_n(b)$ defines a facet of CUT_n, then $\max(|b_i| : 1 \leq i \leq n) \leq \frac{2^{n-2}(n-1)!}{n+1}$. Let β_n denote the maximum absolute value of an $n \times n$ determinant with binary entries; then, the following bound $\max(|b_i| : 1 \leq i \leq n) \leq \beta_{n-1}$ can be shown in an elementary way ([12]).

Example 8. Consider the hypermetric inequality $\mathrm{Hyp}_7(b)$ for $b = (3,1,1,1,-1,-2,-2)$. This is an example of a hypermetric inequality that defines a facet of HYP_7, but not of CUT_7.

Indeed, there are 20 affinely independent cut semimetrics satisfying the hypermetric inequality $\mathrm{Hyp}_7(b)$ at equality. The truncated distance of the graph G_9 shown in Figure

3 (taking value 1 on the edges of G_9 and value 2 on the pairs that are not edges) also satisfies the hypermetric inequality $\mathrm{Hyp}_7(b)$ at equality (for this, label the nodes of G_9 as 1,2,3,4,5,6,7 if their degrees in the complement $\bar{G}_9$ of G_9 are 3,2,2,2,5,1,1, respectively). So, this distance together with the 20 cut semimetrics form a set of 21 affinely independent distances satisfying $\mathrm{Hyp}_7(b)$ at equality.
Note also that the truncated distance of the graph G_{12} (with a suitable labeling of its nodes) (resp. G_{11}, G_{10}) satisfies the hypermetric inequality $\mathrm{Hyp}_7(-3,1,1,1,1,-2,2)$ (resp. $\mathrm{Hyp}_7(3,-1,-1,-1,1,-2,2)$, $\mathrm{Hyp}_7(-3,1,-1,-1,1,2,2)$) at equality. In fact, the hypermetric inequalities $\mathrm{Hyp}_7(-3,1,1,1,1,-2,2), \mathrm{Hyp}_7(3,-1,-1,-1,1,-2,2), \mathrm{Hyp}_7(-3,1,-1,-1,1,2,2)$ are switchings of $\mathrm{Hyp}_7(b)$ by the cuts $\delta(\{1,5,7\})$, $\delta(\{1,6\})$, $\delta(\{2\})$, respectively (see Section 7.3 for the definition of switching).

The complete characterization of the hypermetric facets $\mathrm{Hyp}_n(b)$ for CUT_n is known for the following classes of parameters $b = (b_1, \ldots, b_n)$:
(i) $b_1 \geq \ldots \geq b_p > 0 > b_{p+1} \geq \ldots \geq b_n$ with $b_{n-1} = -1$ (i.e. all negative b_i's except at most one are equal to -1) ([29], [36]).
(ii) $b_i \in \{w, -w, 1, -1\}$ for all $i = 1, \ldots, n$, for some integer $w \geq 2$ ([37]).
For instance, in case (i), $Hyp_n(b_1, \ldots, b_p, -1, \ldots, -1)$ defines a facet of CUT_n for all $3 \leq p \leq n-3$, $b_1, \ldots, b_p \geq 1$ and $b_1 + \ldots + b_p - (n-p) = 1$.

One of the main tools for constructing hypermetric facets is a lifting procedure permitting to obtain a hypermetric facet of CUT_{n+1} from a given hypermetric facet of CUT_n. For instance, if $\mathrm{Hyp}_n(b)$ defines a facet of CUT_n, then $\mathrm{Hyp}_{n+1}(b,0)$ defines a facet of CUT_{n+1} (this is 0-lifting).

Recall that a valid inequality $v^T x \leq 0$ defines a **simplicial face** of CUT_n if the semimetrics $\delta(S)$ satisfying the equality $v^T x = 0$ are affinely inedependent. For example, $\mathrm{Hyp}_3(1,1,-1)$, $\mathrm{Hyp}_4(1,1,-1,0)$, $\mathrm{Hyp}_5(1,1,1,-1,-1)$ define simplicial facets of CUT_3, CUT_4, CUT_5, respectively. More generally, $\mathrm{Hyp}_n(n-4,1,1,-1,\ldots,-1)$ defines a simplicial facet of CUT_n for all $n \geq 3$. In fact, for $b = (a, n-5-a, 1, 1, -1, \ldots, -1)$ with $a \geq \frac{n-5}{2}$, $n \geq 6$, $\mathrm{Hyp}_n(b)$ defines a facet of CUT_n if and only if $a \leq n-4$, and $\mathrm{Hyp}_n(b)$ defines a simplicial face of CUT_n if and only if $a \geq n-4$ ([36]).

