Well-Ordering of Algebra and Kruskal's Theorem

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Abstract

We define well-partial-orderings on abstract algebras and give their order types. For every ordinal in an initial segment of Bachmann hierarchy there is one and only one (up to isomorphism) algebra giving the ordinal as order type. As a corollary, we show Kruskal-type theorems for various structures are equivalent to well-orderedness of certain ordinals.

1 Introduction

The main theorem we prove in this paper is that the class of algebras yields a system of ordinal notations. Before making the assertion precise, we enumerate the consequences derived from the theorem: (1) we give the order types of well-partial-orders on abstract algebras; (2) Kruskal-type theorems for abstract algebras are shown to be equivalent to well-orderedness of ordinals; (3) we calculate the order types of lexicographic path orderings for abstract algebras.

The notion of well-partial-order (and generally well-quasi-order) began to appear around 1950 in the literatures, for example, by Erdös and Rado, and Higman. After the celebrated work by Higman [19], this simple notion has found a large number of applications in the fields of algebras, combinatorics, mathematical logic, and computer science. One of the most elegant is Kruskal's theorem [22, 27], asserting the class of finite trees is a well-partial-order with respect to the topological embedding. In the 80's, the theorem and its extension due to Friedman were used to prove the graph minor theorem by Robertson and Seymour [31] in combinatorics, and to give independence results for strong segments of second order arithmetic [13, 35] in mathematical logic, and to give useful methods to prove the termination of programs [7] in computer science.

We define a class of (abstract) algebras and partial orders (called embeddings) on them. The class of algebras is generated from an empty set and singletons by disjoint sum $+$, direct product $\times$, and the least fixpoint without nested recursion. The embedding on each algebra is defined as the divisibility ordering by Higman [19]. It is not hard to see that the embeddings are well-partial-orders.

A well-partial-order is by definition a partially ordered set that has no infinite bad sequences (See Section 2 for definition). Therefore the collection of all finite bad sequences of the well-partial-order makes a well-founded tree. Important among the properties of the well-partial-orderings is the order type, which is by definition simply the order type of the associated well-founded tree of all bad sequences. In section 2, we give two other characterizations of the order type. The one is the least ordinal reifying the well-partial-ordering, and the other is the greatest ordinal of linearizations. The reification proves well-partial-orderedness from well-orderedness, and conversely the linearization proves well-orderedness from well-partial-orderedness. Therefore knowing the order type, we may prove well-partial-orderedness from well-orderedness of the order type, and vice versa.

The substantial part of this paper is devoted to calculating the order types of the well-partial-orders of the algebra embeddings. The result is interesting in its own right. The following is our main theorem:

> *The map assigning the order type to each algebra is a bijection from the class of algebras to an initial segment of the class of ordinals.*

The initial segment is the class of ordinals up to $\varphi(\Omega^\omega, 0)$ in Bachmann hierarchy. Given two algebras, the order types of the embeddings on them are always different unless they have an isomorphism determined by simple rules. Furthermore for each ordinal α less than $\varphi(\Omega^\omega, 0)$ there is a (unique up to isomorphism) algebra whose embedding gives the order type α.

From this theorem several consequences are derived. The most immediate is that one may regard the class of algebras as a system of ordinal notations. This system has some new features in comparison with the traditional systems. Notably every notation has a meaning in our system. For example, consider the ordinal Γ_0, which has a meaning as the least strongly critical ordinal [16]. But what has a meaning is the ordinal itself, not the notation assigned to it. So the notations assigned to Γ_0 differ in one system to others. In Bachmann hierarchy it is denoted by $\varphi(\Omega, 0)$; in Buchholz notation [3] $\psi_0\Omega^\Omega$, etc. In our system, it is denoted by $\mu X.\, 2X^2 + 1$, which is because the order type of the algebra $\mu X.\, 2X^2 + 1$ is Γ_0. There is still freedom of notations since $\mu X.\, 2X^2 + 1$ is not a unique notation for the algebra. However what is important is that we find an entity, other than the class of ordinals, naturally having a well-order thereon.

Another consequence is that we establish the equivalence of well-partial-orderedness of the algebra embeddings and well-orderedness of the ordinals that are the order types of the embeddings. The merit of this equivalence is apparent if one is concerned with proof theory. After a work of Gentzen on first order arithmetic, it is well-known that well-orderedness of large ordinals is independent from logical systems, especially fragments of second order arithmetic. For example, the ordinal ϵ_0 is independent from system ACA_0, which is a conservative extension of Peano arithmetic. Therefore our main theorem implies that well-partial-orderedness of the embedding on the algebra $\mu X.\, X^2 + 1$ of binary trees is independent from ACA_0. In other words, Kruskal's theorem for binary trees is unprovable in system ACA_0. Several results are found in Section 6.

For the calculation of lower bounds of order types of algebra embeddings, we use a version of recursive path ordering, which is a family of well-orderings most often used to prove the terminations of term rewriting systems. We associate a well-ordering called Ackermann ordering to each algebra. The ordering may be regarded as a lexicographic path ordering if we view the algebra as the collection of terms. We give the order types of the orderings in Section 6. The order types are greater than those of the recursive path orderings that are defined using multiset orderings.

2 Preliminaries

In this section, we give definitions for well-partial-orders and classic results.

A *partial order* is a set endowed with a binary relation $\trianglelefteq$ that is reflexive, transitive and anti-symmetric. A *total order* is a partial order where every two elements a, b are comparable; namely, one of $a \trianglelefteq b$ and $b \trianglelefteq a$ holds. A *bad sequence* in a partial order A is a sequence $\langle a_0, \ldots, a_n(, \ldots) \rangle$ (finite or infinite) of members of A satisfying $\forall i < j.\, a_i \ntrianglelefteq a_j$.

2.1 Definition
A *well-partial-order* is a partial order that has no infinite bad sequences. □

The order-reflecting maps are important as morphisms of well-partial-orders, since they send bad sequences to bad sequences. Here a map $f : A \to B$ of partial orders reflects order if and only if $f(a) \trianglelefteq_B f(a')$ implies $a \trianglelefteq_A a'$ for all a, a' of A. As an immediate consequence, every order-reflecting map reflects also the property of being a well-partial-order, namely the following holds:

2.2 Proposition
If $f : A \to B$ is an order-reflecting map of partial-orders and B is a well-partial-order, then A is a well-partial-order. □

Among well-partial-orders the following two are well-known. The one is Higman embedding on finite lists of members of a partial order and the other is the (homeomorphic) tree embedding of finite trees. First we give the definitions of these embeddings.

2.3 Definition
Let $\langle A, \prec \rangle$ be a partial order and A^* the set of finite lists of members of A.

The *Higman embedding* $\trianglelefteq_{hig}$ is a partial order on A^* defined as follows: $\langle a_0, \ldots, a_{m-1}\rangle \trianglelefteq_{hig} \langle a'_0, \ldots, a'_{n-1}\rangle$ if and only if there is a strictly monotonic map $f : m \to n$ such that $a_i \prec a'_{f(i)}$ for all $i < m$. $\square$

A finite *ordered* tree is a finite tree with root where for each node there is a linear order on the set of immediate successors. A finite *non-ordered* tree is a finite tree with root without orders on immediate successors. We denote a finite ordered tree t by $t\langle t/0, \ldots, t/(n-1)\rangle$ where $t/0, \ldots, t/(n-1)$ are the immediate successors of the root in this order. As for non-ordered trees, we use the notation $t\{t/0, \ldots, t/(n-1)\}$.

2.4 Definition
The *tree embedding* $\trianglelefteq_T$ is a partial order on the set T of finite ordered trees, the order defined as follows: $t\langle t/0, \ldots, t/(m-1)\rangle \trianglelefteq_T t'\langle t'/0, \ldots, t'/(n-1)\rangle$ if and only if either

(i) $t \trianglelefteq_T t'/j$ for some $j < n$; or
(ii) there is a strictly monotonic function $f : m \to n$ such that $t/i \trianglelefteq_T t/f(i)$ for all $i < m$. $\square$

If the node degree is fixed (e.g. binary trees) the assertion (ii) is replaced simply by that $t/i \trianglelefteq_T t'/i$ for all $i < m$. The tree embedding is also defined for non-ordered finite trees by imposing the condition that f is injection in place that f is strictly monotonic. The following two theorems assert that the Higman embedding and the tree embedding are well-partial-orders.

2.5 Theorem
(i) (Higman's Lemma) If A is a well-partial-order, then A^* is a well-partial-order with respect to the Higman embedding.
(ii) (Kruskal's Theorem) The set of finite ordered trees is a well-partial-order with respect to the tree embedding. $\square$

Vazsonyi's conjecture was well-partial-orderedness for non-ordered trees, but Kruskal indeed proved the theorem for ordered trees and derived Vazsonyi's conjecture from that [22]. For a simpler proof by Nash-Williams using the so-called minimal bad sequence argument, we refer the reader to [27, 16, 35]. The proof is worth comment in two respects; it uses a non-constructive argument in an essential way, and also an impredicative argument. In fact, to formalize the proof one needs a fragment of second order arithmetic having Π^1_1-comprehension axiom or its substitute, e.g. bar induction on recursive well-founded relation.

2.6 Notation
$Bad(A)$ denotes the tree of all finite bad sequences of a given partial order A. $\square$

The tree $Bad(A)$ is well-founded if and only if A is a well-partial-order. So the structure of the well-founded tree $Bad(A)$ has a lot of information on the structure of well-partial-order A. In particular, the order type $|Bad(A)|$ of the well-founded tree is of great importance. Let T be a well-founded tree. We assign an ordinal $|\sigma|$ to each node σ by the following definition:

$$|\sigma| = \sup \{|\sigma'| + 1 \mid \sigma' \text{ is an immediate successor of } \sigma\}.$$

Then the order type $|T|$ of well-founded tree T is defined by the ordinal $|\langle\ \rangle|$ assigned to the root $\langle\ \rangle$ of T.

2.7 Definition

The *order type* of a well-partial-order A is the order type $|Bad(A)|$ of the well-founded tree $Bad(A)$. $\square$

Note that well-orderedness of $|T|$ is equivalent to well-foundedness of T. Hence well-orderedness of the tree $|Bad(A)|$ is equivalent to well-partial-orderedness of A. It is difficult, however, to calculate the ordinal $|Bad(A)|$ concretely. The main techniques are the reification [36] and the linearization, where the reification gives upper bounds and the linearization lower bounds.

2.8 definition

Let A be a partial order and α an ordinal.

A *reification* of A by α is a map $r : Bad(A) \to \alpha + 1$ satisfying $\sigma \subset \tau \Rightarrow r(\sigma) > r(\tau)$. (Notation: $\sigma \subset \tau$ denotes that the sequence σ is a proper initial segment of τ.) $\square$

It is immediate to see that if a partial order A has a reification by an ordinal α, then $WO(\alpha)$ implies $Wpo(A)$. An example of reification is the assignment $|\cdot|$ given above. In fact, by recursion on the well-founded tree $Bad(A)$, we can show that the reification $|\cdot|$ is the least one: if r is a reification then $|\sigma| < r(\sigma)$ for every node σ of $|Bad(A)|$. Namely the following proposition holds.

2.9 Proposition

If a partial order A has a reification by an ordinal α, then $|Bad(A)| \leq \alpha$. $\square$

Here for the inequality $\leq$, we cannot drop the equality. The assignment $|\cdot|$ gives a reification by the ordinal $|Bad(A)|$. This proposition shows that reifications provide upper bounds of $|Bad(A)|$. As for lower bounds, they are obtained by linearizations.

