Cut Cones IV : Lattice Points

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Abstract

Let $R_+(\mathcal{K}_n), Z(\mathcal{K}_n), Z_+(\mathcal{K}_n)$ be, respectively, the cone over R, the lattice and the cone over Z, generated by all cuts of the complete graph on n nodes. For $i \geq 0$, let $A_n^i := \{d \in R_+(\mathcal{K}_n) \cap Z(\mathcal{K}) :$ d has exactly i realizations in $Z_+(\mathcal{K}_n)\}$. We show that A_n^i is infinite, except undecided case $A_6^0 \neq \emptyset$ and empty A_n^i for $i = 0,\ n \leq 5$ and for $i \geq 2,\ n \leq 3$. The set A_n^1 contains $0, 1, \infty$ of nonsimplicial points for $n \leq 4,\ n = 5,\ n \geq 6$, respectively. On the other hand, there exists a finite number $t(n)$ such that $t(n)d \in Z_+(\mathcal{K}_n)$ for any $d \in A_n^0$; we estimate also such scales for classes of points. We construct families of points of A_n^0 and $Z_+(\mathcal{K}_n)$, especially on a 0-lifting of a simplicial facet, and points $d \in R_+(\mathcal{K}_n)$ with $d_{i,n} = t$ for $1 \leq i \leq n-1$.

1 Introduction

We study here integral points of cones. Suppose there is a cone C in R^n which is generated by its extreme rays $e_1, e_2, ..., e_m$, all $e_i \in Z^n$.

Let d be a linear combination,

$$d = \sum_{1 \leq i \leq m} \lambda_i e_i. \tag{1}$$

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We call the expression a K-realization of d if $\lambda_i \in K$, $1 \leq i \leq m$, and K is either R_+ or Z or Z_+.

If $\lambda_i \geq 0$ for all i, then $d \in C$, and (1) is a R_+-realization of d. If λ_i is an integer for all i, then $d \in L$ where L is a lattice generated by the integral vectors e_i, $1 \leq i \leq m$, and (1) is a Z-realization of d. Obviously $L \subseteq Z^n$. If $\lambda_i \geq 0$ and is integral for all i, then we call the point d an h-point of C. Hence h-points are the points having a Z_+-realization. A point $d \in C \cap L$ is called quasi-h-point if it is not an h-point. In other words, d is a quasi-h-point if it has R_+- and Z-realizations but no Z_+-realization.

We consider cut cones, i.e. those where e_i are cut vectors. Here are given examples of cut cones having or having no quasi-h-points. We prove that some points are quasi-h-points. We study scales, multiplying by which, a point has Z_+-realizations.

In fact, those problems are related to feasibility problems of the following integer program

$$\{A\lambda = d,\ \lambda \in Z_+^m\}, \tag{2}$$

where A is the $n \times m$ matrix whose columns are the vectors e_i.

2 Definitions and notations

Set $V_n = \{1, ..., n\}$, $E_n = \{(i,j) : 1 \leq i < j \leq n\}$, then $K_n = (V_n, E_n)$ denotes the complete graph on n points. Denote by $P_{(i_1, i_2, \ldots, i_k)} = P_k$ the path in K_n going through the vertices $i_1, i_1, ..., i_k$.

For $S \subseteq V_n$, $\delta(S) \subseteq E_n$ denote the *cut* defined by S, with $(i,j) \in \delta(S)$ if and only if $\mid S \cap \{ij\} \mid = 1$. Since $\delta(S) = \delta(V_n - S)$, we take S such that $n \notin S$. The incidence vector of the cut $\delta(S)$ is called a *cut vector* and, by abuse of language, is also denoted as $\delta(S)$. Besides, $\delta(S)$ determines a distance function (in fact, a semimetric) $d_{\delta(S)}$ on points of V_n as follows: $d_{\delta(S)}(i,j) = 1$ if $(i,j) \in \delta(S)$, otherwise the distance between i and j is equal to 0. For simplicity sake, we set $\delta(\{i, j, k, ...\}) = \delta(i, j, k, ...)$.

Denote by $\mathcal{K}_n$ the family of all nonzero cuts $\delta(S), S \subseteq V_n$. For any family $\mathcal{K} \subseteq \mathcal{K}_n$ define the cone $C(\mathcal{K}) := R_+(\mathcal{K})$ as the conic hull of cuts in $\mathcal{K}$. The cone $C(\mathcal{K})$ lies in the space $R(\mathcal{K})$ spanned by the set $\mathcal{K}$. We set $C_n := C(\mathcal{K}_n)$.

So, each point $d \in C(\mathcal{K})$ has a representation $d = \sum_{\delta(S) \in \mathcal{K}} \lambda_S \delta(S)$. Since $\lambda_S \geq 0$, the representation is called *R_+-realization* of d. The number $\sum_{\delta(S) \in \mathcal{K}} \lambda_S$ is called *the size* of the R_+-realization.

The lattice $L(\mathcal{K}) := Z(\mathcal{K})$ is the set of all integral linear combinations of cuts in $\mathcal{K}$. Let $L_n = L(\mathcal{K}_n)$. The lattice L_n is easily characterized; namely, $d \in L_n$ if and only if d satisfies the following *condition of evenness*

$$d_{ij} + d_{ik} + d_{jk} \equiv 0 \pmod 2), \text{ for all } 1 \le i < j < k \le n. \quad (3)$$

So, $2Z^{n(n-1)/2} \subset L_n \subset Z^{n(n-1)/2}$.

The points of $L(\mathcal{K})$ with nonnegative coefficients, i.e. the points of $Z_+(\mathcal{K})$ are called *h-points*. We denote the set of h-points of the cone $C(\mathcal{K})$ by $hC(\mathcal{K})$. For $d \in Z_+(\mathcal{K})$, any decomposition of d as nonnegative integer sum of cuts is called a Z_+*-realization* of d. An h-point of C_n is (seen as a semimetric) exactly isometrically *embeddable into a hypercube* (or *h-embeddable*) semimetric. This explains the name of an h-point.

For $d \in C_n$, define

$$s(d) := \text{ minimum size of } R_+ - \text{realizations of } d,$$

$$z(d) := \text{ minimum size of } Z_+\text{-realizations of } d \text{ if any.}$$

Let $d(G)$ be the shortest path metric of a graph G. We set

$$z_n^t := z(2td(K_n)).$$

For this special case,$G = K_n$, $s(d) = s(2td(K_n))$ is equal to $a_n^t := \frac{tn(n-1)}{\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil}$.

A point d is called a *quasi-h-point* of $C(\mathcal{K})$ if $d \in A(\mathcal{K}) := C(\mathcal{K}) \cap L(\mathcal{K}) - Z_+(\mathcal{K})$.

Recall (see [16]), that a *Hilbert basis* is a set of vectors $e_1, ..., e_k$ with the property that each vector lying in both, the lattice and the cone, generated by $e_1, ..., e_k$, is a nonnegative integral combination of these vectors. $A(\mathcal{K}) = \emptyset$ would mean that $\mathcal{K}$ is a Hilbert basis of $C(\mathcal{K})$. Actually, $\mathcal{K}$ would be the minimal Hilbert basis of $C(\mathcal{K})$ if it is a Hilbert basis, since $\delta(S)$ does not belong to $R_+(\mathcal{K}_n - \delta(S))$ for any $\delta(S) \in \mathcal{K}_n$ (see [4]).

Define

$$A^i(\mathcal{K}) := \{d \in C(\mathcal{K}) \cap L(\mathcal{K}) : d \text{ has exactly } i \ Z_+ - \text{realizations}\},$$

$$A_n^i := A^i(\mathcal{K}_n).$$

So, above defined set $A(\mathcal{K})$ is $A^0(\mathcal{K})$. Define

$$\eta^i(d) := min\{t \in Z_+ : td \text{ has} > i \ Z_+\text{-realizations}\} =$$

$$= min\{t \in Z_+ : td \notin A^k(\mathcal{K}) \text{ for all } 0 \le k \le i\}.$$

A cone $C = R_+(\mathcal{K})$ is said to be *simplicial* if the set $\mathcal{K}$ is linearly independent; a point $d \in C$ is said to be *simplicial* if d lies on a simplicial face of C, i.e. if d admits unique R_+-realization.

Call $e(\mathcal{K}) := |\mathcal{K}|$ mnus dimension of $\mathcal{K}$, the *excess* of $\mathcal{K}$. Set

$$\mathcal{K}_n^l = \{\delta(S) \in \mathcal{K}_n : |S| = l \text{ or } \ n - |S| = l\}.$$

For even n we set also

$$Even\mathcal{K}_n = \{\delta(S) \in \mathcal{K}_n : |S|, n - |S| \equiv 0 \pmod 2\},$$

$$Odd\mathcal{K}_n = \{\delta(S) \in \mathcal{K}_n : |S|, n - |S| \equiv 1 \pmod 2\}.$$

For a subset $T \subseteq V_n$ denote

$$EvenT\mathcal{K}_n = \{\delta(S) \in \mathcal{K}_n : |S \cap T| \equiv 0 \pmod 2\},$$

$$OddT\mathcal{K}_n = \{\delta(S) \in \mathcal{K}_n : |S \cap T| \equiv 1 \pmod 2\}.$$

So $Even\mathcal{K}_n = EvenT\mathcal{K}_n$, $Odd\mathcal{K}_n = OddT\mathcal{K}_n$ for $T = V_n, n$ even.

Remark that $\mathcal{K}_{2m}^m = \{\delta(S) \in \mathcal{K}_{2m}^m : 1 \notin S\} = \{\delta(S) \in \mathcal{K}_{2m}^m : 1 \in S\}$.

Denote by $\mathcal{K}_n^{i,j}$,$\mathcal{K}_n^{\neq i}$, $\mathcal{K}_n^{\not\equiv i \pmod a}$ the families of $\delta(S) \in \mathcal{K}_n$ with $|S| \in \{i, j, n-i, n-j\}$, $|S| \notin \{i, n-i\}$, $\min\{|S|, n-|S|\} \not\equiv i \pmod a$, respectively.

We write C_b^a for $C(\mathcal{K}_b^a)$ where a and b are indexes or sets of indexes.

3 Families of cuts $\mathcal{K}$ with $A(\mathcal{K}) = \emptyset$

Of course $A(\mathcal{K}) = \emptyset$ if $e(\mathcal{K}) = 0$, i.e. if the cone $C(\mathcal{K})$ is simplicial. It is easy to see that $C(\mathcal{K}_n^l)$ is simplicial if and only if either $l = 1$, or $l = 2$, or $(l, n) = (3, 6)$. Also $e(\mathcal{K}_3) = 0$.