Several generalizations of hypermetric inequalities have been proposed. They are of the form $\sum_{1 \leq i<j \leq n} b_i b_j x_{ij} - \sum_{ij \in E(G)} x_{ij} \leq 0$, where $b \in \mathbb{Z}^n$ and G is a subgraph (eventually edgeweighted) of K_n. So, hypermetric inequalties are the case when $\sum_{1 \leq i \leq n} b_i = 1$ and G is the empty graph. When $\sum_{1 \leq i \leq n} b_i = 2r+1$ is odd and G is an antiweb (resp. the suspension of a tree), we have the clique-web inequalities (considered in [1], [36], [37], [38]) (resp. the suspended-tree inequalities, considered in [18]). Further generalizations of suspended-tree inequalities are considered in [63].

For instance, let $b \in \mathbb{Z}^n$ with $b_1, \ldots, b_p > 0 > b_{p+1}, \ldots, b_n$, $\sum_{1 \leq i \leq n} b_i = 3$, $p \geq 3$ and let C be a cycle on the nodes $(1, \ldots, p)$. Then, the inequality

$$\sum_{1 \leq i < j \leq n} b_i b_j x_{ij} - \sum_{ij \in E(C)} x_{ij} \leq 0$$

is valid for CUT_n (it is the case $r = 1$ of the clique-web inequalities); it defines a facet of CUT_n, for instance, if $p \geq 5$ and $b_{p+1} = \ldots = b_n = -1$.

Also, let $b \in \mathbb{Z}^n$ with $\sum_{1 \leq i \leq n} b_i = 2r + 1$, $b_1, \ldots, b_p > 0 > b_{p+1}, \ldots, b_n$, $3 \leq p \leq n - 1$, let T be a tree spanning the nodes $(2, \ldots, p)$ and, for $i \in \{2, \ldots, p\}$, let $deg_T(i)$ denote the degree of node i in T. Then, the inequality

$$\sum_{1 \leq i < j \leq n} b_i b_j x_{ij} - \frac{r(r+1)}{2}\Big(\sum_{2 \leq i \leq p} (2 - deg_T(i)) x_{1i} + \sum_{ij \in E(T)} x_{ij}\Big) \leq 0$$

is valid for CUT_n (it is the suspended-tree inequality).

7.2 Analogues of hypermetric inequalities for other cut families

We indicate how hypermetric inequalities can be modified in order to obtain valid inequalities for other cut families.

We recall some definitions. Let $X = \{1, \ldots, n\}$ and let $T \subseteq X$ with $|T|$ even. The cut $\delta(S)$ is called an **even T-cut** (resp. **odd T-cut**) if $|S \cap T|$ is even (resp. odd). If n is even and $T = X$, then an even T-cut is simply called an **even cut**.

Even cuts can generalized to t-ary cuts as follows. Let $t \geq 2$ be an integer and suppose $n \equiv 0 \pmod{t}$. The cut $\delta(S)$ is called a **t-ary cut** if $|S| \equiv 0 \pmod{t}$. Hence, 2-ary cuts are just even cuts.

Let $S_1, \ldots, S_k$ be a partition of X into k parts. The **multicut** $\delta(S_1, \ldots, S_k)$ consists of the edges ij of K_n whose endnodes i, j belong to distinct classes of the partition. So, for $k = 2$, we have the usual notion of cut. The multicut $\delta(S_1, \ldots, S_k)$ is said to be **even** if $|S_1|, \ldots, |S_k|$ are all even.

We now indicate analogues of hypermetric inequalities that are valid for even T-cuts, t-ary cuts, multicuts, and even multicuts.

• ([39]) Suppose $T \subseteq X$ with $|T|$ even. Let $b \in \mathbb{Z}^n$ such that b_i is odd for all $i \in T$, b_i is even for all $i \in X - T$ and $\sum_{1 \leq i \leq n} b_i = 2$. Then, the inequality

$$\sum_{1 \leq i < j \leq n} b_i b_j x_{ij} \leq 0 \tag{35}$$

is satisfied by all even T-cuts. In particular, if $b_i = \pm 1$ for all i, then the inequality (35) defines a facet of the even cut cone (the cone generated by all even cuts of K_n).

Let $t \geq 2$ be an integer and suppose $n \equiv 0 \pmod t$. Let $b \in \mathbb{Z}^n$ such that $b_i \equiv \beta \pmod t$ for all $i = 1, \ldots, n$, where $\beta \in \{1, 2, \ldots, t-1\}$, and $\sum_{1 \leq i \leq n} b_i = t$. Then, the inequality (35) is satisfied by all t-ary cuts.

• ([49], [34]) Let $b \in \mathbb{Z}^n$ with $\sigma = \sum_{1 \leq i \leq n} b_i \geq 1$. The inequality

$$\sum_{1 \leq i < j \leq n} b_i b_j x_{ij} \leq \frac{\sigma(\sigma - 1)}{2}$$

is satisfied by all multicuts. For instance, if $b_i = \pm 1$ for all i, then the above inequality defines a facet of the multicut polytope (the convex hull of all multicuts of K_n). (Note that the multicut cone coincides with the cut cone.) Another generalization of hypermetric inequalities to multicuts is presented in [22].