2.10 Definition
Let $\langle A, \trianglelefteq \rangle$ be a partial order.

A *linearization* of A is a total order $\sqsubseteq$ on A finer than $\trianglelefteq$, namely, the total order $\sqsubseteq$ for which $a \trianglelefteq a' \Rightarrow a \sqsubseteq a'$ holds. $\square$

If $\langle A, \sqsubseteq \rangle$ is a linearization of a well-partial-order $\langle A, \trianglelefteq \rangle$, then $\langle A, \sqsubseteq \rangle$ becomes a well-order. It is this property that was used in [7] to prove that the simplification ordering terminates. The following proposition asserts that the linearization gives lower bounds of the order types of well-partial-orders.

2.11 Proposition
If a well-partial-order A has a linearization of order type α, then $\alpha \leq Bad(A)$.

(*Proof*) Let T be the well-founded tree of finite descending sequences with respect to the total order given by the linearization. Then $T \subseteq Bad(A)$ and so $\alpha = |T| \leq |Bad(A)|$. $\square$

In this proposition, we cannot omit the equality of $\leq$, In fact, the following is proved in [6]. Let $|Lin(A)|$ be the ordinal

$$\sup\{|\langle A, \sqsubseteq \rangle| \,:\, \langle A, \sqsubseteq \rangle \text{ is a linearization}\}.$$

De Jongh and Parikh proved that there is a linearization of order type $|Lin(A)|$. In other words, sup can be replaced by max. We can show that $|Lin(A)|$ is equal to $|Bad(A)|$, and so there is a linearization of order type $|Bad(A)|$.

Therefore if one finds an ordinal α giving both reification and linearization, then the order type $|Bad(A)|$ of well-partial-order A turns out to be equal to the ordinal α. In addition, well-partial-orderedness of A is equivalent to well-orderedness of α in the logic necessary to prove the reification and the linearization.

In order to calculate the order types of well-partial-orders, we need some ordinal notations. In later sections, we will use the class of algebras as a system of ordinal notations. We will compare the system with the Bachmann hierarchy. We refer the reader to [17] for definitions and basic properties of Bachmann hierarchy. $\langle \varphi(\alpha, \text{-}) \rangle$ denotes the family of normal functions in the hierarchy. We use also the family $\langle \overline{\varphi}(\alpha, \text{-}) \rangle$ modified so that $\overline{\varphi}(\text{-}, \text{-})$ becomes one-to-one. We use $\oplus$ for the natural sum of ordinals and $\otimes$ for the natural product. For the definition, see [36]. Therein we can find also the definition of additively indecomposable ordinals and multiplicatively indecomposable ordinals.

3 Discontinuity of Higman Embeddings

In this section we give an analysis of Higman embedding from the calculation of order types. The results in this section will be derived from more general theorems in later sections. A strange property of Higman embedding, however, leads us to what we want to prove later. The anomaly we show here is that, whereas the definition of Higman embedding is completely uniform on the base partial order, the order type of Higman embedding is not continuous on the order type of the base well-partial-order.

For Higman's lemma, proofs without using the minimal bad sequence argument were known to many researchers [33, 36, 26, 30, 5]. Schütte and Simpson gave a reification of Higman's embedding [33, 36]. They proved that if the base partial order A is a well-partial-order and has a reification by ordinal α, then the Higman embedding has a reification $\omega^{\omega^{\alpha+1}}$. By a closer inspection, however, we see that this last ordinal is not the least reification ordinal in most cases.

We give the exact order types for Higman embedding by analyzing the proof by Schütte and Simpson. We show the order types are $\omega^{\omega^{\alpha}}$ in some cases, and $\omega^{\omega^{\alpha+1}}$ in the other. If the reification ordinal of the base partial order is of the form $\epsilon_\gamma + n$, epsilon number plus a finite number, then the order type of Higman embedding must be $\omega^{\omega^{\alpha+1}}$; otherwise $\omega^{\omega^{\alpha}}$ (if α is finite, then $\omega^{\omega^{-1+\alpha}}$). This means that the order types of Higman embedding fills all multiplicatively indecomposable ordinals *except* epsilon numbers. Therefore there are no well-partial-orders whose Higman embedding yields an epsilon number as the order type.

This observation is our start point. In later sections, we show these gaps at epsilon numbers are filled with other structures (binary trees, etc.). In fact, we show more: every ordinal up to $\varphi(\Omega^\omega, 0)$ in Bachmann hierarchy is filled up with a structure (called an *algebra*), and furthermore there is no superposition, namely, every ordinal is filled up with a unique algebra. The Higman embedding has gaps at all epsilon numbers, and these gaps are filled with other algebras. For example, the ordinal ϵ_0 is with the algebra of binary trees.

The following is a sketch of the reification of Higman embedding given in [36]. We prove the following theorem.

3.1 Theorem

If A is a well-partial-order of order type α, then Higman embedding on A^* has the order type of the following:

$$\begin{cases} \omega^{\omega^{\alpha-1}} & \textit{if } \alpha = n \textit{ finite} \\ \omega^{\omega^{\alpha}} & \textit{if } \alpha = \beta + n \textit{ where } \beta \textit{ is a limit, not an epsilon number} \\ \omega^{\omega^{\alpha+1}} & \textit{if } \alpha = \beta + n \textit{ where } \beta \textit{ is a limit, an epsilon number} \end{cases} \quad \square$$

3.2 Notation

$s, s_1, \ldots$ for members of A
$S, S_1, \ldots$ for members of $Bad(A)$

$\sigma, \sigma_1, \ldots$ for members of A^*
$\Sigma, \Sigma_1, \ldots$ for members of $Bad(A^*)$ $\square$

For a partial order $\langle A, \trianglelefteq \rangle$ and $s \in A$,

$$A_s =_{def} \{t \in A \mid s \ntrianglelefteq t\};$$

and for $S \in Bad(A)$,

$$A_S =_{def} \{t \in A \mid S\hat{\ }\langle t \rangle \in Bad(A)\}.$$

Note that for the sequence $\langle s \rangle$ of length one, $A_s = A_{\langle s \rangle}$ holds. To each $\Sigma \in Bad(A^*)$, we associate a set of the form $\coprod_i \prod_j B_{ij}$ (finite disjoint union of finite product) where the summand $\prod_j B_{ij}$ is of the form

$$(A_{S_1})^* \times A_{S'_1} \times (A_{S_2})^* \times A_{S'_2} \times \cdots \times A_{S'_{m-1}} \times (A_{S_m})^* \qquad (m \geq 1),$$

as well as a one-to-one map $h_\Sigma : (A^*)_\Sigma \hookrightarrow \coprod_i \prod_j B_{ij}$. If the bad sequence Σ is extended to $\Sigma\hat{\ }\langle \sigma \rangle$ where $\sigma \in (A^*)_\Sigma$, a new set $\coprod_i \prod_j B'_{ij}$ is defined by decomposing the summand such that $h_\Sigma(\sigma) \in \prod_j B_{i_0 j}$ into a disjoint sum as in Figure 1. There we suppose

$$\begin{aligned} h_\Sigma(\sigma) = \langle \sigma_1, s_{n_1}, \sigma_2, s_{n_2}, \ldots, s_{n_{m-1}}, \sigma_m \rangle \\ \in (A_{S_1})^* \times A_{S'_1} \times (A_{S_2})^* \times A_{S'_2} \times \cdots \times A_{S'_{m-1}} \times (A_{S_m})^*. \end{aligned}$$

and after the decomposition, the product $\times$ is distributed over the disjoint sum $+$ in order to maintain the form of $\coprod_i \prod_j B_{ij}$.

The reification given in [36] is as follows. Suppose the base well-partial-order has a reification $|\ | : Bad(A) \to \alpha{+}1$. Let $(A^*)\check{_\Sigma}$ denote the set $\coprod_i \prod_j B_{ij}$ associated to $\Sigma \in Bad(A)$. If we find an ordinal assignment $|(A^*)\check{_\Sigma}|$ to each $(A^*)\check{_\Sigma}$ so that $|(A^*)\check{_\Sigma}| > |(A^*)\check{_{\Sigma\hat{\ }\langle\sigma\rangle}}|$ then the required reification is obtained by the mapping $\Sigma \mapsto |(A^*)\check{_\Sigma}|$. The assignment satisfying this condition is given as follows: for each $(A^*)\check{_\Sigma} = \coprod_i \prod_j B_{ij}$ we assign an ordinal by $|(A^*)\check{_\Sigma}| = \bigoplus_i \bigotimes_j |B_{ij}|$ where $\oplus$ is a natural sum and $\otimes$ is a natural product. Further, to each multiplicand B_{ij} is assigned an ordinal according to whether B_{ij} has the form A_S or $(A_S)^*$ by the following equation:

$$|B_{ij}| \quad = \quad \begin{cases} \omega^{\omega^{|S|}} & \text{if } B_{ij} \text{ is } A_S \\ \omega^{\omega^{|S|+1}} & \text{if } B_{ij} \text{ is } (A_S)^*. \end{cases}$$

$|(A^*)\check{_\Sigma}| > |(A^*)\check{_{\Sigma\hat{\ }\langle\sigma\rangle}}|$ is easily checked using the fact that the ordinals of the form ω^{ω^x} are multiplicatively indecomposable.

It is possible, however, to assign smaller ordinals. All that is required for the assignment is that $|A_S|$ is additively indecomposable, $|(A_S)^*|$ is multiplicatively indecomposable, and $|A_S| < |(A_S)^*|$. Therefore we can assign $\omega^{|S|}$ to A_S and $\omega^{\omega^{|S|}}$ to $(A_S)^*$ *unless* $|S|$ is an epsilon number, in which case $|A_S| < |(A_S)^*|$ fails. To handle this case, define an ordinal function $(\ )^\dagger$ by the following:

$$\begin{array}{llllll}
((A_{S_1})^*)_{\sigma_1} \times A_{S'_1} & \times\, (A_{S_2})^* & \times\, A_{S'_2} & \times \cdots \times\, A_{S'_{m-1}} & \times\, (A_{S_m})^* \\
+\ (A_{S_1})^* \times (A_{S'_1})_{s_{n_1}} & \times\, (A_{S_2})^* & \times\, A_{S'_2} & \times \cdots \times\, A_{S'_{m-1}} & \times\, (A_{S_m})^* \\
+\ (A_{S_1})^* \times A_{S'_1} & \times ((A_{S_2})^*)_{\sigma_2} & \times\, A_{S'_2} & \times \cdots \times\, A_{S'_{m-1}} & \times\, (A_{S_m})^* \\
+\ (A_{S_1})^* \times A_{S'_1} & \times\, (A_{S_2})^* & \times (A_{S'_2})_{s_{n_2}} & \times \cdots \times\, A_{S'_{m-1}} & \times\, (A_{S_m})^* \\
\quad \vdots & & & & \\
+\ (A_{S_1})^* \times A_{S'_1} & \times\, (A_{S_2})^* & \times\, A_{S'_2} & \times \cdots \times (A_{S'_{m-1}})_{s_{n_{m-1}}} & \times\, (A_{S_m})^* \\
+\ (A_{S_1})^* \times A_{S'_1} & \times\, (A_{S_2})^* & \times\, A_{S'_2} & \times \cdots \times\, A_{S'_{m-1}} & \times ((A_{S_m})^*)_{\sigma_m}
\end{array}$$

where (with an abuse of notation), for $\sigma_i = \langle s_{n_{i-1}+1}, s_{n_{i-1}+2}, \ldots, s_{n_i-1}\rangle$,

$$\begin{aligned}
((A_{S_i})^*)_{\sigma_i} \ =\ & \\
& (A_{S_i\hat{}\langle s_{n_{i-1}+1}\rangle})^* \\
& + (A_{S_i\hat{}\langle s_{n_{i-1}+1}\rangle})^* \times A_{S_i} \times (A_{S_i\hat{}\langle s_{n_{i-1}+2}\rangle})^* \\
& \ \vdots \\
& + (A_{S_i\hat{}\langle s_{n_{i-1}+1}\rangle})^* \times A_{S_i} \times (A_{S_i\hat{}\langle s_{n_{i-1}+2}\rangle})^* \times \cdots \times A_{S_i} \times (A_{S_i\hat{}\langle s_{n_i-1}\rangle})^*
\end{aligned}$$