Note that $e(\mathcal{K}_n) = 2^{n-1} - 1 - \binom{n}{2}$

Some examples of $\mathcal{K}$ with a positive excess but with $A(\mathcal{K}) = \emptyset$ are:

a) $\mathcal{K}_4$, $\mathcal{K}_5$ with the excess 1 and 5, respectively. The first proof was given in [3]; details of the proof see in [10], where, for any $d \in C_n \cap L_n,\ n = 4, 5$, the explicit Z_+-realization of d is given.

b) $Odd\mathcal{K}_6$ with the excess 1. For proof see [10].

c) (See the case $n = 5$ of Theorem 6.2 below) The family of cuts (with excess 5) on a facet of $C(\mathcal{K}_6)$ which is a 0-lifting of a simplicial 5-gonal facet of $C(\mathcal{K}_5)$.

But $\mathcal{K}_n^{1,2}$ of excess n has $A(\mathcal{K}) \neq \emptyset$ for $n \geq 6$. Below we give some examples of $\mathcal{K}$ with $A(\mathcal{K}) \neq \emptyset$ which are, in a way, close to the above examples of $\mathcal{K}$ with $A(\mathcal{K}) = \emptyset$.

Denote by $Q(b)$ the linear form $\sum_{1 \leq i < j \leq n} b_i b_j x_{ij}$ for $b \in Z^n$. If $\sum_{i=1}^n b_i = 1$, the inequality $Q(b) \leq 0$ is called *hypermetric inequality*. Call $d \in R^{n(n-1)/2}$ a *hypermetric* if it satisfies all hypermetric inequalities. It is valid for $C(\mathcal{K}_n)$, (see [3]). We denote the hypermetric inequality by $Hyp_n(b)$. For large classes of parameters b (see [4], [6]) $Hyp_n(b)$ is a facet of $C(\mathcal{K}_n)$. The only known case when a hypermetric face is simplicial is (up to permutation) $Hyp_n(1^2, -1^{n-3}, n-4), n \geq 3$, and (its "switching" in terms of [6]) $Hyp_n(-1, 1^{n-2}, -(n-4))$. Call the facet $Hyp_n(1^2, -1^{n-3}, n-4)$ the *main n-facet*. Call the facet $Hyp_n(1^2, 0^k, -1^{n-k-3}, n-k-4)$ the *k-fold 0-lifting* of the main (n-k)-facet. It is a facet of $C(\mathcal{K}_n)$, because every k-fold 0-lifting of a facet of C_{n-k} is a facet of C_n (see [4]). A 1-fold 0-lifting we call simply 0-lifting. Up to a permutation we have:

the unique type of facets of $C(\mathcal{K}_3)$ is the main 3-facet (triangle inequality);

the unique type of facets of $C(\mathcal{K}_4)$ is the main 4-facet (which is the 0-lifting $Hyp_4(-1, 1^2, 0)$ of a main 3-facet);

all facets of $C(\mathcal{K}_5)$ are 2-fold 0-liftings of a main 3-facet (i.e. 0-lifting of a main 4-facet), and the main 5-facet $Hyp_5(1^3, -1^2)$, called the *pentagonal* facet;

all facets of $C(\mathcal{K}_6)$ are: 2-fold 0-liftings of a main 4-facet, 0-lifting of a main 5-facet, the main 6-facet $Hyp_6(2, 1, 1, -1^3)$ and its "switching" $Hyp_6(-2, -1, 1^4)$.

Lemma 3.1. *If $\mathcal{K}$ is a family of cuts $\delta(S)$, $|S| \leq \frac{n}{2}$, lying on a face F of C_n, then the family*

$$\mathcal{K}' = \mathcal{K} \cup \{\delta(\{n+1\})\} \cup \{\delta(S \cup \{n+1\}) : \delta(S) \in \mathcal{K}\}$$

is the family of cuts lying on a 0-lifting of the face F. If, for above $\mathcal{K}$, $C(\mathcal{K})$ is a simplicial facet of C_n), we obtain, for $n \geq 4$,

$$e(\mathcal{K}') = |\mathcal{K}'| - dim\mathcal{K}' = (2|\mathcal{K}| + 1) - dim\mathcal{K}' =$$

$$= 2(\binom{n}{2} - 1) + 1 - (\binom{n+1}{2} - 1) = n(n-3)/2.$$

□

Recall that $A(\mathcal{K}) = \emptyset$ for $\mathcal{K} = \mathcal{K}_5, \mathcal{K}_6^1, \mathcal{K}_6^2, \mathcal{K}_6^3, \mathcal{K}_6^{1,3} = Odd\mathcal{K}_6$ and for the family of any (except triangle) facet of $\mathcal{K}_6$, since $\mathcal{K}_6^i$ is simplicial for $i = 1, 2, 3$, and $\mathcal{K}_5$, $Odd\mathcal{K}_6$ are examples given in the beginning of this section.

4 Antipodal extension

A fruitful method of obtaining quasi-h-points is the *antipodal extension operation* at the point n. For $d \in R^{n(n-1)/2}$ we define $ant_\alpha d \in R^{n(n+1)/2}$ by

$$\begin{aligned}
&(ant_\alpha d)_{ij} = d_{ij} \text{ for } 1 \le i < j \le n,\\
&(ant_\alpha d)_{n,n+1} = \alpha,\\
&(ant_\alpha d)_{j,n+1} = \alpha - d_{jn} \text{ for } 1 \le j \le n-1.
\end{aligned}$$

For $\mathcal{K} \subseteq \mathcal{K}_n$, define

$$ant\mathcal{K} = \{ant_1\delta(S) : \delta(S) \in \mathcal{K}\} \cup \{\delta(n+1)\}.$$

Note that

$$ant_1\delta(S) = \delta(S) \text{ if } \{n\} \in S, \text{ and } ant_1\delta(S) = \delta(S \cup \{n+1\}) \text{ if } \{n\} \notin S.$$

Hence

$$ant\mathcal{K} = \{\delta(S) : \delta(S) \in \mathcal{K}, n \in S\} \cup \{\delta(S \cup \{n+1\}) : \delta(S) \in \mathcal{K}, \{n\} \notin S$$

Observe that if $d \in C(\mathcal{K})$ and $d = \sum_{\delta(S)\in\mathcal{K}} \lambda_S \delta(S)$, then

$$\begin{aligned}
ant_\alpha d &= \sum_{\delta(S)\in\mathcal{K}} \lambda_S ant_\alpha \delta(S) + \alpha(1 - \sum_S \lambda_S)\delta(n+1)\\
&= \sum_{\delta(S)\in\mathcal{K}} \lambda_S ant_1 \delta(S) + (\alpha - \sum_S \lambda_S)\delta(\{n+1\}). \qquad (4)
\end{aligned}$$

Also if

$$ant_\alpha d = \sum_{\delta(S)\in\mathcal{K}} \lambda_S ant_1\delta(S) + \lambda_0 \delta(n+1),$$

then $\alpha = \sum_S \lambda_S + \lambda_0$, and $d = \sum_{\delta(S)\in\mathcal{K}} \lambda_S\delta(S)$ is the projection of $ant_\alpha(d)$ on $R^{n(n-1)/2}$.

So $ant_\alpha d \in R(ant\mathcal{K})$ if and only if $d \in R(\mathcal{K})$.

Note that the cone $R(ant\mathcal{K})$ is the intersection of the triangle facets $Hyp_{n+1}(1^2, -1_j, 0^{n-2})$, where $b_n = b_{n+1} = 1$, $b_j = -1$ and $b_i = 0$ for $i \neq j$, $1 \le i \le n-1$.

Proposition 4.1 (Proposition 2.6 of [8])

$$(i) ant_\alpha d \in L_{n+1} \text{ if and only if } d \in L_n \text{ and } \alpha \in Z,$$

$$(ii) ant_\alpha d \in C_{n+1} \text{ if and only if } d \in C_n \text{ and } \alpha \ge s(d),$$

$$(iii) ant_\alpha d \in hC_{n+1} \text{ if and only if } d \in hC_n \text{ and } \alpha \geq z(d),$$

(iv) $ant_\alpha d$ is a simplicial point of C_{n+1} if and only if d is a simplicial point of C_n and $\alpha \geq s(d)$. □

Clearly, $s(ant_\alpha d) = \alpha$ if $ant_\alpha d \in C_{n+1}$ and $z(ant_\alpha d) = \alpha$ if $ant_\alpha d \in hC_{n+1}$. Also $ant_\alpha d \in A^i_n$ for $i > 0$ if and only if $d \in A^i_n$, $\alpha \in Z_+$, $\alpha \geq z(d)$.

Proposition 4.1 implies obviously the following important

Corollary 4.2 *Let* $d \in hC_n$, *and let* α *be an integer such that* $s(d) \leq \alpha < z(d)$. *Then* $ant_\alpha d \in A(ant\mathcal{K}_n) \subset A^0_{n+1}$, *i.e.* $ant_\alpha d$ *is a quasi-h-point in* C_{n+1}.

5 Spherical t-extension and gate extension

Let $d \in C_{n+1}$. We write $d = (d^0, d^1)$, where

$$d^0 = \{d_{ij} : 1 \leq i < j \leq n\}, \; d^1 = \{d_{i,n+1} : 1 \leq i \leq n\}.$$

A point $d \in C_{n+1}$ is called the *spherical t-extension* or simply *t-extension* of the point $d^0 \in C_n$ if $d = (d^0, d^1)$ and $d^1_{i,n+1} = t$ for all $i \in V_n$. We denote the spherical t-extension of d^0 by $ext_t d^0$.

Let j_n be the n-vector all of whose components are equal to 1. Then for the t-extension (d^0, d^1), we have $d^1 = tj_n$.

Proposition 5.1. $ext_t d$ *is a hypermetric if and only if*

(i) d *is a hypermetric,*

(ii) $t \geq (\sum b_i b_j d_{ij})/\Sigma(\Sigma - 1)$

for all integers $b_1, ..., b_n$ *with* $\Sigma := \sum_1^n b_i > 1$ *and g.c.d.*$b_i = 1$.