•([39]) Let $b \in \mathbb{Z}^n$ with $\sigma = \sum_{1 \leq i \leq n} b_i \geq 2$. The inequality

$$\sum_{1 \leq i < j \leq n} b_i b_j x_{ij} \leq \frac{\sigma(\sigma - 2)}{2}$$

is satisfied by all even multicuts.

7.3 Hypermetric inequalities for the cut polytope

The cut polytope $\mathrm{CUT}_n^{\square}$ is defined as the convex hull of all cut semimetrics $\delta(S)$ for $S \subseteq X$. This polytope has been extensively studied since it plays a central role for the resolution of the maximum cut problem in combinatorial optimization.

In fact, $\mathrm{CUT}_n^{\square}$ is, in a sense, equivalent to the cut cone CUT_n. Indeed, all the facets of $\mathrm{CUT}_n^{\square}$ containing a given vertex $\delta(S)$ of $\mathrm{CUT}_n^{\square}$ can be obtained from the facets of CUT_n (that is, the facets of $\mathrm{CUT}_n^{\square}$ containing the origin $\delta(\emptyset) = 0$) by some simple reflection, called **switching** ([16]).

Switching acts on inequalities as follows. Let $v^T x \leq \alpha$ be a valid inequality for $\mathrm{CUT}_n^{\square}$ and let $\delta(S)$ be a cut. Define $v^S \in \mathbb{R}^{\binom{n}{2}}$ by $v_{ij}^S = -v_{ij}$ if $\delta(S)_{ij} = 1$ and $v_{ij}^S = v_{ij}$ otherwise. Then, the inequality $(v^S)^T x \leq \alpha - v^T \delta(S)$, obtained by **switching** $v^T x \leq \alpha$ by the cut $\delta(S)$, is valid for $\mathrm{CUT}_n^{\square}$.

Given $b \in \mathbb{Z}^n$, set $\sigma = \sum_{1 \leq i \leq n} b_i$ and $\gamma = \min(|\sigma - 2\sum_{i \in S} b_i| : S \subseteq X)$, called the **gap** of the b_i's. Then, the inequality

$$\sum_{1 \leq i < j \leq n} b_i b_j x_{ij} \leq \frac{\sigma^2 - \gamma^2}{4}$$

is valid for $\mathrm{CUT}_n^{\square}$; it is called a **gap inequality** ([52]).

Note that the gap inequalities with $\sigma = 0$ are exactly the inequalities of negative type and the gap inequalities with $\sigma = 1$ are the hypermetric inequalities. Moreover, the gap inequalities with σ odd and such that $\sum_{i \in A} b_i = \frac{\sigma-1}{2}$ for some $A \subseteq X$, are exactly the switchings of the hypermetric inequalities; they are of the form $\sum_{1 \leq i < j \leq n} b_i b_j x_{ij} \leq \frac{\sigma^2-1}{4}$.

Let G be a graph on n nodes and let $mc(G)$ denote the maximum cardinality of a cut in G, i.e. $mc(G) = \max(\sum_{ij \in E(G)} x_{ij} : x \in \mathrm{CUT}_n^{\square})$. Some upper bounds for $mc(G)$ are known. Let $L(G)$ denote the Laplacian matrix of G, $L(G)$ is the $n \times n$ matrix whose ij-th entry is $deg_G(i)$ if $i = j$, -1 if $ij \in E(G)$ and 0 otherwise. Set

$$\varphi(G) = \frac{n}{4} \min(\lambda_{\max}(L(G) + diag(u)) : u \in \mathbb{R}^n, \sum_{1 \leq i \leq n} u_i = 0)$$

where $diag(u)$ is the diagonal matrix with diagonal entries $u_1, \ldots, u_n$ and $\lambda_{\max}(L(G) + diag(u))$ is the largest eigenvalue of the matrix $L(G) + diag(u)$. Set

$$\psi(G) = \max(\frac{1}{2} Trace(AY) : \frac{1}{2} J - Y \text{ is positive semidefinite and } Y_{ij} = 0 \text{ for } 1 \leq i \leq n)$$

where J is the $n \times n$ matrix with all entries equal to 1. Then, $mc(G) \leq \varphi(G)$ ([27]) and $mc(G) \leq \psi(G)$ ([60]). In fact, by general duality theory, these two bounds coincide, i.e. $\varphi(G) = \psi(G)$ ([55]).

It is easy to see that

$$\psi(G) = \max(\sum_{ij \in E(G)} : x \text{ satisfies the inequalities (36) for all } b \in \mathbb{Z}^n).$$

$$\sum_{1 \leq i < j \leq n} b_i b_j x_{ij} \leq \frac{\sigma^2}{4} \tag{36}$$

The inequalities (36) are clearly valid for the cut polytope $\mathrm{CUT}_n^{\square}$, but they are never facet defining since they are dominated by the gap inequalities. While optimization over the convex body defined by the gap inequalities is probably hard, optimization over its relaxation by the inequalities (36) can be done in polynomial time. Compare with the complexity results about hypermetric inequalities from Remark 4.12.

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