Figure 1

$$\alpha^\dagger \ = \ \begin{cases} \alpha - 1 & \text{if } \alpha \text{ is finite} \\ \alpha & \text{if } \alpha = \beta + n \text{ and } \beta \text{ is a limit, not an epsilon number} \\ \alpha + 1 & \text{if } \alpha = \beta + n \text{ and } \beta \text{ is a limit, an epsilon number.} \end{cases}$$

This function $(\ )^\dagger$ simply skips all epsilon numbers. Then a new assignment to the multiplicands B_{ij} is given by the equation

$$|B_{ij}| \ = \ \begin{cases} \omega^{|S|^\dagger} & \text{if } B_{ij} \text{ is } A_S \\ \omega^{\omega^{|S|^\dagger}} & \text{if } B_{ij} \text{ is } (A_S)^*. \end{cases}$$

3.3 Lemma
The mapping $\Sigma \mapsto |(A^*)\check{_\Sigma}|$ *yields a reification* $Bad(A^*) \to \omega^{\omega^{\alpha^\dagger}} + 1$ *where* α *is the order type of the base well-partial-order* A. □

In turn, to show that these ordinals $\omega^{\omega^{\alpha^\dagger}}$ are exactly the order types of Higman embedding, we must give linearizations yielding the same ordinals as those of reifications. If the order type of the base well-partial-order is not of the form $\epsilon_\gamma + n$, then the required linearization is given by the recursive path ordering on monadic terms (kachinuki ordering) [23, 32, 25]. The *kachinuki ordering* is a linear ordering $\sqsubset$ on the set A^* of finite lists of a linear order A, the order $\sqsubset$ defined as follows: $\langle s_0, \ldots, s_{m-1}\rangle \sqsubset \langle t_0, \ldots, t_{n-1}\rangle$ if and only if one of the following holds:

(i) $s_0 = t_0$, $\langle s_1, \ldots \rangle \sqsubset \langle t_1, \ldots \rangle$
(ii) $s_0 \prec t_0$, $\langle s_1, \ldots \rangle \sqsubset \langle t_0, \ldots \rangle$
(iii) $s_0 \succeq t_0$, $\langle s_0, \ldots \rangle \sqsubseteq \langle t_1, \ldots \rangle$

In (iii), $\succeq$ denotes $\succ \cup =$ and likewise for $\sqsubseteq$. By calculation, one can show that the order type of kachinuki ordering is the ordinal $\omega^{\omega^{-1+\alpha}}$ [32]. Hence, except the case $\alpha = \epsilon_\gamma + n$, the kachinuki ordering gives the same ordinal $\omega^{\omega^{\alpha^\dagger}}$ as the one given by the reification. For the exceptional case, we must create another linearization providing the maximum order type. If the order type α of the base well-partial-order is an epsilon number ϵ_γ, then the following ordering $\sqsubset'$ gives the required linearization: for $\sigma, \tau \in A^*$, the order relations $\sigma \sqsubset' \tau$ holds if and only if either $|\sigma| < |\tau|$ or both $|\sigma| = |\tau|$ and σ is smaller than τ by lexicographic ordering, where $|\sigma|$ denotes the length of the sequence σ. Then the order type of $\sqsubset'$ is $(\epsilon_\gamma)^\omega = \omega^{\omega^{\epsilon_\gamma + 1}}$. For the case $\alpha = \epsilon_\gamma + n + 1$ $(n \geq 0)$, by induction , we may assume there is an order isomorphism $|\cdot|_n$ from $(A \backslash \{m_A\})^*$ to $\omega^{\omega^{\epsilon_\gamma + n + 1}}$, where m_A is the largest element of the linear order A. Then the sequence $\sigma = \langle \sigma_1, m_A, \ldots, \sigma_{k-1}, m_A, \sigma_k \rangle$ of A^* is carried to

$$|\sigma|_{n+1} \; = \; (\omega^{\omega^{\epsilon_\gamma + n + 1}})^{k-1} \cdot |\sigma_1|_n + \cdots + (\omega^{\omega^{\epsilon_\gamma + n + 1}}) \cdot |\sigma_{k-1}|_n + |\sigma_k|_n,$$

giving the order isomorphism from A^* to $\omega^{\omega^{\epsilon_\gamma + n + 2}}$. Therefore the following lemma holds.

3.4 Lemma
Higman embedding on A^* has a linearization of order type $\omega^{\omega^{\alpha^\dagger}}$ where α is the order type of the base well-partial-order A. $\square$

Theorem 3.1 is an immediate consequence of Lemmata 3.3 and 3.4. From this observation, we see that the order types $|Bad(A^*)|$ of Higman embedding has gaps at epsilon numbers as well as all ordinals that are not multiplicatively indecomposable. In the following sections, we show how these gaps are filled by other algebras uniquely.

4 Algebra Embedding

In his seminal paper [19], Higman studied the divisibility ordering on abstract algebras, and showed minimal algebras are well-partial-orders if the set of operators is ordered by a well-partial-ordering. In particular, if the number of operators is finite, the divisibility orderings are always well-partial-orders. Higman embedding is a special case of this general observation.

Our definition of algebras is almost on the same line of the minimal algebras. We include the disjoint sum and the direct product to the definition of algebras. Moreover we allow a set of generators, if the set itself is an algebra. In short, the class of algebras is the smallest class generated from empty sets and singletons by the disjoint sum $+$, the direct product $\times$, and the least fixpoint operator μ using one algebra variable.

Following the definition of algebras and their terms, we will define a partial ordering on each algebra, called an *algebra embedding*. This partial order is exactly the divisibility ordering of Higman. So every algebra embedding turns out to be a well-partial-order. Our goal is to calculate the order type of these algebra embeddings. From this calculation, one unexpected property is shown: for each ordinal less than $\varphi(\Omega^\omega, 0)$, there is an algebra giving the ordinal as the order type. In addition, such an algebra is unique up to isomorphism (Theorem 6.19). In Section 5, we give the upper bounds of the order types by providing reifications, and in Section 6, the lower bounds by linearizations.

4.1 Definition

Algebras are generated by the following rules:

$$\emptyset \qquad i:1 \qquad X$$

$$\frac{A \quad B}{A+B} \qquad \frac{A \quad B}{A\times B} \qquad \frac{A}{\mu X.\, A.}$$

Here i is the identifier of the singleton, and we impose on $A+B$ and $A\times B$ the condition that all identifiers occurring in them are distinct. X is the only one algebra variable. This means in our setting only single recursion is allowed □

The intended meaning of the connectives defining algebras should be almost clear. For example, $i:1$ is a singleton containing a unique i. We have only one algebra variable X, and so it is impossible to define a many sorted algebra with mutual recursion. The extension to this direction allows us more complicated structures as the tree embedding with gap condition, and will be handled in a forthcoming paper.

If all occurrences of the variable X are within the scopes of μ-operators, then the algebra is called *closed*; otherwise *open*. Note that though we have only single recursion, there is no restriction to use an algebra already constructed as a part of another algebra, e.g., $\mu X.\, X \times (\mu X.\, X + (i:1)) + (i':1)$.

4.2 Definition

Let A be an algebra.

t is a *term* of sort A iff $t : A$ is derived by the following rules:

$$i:1$$

$$\frac{a:A}{\iota a : A+B} \qquad \frac{b:B}{\iota' b : A+B}$$

$$\frac{a : A \quad b : B}{\langle a, b\rangle : A \times B}$$

$$\frac{a : \eta A}{\gamma a : \mu X.A} \qquad (\eta \text{ is the substitution } [(\mu X.\, A)/X]).$$

To be precise, the first rule should be written $i : (i : 1)$. Namely i is a unique element of the singleton algebra $i : 1$. ι and ι' are injections associated to each pair of algebras A and B. $\langle\cdot,\cdot\rangle$ is a pairing function associated to each pair of algebras A and B. And γ is the constructor associated to each initial algebra $\mu X.\, A$. $\Box$

4.3 Remark

In the last rule, the substitution ηA should be done without renaming of identifiers. For example, if A is $X + (i : 1)$, the substitution ηA yields $(\mu X.\, X + (i : 1)) + (i : 1)$. This last is not an algebra in an exact sense, since there are common identifiers i therein. For simplicity, we call such objects also algebras. $\Box$

We omit identifiers of singletons 1 if the distinction of the occurrences are clear from the context.

4.4 Remark

If A is an initial algebra of the form $\mu X.\, C_1 X^{n_1} + \ldots C_p X^{n_p}$ with closed algebras $C_1, \ldots, C_p$, then A is isomorphic to the set of terms generated by the following BNF:

$$a \quad ::= \quad \mathtt{const}_1(c_1, \overbrace{a, \ldots, a}^{n_1 \text{ copies}}) \quad | \quad \cdots \quad | \quad \mathtt{const}_p(c_p, \overbrace{a, \ldots, a}^{n_p \text{ copies}})$$
$$(c_i \text{ is a term of sort } C_i)$$

where $\mathtt{const}_k$ is any symbol uniquely associated to the algebra A and the summand. To see the isomorphism, identify $\mathtt{const}_k(c_k, a_1, \ldots, a_{n_k})$ with $\gamma \circ \iota_k\langle c_k, a_1, \ldots, a_{n_k}\rangle$ where γ is the constructor associated to A and ι_k is the k-th injection of $C_k A^{n_k}$ to $C_1 A^{n_1} + \cdots + C_k A^{n_k} + \cdots + C_p A^{n_p}$. $\Box$

4.5 Example

(i) Natural numbers $N = \mu X.\, X + 1$. The terms of sort N are generated by the BNF $n ::= \mathtt{succ}(n) \mid \mathtt{zero}$.

(ii) Finite lists of A, $A^* = \mu X.\, AX + 1$. The terms are generated by the rule $l ::= \mathtt{push}(a, l) \mid \mathtt{emp}$ where $a : A$.

(iii) Finite lists of A entailed with B, $A^* B = \mu X.\, AX + B$. The terms are generated by the rule $l ::= \mathtt{push}'(a, l) \mid \mathtt{tail}(b)$ where $a : A$ and $b : B$. The terms of this sort may be written in the form $\langle a_1, \ldots, a_k; b\rangle$ $(k \geq 0)$.

(iv) Binary trees $B = \mu X.\, X^2 + 1$. The terms are generated by the rule $t ::= \texttt{cons}(t, t) \mid \texttt{nil}$ $\square$

4.6 Notation

Let s and t be two term, not necessarily of the same sort.

$s \subseteq t$ denotes that s is a subterm of t in a usual sense. s may be equal to t. We write $s \subset t$ if s is a proper subterm of t. $\square$

Next we define an embedding on each algebra A. The properties of this embedding are the center of our interest. Among the rules in the following definition, the projection rule is important and makes the embedding correspond to the divisibility ordering of [19]. Other congruence rules are generally required to partially ordered algebras [19, 15] (in [15] this property of preserving order is called isotony; also antitony occurs in ordered algebraic structures).