Proof. If $ext_t d$ is hypermetric, then $\sum b_i b_j (ext_t d)_{ij} \leq 0$ for any $b_1, ...,$-$b_n, b_{n+1} \in Z_+$ with $\sum b_i = 1$, i.e.

$$\sum_{1 \leq i < j \leq n} b_i b_j d_{ij} + \sum_{1 \leq i \leq n} b_i b_{n+1} t \leq 0.$$

Since $b_{n+1} = 1 - \Sigma$, the second term is equal to $-t\Sigma(\Sigma - 1)$. We obtain (i) if $b_{n+1} = 0$ or 1; otherwise $\Sigma(\Sigma - 1) \neq 0$, and we get (ii). □

Corollary 5.2. $ext_t d$ is a semimetric if and only if d is a semimetric and $t \geq \frac{1}{2} max_{(ij)} d_{ij}$.

In fact, apply (ii) above to the case $b_i = b_j = 1$, $b_{n+1} = -1$ and $b_k = 0$ for other b's.

Similarly to Proposition 5.1, one can check that $ant_t d$ is a hypermetric (a semimetric) if and only if d is a hypermetric (a semimetric, respectively) and

$$t \geq \Big(\sum_{1 \leq i < j \leq n} b_i b_j d_{ij} \Big) / \Sigma(\Sigma - 1) + \sum_{1}^{n} b_i d_{in} / \Sigma$$

for any integers $b_1, ..., b_n$ with $\Sigma := \sum_1^n b_i > 1$ and g.c.d.$b_i = 1$

$(t \geq \frac{1}{2} max_{1 \leq i < j \leq n-1}(d_{ij} + d_{in} + d_{jn})$, respectively).

There is another operation, similar to antipodal extension operation. We call it the *gate extension operation* at the point n (called *gate*). For $d \in R^{n(n-1)/2}$, define $gat_\alpha d \in R^{n(n-1)/2}$ by

$$\begin{aligned}
&(gat_\alpha d)_{ij} = d_{ij} \text{ for } 1 \leq i < j \leq n, \\
&(gat_\alpha d)_{n,n+1} = \alpha, \\
&(gat_\alpha d)_{i,n+1} = \alpha + d_{in} \text{ for } 1 \leq i \leq n-1.
\end{aligned}$$

The following identity shows that $gat_\alpha d$ is, in a sense, a complement of $ant_\alpha d$.

$$ant_\alpha d + gat_{2t-\alpha} d = 2ext_t d. \tag{5}$$

Recall that we take S in $\delta(S)$ such that $n \notin S$. Hence, for $\mathcal{K} \subseteq \mathcal{K}_n$, we have

$$gat\mathcal{K} = \mathcal{K} \cup \{\delta(n+1)\}.$$

Actually, $ant\mathcal{K}_n = OddT\mathcal{K}_{n+1}$, $gat\mathcal{K}_n = \{\delta(n+1)\} \cup EvenT\mathcal{K}_{n+1}$, for $T = \{n, n+1\}$.

Note that the cone $R_+(gat\mathcal{K})$ is the intersection of the triangle facets $Hyp_{n+1}(1_i, 0^{n-2}, -1, 1_{n+1})$, where $b_i = b_{n+1} = 1$, $b_n = -1$, $b_j = 0$ for $j \neq i$, $1 \leq j \leq n-1$.

It is clear that any R_+-realization of $gat_\alpha d$ (if it belongs to C_{n+1}) has the form $\sum_S \lambda_S \delta S + \alpha \delta(n+1)$ where $n+1 \notin S$, and where the above realization is any R_+-realization of d. So, $gat_\alpha d \in L_{n+1}(C_{n+1}, hC_{n+1}, A^i_{n+1}$, respectively) if and only if $d \in L_n(C_n, hC_n, A^i_n$, respectively) and $\alpha \in Z(R_+, Z_+, Z$, respectively).

Also $gat_\alpha d$ is a hypermetric (a metric) if and only if $\alpha \in R_+$ and d is a hypermetric (a metric, respectively).

Hence if $\alpha \in Z_+$, we have

$$gat_\alpha d \in A^i_{n+1} \Longleftrightarrow d \in A^i_n. \tag{6}$$

In particular, $gat_\alpha d$ is a quasi-h-point if and only if d is.

The following facts are obvious.

1. If d_i is the t_i-extension of d_i^0, $i = 1, 2$, then $d_1 + d_2$ is the $(t_1 + t_2)$-extension of $d_1^0 + d_2^0$.

2. If d^0 lies in a facet of the cut cone, then the t-extension of d^0 lies in the 0-lifting of the facet.

We call a point $d \in C_n$ *even* if all distances d_{ij} are even.

Let $d = \sum_S \lambda_S \delta(S)$ be a Z_+-realization of an h-point d. We call the realization *(0,1)-realization* ($2Z_+$*-realization*) if all λ_S are equal to 0 or 1 (are even, respectively). We have

Fact. Let d be an h-point. Then $d = d_1 + d_2$, where d_1 has a (0,1)-realization, and d_2 has an $2Z_+$-realization.

Obviously, if d has an $2Z_+$-realization, then d is even. But if d is even, it can have no $2Z_+$-realizations.

The following Proposition 5.3 is an analog of Proposition 4.1.

Proposition 5.3. *(i)* $ext_t d \in L_{n+1}$ *if and only if* $d \in 2Z^{n(n-1)/2}$ *and* $t \in Z$*,*

(ii) $ext_t d \in C_{n+1}$ *if* $d \in C_n$ *and* $2t \geq s(d)$*,*

(iii) suppose that d *has* $2Z_+$*-realizations, and let* $z_{even}(d)$ *denote their minimal size; then* $ext_t d \in hC_{n+1}$ *if* $d \in hC_n$ *and* $2t \geq z_{even}(d)$*.*

Proof. (i) is implied by the trivial equality $d_{i,n+1} + d_{j,n+1} + d_{ij} = 2t + d_{ij}$, $1 \leq i < j \leq n$.

From (5) we have $ext_t d = \frac{1}{2}(ant_\alpha d + gat_{2t-\alpha} d)$. Taking $\alpha = s(d)$ and applying (ii) of Proposition 4.1 we get (ii).

Taking $\alpha = z_{even}(d)$, applying (iii) of Proposition 4.1 and using that $ant_{z_{even}} d, gat_{2t-z_{even}(d)} d \in 2Z_+(\mathcal{K}_{n+1})$, we get (iii). □

Define $ext_t^m d = ext_t(ext_t^{m-1} d)$, where $ext_t^1 d = ext_t d$.

Proposition 5.4. *If* $2t \geq s(d)$*, then* $ext_t^m d \in C_{n+m}$ *for any* $m \in Z_+$*, and*

$$max(s(ext_t^{m-1} d),\ 2t - \frac{t}{\lceil m/2 \rceil}) \leq s(ext_t^m d) \leq 2t - 2^{-m}(2t - s(d)).$$

Proof. From Proposition 5.3(ii) we get

$$s(ext_t d) \leq \frac{1}{2} s(ant_{s(d)} d + gat_{2t-s(d)} d) = t + \frac{1}{2} s(d) \leq 2t.$$

By induction on m, we obtain that $ext_t^m d \in C_{n+m}$ for all $m \in Z_+$, and the upper bound for $s(ext_t^m d)$.

The lower bound is implied by the fact that the restriction of $ext_t^m d$ on m extension points is $td(K_m)$. Since $s(td(K_m)) = \frac{1}{2} a_m^t$ (see Section 2), we have

$$s(ext_t^m) \geq s(td(K_m)) = \frac{1}{2} \frac{tm(m-1)}{\lfloor m/2 \rfloor \lceil m/2 \rceil} = 2t - \frac{t}{\lceil m/2 \rceil}.$$

□

Remark. So, if $s(d) \le 2t$, then $\lim_{m\to\infty} s(ext_t^m d) = 2t$.

Probably, there exist $m_0 = m_0(t,d)$ such that $s(ext_t^m d) = 2t$ for $m \ge m_0$.

We conjecture that $ext_t^m d \notin C_{n+m}$ for $m > m_1$ if $s(d) > 2t$.

For example, if $t = 1$ and $d = d(G)$ ($d(G)$ is the shortest path metric of the graph G) , then it can be proved that $m_1 = 2$.

If the conjecture is true, then

$$s(d) = 2min\{t : ext_t^m d \in C_{n+m} \text{ for all } m \in Z_+\}.$$

Recall, that Proposition 4.1(ii) implies

$$s(d) = min\{\alpha : ant_\alpha d \in C_{n+1}\}.$$

In terms of $ext_n^m d$ we have also analogs of (i) and (iii) of Proposition 4.1.

Proposition 5.5.

(i) $ext_t^m d \in L_{n+m}$ for all $m \in Z_+$ if and only if $d \in 2Z^{n(n-1)/2}$ and t is even.

(iii) $ext_t^m d \in hC_{n+m}$ for all $m \in Z_+$ if and only if t is an even positive integer, and $d = td(K_n)$.

Proof. The evenness of t follows from $ext_t^3 d \in L_{n+3}$. So, (i) is implied by Proposition 5.3(i).

Recall the result of [5] that $t\sum_1^n \delta(i)$ is the unique Z_+-realization of $td(K_n)$ for even t and $m \ge \frac{t^2}{4} + \frac{t}{2} + 3$. Using this fact, we get that any Z_+-realization of $ext_t^m d$ contains $t/2$ cuts $\delta(i)$ for some i if m is large enough. So, $d = ext_t d'$ for some $d' \in hC_{n-1}$, etc. □

6 Quasi-h-points of 0-lifting of the main facet

Consider the main facet

$$F_0(n) = Hyp_n(1^2, -1^{n-3}, n-4) = Hyp_n(b^0),$$

where $b_1^0 = b_2^0 = 1,\ b_i^0 = -1,\ 3 \le i \le n-1,\ b_n^0 = n-4$. The cut vectors $\delta(S)$ lying in the facet are defined by equations $b(S) \equiv \sum_{i\in S} b_i = 0$ or 1. We take S not containing n. Then $S \in \mathcal{S}$, where

$$\mathcal{S} = \{\{1\},\{2\},\{1i\},\{2i\}\{12i\}\ (3 \le i \le n-1), \{12ij\}\ (3 \le i < j \le n-1)\}.$$

We set

$$m = |\mathcal{S}| = \frac{n(n-1)}{2} - 1.$$

Every n-facet contains at least m cut vectors. Since the main n-facet contains exactly m cuts, it is simplicial.

The 0-lifting of the main facet is the facet

$$F(n) = Hyp_{n+1}(1^2, -1^{n-3}, n-4, 0).$$

Besides the above cuts $\delta(S), S \in \mathcal{S}$, it contains, according to Lemma 3.1, only the cuts $\delta(S \cup \{n+1\}), S \in \mathcal{S}$, and $\delta(n+1)$.