4.7 Definition

The *embedding* $\trianglelefteq_A$ is a binary relation (in fact, a partial order) on the set of terms of an algebra A given by one projection rule and five congruence rules as follows:

(*projection*)

$$\frac{a \trianglelefteq_A a'^{\circ}}{a \trianglelefteq_A a'} \qquad \text{if } a'^{\circ} \text{ is a proper subterm of } a' \text{ (i.e., } a'^{\circ} \subset a' \text{) having sort } A.$$

(*congruence*)

$$i \trianglelefteq_1 i$$

$$\frac{a \trianglelefteq_A a'}{\iota a \trianglelefteq_{A+B} \iota a'} \qquad \frac{b \trianglelefteq_B b'}{\iota' b \trianglelefteq_{A+B} \iota' b'}$$

$$\frac{a \trianglelefteq_A a' \qquad b \trianglelefteq_B b'}{\langle a, b\rangle \trianglelefteq_{A\times B} \langle a', b'\rangle}$$

$$\frac{b \trianglelefteq_{B[A]} b'}{\gamma b \trianglelefteq_A \gamma b'} \qquad \text{where } A = \mu X.\, B[X]. \qquad \square$$

4.8 Remark

The reason we imposed the condition that the identifiers of singletons should all be distinct is as follows: the intended meaning of $A + B$ is a disjoint sum of two partial orders A and B. Therefore the terms of A and the terms of B are incomparable. Consider an algebra $C = (\mu X.\, X + 1) + 1$, which should be a disjoint sum of the set of natural numbers and a singleton. If we suppose, however, two singletons therein have the same term i, then $\iota'(i)$ and $\iota \circ \gamma \circ \iota'(i)$

are both terms of C, where i in the first term comes from the last 1 of C, and i in the second from the first 1. By the projection rule of the embedding,

$$\iota'(i) \quad \trianglelefteq_C \quad \iota \circ \gamma \circ \iota'(i).$$

Hence two summands $\mu X. X + 1$ and 1 of C are not disjoint, contradicting the intended meaning. Therefore we force the identifiers of singletons to be all distinct (even if we omit the identifiers for simplicity). $\square$

4.9 Example

(i) The embedding on the algebra N of natural numbers is the linear order on natural numbers

$$\mathtt{zero} \quad \trianglelefteq_N \quad \mathtt{succ}(\mathtt{zero}) \quad \trianglelefteq_N \quad \mathtt{succ}(\mathtt{succ}(\mathtt{zero})) \quad \trianglelefteq_N \quad \cdots$$

(ii) The embedding on the algebra A^* of finite lists of A is exactly the same as Higman embedding (Definition 2.3).

(iii) The embedding on the algebra B of binary trees is exactly the tree embedding on binary trees (Definition 2.4 and the remark that follows). $\square$

In the theory of well-partial-order, order-reflecting maps play an important role as mentioned in Section 2. Since the algebra embedding is defined using subterms, we need the reflection of the property of being subterms in order to show some naturally arising maps are order-reflecting. The following definition of anti-createdness is the formalization of the reflection for subterms.

4.10 Definition

Let $A[X], B[X]$ be algebras with a free variable X, and C, D closed algebras (or in general arbitrary sets). Suppose g is a function from C to D and f a function from $A[C]$ to $B[D]$.

f *anti-creates subterms* with respect to g if and only if

$$\forall a : A[C] \;\; \forall d : D \text{ such that } d \subset fa \;\; \exists c : C \qquad \begin{array}{ccc} c & \overset{g}{\mapsto} & d \\ \cap & & \cap \\ a & \underset{f}{\mapsto} & fa. \end{array}$$

In other words, there are no new created subterms $d \subset fa$ other than the images of some subterms $c \subset a$. $\square$

4.11 Remark

We may be interested in the case that f and g are partial functions. The definition of anti-createdness works if we force the quantifiers $\forall a$ and $\exists c$ to range over the domains of the partial functions. $\square$

We can extend the algebras by adjoining a class of sets as atomic sorts. We should regard the elements of the sets as atomic terms, which have no proper subterms. By this extension, every open algebra $A[X]$ may be naturally regarded as an endofunctor on the category **Set** of small sets. We can immediately prove that $A[f]$ anti-creates subterms with respect to every function f.

We are interested especially in the anti-createdness of the following two special cases:

(i) Let $A[X]$ and $B[X]$ be two open algebras and f_X a natural transformation from $A[X]$ to $B[X]$. We say the transformation f_X *anti-creates subterms* if functions f_X anti-create subterms with respect to identity functions id_X for all sets X.

(ii) Let C and D be initial algebras and f a function from C to D. We say f *anti-creates subterms* if f anti-creates subterms with respect to f itself.

The following proposition and corollary are the principal result with regard to order-reflecting maps. Namely natural transformations between open algebras induce order-reflecting maps of initial algebras, provided the natural transformations anti-create subterms.

4.12 Proposition
Let $f_X : A[X] \to B[X]$ be a natural transformation between two open algebras.

If f_X anti-creates subterms and the function f_X for each X reflects order, then the natural transformation f_X induces a function $f^\mu : \mu X.\, A[X] \to \mu X.\, B[X]$ that anti-creates subterms and reflects order.

(*Proof*) The value $f^\mu(\gamma a)$ is defined by $\gamma(B[f^\mu](f_{\mu X.A}(a)))$ by induction on the construction of γa. The anti-createdness is necessary to show that f^μ reflects order. □

The condition imposed on f_X is stronger than necessary. All we need is that $f_{\mu X.A}$ exists, reflects order and anti-creates subterms with respect to $id_{\mu X.A}$. However, the following corollary requires a further condition, which is derived from the hypothesis that f_X is a natural transformation.

4.13 Corollary
Let f_X be a natural transformation from an open algebra $A[X]$ to an open algebra $B[X]$.

If f_X is a natural isomorphism where both f_X and its inverse f_X^{-1} anti-create subterms and reflect order, then f^μ is an isomorphism where both f^μ and its inverse anti-create subterms and reflect order. □

From this corollary, we know that if a natural transformation and its inverse between two open algebras reflect order and anti-create subterms, then the isomorphism is applicable even within the scope of μ-operators. In the following we give such isomorphisms.

4.14 Lemma
The following are natural isomorphisms such that both the isomorphisms and their inverses reflect order and anti-create subterms.

$$
\begin{array}{ll}
(A+B)+C \xrightarrow{\sim} A+(B+C) & \\
(A\times B)\times C \xrightarrow{\sim} A\times(B\times C) & \\
A+B \xrightarrow{\sim} B+A & \\
A\times B \xrightarrow{\sim} B\times A & \\
\emptyset + A \xrightarrow{\sim} A & \emptyset\times A \xrightarrow{\sim} \emptyset \\
1\times A \xrightarrow{\sim} A & \\
A(B+C) \xrightarrow{\sim} AB+AC & \\
\mu X.\, A \xrightarrow{\sim} A & \textit{whenever } A \textit{ does not contain free } X \\
\mu X.\, A \xrightarrow{\sim} \emptyset & \textit{whenever } A[X:=\emptyset] \xrightarrow{\sim} \emptyset \\
\mu X.\, XA_1+\cdots+XA_n+B \xrightarrow{\sim} (\mu X.\, XA_1+\cdots XA_n+1)B & \\
 & \textit{where } A_1,\ldots,A_n, \textit{ and } B \textit{ are closed algebras} \quad \square
\end{array}
$$

The last isomorphism becomes easier to see if we write it $\mu X.\, XA+B \xrightarrow{\sim} (\mu X.\, XA+1)B$ with $A = A_1+\cdots+A_n$. The left hand side is the algebra of finite lists of A entailed with B (Example 4.5 (iii)). The terms of this algebra can be written in the form $\langle a_1,\ldots,a_m;b\rangle$. Then the operation $\langle a_1,\ldots,a_m;b\rangle \mapsto \langle\langle a_1,\ldots,a_m\rangle, b\rangle$ yields an isomorphism from $\mu X.\, XA+B$ to the direct product $(\mu X.\, XA+1)B$ of the algebra $A^* = \mu X.\, XA+1$ of finite lists of A and the algebra B. The rule $\mu X.A \xrightarrow{\sim} \emptyset$ should be clear if one sees that $\emptyset$ is the least fixpoint of the operator $A[X]$ if $A[\emptyset]\cong\emptyset$.

We view the isomorphisms in the previous lemma as rewriting rules from the left hand sides to the right, except first four isomorphisms that assert associativity and commutativity for disjoint sum and direct product. So the rewriting rules are up to associativity and commutativity of $+$ and $\times$. We stress once more those isomorphisms remain valid even within the scopes of μ-operators.

4.15 Definition
A *normal form* of an algebra A is an algebra that is isomorphic to A by the isomorphisms in the previous lemma, and that has no redex if one views the isomorphisms as rewriting rules (up to associativity and commutativity of $+$ and $\times$). $\square$

4.16 Lemma
The rewriting rules are confluent and strongly terminating. Therefore every algebra has a unique normal form. In addition, the function yielding a normal form for each algebra is primitive recursive. □

An algebra is in normal form if and only if it has the form $\coprod_i \prod_j B_{ij}$ where each B_{ij} is an initial algebra in normal form. Note that the rewriting rules are given in a form of normal conditional rewriting [10] due to the side condition $A[\emptyset] \xrightarrow{\simeq} \emptyset$ of the rule $\mu X.\, A[X] \xrightarrow{\simeq} \emptyset$. It is possible, however, to check the voidness of the algebra independently from the reductions. Therefore the rewriting rules may be presented as a system of usual term rewriting (though associativity and commutativity of disjoint sum and direct product are still manipulated implicitly). Then the reduction can be carried out along with the construction of algebras. Hence the primitive recursiveness of the function inducing the normal forms is almost evident.

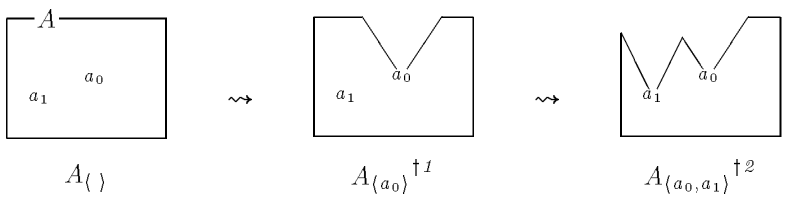
Every initial algebra is isomorphic to an algebra written in the form of polynomials $\mu X.\, X^n C_n + \cdots X C_1 + C_0$ where each C_i is a closed algebra. Note that, however, in normal form the monomial $X^i(D + D')$ should be decomposed into $X^i D + X^i D'$.

4.17 Remark
In Corollary 6.20, we will show that the above isomorphisms completely determine the equivalence of algebras with respect to the morphisms respecting the embeddings. Therefore the equivalence is primitive recursively decidable by the comparison of normal forms. □

5 Reification of Algebra Embedding

In this section, we give a reification for the embedding relations $\trianglelefteq_A$ of each algebra A. The crucial idea is to use the class of algebras itself as a system of ordinal notations. To this end, we provide a well-ordering with the class of algebras and associate a descending sequence of algebras with each bad sequence of an algebra A. We give also the order isomorphism from the well-ordering of the algebras to the segment of Bachmann hierarchy up to $\varphi(\Omega^\omega, 0)$.

The reification to be given shows that well-partial-orderedness of each algebra reduces to well-orderedness of some associated ordinal. The reduction is carried out in a weak fragment of second order arithmetic, e.g. in system RCA_0 [14, 34, 37], even intuitionistically. Therefore if one has an elementary proof of well-orderedness of the associated ordinal, then also well-partial-orderedness of the algebra is proved in an elementary method (we do not single out which is more elementary). Furthermore if the proof of well-ordering is constructive, so is the proof of the well-partial-orderedness.



†1: a_0 is not in $A_{\langle a_0\rangle}$. †2: Neither a_0 nor a_1 is in $A_{\langle a_0,a_1\rangle}$.

Figure 2

5.1 Notation

Let A be a partial order and a an element of A.