Note that $A(\mathcal{K}) = \emptyset$ for the main n-facet (as for any simplicial $C(\mathcal{K})$).

Now we consider even points having no $2Z_+$-realization. The simplest such points are points having a (0,1)-realization. We call these points *even (0,1)-points.*

Let $d^0 \in F_0(n)$ be an even h-point, and let $\sum_{S \in \mathcal{S}_0} \lambda_S \delta(S)$ be one of its Z_+-realizations. Consider a minimal set of comparisions mod 2 that λ_S's have to satisfy. The comparisions are implied by the conditions $d_{ij} \equiv 0$ for all pairs (ij). Since $d^0 \in L_n$, we have $d_{ij} \equiv d_{ik} + d_{jk} \pmod 2$ for all ordered triples (ijk). Hence independent comparisions are implied by the comparisions $d_{in} \equiv 0 \pmod 2$, $1 \le i \le n-1$. The comparisions are as follows. (For simplicity sake, we set $\lambda_{\{ij\dots\}} = \lambda_{ij\dots}$ and omit the indication (mod 2)).

$$\lambda_{1i} + \lambda_{2i} + \lambda_{12i} + \sum_{3 \le j \le n-1, j \ne i} \lambda_{12ij} \equiv 0, \ 3 \le i \le n-1,$$

$$\lambda_1 + \sum_{3 \le i \le n-1} (\lambda_{1i} + \lambda_{12i}) + \sum_{3 \le i < j \le n-1} \lambda_{12ij} \equiv 0, \tag{7}$$

$$\lambda_2 + \sum_{3 \le i \le n-1} (\lambda_{2i} + \lambda_{12i}) + \sum_{3 \le i < j \le n-1} \lambda_{12ij} \equiv 0.$$

The system of comparisions (7) has $n-1$ equations with $m = n(n-1)/2-1$ unknowns. Hence the number of (0,1)-solutions distinct from the trivial zero solution is equal to $2^{m-(n-1)} - 1 = 2^{\binom{n-1}{2}-1} - 1$.

This shows that all points of $F_0(3)$ have $2Z_+$-realizations. The only even (0,1)-points of $F_0(4)$ are 2 points $2d(K_3)$ with $d_{13} = 0$ or $d_{23} = 0$, and the point $2d(K_4 - P_{(1,2)})$. There are 31 even (0,1)-points in $F_0(5)$.

Since there are exponentially many even (0,1)-points in $F_0(n)$, we consider points of the following type and call them *special.*

For these points the coefficients λ_S are as follows.

$$\lambda_1 = a_1, \ \lambda_2 = a_2, \ \lambda_{1i} = b_1, \ \lambda_{2i} = b_2, \ \lambda_{12i} = c_1, \ 3 \le i \le n-1,$$

$$\lambda_{12ij} = c_2,\ 3 \le i < j \le n-1.$$

Here $a_i, b_i, c_i,\ i = 1,2$, are equal to 0 or 1.

If we set

$$k = n-3,\ l = \frac{n(n-1)}{2},$$

then for the special points (7) takes the form

$$b_1 + b_2 + c_1 + (k-1)c_2 \equiv 0,$$

$$a_1 + k(b_1 + c_1) + \frac{k(k-1)}{2} c_2 \equiv 0,$$

$$a_2 + k(b_2 + c_1) + \frac{k(k-1)}{2} c_2 \equiv 0.$$

Since we have 3 equations for 6 variables, we can express 3 variables a_1, a_2, c_1 through other 3 variables b_1, b_2, c_2.

There are 4 families of the solutions of the system depending on what is $k \pmod 4$. The solutions are as follows (undefined equivalences are taken by (mod 2)).

$$k \equiv 0 \pmod 4,\ a_1 = a_2 = 0,\ c_1 \equiv b_1 + b_2 + c_2,$$

$$k \equiv 1 \pmod 4,\ a_1 = b_2,\ a_2 = b_1, c_1 \equiv b_1 + b_2,\ c_2 \text{ arbitrary},$$

$$k \equiv 2 \pmod 2,\ a_1 = a_2 = c_2,\ c_1 \equiv b_1 + b_2 + c_2,$$

$$k \equiv 3 \pmod 4,\ a_1 \equiv b_2 + c_2,\ a_2 \equiv b_1 + c_2,\ c_1 \equiv b_1 + b_2.$$

In each case we obtain 7 nontrivial special even (0,1)-point.

Taking in attention the definition of $\mathcal{S}$, for $a = 0, \pm$, we denote by $\lambda^a_{ik},\ \lambda^a_k$ the k-vectors with the components $\lambda^a_{ij},\ 3 \le j \le n-1,\ i = 1,2,\ \lambda^a_{12j},\ 3 \le j \le n-1$, respectively. Similarly, λ^a_l is the l-vector with the components $\lambda^a_{12ij},\ 3 \le i < j \le n-1$.

In this notation a special point d^0 has a (0,1)-realization λ^0 such that $\lambda^0_i = a_i,\ \lambda^0_{ik} = b_i j_k,\ i = 1,2,\ \lambda^0_k = c_1 j_k$ and $\lambda^0_l = c_2 j_l$.

Recall that special points are simplicial. Therefore their size is equal to $\sum_{S \in \mathcal{S}} \lambda_S$. We show below that the t-extension of 2 special points with $(a_1, a_2, b_1, b_2, c_1, c_2) = (1,1,0,0,0,1)$ and $(0,1,0,1,1,1)$ are quasi-h-points for $n \equiv 2 \pmod 4$.

For $n = 6$ the points d^0 are $d(K_6 - P_3)$ and $ant_{10}(ext_4 d(K_4))$. Another example of $d \in A^0_7$ is $ant_6(ext_3 d(K_5)) = d^{5,3}$ in terms of Corollary 6.6 below.

Proposition 6.1. *Let d^0 be one of the 7 special points of the main facet $F_0(n)$. Let t be a positive integer such that $t \geq \frac{1}{2}\sum_{S\in\mathcal{S}} \lambda^0_S$. Then the t-extension of d^0 is an h-point if $n \not\equiv 2 \pmod 4$, and if $n \equiv 2 \pmod 4$, then there is 2 points d^0 such that its t-extension is a quasi-h-point, namely the points with $(a_1, a_2, b_1, b_2, c_1, c_2) = (1, 1, 0, 0, 0, 1)$ and $(0, 1, 0, 1, 1, 1)$.*

Proof. Recall that we can take $\mathcal{S}$ such that $n \notin S$ for all $S \in \mathcal{S}$.

We apply the equation (2) to the t-extension d. In the case the matrix A takes the form

$$A = \begin{pmatrix} B & B & 0 \\ D & \overline{D} & j_n \end{pmatrix}$$

Here the first m columns correspond to sets $S \in \mathcal{S}$, the next m columns correspond to sets $S \cup \{n+1\}$, $S \in \mathcal{S}$, and the last $(2m+1)$st column corresponds to $\{n+1\}$. The size of the matrix B is $\binom{n}{2} \times m$, and $D, \overline{D}$ are $n \times m$ matrices such that $D + \overline{D} = J$, where J is the matrix all of whose elements are equal to 1. Each column of the matrix J is the vector j_n consisting of n 1's. In this notations, we can write J as the direct product $J = j_n \times j_m^T$. Hence for any m-vector a we have $Ja = (j_m, a)j_n$.

The rows of D and $\overline{D}$ are indexed by pairs $(i, n+1)$, $1 \leq i \leq n$. The S-column of the matrix D is the $(0,1)$-indicator vector of the set S. Since $n \notin S$ for all $S \in \mathcal{S}$, the last row of D consits of 0's only.

We look out solutions of the system (2) for this matrix A such that λ is a nonnegative integral (2m+1)-vector. We set

$$\mu_S = \lambda_{S\cup\{n+1\}},\ S \in \mathcal{S},\ \gamma = \lambda_{\{n+1\}}.$$

Then the system (2) takes the form

$$B(\lambda + \mu) = d^0,$$

$$D(\lambda - \mu) + (\gamma + (j_m, \mu))j_n = d^1,$$

Now, if we set $\lambda^+ = \lambda + \mu$, $\lambda^- = \lambda - \mu$, $\gamma_1 = \gamma + (j_m, \mu)$, and recall that $d^1 = tj_n$, we obtain the equations

$$B\lambda^+ = d^0,$$

$$D\lambda^- + \gamma_1 j_n = tj_n. \tag{8}$$

Recall that the last row of D is the 0-row. Hence the last equation of the system (8) gives $\gamma_1 = t$, and the equation (8) takes the form

$$D\lambda^- = 0.$$

A solution $(\lambda^+, \lambda^-, \gamma_1)$ is feasible if the vector (λ, μ, γ) is nonnegative. Since

$$\lambda = \frac{1}{2}(\lambda^+ + \lambda^-),\ \mu = \frac{1}{2}(\lambda^+ - \lambda^-),\ \text{and}\ \gamma = t - (j_m, \mu),$$

a solution $(\lambda^+, \lambda^-, \gamma_1)$ is feasible if

$$\lambda^+ \geq 0,\ |\lambda^-| \leq \lambda^+,\ \text{and}\ t \geq (j_m, \mu). \tag{9}$$

Since the main facet $F_0(n)$ is simplicial, the system $B\lambda^+ = d^0$ has the full rank m such that $\lambda^+ = \lambda^0$ is the unique solution.

We try to find an integral solution for λ^-. By (9), we have that $|\lambda^-| \leq \lambda^0$. This implies that $\lambda^-_S \neq 0$ only for sets S where $\lambda^0_S \neq 0$. Since λ^0 is a (0,1)-vector, an integral λ^-_S takes the value 0 and ± 1 only.

We write explicitely the matrix $(D, j_n) \equiv D_n$.

$$D_n = \begin{pmatrix} 1 & 0 & j_k^T & 0 & j_k^T & j_l^T & 1 \\ 0 & 1 & 0 & j_k^T & j_k^T & j_l^T & 1 \\ 0 & 0 & I_k & I_k & I_k & G_k & j_k \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The first, the second and the last rows of the matrix D_n are indexed by the pairs $(1, n+1), (2, n+1)$ and $(n, n+1)$, respectively. The third row consists of matrices with k rows corresponding to the pairs $(i, n+1)$ with $3 \leq i \leq n-1$. The columns of D_n are indexed by sets $S \in \mathcal{S}_0 \cup \{n+1\}$ in the following sequence $\{1\}, \{2\}, \{1i\}, \{2i\}, \{12i\},\ 3 \leq i \leq n-1,\ \{12ij\},\ 3 \leq i < j \leq n-1,\ \{n+1\}$. I_k is the $k \times k$ unite matrix, and G_k is $k \times l$ incidence matrix of the complete graph K_k. G_k contains exactly two 1's in each column, i.e. $j_k^T G_k = 2j_l^T$. The matrix $D_{n'}$ is an obvious submatrix of D_n, for $n' < n$.

In the above notation, the equation $D\lambda^- = 0$ takes the form

$$\lambda^-_i + j_k^T(\lambda^-_{ik} + \lambda^-_k) + j_l^T\lambda^-_l = 0,\ i = 1, 2,$$

$$\lambda^-_{1k} + \lambda^-_{2k} + \lambda^-_k + G_k\lambda^-_l = 0.$$

Since $j_k^T G_k = 2j_l^T$, the last equality implies that

$$j_k^T(\lambda^-_{1k} + \lambda^-_{2k} + \lambda^-_k) + 2j_l^T\lambda^-_l = 0.$$

Hence the above system implies

$$\lambda^-_1 + \ \lambda^-_2 + j_k^T\lambda^-_k = 0.$$

Recall that we look out a $(0,\pm1)$-solution. Note that if $\lambda^+_S = 1$ and $\lambda^-_S = 0$, then $\lambda_S = \mu_S = \frac{1}{2}$ is nonintegral. Hence we shall look out a solution such that $\lambda^-_S = \pm\lambda^0_S$. So, such a solution is nonzero there where λ^0_S is nonzero.

The main part of above equations is contained in the term $G_k\lambda^-_l$. We can treat the (±1)-variables $(\lambda^-)_{ij} \equiv \lambda^-_{12ij}$ as labels of edges of the complete graph K_n. Now the problem is reduced to finding such labelling of edges of K_n that the sum of labels of edges incident to a given vertex is equal to a prescribed value, usually equal to 0 or ±1. The existence of such a solution depends on a possibility of factorization of K_n into circuits and 1-factors.

Corresponding facts can be found in [14], Theorems 9.6 and 9.7.

A tedious inspection shows that a feasible labelling exists for each of the 7 special points if $n \not\equiv 2 \pmod 4$ (i.e. if $k \not\equiv 3 \pmod 4$), and for 5 special points if $n \equiv 2 \pmod 4$. For other 2 points with $(a_1,a_2,b_1,b_2,c_1,c_2) = (1,1,0,0,0,1)$ and $(0,1,0,1,1,1)$ there is no feasible solution, i.e. there are S such that $\lambda^-_S = 0 \neq \pm\lambda^0_S$.

Now the assertion of the proposition follows. □

In the table below t-extensions of some special points are given explicitly. The last column of the table gives a point of A^0_{4m-1} for any $m \geq 2$.

$n \pmod 4 \equiv$	3	0	1	2
d_{12}	$n-3$	0	$n-1$	2
d_{1i} $3 \leq i \leq n-1$	$\binom{n-4}{2}+1$	$\binom{n-4}{2}$	$\binom{n-4}{2}+2$	$\binom{n-4}{2}+1$
d_{2i} $3 \leq i \leq n-1$	$\binom{n-3}{2}$	$\binom{n-4}{2}$	$\binom{n-3}{2}+1$	$\binom{n-4}{2}+1$
$d_{ij} (i \neq j)$ $(3 \leq i,j \leq n-1)$	$2(n-4)$	$2(n-5)$	$2(n-4)$	$2(n-5)$
d_{1n}	$\binom{n-3}{2}$	$\binom{n-3}{2}$	$\binom{n-3}{2}+1$	$\binom{n-3}{2}+1$
d_{2n}	$\binom{n-2}{2}$	$\binom{n-3}{2}$	$\binom{n-2}{2}+1$	$\binom{n-3}{2}+1$
d_{in} $3 \leq i \leq n-1$	$n-3$	$n-4$	$n-3$	$n-4$
$d_{in+1} (i \neq n+1)$	$\binom{n-2}{2}/2$	$\binom{n-3}{2}/2$	$(\binom{n-2}{2}+3)/2$	$(\binom{n-3}{2}+3)/2$

Remarks.

a) For the smallest possible $n \equiv 2 \pmod 4$, and $n \geq 6$, (i.e. for $n = 6$) distance d is the 3-extension of $d_6 = 2d(K_6 - P_{(1,6,2)})$, corresponding to the special point (1,1,0,0,0,1). On the other hand, the 3-extension of $2d(K_5 - P_{(1,2,5)})$ by the point 6 is an h-point.

For $n \equiv 0$ and $n \equiv 3 \pmod 4$ this d is an antipodal extension at the point n, i.e. $d_{in} + d_{2i} = d_{2n}$ for all i.

b) If we consider λ_l^0 such that $\lambda_{12ij}^0 = 0$ or 1, then the problem is reduced to a factorization of the graph whose edges are pairs (ij) such that $\lambda_{12ij}^0 \neq 0$.

c) In fact, we can take t slightly less. By (9), we must have $t \geq (j_m, \mu)$. Let r be the number of $S \in \mathcal{S}_0$ such that $\lambda_S = 1$. Then $(j_m, \mu) \leq \frac{1}{2}(\sum_{S \in \mathcal{S}_0} \lambda_S^0 - r)$.

Proposition 6.2. *Let $\mathcal{K}$ be the family of cuts lying on the 0-lifting $F(n)$ of the main facet $F_0(n)$. Then $A(\mathcal{K}) = 0$ if and only if $n \leq 5$.*

Proof. By Lemma 6.1, $F(6)$ has quasi-h-points, and (6) implies that quasi-h-points exist in all $F(n)$ for $n > 6$. We prove that there is no quasi-h-point on $F(n)$ for $n \leq 5$.

We use the above notations and the equations $B(\lambda + \mu) = d^0$, $D_n(\lambda - \mu) + \gamma_1 j_n = d^1$. The first equation has the unique solution $\lambda + \mu = \lambda^0$. Hence $2D_n\lambda - D_n\lambda^0 + \gamma_1 j_n = d^1$, where $\gamma_1 = \gamma + (j_{m_0}, \lambda^0) - (j_{m_0}, \lambda)$. The last row gives $\gamma_1 = d_{n,n+1}$. Hence the i-th row of the equation with D_n takes the form

$$(D_n\lambda)_i = \frac{1}{2}((D_n\lambda^0)_i + d_{i,n+1} - d_{n,n+1}).$$

It can be shown that the condition of evenness (3) implies that the right hand side is an integer for $n \leq 5$. Moreover, for $n \leq 5$, the matrix D_n is unimodular, i.e. $|detD'| \leq 1$ for each $n \times n$ submatrix D' of D_n. Therefore any solution λ is an integer. This implies that μ and $\gamma = d_{n,n+1} - (j_{m_0}, \mu)$ are integers, too.

So, all points $d \in L_{n+1} \cap F(n)$ have a Z_+-realization (λ, μ, γ) for $n \leq 5$. □

Now we give some other examples of Z_+-realizations of t-extensions of even h-points.

Using the fact that $\sum_{i \in V_n} \delta(i)$ is the unique Z_+-realization of $2d(K_n)$ for $n \neq 4$, (see [5]), we obtain

Lemma 6.3. *The only Z_+-realizations of $ext_t(2d(K_n))$, $n \geq 5$, $t \in Z_+$, are*

$$(1) \ \sum_{i \in V_n} \delta(i) + (t-1)\delta(n+1) \text{ for } t \geq 1,$$

$$(1') \ \sum_{i \in V_n} \delta(i, n+1) + (t-n+1)\delta(n+1) \text{ for } t \geq n-1.$$

Proof. Note that $d^0 = 2d(K_n)$ is an even (0,1)-point of C_n. The coefficients of its (0,1)-realization λ^0 are as follows: $\lambda^0_S = 1$ if $S = \{i\}$, $1 \leq i \leq n-1$, or $S = V_{n-1}$, and $\lambda^0_S = 0$ for other S. (Recall that we use S such that $n \notin S$.) Since it is unique Z_+-realization of d^0, the equation $B\lambda^+ = d^0$ has the unique integral solution $\lambda^+ = \lambda^0$.

Submatrix of D consisting of columns corresponding S with $\lambda^+_S \neq 0$, and without the last zero row, has the form $D = (I_{n-1}, j_{n-1})$. Hence the unique (± 1)-solutions of the equation $D\lambda^- = 0$ are as follows: 1) $\lambda^-_i = 1$, $1 \leq i \leq n-1$, $\lambda^-_{V_{n-1}} = -1$, and 2) $\lambda^-_i = -1$, $1 \leq i \leq n-1$, $\lambda^-_{V_{n-1}} = 1$.

Since $(j_m, \mu) = 1$ in the first case, and $(j_m, \mu) = n-1$, in the second case, we have $\gamma = t-1$, and $\gamma = t-n+1$, respectively. These solutions give the above Z_+-realizations (1) and (1'). □

If we define $d^{n,t} = ant_{2t}ext_t(2d(K_{n-1}))$, we obtain

$$d^{n,t}_{ij} = 2,\ 1 \leq i < j \leq n-1,\ d_{i,n} = d_{i,n+1} = t,\ 1 \leq i \leq n,\ d_{n,n+1} = 2t.$$

If we apply (4) to (1) and (1') of Lemma 6.3 (where n is interchanged by $n-1$), we obtain (2) and (2) with n and $n+1$ interchanged of Lemma 6.4 below. Summing these two expressions, we obtain the symmetric expression (3) of the lemma.

Lemma 6.4. *For $d^{n,t}$ the following holds*

$$(2)\ d^{n,t} = \sum_{i \in V_{n-1}} \delta(i, n+1) + (t-1)\delta(n) + (t-n+2)\delta(n+1),$$

$$(3)\ 2d^{n,t} = \sum_{i \in V_{n-1}} (\delta(i,n) + \delta(i,n+1)) + (2t-n+1)(\delta(n) + \delta(n+1)).$$

Lemma 6.5. *For $n \geq 6$, $d^{n,t}$ is h-embeddable if and only if $t \geq n-2$. Moreover, for $t \geq n-2$, the only Z_+-realizations are (2) and its image under the transposition (n,n+1).*

Proof. In fact, if we use Lemma 6.4, then the restrictions of an h-embedding of $d^{n,t}$ onto $V_{n+1} - \{n\}$ and V_n has to be of the form (1) and (1') or (1') and (1). □

The realizations (2) and (3) of Lemma 6.4 imply

Corollary 6.6. *$d^{n,t}$ is a quasi-h-point of C_n and $(antC_n) \cap C^{1,2}_{n+1}$ having the scale 2 if $\lceil \frac{n-1}{2} \rceil \leq t \leq n-3$, $n \geq 5$.*

In fact, for $n = 7$ we have to prove only that $2d(K_6 - P_{(5,6)})$ is an quasi-h-point of scale 2, and it will be done Section 7. For $n \geq 8$ we use (2), (3) and Lemma 6.4.