A_a denotes the the suborder $\{x \in A \mid a \not\trianglelefteq_A x\}$ of A. $\square$

Given a bad sequence $\langle a_0, a_1, \ldots\rangle$ of a partial order A, we associate a sequence $A, A_{\langle a_0\rangle}, A_{\langle a_0,a_1\rangle}, \ldots$ of suborders of A. Here the suborder A_σ is defined for every finite sequence σ of members of A by

$$\begin{array}{lcl} A_{\langle\,\rangle} & = & A \\ A_{\sigma\hat{}\langle a\rangle} & = & (A_\sigma)_a \end{array}$$

(see Figure 2). Then the sequence $A_{\langle\,\rangle}, A_{\langle a_0\rangle}, A_{\langle a_0,a_1\rangle}, \ldots$ are decreasing with respect to the strict inclusion $\supset$ of sets. Therefore the following observation follows:

> A partial order A is a well-partial-order if and only if the decreasing sequence $A_{\langle\,\rangle}, A_{\langle a_0\rangle}, A_{\langle a_0,a_1\rangle}, \ldots$ associated to each bad sequence $\langle a_0, a_1, \ldots\rangle$ eventually terminates after finite steps.

5.2 Proposition

(i) $1_i = \emptyset$

(ii) $(A+B)_{\iota a} = A_a + B$, *and* $(A+B)_{\iota b} = A + B_b$.

(iii) $(A \times B)_{\langle a,b\rangle} = (A_a \times B) \cup (A \times B_b)$.

(iv) $A_a = \gamma(B[A_a])_b$ *where* $A = \mu X.\, B[X]$ *and* $a = \gamma b$. *Here the suborder* $\gamma(B[A_a])_b$ *of* A *is defined by* $\{\gamma b' : A \mid b' \in (B[A])_b$ *and* $\forall a'^{\circ} \subset \gamma b'.\, a' : A \Rightarrow a'^{\circ} \in A_a\}$. $\square$

The meaning of (iv) may be clearer if one observes that A_a is a least fixpoint of the operator $X \mapsto (B[X])_b$ in the complete lattice of all subsets of A. What we next do is to translate the above suborders A_a to algebras $A_a^{\vee}$. As seen below, the translation is simple: disjoint sum to disjoint sum, direct product to direct product, least fixpoint to least fixpoint. The only point to remark is that union is translated to disjoint sum.

5.3 Definition

An algebra $A_a^{\vee}$ is associated to each algebra A in normal form and a term a of sort A by the following definition:

(i) $1_i^{\vee} = \emptyset$.

(ii) $(A+B)_{\iota a}^{\vee} = A_a^{\vee} + B$

$(A+B)_{\iota' b}^{\vee} = A + B_b^{\vee}$

(iii) $(A \times B)_{\langle a,b\rangle}^{\vee} = A_a^{\vee} \times B + A \times B_b^{\vee}$

(iv) $(\mu X.\, B[X])_{\gamma b}^{\vee} = \mu X.\, (B[X])_b^{\vee}$ where $(B[X])_b^{\vee}$ is carried out in such a way that if $X_{a^\circ}^{\vee}$ is encountered for some a° in the process of calculation, then it is replaced by $A_{a^\circ}^{\vee}$ (see Example below). $\square$

5.4 Example

(i) Let B be the algebra $\mu X.\, X^2+1$ of binary trees and $\mathtt{cons}(t/0, t/1)$ a term of sort B (see Example 4.5 (iv)).

$$\begin{aligned}
B_{\mathtt{cons}(t/0,t/1)}^{\vee} &= \mu X.\, (X^2+1)_{\iota\langle t/0,t/1\rangle}^{\vee} \\
&= \mu X.\, (X^2)_{\langle t/0,t/1\rangle}^{\vee} + 1 \\
&= \mu X.\, X B_{t/1}^{\vee} + B_{t/0}^{\vee} X + 1 \\
&= (B_{t/0}^{\vee} + B_{t/1}^{\vee})^*
\end{aligned}$$

Observe that $X_{t/0}^{\vee}$ and $X_{t/1}^{\vee}$ are replaced by $B_{t/0}^{\vee}$ and $B_{t/1}^{\vee}$ respectively.

(ii) Let A^* be the algebra $\mu X.\, AX+1$ of finite lists of A and $\langle a_0, a_1 \ldots, a_{n-1}\rangle$ a term of sort A^*.

$$\begin{aligned}
(A^*)_{\langle a_0,\ldots,a_{n-1}\rangle}^{\vee} &= \mu X.\, A_{a_0}^{\vee} X + A(A^*)_{\langle a_1,\ldots a_{n-1}\rangle}^{\vee} + 1 \\
&\cong (A_{a_0}^{\vee})^* + (A_{a_0}^{\vee})^* A (A^*)_{\langle a_1,\ldots a_{n-1}\rangle}^{\vee} \\
&\;\;\vdots \\
&\cong (A_{a_0}^{\vee})^* \\
&\quad + (A_{a_0}^{\vee})^* A (A_{a_1}^{\vee})^* \\
&\quad + \cdots \\
&\quad + (A_{a_0}^{\vee})^* A (A_{a_1}^{\vee})^* A \cdots A (A_{a_{n-1}}^{\vee})^*
\end{aligned}$$

If we replace $(A_{a_i}^{\vee})^*$ by $(A_{a_i})^*$ then this becomes equal to $((A_{S_i})^*)_{\sigma_i}$ in Figure 2 by transformation $A_{S_i} \mapsto A$ and $\sigma_i \mapsto \langle a_0, \ldots, a_{n-1}\rangle$. By applying this repeatedly, the definition of $(A^*)^{\check{}}_{\Sigma}$ given in Section 3 due to Simpson [36] (where Σ is a bad sequence of terms of sort A^*) turns out to be equal to the one in this section. $\square$

The following lemma is almost evident, but the most important for the reification.

5.5 Lemma
Let A be an algebra in normal form and a a term of sort A.

There is a $\trianglelefteq$-reflecting map (thus injection) $f_a : A_a \rightarrowtail A^{\check{}}_a$.

(*Proof*) Induction on the construction of a. Prove at the same time if A is written $B[C]$ for some algebra $B[X]$ and C, then f_a anti-creates subterms with respect to id_C. For the case A is an initial algebra $\mu X.\, B[X]$, the following observation is used: supposed $a = \gamma b$ where $b : B[A]$, the map $f_b : (B[A])_b \to (B[A])^{\check{}}_b = B^{\check{}}_b[A]$ is also a map from $(B[A_a])_b$ to $B^{\check{}}_b[A_a]$ by anti-createdness. $\square$

Recall that the transformation of A_a into $A^{\check{}}_a$ changes union to disjoint sum. Hence even if $a' \trianglelefteq a'$ holds in A_a it is not necessarily the case that $f_a(a') \trianglelefteq f_a(a'')$ in $A^{\check{}}_a$ since they may lie in different summands. So f_a reflects embedding but does not preserve it.

5.6 Remark
(i) Since union is transformed to disjoint sum, if the union has a non-void meet then f_a must choose summands for the terms in the meet. If we are concerned with constructive logic, we require the canonical choice of summands. The designing of the canonical choice is possible since $a' \in A_a$ is a primitive recursive predicate in A, a and a'.

(ii) If we are concerned with Reverse Mathematics, the lemma should be proved in a fragment of second order arithmetic that is as weak as possible. The lemma proves by induction that f_a anti-creates subterms and reflects embeddings. These two notions are formalized by Π^0_1-formulas, and therefore the lemma is provable in the most basic system RCA_0, since this system has Σ^0_1-induction that induces Π^0_1-induction. $\square$

5.7 Corollary
Let A be an algebra in normal form and $\langle a_0, a_1, \ldots, a_{n-1}\rangle$ a bad sequence of A.

There is a $\trianglelefteq$-reflecting injection

$$A_{\langle a_0,\ldots,a_{n-1}\rangle} \rightarrowtail A^{\check{}}_{\langle a_0,\ldots,a_{n-1}\rangle}.$$

Here

$$A_{\langle a_0,\ldots,a_{n-1}\rangle} = (\cdots((A_{a_0})_{a_1})\cdots)_{a_{n-1}}$$
$$A^{\check{}}_{\langle a_0,\ldots,a_{n-1}\rangle} = (\cdots((A^{\check{}}_{a_0})^{\check{}}_{a_1})\cdots)^{\check{}}_{a_{n-1}}\ .$$

In the right hand side of the second equation, for example, a_1 *is identified with its image under* $f_{a_0} : A_{a_0} \to A^{\check{}}_{a_0}$. □

By this corollary, we can associate to each bad sequence $\langle a_0, a_1, \ldots\rangle$ of an algebra A a sequence of algebras, $A, A^{\check{}}_{\langle a_0\rangle}, A^{\check{}}_{\langle a_0,a_1\rangle}, \ldots$. Therefore showing A is a well-partial-order is equivalent to saying that all sequences $A, A^{\check{}}_{\langle a_0\rangle}, A^{\check{}}_{\langle a_0,a_1\rangle}, \cdots$ associated to bad sequences of A eventually terminate after finite steps. This last is proved by providing a well-ordering with the class of algebras so that the sequences are strictly decreasing.

The basic idea is to regard the expressions of algebras as finite trees. Suppose given a closed algebra in normal form. If the algebra is of the form $A_0+A_1+\cdots+A_{n-1}$, then it is regarded as a tree $+\{A_0, A_1, \ldots, A_{n-1}\}$ having n immediate successors $A_0, A_1, \ldots, A_{n-1}$ of the root, and labeled by $+$ on the root node. Likewise for $A_0 \times A_1 \times \cdots \times A_{n-1}$. If the algebra is an initial algebra equivalent to $\mu X.\, X^n A_n + \cdots + XA_1 + A_0$ (if there are many homogeneous monomials $X^iC + \cdots + X^iC'$ then identify it with a single monomial $X^i(C + \cdots C')$), then it is regarded as a tree $\mu\langle A_n, \ldots, A_1, A_0\rangle$ labeled by μ on the root. Note that for the nodes labeled by $+$ or $\times$ there are no orders on immediate successors, while for the nodes labeled by μ there are orders. Following the recursive path ordering with status [24, 20], we define a well-order on the class of algebras incorporating both multiset path ordering and lexicographic path ordering depending on the labels on the nodes. The precedence order of the labels is $\emptyset \prec 1 \prec + \prec \times \prec \mu$.

In the following definition, the term *subalgebra* is used to denote the subexpressions of algebras (like subterms), not the subalgebra in the theory of universal algebra. We need multiset ordering $<^\circ$ on multisets [9] and lexicographic ordering $<^*$ on finite lists of variable lengths. This last ordering is defined as follows: $\langle a_m, \ldots, a_1, a_0\rangle <^* \langle a'_n, \ldots, a'_1, a'_0\rangle$ if and only if either (i) $m < n$; or (ii) $m = n$ and there is $j \le m$ such that $a_m = a'_m$, $a_{m-1} = a'_{m-1}$, …, $a_{j+1} = a'_{j+1}$ and $a_j < a'_j$. This ordering corresponds to $<_r$ in [10].

5.8 Definition

Let A, B be closed algebras regarded as finite trees by the procedure mentioned above. Suppose $\alpha, \beta \in \{\emptyset, 1, +, \times, \mu\}$ are the labels on the roots of A, B respectively. The set of labels has the precedence order $\emptyset \prec 1 \prec + \prec \times \prec \mu$.

The binary relation $A < B$ is defined by the following:

(i) $A < B$ if there is a proper subalgebra B° of B such that $A \le B^\circ$. Here $A \le B$ means $A < B$ or $A = B$.

(ii) $A < B$ if $\alpha \prec \beta$ and for all proper subalgebra A°, it holds that $A^\circ < B$.

(iii) $A < B$ if $\alpha = \beta$ and one of the following is the case:

(a) The label α is either $+$ or $\times$, and $\{A_0, A_1, \ldots, A_{m-1}\} <^\circ \{B_0, B_1, \ldots, B_{n-1}\}$ by multiset ordering, where A_i, B_j are immediate successors of the root of the trees A, B respectively.

(b) The label α is μ, and $\langle A_m, \ldots, A_1, A_0\rangle <^* \langle B_n, \ldots, B_1, B_0\rangle$ by lexicographic ordering, where A_i, B_j are immediate successors of the root of the trees A, B respectively.

(iv) $A < B$ holds only when one of (i) through (iii) applies. □

Another view for $\langle A_m, \ldots, A_0\rangle <^* \langle B_n, \ldots, B_0\rangle$ is to regard X as greater than all closed algebras and compare $X^m A_m + \cdots + A_0$ and $X^n B_n + \cdots + B_0$. This approach has an advantage that there is no need to transform initial algebras to the form $\mu X. X^n C_n + \cdots + C_0$. The following proposition is proved by the usual argument that derives a contradiction assuming there is a minimal infinite descending sequence.

5.9 Proposition
The binary relation $<$ is a well-order on the class of algebras in normal form. Let us denote by $|A|$ the ordinal corresponding to an algebra A. □

5.10 Proposition
If A is an algebra in normal form, then $A^{\vee}_a < A$ for every $a : A$. □

The proof is easy by definition of $A^{\vee}_a$. We remark that if A is not in normal form then this proposition does not necessarily hold. For example, consider $A = 2 \times 3$. Then $A^{\vee}_a = 1 \times 3 + 2 \times 2 = 7 > 6$ for all $a : A$.

Recall that we associated a sequence $A, A^{\vee}_{\langle a_0\rangle}, A^{\vee}_{\langle a_0, a_1\rangle}, \ldots$ of algebras to each bad sequence $\langle a_0, a_1, \ldots\rangle$ of A. By the last proposition, we have the decreasing sequence $A > A^{\vee}_{\langle a_0\rangle} > A^{\vee}_{\langle a_0, a_1\rangle} > \cdots$. This sequence must be finite since the order $<$ is a well-order on the class of algebras in normal form by Proposition 5.9. Therefore the mapping $\langle a_0, \ldots, a_{n-1}\rangle \mapsto |A^{\vee}_{\langle a_0, \ldots, a_{n-1}\rangle}|$ yields a reification of the partial order A by the ordinal $|A|$. This observation leads us to the following theorem, which is a main theorem of this section.

5.11 Theorem
Every algebra is a well-partial-order with respect to the embedding $\trianglelefteq_A$ and the order type $|Bad(A)|$ of the well-partial-order is upper bounded by the ordinal $|A|$. □

5.12 Remark
The argument used in this section is quite elementary except the proof that the order $<$ on the class of algebras is a well-order. So the reduction of well-partial-

orderedness of an algebra A to well-orderedness of the ordinal $|A|$ is proved in system RCA_0, even intuitionistically (see Remark 5.6). $\square$

In order to compare the upper bound $|A|$ of the algebra embedding with a traditional ordinal notation, we give an order isomorphism from the class of algebras in normal form to an initial segment of Bachmann hierarchy. We refer the reader to [17] for a comprehensive definition of Bachmann hierarchy. The family of normal functions of the hierarchy is denoted by $\langle \varphi_\alpha(\text{-})\rangle$ where α ranges over all ordinals less than $\epsilon_{\Omega+1}$ (Ω is the first regular ordinal greater than ω). For our purpose, it suffices to consider the indices α up to Ω^ω. We choose as the base normal function $\varphi_0(\text{-})$ the enumeration ω^{ω^x} of multiplicatively indecomposable ordinals. This choice is because our definition of algebras has connectives $+$ and also $\times$, and so every ordinal between consecutive multiplicatively indecomposable ordinals is constructed in terms of these two connectives. The family $\langle \overline{\varphi}_\alpha \rangle$ is a modification of the hierarchy so that ordinals have unique notations [17]. The reader might observe that the recursion relation of $\overline{\varphi}$ [17] has the same outlook as the lexicographic path ordering (also the ordinal function ϑ in [29] has the same relation). We first employ an auxiliary ordinal function $\overline{\mu}$ from Ω^ω to Ω^ω in order to make the description easier.

5.13 Definition
The function $\overline{\mu}$ is a map from Ω^ω to Ω^ω, the map defined as in Figure 3. There $\alpha < \Omega^\omega$ is supposed to be in Cantor normal form $\Omega^n\alpha_n + \cdots + \Omega\alpha_1 + \alpha_0$ to base Ω. $\square$

Cantor Normal Form of α	Value of $\overline{\mu}(\alpha)$
$\alpha < \Omega$	α
$\alpha = \Omega\alpha_1 + \alpha_0$	$\overline{\varphi}(0, -1 + \alpha_1) \otimes (1 + \alpha_0)$
$\alpha = \Omega^2 + \Omega\alpha_1 + \alpha_0$	$\overline{\varphi}(1 + \alpha_1, \alpha_0)$
$\alpha = \Omega^2\alpha_2 + \Omega\alpha_1 + \alpha_0$ $(\alpha_2 > 1)$	$\overline{\varphi}(\Omega(-1 + \alpha_2) + \alpha_1, \alpha_0)$
$\alpha \geq \Omega^3$	$\overline{\varphi}(\Omega^{n-1}\alpha_n + \cdots + \Omega\alpha_2 + \alpha_1, \alpha_0)$

Figure 3

Note that if every coefficient α_i of the Cantor normal form is less than $\varphi(\Omega^\omega, 0)$ then $\overline{\mu}(\alpha)$ is also less than $\varphi(\Omega^\omega, 0)$. The definition of $\overline{\mu}$ may appear complicated, but it is a kind of adjustment for the case $\alpha < \Omega^2$. An order isomorphism of the class of algebras into an initial segment of Bachmann hierarchy up to $\varphi(\Omega^\omega, 0)$ is defined as follows:

5.14 Proposition

By the following equations, the class of algebras in normal form becomes order isomorphic to the ordinal $\varphi(\Omega^\omega, 0)$.

$$
\begin{array}{lcll}
|\emptyset| & = & 0 & \\
|1| & = & 1 & \\
|A+B| & = & |A| \oplus |B| & \textit{(natural sum)} \\
|A \times B| & = & |A| \otimes |B| & \textit{(natural product)} \\
|\mu X.\, X^n C_n + \cdots + X C_1 + C_0| & & & \\
 & = & \overline{\mu}(\Omega^n |C_n| + \cdots + \Omega |C_1| + (-1 + |C_0|)) & \square
\end{array}
$$

6 Linearization of Algebra Embeddings

In the last section, the upper bounds of order types of algebra embeddings are given by the well-ordering on the class of algebras. Namely for each algebra A the order type of $Bad(A)$ is upper bounded by $|A|$, which is the ordinal corresponding to A in the well-ordering of the class of algebras.

The lower bounds of well-partial-orders are given by linearizations, as seen in Preliminaries. In this section, we give a uniform way to linearize the algebra embeddings, called Ackermann orderings. We show for almost all algebras A an Ackermann ordering gives a linearization of order type $|A|$. For exceptional cases, a modification of the Ackermann ordering yields the order type $|A|$. Therefore the ordinals $|A|$ are lower bounds of algebra embeddings as well as their upper bounds. So we derive that the order type of embedding of an algebra A is exactly the ordinal $|A|$.

Ackermann orderings on algebras depend on the orders of summands and multiplicands. That is to say, if one exchanges the order of summands from $A + B$ to $B + A$ then the associated Ackermann orderings are in general different, and likewise for $A \times B$. Given an algebra, therefore, there are several associated Ackermann orderings according to the permutations of summands and multiplicands.

6.1 Convention

In this section, $+$ and $\times$ are non-commutative unless otherwise mentioned. $\square$

6.2 Definition

Let A be a closed algebra where the order of summands and multiplicands is fixed.

The *Ackermann ordering* on A is the binary relation $<_A$ between terms of A defined as follows:

(i) $A = \emptyset$ or 1.
$<_A$ is void.

(ii) $A = B + C$.
$a <_A a'$ if and only if one of
(1) $a = \iota b$ and $a' = \iota' c$;
(2) $a = \iota b, a' = \iota b'$ and $b <_B b'$;
(3) $a = \iota' c, a' = \iota' c'$ and $c <_C c'$.

(iii) $A = B \times C$.
$\langle b, c\rangle <_A \langle b', c'\rangle$ if and only if either
(1) $c <_C c'$;
(2) $c = c'$ and $b <_B b'$.

(iv) $A = \mu X.\, C[X]$.
$\gamma c <_A \gamma c'$ (where $c, c' : C[A]$) if and only if either
(1) $\gamma c \leq_A a'$ for some proper subterm $a' : A$ of $\gamma c'$. Here $\leq_A$ is $<_A$ or $=$.
(2) $c <_{C[A]} c'$ and $a <_A \gamma c'$ for all proper subterm $a : A$ of γc. $\square$

6.3 Notation
We write $Ack(A)$ if we emphasize that we are concerned with the Ackermann ordering on an algebra A. $\square$

Our definition of Ackermann ordering is based on the same idea of the orderings in [1, 10, 21]. Note that the Ackermann ordering associated to a direct product algebra $A \times B$ is the lexicographic ordering, but it compares from right to left in the reverse order to the usual lexicographic ordering. This choice is in order to accommodate our ordering to the traditional ordinal notations.

6.4 Proposition
The reflexive closure $\leq_A$ of Ackermann ordering $<_A$ is a total order and is a linearization of the algebra embedding $\trianglelefteq_A$, that is,

$$a \trianglelefteq_A a' \quad \Rightarrow \quad a \leq_A a'. \qquad \square$$

It is convenient to introduce an alternative way to construct initial algebras. This is for reducing the number of exceptional cases in the argument below. The difference is that we assume the existence of the least element.

6.5 Definition
$\tilde{\mu} X.\, B[X]$ is an alternative form of an initial algebra whose terms are generated by the following rules (where $A = \tilde{\mu} X.\, B[X]$):

$$0_A : A \qquad\qquad \frac{b : B[A]}{\gamma b : A}$$

where 0_A is a constant term associated to each (occurrence of) the algebra A and γ is the associated constructor.

The Ackermann ordering on $A = \tilde{\mu}X.\, B[X]$ is defined as follows: $a <_A a'$ if and only if either

(i) $a = 0_A$ and $a' \neq 0_A$ (i.e., 0_A is the least element)
(ii) $a \leq_A a'^{\circ}$ for some proper subterm a'° of a' having sort A.
(iii) $a = \gamma b$, $a' = \gamma b'$, $b <_{B[A]} b'$ and $a^{\circ} < a'$ for all proper subterm a° of a having sort A. $\square$

The initial algebra $\tilde{\mu}X.\, B[X]$ is translated to the former notation by $\tilde{\mu}X.\, B[X] = \mu X.\, 1 + B[X]$. It is easy to see these two are order isomorphic with respect to the associated Ackermann orderings.

The following are a set of isomorphisms respecting Ackermann orderings. There we say a morphism $f : A \rightarrow B$ preserves Ackermann orderings if it satisfies $a <_A a' \Rightarrow f(a) <_B f(a')$. As in the case of algebra embeddings, we regard those isomorphisms as rewriting rules from left to right.