Remark. $d^{n-1,2} = 2d(K_n - P_2)$ and it is a quasi-h-point for $n \geq 6$. Its scale lies in the segment $[\lceil \frac{n}{4} \rceil, \frac{n}{2})$. $d^{n-1,2} \in Z(ant\mathcal{K}_{n-1} \cap \mathcal{K}_n^{1,2})$ (see Remark c) after Lemma 7.1 below) for $n \geq 6$, but $d^{n-1,2} \in R_+(ant\mathcal{K}_{n-1} \cap \mathcal{K}_n^{1,2})$ only for $n = 6$.

The cone $(antC_{n-1}) \cap C_n^{1,2}$ has excess 1. It has $2n-2$ cuts $\delta(i, n-1), \delta(i, n), \delta(n-1), \delta(n)$, for $i \in V_{n-2}$, its dimension is $2n-3$, and there is the following unique linear dependency

$$\sum_{i \in V_{n-2}} \delta(i, n-1) + (n+4)\delta(n) = \sum_{i \in V_{n-2}} \delta(i, n) + (n-4)\delta(n-1).$$

The sides of above equation differ only by the transposition (n-1,n).

The number of quasi-h-points in $(antC_{n-1}) \cap C_n^{1,2}$ is 0 for $n = 5$ (since it is so for the larger cone C_5) and $\geq n - 2 - \lceil \frac{n}{2} \rceil = \lfloor \frac{n}{2} \rfloor - 2$, which is implied by Corollary 6.6. Perhaps, it is exactly 1 for $n = 6, 7$.

7 Cones on 6 points

Consider the following cones generated by cut vectors on 6 points:

$$C_6,\ C_6^1,\ C_6^2 = EvenC_6,\ C_6^3,\ C_6^{1,2},\ C_6^{1,3} = OddC_6,\ C_6^{2,3},\ antC_5.$$

Recall (see Section 3) that the facets of C_6 are up to permutations of V_6 as follows:

a) 3-fold 0-lifting of the main 3-facet, 3-gonal facet $Hyp_6(1^2, -1, 0^3)$,

b) 0-lifting of the main 5-facet, 5-gonal facet $Hyp_6(1^3, -1^2, 0)$,

c) the main 6-facet and its "switching" (7-gonal simplicial facets) $Hyp_6(2, 1^2, -1^3)$ and $Hyp_6(-2, -1, 1^4)$.

Let

$$d_6 := 2d(K_6 - P_{(5,6)}).$$

Recall that (up to permutations) d_6 is the only known quasi-h-point of C_6.

The following lemma is useful for what follows. It can be checked by inspection. Recall that $V_n = \{1, 2, ..., n\}$.

Lemma 7.1. *(1) All Z_+-realizations of $2d_6$ are*

$$1a)\ 2d_6 = \sum_{i \in V_4} (\delta(i,5) + \delta(i,6)) \in Z_+(\mathcal{K}_6^2) = Z_+(Even\mathcal{K}_6),$$

$$1b)\ 2d_6 = (\delta(5) + \delta(6)) + \sum_{i \in V_3} (\delta(i,4,5) + \delta(i,4,6)) \in$$

$$Z_+(\mathcal{K}_6^{1,3}) = Z_+(Odd\mathcal{K}_6),$$

$$1c)\ 2d_6 = \delta(5) + \delta(j,5) + \sum_{i \in V_4 - \{j\}} (\delta(i,j,6) + \delta(i,6)) \text{ for } j \in V_4.$$

(2) Some representations of $d_6 = 2d(K_6 - P_{(5,6)})$ in L_6 are

$$2a)\ d_6 = \delta(5) + \sum_{i \in V_4} \delta(i,6) - \delta(6) \in L_6^{1,2},$$

$$2b)\ d_6 = 2\delta(5) + 2\delta(6) + \sum_{i \in V_4} \delta(i) - \delta(5,6) \in L_6^{1,2},$$

$$2c)\ d_6 = \sum_{i \in V_4} \delta(V_4 - \{i\}) - \delta(5,6) -$$

$$\sum_{i \in V_4} (\delta(i,i+1,6) - \delta(i,i+1)) \in L_6^{2,3}.$$

Here $i+1$ is taken by mod 4. □

Remarks.

a) The projection of 2a) onto $V_6 - \{1\}$ gives the Z_+-realization $2d(K_5 - P_{(5,6)}) = \delta(5) + \sum_{i=2,3,4} \delta(i,6)$; it and its permutation by the transposition (5,6) are the only Z_+-realizations of the above h-point.

b) "Small" pertubations of d_6 do not produce other quasi-h-points. For example, one can check that

$$d_6 + \delta(1,2) = \delta(1) + \delta(2) + \delta(6) + \delta(1,2,5) + \delta(3,5) + \delta(4,5);$$

it and its permutation by the transposition (5,6) are the only Z_+-realizations of this h-point.

c) Actually, 2a) is the case $n = 5, \alpha = 4$ of

$$ant_\alpha(2d(K_n)) = \delta(n) + \sum_{i \in V_{n-1}} \delta(i, n+1) - (n-\alpha)\delta(n+1) =$$

$$\sum_{i \in V_{n+1}} \delta(\{i\}) + (\frac{\alpha}{2} - 1)(\delta(\{n\}) + \delta(\{n+1\}) - \delta(\{n, n+1\})).$$

d) One can check that $L_n^{\neq 1} \subset L_n$ strictly, and $2Z^{15} \subset L_6^{\neq}$ strictly. Note that $L_6^{2,3} = L_6^{\neq 1}$. On the other hand, $L_n^{i,j} = L_n$ if and only if $(i,j) = (1,2)$.

e) By 1a) and 1b) of Lemma 7.1 we have

$$2d_6 \in hC_6^2 \text{ and } 2d_6 \in hC_6^{1,3},$$

$$\text{but } 2d_6 \notin L_6^2 \cup L_6^{1,3} = L(Even\mathcal{K}_6) \cup L(Odd\mathcal{K}_6).$$

We call a subcone of C_n a *cut subcone* if its extreme rays are cuts.

Lemma 7.2. *Let $d \in A(\mathcal{K})$ and let $\mathcal{K}(d)$ be the set of cuts of a minimal cut subcone of C_n containing d. Then*

(i) $d \in A(\mathcal{K}')$ for any $\mathcal{K}'$ such that $\mathcal{K}(d) \subseteq \mathcal{K}' \subseteq \mathcal{K}$,

(ii) $e(\mathcal{K}') = 1$ implies $\mathcal{K}' = \mathcal{K}(d)$.

Proof. In fact, $d \notin Z_+(\mathcal{K}(d))$ implies $d \notin Z_+(\mathcal{K}')$, and $d \in Z(\mathcal{K}(d)) \cap C(\mathcal{K}(d))$ implies $d \in Z(\mathcal{K}') \cap C(\mathcal{K}')$, and (i) follows. If $e(\mathcal{K}') = 1$, then any proper cut subcone of $C(\mathcal{K})$ is simplicial and has no quasi-h-points. □

Now we remark that the cone $C_6^{1,2} \cap antC_5$ has excess 1, since it has dimension 9 and contains 10 cuts $\delta(5), \delta(6), \delta(i,5)\delta(i,6)$, $1 \leq i \leq 4$, with the unique linear dependency

$$\sum_{i \in V_n} (\delta(i,5) - \delta(i,6)) = 2(\delta(5) - \delta(6)).$$

Proposition 7.3. $d_6 = 2d(K_6 - P_2) \in A(\mathcal{K}_6)$ *and it is a quasi-h-point of the following proper subcones of C_6: $C_6^{1,2}$, $C_6^{2,3}$, $antC_5$, the triangle facet $Hyp(1^2, -1, 0^3)$ and $C_6^{1,2} \cap antC_5$ (which is a minimal cut subcone of C_6 containing d).*

Proof. The point d_6 , is the antipodal extension $ant_4(d_5)$ of the point $d_5 := 2d(K_5)$. The minimum size of Z_+-realizations of d_5 is equal to $z(d_5) = z_5^1 = 5$, since the only its Z_+-realization is the following decomposition $2d(K_5) = \sum_{i=1}^5 \delta(i)$.

The minimum size of R_+-realizations of d_5 is $s(d_5) = a_5^1 = 10/3$ which is given by the R_+-realization $d_5 = \frac{1}{3}\sum_{1 \leq i < j \leq 5} \delta(ij)$.

Since $10/3 < 4 < 5$, we deduce that $d_6 = 2d(K_6 - P_{\{5,6\}}) \notin Z_+(C_6)$.

But $d_6 \in C_6 \cap L_6$, from (1) and (2) of Lemma 7.1. So, $d_6 \in A_6^0$. Now, from 1a) and 2) of the same lemma, we have $d_6 \in C(\mathcal{K}_6^{1,2} \cap ant\mathcal{K}_5) \cap L(\mathcal{K}_6^{1,2} \cap ant\mathcal{K}_5)$, and so, using (ii) of Lemma 7.2, we get that $\mathcal{K}_6^{1,2} \cap ant\mathcal{K}_5$ is a minimal subcone $\mathcal{K}(d)$.

Using (i) of Lemma 7.2, and the fact that $antC_5$ is the intersection of some triangular facets, we get the assertion of Proposition 7.3 for $C_6^{1,2}$, $antC_5$ and the triangle facet. Finaly, 1a) and 2c) of Lemma 7.1 imply that $d_6 \in A(\mathcal{K}_6^{2,3})$. □

Remarks.

a) On the other hand, the following subcones $C(\mathcal{K})$ of C_6 have $A(\mathcal{K})=\emptyset$: 5 simplicial cones C_6^i, $i=1,2,3$, both 7-gonal facets, and nonsimplicial cones: C_5, $C_6^{1,3}=OddC_6$ and 5-gonal facet.

b) Nonsimplicial cones $C_6, C_6^{1,2}, C_6^{2,3}, C_6^{1,3}$, $C_5, antC_5$, $Hyp_6(1^2,-1,0^3)$, $Hyp_6(1^3,-1^2,0)$ have excess 16,6,10,1,5,5,9,5, respectively. The cones C_6,-$C_6^{1,2}, C_6^{2,3}, C^{1,3}, C_5$ have, respectively, 210,495,780,60,40 facets and the facets are partitioned, respectively, into 4,5,8,1,2 classes of equivalent facets up to permutatons.