6.6 Proposition
The following nine natural transformations are isomorphisms where both the isomorphisms and their inverses preserve Ackermann orderings.

$$
\begin{array}{lcll}
\emptyset + A & \xrightarrow{\sim} & A & \\
A + \emptyset & \xrightarrow{\sim} & A & \\
\emptyset \times A & \xrightarrow{\sim} & \emptyset & \\
A \times \emptyset & \xrightarrow{\sim} & \emptyset & \\
1 \times A & \xrightarrow{\sim} & A & \\
A \times 1 & \xrightarrow{\sim} & A & \\
A(B + C) & \xrightarrow{\sim} & AB + AC & \\
\tilde{\mu}X.\, A & \xrightarrow{\sim} & 1 + A & \textit{whenever } A \textit{ is closed} \\
\tilde{\mu}X.B[X] + C & & & \\
 & \xrightarrow{\sim} & (\tilde{\mu}X.\, B[X]) + (\tilde{\mu}X.\, B^1[X])C & \\
 & & & \textit{if } B[X] \textit{ is of degree } 1 \textit{ and } C \textit{ is closed.}
\end{array}
$$

In the last transformation $B^1[X]$ is obtained from $B[X]$ by erasing all constant summands. For example, if $B[X] = F + DXE + F' + D'XE'$ then $B^1[X] = DXE + D'XE'$. $\square$

Note that $(A + B)C$ is not isomorphic to $AC + BC$. Also there is no need for the rule $\tilde{\mu}X.\, B[X] \xrightarrow{\sim} \emptyset$ since $\tilde{\mu}X.\, B[X]$ is always non-empty.

Next we calculate the order types of Ackermann orderings. It suffices to consider the algebras having no redices for the rewriting rules in the previous proposition.

The full exposition is lengthy. So we will be content with a brief description how the calculation is carried out. The first thing is to associate to each closed algebra A a sequence that is strictly increasing and unbounded with respect to the Ackermann ordering $<_A$. Later we will give the ordinals corresponding to the components of the sequence so that the supremum of those ordinals provide the order types of $<_A$.

6.7 Definition

Let A be a closed algebra having no redices for the rewriting rules defined by the previous proposition.

A sequence $\langle a[n]\rangle_{n\in\omega}$ is associated to A by the following definition (we follow the convention that $\langle c[n]\rangle_{n\in\omega}$ is the sequence associated to C, etc.):

(o) $\emptyset$ and 1 have no associated sequences.

(i) For a disjoint sum algebra $A = B + C$,

$$a[n] \;=\; \iota'(c[n])$$

(ii) For a product algebra $A = B \times C$,

$$a[n] = \langle 0_B, c[n]\rangle \qquad \text{where } 0_B \text{ denotes the least element of } B.$$

(iii) For an initial algebra, the sequence is manipulated by the following four cases (a) through (d).

(a) If A is of the form $\tilde{\mu}X.\, B[X] + C[X]\cdot D$ ($C[X]$ may be void), then

$$\begin{aligned} a[0] &= 0_A \\ a[n+1] &= \gamma^D\langle 0_{C[A]}, d[n]\rangle \end{aligned}$$

where $\gamma^D : C[A]\cdot D \to A$ is the composition of the constructor γ preceded by the injection from $C[A]\cdot D$ to $B[A] + C[A]\cdot D$.

(b) If A is of the form $\tilde{\mu}X.\, B[X] + C[X]\cdot X$ ($C[X]$ may be void), then

$$\begin{aligned} a[0] &= 0_A \\ a[n+1] &= \gamma^X\langle 0_{C[A]}, a[n]\rangle \qquad \text{where } \gamma^X : C[A]\cdot A \to A. \end{aligned}$$

(c) If A is of the form $\tilde{\mu}X.\, B[X] + C[X]\cdot D + E + 1$ ($C[X]$ must contain X),

$$\begin{aligned} a[0] &= \gamma^1(i) \qquad\qquad \text{where } \gamma^1 : 1 \to A \\ a[n+1] &= \gamma^D\langle c_0, d[n]\rangle \end{aligned}$$

where $c_0 : C[A]$ is defined as follows: if $C[A] = C_1 \times \cdots \times C_n$ then the i-th component of c_0 is the least element 0_{C_i} for all C_i except the leftmost occurrence of A, for which the component is set $a[0]$. Namely if $C[X] = C_1 \cdots X \cdots C_n$ where the designated X is the leftmost occurrence of X, then c_0 is put $\langle 0_{C_1}, \ldots, a[0], \ldots, 0_{C_n}\rangle$ where the component $a[0]$ corresponds to the occurrence of X. The component $a[0]$ in the definition of $a[n{+}1]$ ensures $a[0] <_A a[n+1]$.

(d) If A is of the form $\tilde{\mu}X.\, B[X] + C[X] \cdot X + E + 1$ ($C[X]$ may be void),

$$\begin{array}{lcl} a[0] & = & \gamma^1(i) \\ a[n+1] & = & \gamma^X \langle 0_{C[A]}, a[n] \rangle \quad \square \end{array}$$

6.8 Proposition

Let A be a closed algebra with the associated sequence $\langle a[n] \rangle_{n\in\omega}$.

The sequence $\langle a[n] \rangle_{n\in\omega}$ is strictly increasing and unbounded with respect to Ackermann ordering $<_A$. $\square$

In the above we associate a sequence of terms to each closed algebra A. In the following we associate a sequence of algebras to each closed algebra A. The proposition after the definition shows that these two sequences are related intimately.

6.9 Definition

Let A be a closed algebra having no redices for the rewriting rules defined in Proposition 6.6

The *fundamental sequence* $\langle A_n^{\wedge} \rangle_{n\in\omega}$ is a sequence of closed algebras associated to each A as follows (the cases correspond to those in the Definition 6.7):

(o) $\emptyset$ and 1 has no fundamental sequences.

(i) $\begin{array}{lcl} A & = & B + C \\ A_n^{\wedge} & = & B + C_n^{\wedge} \end{array}$

(ii) $\begin{array}{lcl} A & = & B \times C \\ A_n^{\wedge} & = & B \times C_n^{\wedge} \end{array}$

(iii) A is an initial algebra.

(a) $\begin{array}{lcl} A & = & \tilde{\mu}X.\, B + CD \\ A_0^{\wedge} & = & \emptyset \\ A_{n+1}^{\wedge} & = & \tilde{\mu}X.\, B + CD_n^{\wedge} \end{array}$

(b) $\begin{array}{lcl} A & = & \tilde{\mu}X.\, B + CX \\ A_0^{\wedge} & = & \emptyset \\ A_{n+1}^{\wedge} & = & \tilde{\mu}X.\, B + CA_n^{\wedge} \end{array}$

(c) $\begin{array}{lcl} A & = & \tilde{\mu}X.\, B + CD + E + 1 \\ A_0^{\wedge} & = & \tilde{\mu}X.\, B + CD + E \\ A_{n+1}^{\wedge} & = & \tilde{\mu}X.\, B + CD_n^{\wedge} + A_0^{\wedge} + 1 \end{array}$

(d) $\begin{array}{lcl} A & = & \tilde{\mu}X.\, B + CX + E + 1 \\ A_0^{\wedge} & = & \tilde{\mu}X.\, B + CX + E \\ A_1^{\wedge} & = & \tilde{\mu}X.\, B + CA_0^{\wedge} + 1 \\ A_{n+2}^{\wedge} & = & \tilde{\mu}X.\, B + CA_{n+1}^{\wedge} \end{array}$ $\square$

For an algebra A and a term a of sort A, let $A_{<a}$ denote the suborder $\{x : A \mid x <_A a\}$ of A.

6.10 Proposition
(i) $Ack(A_{<a[n]})$ *is order isomorphic to* $Ack(A^{\wedge}_n)$.
(ii) $Ack(A)$ *is order isomorphic to* $\sup_{n\in\omega}(Ack(A^{\wedge}_n))$. $\square$

(i) is proved by giving morphisms preserving Ackermann orderings from $A_{<a[n]}$ to $A^{\wedge}_n$, and from $A^{\wedge}_n$ to $A_{<a[n]}$. (ii) is an immediate consequence of (i) since the sequence $\langle a[n]\rangle_{n\in\omega}$ is unbounded. This assertion (ii) allows us to calculate the order type of Ackermann ordering $Ack(A)$ from the smaller order types of $Ack(A^{\wedge}_n)$.

6.11 Definition
The function $\mu : \Omega^\omega \to \Omega^\omega$ has the same definition as $\overline{\mu}$ of Definition 5.13 except that all $\overline{\varphi}$ therein is replaced by φ and that $\mu(\Omega\alpha_1+\alpha_0) = \varphi(0, -1+\alpha_1)\times(1+\alpha_0)$ (natural product is replaced by ordinary product). $\square$

Now we provide the order type $A^\star$ of Ackermann ordering $Ack(A)$. In the following, we say an initial algebra $\tilde{\mu}X.\,B[X]$ has *degree* n if the largest number of X in a single summand of $B[X]$ is n (an analogy of the degrees of polynomials). The *rightmost multiplicand* of $\coprod_{i=1}^{m}\prod_{j=1}^{n_i} B_{ij}$ where B_{ij} is not decomposable any more to the form of a product, is the multiplicand $B_{m\,n_m}$.

6.12 Definition
Let A be a closed algebra. Without loss of generality we assume A has no redices for the rewriting rules given in Proposition 6.6.

An ordinal $A^\star$ is associated to each A by the following

(o) $\emptyset^\star = 0, \quad 1^\star = 1, \quad X^\star = \Omega.$

(i) $(A+B)^\star = A^\star + B^\star.$

(ii) $(A\times B)^\star = A^\star \times B^\star.$

(iii) In case A is an initial algebra,

(a) If $A = \tilde{\mu}X.\,B[X] + C$ is of degree 2 or more (C may be void), let α be the least ordinal satisfying $\max({D_i}^\star) < \mu(B^\star + \alpha)$ where D_i ranges over all proper subalgebras of $\tilde{\mu}X.\,B[X]$ other than the rightmost multiplicand of $B[X]$. Then $A^\star = \mu(B^\star + \alpha + C^\star)$.

(b) If A has degree 1, it has the form $A = \tilde{\mu}X.\,B[X] + E + DXC$ ending with a summand of degree 1 (E may be void), since we assumed A has no redices. Let α be the least ordinal fulfilling $\max\{(\tilde{\mu}X.\,B^1[X])^\star, D^\star\} < \omega^{\omega^{\alpha+1}}$ where

$B^1[X]$ is obtained from $B[X]$ by erasing all constant summands (see Proposition 6.6). If $(\tilde{\mu}X.\, B[X])^\star$ and $E^\star$ are both less than $\omega^{\omega^{\alpha+1}}$ then

$$A^\star \;=\; \omega^{\omega^{\alpha+C^\star}};$$

otherwise

$$\text{if } C = 1 \text{ then } A^\star = \omega^{\delta+1}$$
$$\text{else } \; A^\star = \omega^{\omega^{\gamma+(-1+C^\star)}}$$

where

$$\begin{array}{lcccl} \omega^\delta & \leq & \max\{(\tilde{\mu}X.\, B[X])^\star, E^\star\} & < & \omega^{\delta+1} \\ \omega^{\omega^\gamma} & \leq & \max\{(\tilde{\mu}X.\, B[X])^\star, E^\star\} & < & \omega^{\omega^{\gamma+1}} \quad \Box \end{array}$$

Note that the disjoint sum and the direct product are mapped to the ordinal sum $+$ and the ordinal product $\times$ and in addition the variable X to Ω. Therefore if an initial algebra contains the summand of the form $\cdots C \cdot X \cdots$ where C is closed, this C disappears in $\mu((B[X])^\star)$ since $C^\star \cdot \Omega = \Omega$. Likewise if two summands occur as $D[X] + E[X]$ and the degree of $D[X]$ is smaller than that of $E[X]$ then the entire $D[X]$ disappears. However the closed algebra C or the closed algebras in $D[X]$ may be too large to ignore them. The ordinals α, γ and δ occurring in the definition above reflect this effect of closed algebras that would simply be ignored if there were no adjusting ordinals α, γ, δ.