8 Scales

In this section we consider the scale $\eta^0(ant_\alpha 2d(K_n))$ which is, by Proposition 4.1(iii), equal to $min\{t\in Z_+ : \alpha t\geq z_n^t\}$, especially for two extreme cases $\alpha=4$ and $\alpha=n-1$. The number t below is always a positive integer.

Denote by $H(4t)$ a Hadamard matrix of order $4t$, and by $PG(2,t)$ a projective plane of order t.

It is proved in [5] that $t\sum_1^n\delta(\{i\})$ is the unique Z_+-realization of $2td(K_n)$ if $n\geq t^2+t+3$, and that for $n=t^2+t+2$, $2td(K_n)$ has other Z_+-realizations if and only if there exists a $PG(2,t)$. Below, in $(iv_1)-(iv_3)$ of Theorem 8.1, we reformulate this result in terms of A_n^1, $\eta^1(2d(K_n))$, z_n^t, using the following trivial relations

$$\eta^1(2d(K_n))\geq t+1 \Leftrightarrow 2td(K_n)\in A_n^1 \Leftrightarrow z_n^t=nt \Leftrightarrow$$

$$\Leftrightarrow t\sum_1^n\delta(\{i\}) \text{ is the unique } Z_+\text{-realization of } 2td(K_n).$$

(iii_2) of Theorem 8.1 follows from a result of Ryser (reformulated in terms of z_n^t in Theorem 4.6(1) of [9]) that $z_n^t\geq n-1$ with equality if and only if $n=4t$ and there exists a $H(4t)$.

Theorem 8.1

(i_1) $ant_\beta 2td(K_n)\in C_{n+1}$ if and only if $\beta\geq\frac{tn(n-1)}{\lfloor n/2\rfloor\lceil n/2\rceil}$;
(i_2) $ant_\beta 2td(K_n)\in A^0$ if and only if $\frac{tn(n-1)}{\lfloor n/2\rfloor\lceil n/2\rceil}\leq\beta<z_n^t$, $\beta\in Z_+$;
(i_3) $ant_\beta 2td(K_n)\in hC_{n+1}$ if and only if $\beta\geq z_n^t$, $\beta\in Z_+$;
(i_4) $ant_\alpha 2d(K_n)\in C_{n+1}\cap L_{n+1}$ if and only if $\frac{n(n-1)}{\lfloor n/2\rfloor\lceil n/2\rceil}\leq\alpha$, $\alpha\in Z_+$.

Moreover, if $d = ant_\alpha 2d(K_n) \in C_{n+1} \cap L_{n+1}$, then
(ii_1) either $n = 3, d \in A_3^1$, d is simplicial, $d = ant_3 2d(K_4)$
(so $\eta^i(d) = 1$ for $i \geq 0$),
or $d \in A_n^1$, d is not simplicial, $\alpha \geq n \geq 4$ (so $\eta^0(d) = 1$),
or $d \in A_n^0$ (so $\eta^0(d) \geq 2$),
(ii_2) $\eta^0(d) = min\{t : z_n^t \leq \alpha t\}$.

(iii_1) $\eta^0(ant_4 2d(K_n) = \eta^0(2d(K_{n+1} - P_{(1,2)})) = \eta^0(2d(K_{n\times 2}))$;
(iii_2) $\lceil n/4 \rceil \leq \eta^0(ant_4 2d(K_n)) \leq$
$min\{t \in Z_+ : n \leq 4t$ and there exists a $H(4t)\} < n/2$;
(iii_3) For $n = 4t, 4t-1$, we have $\eta^0(ant_4 2d(K_n)) =$
$\lceil n/4 \rceil = t$ if and only if there exists a $H(4t)$;

(iv_1) $\eta^0(ant_{n-1}2d(K_n)) = \eta^1(2d(K_n)) \leq min\{n-3, \eta^1(2d(K_{n+1}))\}$;
(iv_2) $\left\lceil \frac{1}{2}(\sqrt{4n-7} - 1) \right\rceil = min\{t \in Z_+ : n \leq t^2 + t + 2\}$
$\leq \eta^0(ant_{n-1}2d(K_n))$
$\leq min\{t \in Z_+ : n \leq t^2 + t + 2$ and there exists a $PG(2,t)\}$;
(iv_3) For $n = t^2 + t + 2$, we have $\eta^0(ant_{n-1}2d(K_n)) =$
$\left\lceil \frac{1}{2}(\sqrt{4n-7} - 1) \right\rceil = t$ if and only if there exists a $PG(2,t)$.

Remarks. a) For $i \geq 0$, we have $\eta^{i+1}(2d(K_4)) = i + 1$, but $\eta^i(ant_3(2d(K_4))) = 1$, since $ant_3(2d(K_4))$ is a simplicial point. For $i \geq 0$ and $n \geq 5$, we have $\eta^{i+1}(2d(K_n)) \leq \eta^i(ant_{n-1}(2d(K_n)))$ with equality for $i = 0$ and for some pair (i,n) with $i \geq 1$. Propositions 5.9-5.11 of [9] imply that

$$\eta^{i+1}(2d(K_5)) = \eta^i(ant_4(2d(K_5))) = 2 \text{ for } i = 0, 1;$$

$$\eta^3(2d(K_5)) = \eta^2(ant_4(2d(K_5))) = \eta^4(2d(K_5)) = 3;$$

$$\eta^5(2d(K_5)) = \eta^4(ant_4(2d(K_5))) = \eta^3(ant_4(2d(K_5))) = 4.$$

b) Using the well-known fact that $H(4t)$ exists for $t \leq 106$, we obtain that

$$\eta^0(ant_4(2d(K_n))) = \eta^0(2d(K_{n+1} - P_2)) =$$

$$\eta^0(2d(K_{n\times 2})) = \lceil n/4 \rceil \text{ for } n \in [4, 424];$$

c) Using the well-known fact that $PG(2,t)$, $t \leq 11$, exists if and only if $t \neq 6, 10$, we obtain for $a_n = \eta^0(ant_{n-1}(2d(K_n))) = \eta^1(2d(K_n))$, that $6 \leq a_n \leq 7$ for $33 \leq n \leq 43$, $10 \leq a_n \leq 11$ for $93 \leq n \leq 111$, and $a_n = \left\lceil \frac{1}{2}(\sqrt{4n-7} - 1) \right\rceil$ for all other $n \in [4, 134]$.

d) (iii),(iv) of Theorem 8.1 imply that

$$\eta^0(d(K_{2t\times 2})) \geq 2t \text{ with equality if and only if there exists } H(4t),$$

$$\eta^1(d(K_{t^2+t+2})) \geq 2t \text{ with equality if and only if there exists } PG(2,t).$$

Note also that $a_n \leq n-3$ with equality if and only if $n = 4, 5$.

Proof of (iv_1). *For* $n \geq 4$ *we have*

$$\left\lceil \frac{1}{2}(\sqrt{4n-7}-1) \right\rceil \leq \eta^1(2d(K_n)) = \eta^0(ant_{n-1}(2d(K_n))) \leq n-3.$$

In fact, we have

$$\eta^1(2d(K_n)) = min\{t \in Z_+ : z_n^t < nt\},$$

$$\eta^0(ant_N(2d(K_n))) = min\{t \in Z_+ : z_n^t \leq Nt\},$$

since $2td(K_n)$ has the following Z_+-realization $t\sum_1^n \delta(\{i\})$ of maximal size nt, and since $t(ant_N(2d(K_n))) \in hC_{n+1}$ if and only if $2td(K_n)$ admits a Z_+-realization of size at most Nt. Denote

$$p = \eta^1(2td(K_n)),\ q = \eta^0(ant_{n-1}(2d(K_n))).$$

Then $p \leq q$, because $z_n^q \leq (n-1)q$ implies $z_n^q \leq nq$. Also, $q \leq n-3$, because $2(n-3)d(K_n)$ has the Z_+-realization $\sum_1^{n-1}((n-4)\delta(\{i\}) + \delta(\{i,n\})$ of size $(n-3)(n-1)$. On the other hand, $p \geq q$, because $z_n^p < np$ implies $z_n^p \leq np-(n-3)$, which is proved in Proposition 5.3 of [9]. So $z_n^p \leq np - q \leq np - p$. We have $p \geq \left\lceil \frac{1}{2}(\sqrt{4n-7}-1) \right\rceil$, because otherwise $n \geq p^2+p+3$, and using [5], $2td(K_n)$ has exactly one Z_+-realization, a contradiction with the definition of p.

Theorem 8.2.

$(i)\eta_n^0 < \infty$,
$(ii)\eta_n^{i-1}|\eta_n^i$ for $i \geq 1$, and $\eta_{n-1}^i|\eta_n^i$ for $n \geq 5$,
$(iii)\eta^i(ad) = \lceil \eta^i(d)/a \rceil$ for $d \in C_n \cup L_n,\ i \geq 0,\ a \in Z_+$.

Proof. (i) Define

$$Y = L_n \cap C_n \cap \{\sum \lambda_S \delta(S) : 0 \leq \lambda_S \leq 1\}.$$

Clearly, Y is finite, and one can find λ such that λd is an h-point for every $d \in Y$.

Let $d \in L_n \cap C_n$ has a R_+-realization $d = \sum \mu_S \delta(S)$. Clearly the coefficients μ_S are rational numbers. We have $d = d_1 + d_2$, where $d_1 = \sum \lfloor \mu_S \rfloor \delta(S)$, and $d_2 = \sum(\mu_S - \lfloor \mu_S \rfloor)\delta(S)$. By the construction, d_1 is an h-point. Since $d_2 = d - d_1$ and $d \in L_n \cap C_n$, $d_1 \in L_n \cap C_n$, we obtain $d_2 \in Y$. Hence there is λ such that $\lambda d_2 \in hC_n$, and we obtain that $\lambda d = \lambda d_1 + \lambda d_2$ is an h-point, too.

(iii) Take $\lambda = \eta^i(ad)$, i.e. $\lambda(ad)$ has at least $i+1$ Z_+-realizations. Hence $\lambda a \geq \eta^i(d)$ implies $\lambda \geq \lceil \eta^i(d)/a \rceil$, i.e. $\eta^i(ad) \geq \lceil \eta^i(d)/a \rceil$.

Now, take $\lambda = \lceil \eta^i(d)/a \rceil$. So, $\lambda - 1 < \eta^i(d)/a \leq \lambda \Rightarrow (\lambda-1)a < \eta^i(d) \leq \lambda a$. Hence λad has at least $i+1$ Z_+-realizations, implying that $\lambda \geq \eta^i(ad)$, and so $\lceil \eta^i(d)/a \rceil \geq \eta^i(ad)$.

Remarks. a) $\eta^i_4 = \eta^i(2d(K_4)) = i$ for $i \geq 1$; $\eta^0_n = 1$ if and only if $n = 4, 5$.

b) For $d \notin L_n$ and $\lambda \in Z_+$, we have $\lambda d \in L_n$ implies that λ is even (because $(\lambda d_{ij} + \lambda d_{ik} + \lambda d_{jk})/2 = \lambda(d_{ij} + d_{ik} + d_{jk})/2$). Hence, for $d \in Z^{\binom{2}{n}} - A^0_n$, we have either $d \notin L_n$ (so $\eta^0(d)$ is even), or $\eta^0(d) = 1$ (i.e. $d \in hC_n$). Since $d(G) \notin A^0_n$ for any connected graph G on n vertices (see [13]), we have either $\eta^0(d(G)) = 1$ or $\eta^0(d(G))$ is even. But, for example, $\eta^0(2d(K_{10} - P_2)) = \eta^0(2d(K_{9\times 2})) = 3$.

It will be interesting to see whether η^0_n and $max\{\eta^0(d) : d \in A^0_n\}$ are bounded from above by $const \times n$.

The best known lower bound for the last number is $\eta^0(d(K_n - P_2))$ which belongs to the interval $[2\lceil (n-1)/4 \rceil, n-2]$.

It is proved in [17] that for a graphic metric $d = d(G)$, we have

(i) $\eta^0(d) \leq n-2$ if $d(G) \in C_n$,

(ii) $\eta^0(d) \in \{1, 2\}$, i.e. G is an isometric subgraph of a hypercube or a halved cube if $d(G)$ is simplicial.

9 h-points

Recall that any point of $Z_+(\mathcal{K}_n) = hC_n$ is called an h-point.

A point d is called *k-gonal*, if it satisfies all hypermetric inequalities $Hyp_n(b)$ with $\sum_1^n |b_i| = k$.

The following cases are examples when the conditions $d \in L_n$ and hypermetricity of d imply that d is an h-point.

a) [13], [15]: If $d = d(G)$ and G is bipartite, then 5-gonality of d implies that $d \in hC_n$;

b) [1]: If $\{d_{ij}\} \in \{1,2\}$, $1 \le i < j \le n$, then $d \in L_n$ and 5-gonality of d imply that $d \in hC_n$ (actually, $d = d(K_{1,n-1})$, $d(K_{2,2})$ or $2d(K_n)$ in this case);

c) [2]: If $n \ge 9$ and $\{d_{ij}\} \in \{1,2,3\}$, $1 \le i < j \le n$, then $d \in L_n$ and ≤ 11-gonality of d imply that $d \in hC_n$.

So, the cases a),b),c) are among known cases when the problem of testing membership of d in hC_n can be solved by a polynomial time algorithm. The polynomial testing holds for any $d = d(G)$ (see [17]) and for "generalized bipartite" metrics (see [7] which generalize the cases b) and c) above).

The cases a),b) and c) imply (i),(ii) and (iii), respectively, of

Corollary 9.1 *If $d \in A_n^0$, then*

(i) neither $d = d(G)$ for a bipartite graph G,

(ii) nor $\{d_{ij}\} \in \{1,2\}$, $1 \le i < j \le n$,

(iii) nor $\{d_{ij}\} \in \{1,2,3\}$, $1 \le i < j \le n$, if $n \ge 9$.

A point $d \in Z_+(K_n) = hC_n$ is called *rigid* if d admits a unique Z_+-realization. In other words, d is rigid if and only if $d \in A_n^1$. Clearly, if $d \in hC_n$ is simplicial, then d is rigid. Rigid nonsimplicial points are more interesting. Hence we define the set

$$\tilde{A}_n^1 := \{d \in A_n^1 : d \text{ is not simplicial}\},$$

and call its points *h-rigid.*

Theorem 9.2

$$(i) A_n^0 = \emptyset \text{ for } n \le 5,\ 2d(K_6 - P_2) \in A_6^0, |a_n^0 = \infty \text{ for } n \ge 7,$$
$$(ii) \tilde{A}_n^1 = \emptyset \text{ for } n \le 4,\ \tilde{A}_5^1 = \{2d(K_5)\},\ |\tilde{A}_n^1| = \infty \text{ for } n \ge 6,$$
$$(iii) \text{ for } i \ge 2,\ A_n^i = \emptyset \text{ if } n \le 3,\ |A_n^i| = \infty \text{ if } n \ge 4.$$

Proof. (i) and (ii) The first equalities in (i) and (ii) are implied by results of [3]. The inclusion in (i) is implied by [1]. The second equality in (ii) is proved in [12]. We have $|A_n^0| = \infty$ for $n \ge 7$, because $A_6^0 \ne \emptyset$ and $|A_{n+1}^i| = \infty$ whenever $A_n^i \ne \emptyset$ from(6).

We prove the third equality of (ii): $|\tilde{A}_n^1| = \infty$ for $n \ge 6$. The equality is implied by the fact that $ant_\alpha(2d(K_n)) \in \tilde{A}_{n+1}^1$ for any $n \ge 5$, $\alpha \in Z_+$, $\alpha \ge n$. We prove the inclusion.

Recall that $2td(K_n)$ has the unique Z_+-realization of size tn if $n \ge t^2 + t + 3$. (See [5] or the beginning of Section 8). For $t = 1$ we obtain the equality $z(2d(K_n)) = n$ for $n \ge 5$ Using that $2d(K_n)$ is not simplicial for $n \ge 4$, and (iv) of Proposition 4.1 we obtain the wanted inclusion.

(iii) Since C_3 is simplicial, $A_3^i = \emptyset$ for $i \geq 2$. Consider now $n = 4$. We show that $A_4^i = \{2(i-1)d(K_4) + d : d$ is a simplicial h-point of $C_4\}$. This follows from the fact that the only linear dependency on cuts of C_4 is, up to multiple, $\delta(1) + \delta(2) + \delta(3) + \delta(4) = \delta(1,4) + \delta(2,4) + \delta(3,4)$.

So, $|A_4^i| = \infty$, because there are an infinity of simplicial points, e.g. $\lambda d(K_{2,2})$ for $\lambda \in Z_+$. Finally we use (6). □

Some questions.

a) Is it true that all 10 permutations of $d_6 = 2d(K_6 - P_2)$ are only quasi-h-points of C_6? If yes, then these 10 points and 31 nonzero cuts from $\mathcal{K}_6$ form a Hilbert basis of C_6.

b) Does exist a ray $\{\lambda d : \lambda \in R_+\} \subset C_n$ containing an infinity set of quasi-h-points? Recall that we got in Section 6 examples of rays $\{d^0 + td^1 : t \geq 0\}$ containing infinitely many quasi-h-points.

Lemma 9.3. *Let $d \in A_n^0$, and let $d = ant_\alpha d'$ where $d' \notin A_{n-1}^0$. Then d' is an h-point and $z(d') \geq \lceil s(d') \rceil + 1$.*

Proof. In fact, $d \in C_n \cap L_n$, so $d' \in C_{n-1} \cap L_{n-1}$. But $d' \notin A_{n-1}^0$, so d' is an h-point of C_{n-1}. Hence by Proposition 4.1(ii), $\alpha \in Z_+$, $s(d') \leq \alpha < z(d')$.

Note that for $n \geq 5$ we have $2d(K_{n\times 2}) \in A_{2n}^0$, $2d(K_{n\times 2}) = ant_4 d'$ where $d' \in A_{2n-1}^0$ and $d' = ant_4 d$" for d"$\in A_{2n-2}^0$, etc.

So, d' is neither simplicial point nor an antipodal extension (i.e. $d' \notin R_+(ant\mathcal{K}_{n-2})$), nor $d' \in Z_+(\mathcal{K}_{n-1}^m)$, $m = \lfloor (n-1)/2 \rfloor$,

because in each of these 3 cases we have for an h-point d', $z(d') = s(d')$; it implies also that, by Proposition 4.1(iv), d itself is not simplicial. □

The following proposition makes plausible the fact that the metric $d_6 = 2d(K_6 - P_2)$ is the unique (up to permutations) quasi-h-point of C_6.

Proposition 9.4. *Let $d \in A_6^0$, $d = ant_\alpha d'$ and $d \neq d_6$. Then*

a) both d and d' are not simplicial;

b) $d' \notin R_+(ant\mathcal{K}_4)$, $d' \notin Z_+(\mathcal{K}_5^2)$;

c) $d' \neq \lambda d(G)$ for any $\lambda \in Z_+$ and any graph G on 5 vertices;

d) d' has at least two Z_+-realizations.

Proof. Since $A_5^0 = \emptyset$ by [3], we can apply Proposition 9.3, and a),b) follow. One can see by inspection, that among all 21 connected graphs on 5 vertices, the only graphs G with nonsimplicial $d(G) \in C_6$ are the following 3 graphs: K_5, $K_5 - P_2$, and $K_4.K_2 = K_4$ with an additional vertex adjacent to a vertex of K_4. For these graphs, $\lambda d(G)$ is an h-point if and only if $\lambda \in 2Z_+$.

Since $2d(K_5 - P_2) = ant_4(2d(K_4))$, then, according to b), $d' \neq \lambda d(K_5 - P_2)$.

Since for any $\lambda \in Z_+$ we have $z(2\lambda d(K_4.K_2)) = 5\lambda = s(2\lambda d(K_4.K_2))$, and (by Proposition 9.3) $s(d') < z(d')$, then $d' \neq \lambda d(K_4.K_2)$.

Remains the case $d' = \lambda d(K_5)$. We have $s(d') = \lambda 5/3$, $z(d') = 5$ for $\lambda = 2$ and $z(d') = s(d')$ for $\lambda \in 2Z_+$, $\lambda > 2$.(See Proposition 5.11 of [9]). So $s(d') \leq \alpha < z(d')$ implies $\lambda = 2$, $\alpha = 4$, i.e. exactly the case $d = ant_4(2d(K_5))$. This proves c).

d) follows from the fact (see [12]) that $2d(K_5)$ is the unique nonsimplicial h-point of C_5 with unique Z_+-realization.

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