6.13 Remark

The definition of $A^\star$ is quite complicated when A is an initial algebra, especially, of degree 1. However if one is interested in the algebras of the form $A = \tilde{\mu}X.\, X^n C_n + \cdots + XC_1 + C_0$, then the associated ordinal $A^\star$ has the simple form $\mu(\Omega^n {C_n}^\star + \cdots + \Omega {C_1}^\star + {C_0}^\star)$. $\Box$

The following theorem shows that $A^\star$ is exactly the order type $|Ack(A)|$ of Ackermann ordering on A.

6.14 Theorem

$|Ack(A)| = A^\star$ for every closed algebra A having no redices for the rewriting rules given in Proposition 6.6.

(*Proof*) Prove that $(A_n^\wedge)^\star$ converges to $A^\star$. Therefore by transfinite induction for $A^\star$ we can prove $|Ack(A)| = A^\star$ by (ii) of Proposition 6.10. $\Box$

The order types of the recursive path ordering (the multiset path ordering) seem to be folkloric. If the set of labels is well-ordered by order type α, the associated multiset path ordering is order isomorphic to $\varphi(\alpha, 0)$ in Feferman-Schütte notation [8]. The order types of lexicographic path orderings were seldom mentioned in the literature. Some partial results are found in [28, 10]. From the previous theorem, we know that the order types of the lexicographic path orderings are much greater than the corresponding multiset path orderings.

Recall that Ackermann orderings are linearizations of algebra embeddings. So we have a lower bound for the order type of $Bad(A)$. Consider an initial algebra A of the form $\tilde{\mu}X.\, X^n C_n + \cdots + XC_1 + C_0$. Then the order type $|Bad(A)|$ of the algebra A is upper bounded by $\overline{\mu}(\Omega^n|C_n| + \cdots + \Omega|C_1| + |C_0|)$ by Theorem 5.11 and lower bounded by $\mu(\Omega^n C_n{}^\star + \cdots + \Omega C_1{}^\star + C_0{}^\star)$ by Theorem 6.14. If we assume as the induction hypothesis $|C_i| = C_i{}^\star$, then the difference of the upper bound and the lower bound is only the difference of $\overline{\mu}$ and μ, which comes from the one of $\overline{\varphi}$ and φ. By the recursion relation in [17] we can single out when $\overline{\mu}$ and μ coincide and when they are different as in the following.

In the following proposition, the *lowermost coefficient* of an ordinal $\beta < \Omega^\omega$ means, if we let $\Omega^n\gamma_n + \cdots + \Omega\gamma_1 + \gamma_0$ be a Cantor normal form of β to base Ω, the non-zero ordinal γ_k such that $\gamma_i = 0$ for all $i < k$.

6.15 Proposition

Let $\beta = \beta_0 + m$ be an ordinal less than Ω^ω where β_0 is a limit and m is finite. If $\beta \geq \Omega^2$ then

$$\overline{\mu}(\beta) \quad = \quad \begin{cases} \mu(\beta+1) & \textit{if the lowermost coefficient of } \beta_0 \textit{ is equal to } \mu(\beta_0) \\ \mu(\beta) & \textit{otherwise.} \end{cases}$$

If $\beta = \Omega\gamma$ and $\gamma = \gamma_0 + m$ where γ_0 is a limit and m is finite, then

$$\overline{\mu}(\beta) \quad = \quad \begin{cases} \mu(\Omega(\gamma+1)) & \textit{if } \gamma_0 = \mu(\beta) \\ \mu(\beta) & \textit{otherwise} \end{cases} \qquad \square$$

Therefore $A^\star \neq |A|$ happens only when either

(i) A is of the form $\tilde{\mu}X.\, X^n C_n + \cdots + X^k C_k + m$ ($n \geq 2$, $k \geq 0$ and m is finite) and $C_k{}^\star = \mu(\Omega^n C_n{}^\star + \cdots + \Omega^k C_k{}^\star)$. Then $|A| = \mu(\Omega^n C_n{}^\star + \cdots + \Omega^k C_k{}^\star + 1 + m)$.

(ii) A is of the form $\tilde{\mu}X.\, X(C_1 + m)$ (m is finite) and $C_1{}^\star = \mu(\Omega \cdot C_1{}^\star)$. In this case, $|A| = \mu(\Omega(C_1{}^\star + 1 + m))$.

In these cases, we must find other linearizations yielding the same ordinals as the upper bounds $|A|$. The most difficult is to manipulate the case $k \geq 1$ of (i).

6.16 Lemma

Let A be an initial algebra of the form $\tilde{\mu}X.\, B[X] + X^k C + D$ (degree 2 or more, $k \geq 1$, C, D are closed, and $B[X]$ may be void).

If $A^\star = C^\star$ there is a linearization of the embedding $\trianglelefteq_A$ such that the order type of the linearization is $\mu(B^\star + \Omega^k C^\star + 1 + D^\star)$.

(*Proof*) Put $A' = \tilde{\mu}X. C + B[X] + X^k C + D$, whose Ackermann ordering has order type $\mu(B^\star + \Omega^k C^\star + 1 + D^\star)$. There is a bijection $f : A \to A'$ such that if $a \trianglelefteq_A a'$ then $f(a) \leq_{A'} f(a')$. The inverse image of f gives the required linearization. □

The case (i) above is handled by this lemma if $k \geq 1$. That is to say, the linearization of the lemma applied to the algebra $A = \tilde{\mu}X. X^n C_n + \cdots + X^k C_k + m$ provides the order type $\mu(\Omega^n {C_n}^\star + \cdots \Omega^k {C_k}^\star + 1 + m) = |A|$ as required. If $k = 0$, we simply exchange the order of the summand as $A' = \tilde{\mu}X. C_0 + \Omega^n C_n + \cdots + m$. Then the Ackermann ordering of A' gives a linearization of A with order type $|A|$ by Theorem 6.14. For the case (ii), we consider the algebra $A'' = \tilde{\mu}X. C_1 X + Xm$. The Ackermann ordering on A'' is the linearization of A with order type $|A|$. Therefore we have the following theorem and corollary as the main results of this section.

6.17 Theorem
Let A be a closed algebra in normal form in the sense of Section 4.

There is a linearization of the embedding $\trianglelefteq_A$ such that the order type of the linearization is equal to $|A|$. □

6.18 Corollary
For each closed algebra A, the embedding $\trianglelefteq_A$ is a well-partial-order whose order type $|Bad(A)|$ is equal to $|A|$ (where A is assumed to be reduced to a normal form without loss of generality). □

So the order types $|Bad(A)|$ of algebra embeddings are given effectively by $|A|$. Recall that the mapping $|\cdot|$ is a bijection up to isomorphism respecting embeddings into the class of ordinals. Hence we have the following theorem, which is the main achievement of this paper.

6.19 Theorem
The mapping $A \mapsto |Bad(A)|$ is a bijection from the class of algebras (up to isomorphism) onto an initial segment of ordinals up to $\varphi(\Omega^\omega, 0)$. □

In other words, for each ordinal less than $\varphi(\Omega^\omega, 0)$ there is an algebra whose order type is equal to the ordinal, and furthermore such an algebra is unique up to embedding-reflecting isomorphism. We also note that the isomorphism of algebras is primitive recursively decidable:

6.20 Corollary
The following is primitive recursively decidable: given two closed algebras, determine whether they are isomorphic with respect to embedding-reflecting morphisms.

(*Proof*) If the normal forms of two algebras are distinct, then they have different order types and thus cannot be isomorphic. The reduction to normal forms is primitive recursive (Lemma 4.16). □

We give several examples of the order types $|A| = |Bad(A)|$ of algebras. For the second and the fourth, we can find announcements in [6].

$$\begin{array}{lcll} |\mu X.\, X+1| & = & \omega & \\ |\mu X.\, X^2+1| & = & \epsilon_0 & \\ |\mu X.\, X^2+XN+1| & = & \varphi(\omega,0) & \text{where } N \text{ is } \mu X.\, X+1 \\ |\mu X.\, X^2 2+1| & = & \Gamma_0 & \\ |\mu X.\, X^{n+1}+1| & = & \varphi(\Omega^n,0) & \text{where } n \geq 2 \end{array}$$

Also the result in Section 3 follows since $|\mu X.\, AX+1|$ is equal to $\omega^{\omega^{-1+|A|+1}}$ if $|A|$ is of the form $\epsilon_\gamma + n$, epsilon number plus finite number; and equal to $\omega^{\omega^{-1+|A|}}$ otherwise.

The merit of reducing well-partial-orderedness to well-orderedness becomes apparent when proof theory is combined. After a famous work of Gentzen, ordinal numbers are known to be useful to measure the strength of logical systems. The *proof theoretical ordinal* $|T|$ of a logical system T is the least ordinal α such that well-orderedness of α is not provable in T [3]. The following are known.

$$\begin{array}{lcll} |\mathrm{ACA}_0| & = & \epsilon_0 & \\ |\Sigma^1_1\text{-}\mathrm{DC}_0| & = & \varphi(\omega,0) & [4] \\ |\mathrm{ATR}_0| & = & \Gamma_0 & [14] \end{array}$$

These systems are fragments of second order arithmetic, assuming as basic axioms the comprehension axiom for recursive formulas and the induction axiom (not the induction scheme; the subscript $(\text{-})_0$ comes from this restriction on the induction). This basic system is called RCA_0. System ACA_0 has in addition a comprehension axiom for arithmetical formulas [34, 37]. This system is a conservative extension of Peano arithmetic. The proof theoretical ordinal ϵ_0 follows from this fact. System Σ^1_1-DC_0 has the dependent choice axiom for Σ^1_1-formulas [4]. System ATR_0 has the axiom asserting the existence of Turing jumps along any recursive well-ordering [14, 34, 37].

Therefore well-partial-orderedness of the algebra $\mu X.\, X^2+1$ of binary trees is non-provable in system ACA_0, but provable in any stronger system having a greater proof theoretical ordinal, e.g., Σ^1_1-DC_0, ACA, $\mathrm{ACA}_0 + \Pi^1_1$-Reflection. Likewise well-partial-orderedness of embedding on $\mu X.\, X^2+XN+1$ is non-provable in Σ^1_1-DC_0 and well-partial-orderedness of embedding on $\mu X.\, X^2 2+1$

is independent from ATR_0, while they are provable in any stronger systems than Σ^1_1-DC_0 and ATR_0 (respectively) having greater proof theoretical ordinals.

Another immediate consequence of our main results is that we can use the class of algebras as an ordinal notation up to $\varphi(\Omega^\omega, 0)$. This ordinal notation has some new features in comparison with traditional ordinal notations; (i) every ordinal notation has a meaning. For example, $\mu X. X^2 + 1$ denotes the ordinal ϵ_0 since the order type of the corresponding algebra embedding is equal to ϵ_0; (ii) every notation can be encoded by a Π^1_1-formulas. As is well-known, non-iterated inductive definitions on arithmetical formulas are simulated by the comprehension axiom for Π^1_1 formulas without set parameters [11]. (iii) our system of notations provides an alternative description for Bachmann hierarchy. The Bachmann hierarchy is defined by the Bachmann collection, which is sensitive to the choice of fundamental sequences for limit ordinals. Our theorem shows the hierarchy can admit a definition without using the fundamental sequences. The Feferman-Aczel notation θ [2] also arose for giving the system of notations not depending on fundamental sequences [12].

One might say the obtained notation is up to $\varphi(\Omega^\omega, 0)$ (smaller than Howard ordinal $\varphi(\epsilon_{\Omega+1}, 0)$) and is too weak to analyze impredicative logical system. However we hope to show in a forthcoming paper this bound can be extended to $\theta\Omega_\omega 0 = |\Pi^1_1\text{-}\mathrm{CA}_0|$ by allowing mutual recursions.

7 Embedding on Finite Trees

We turn to finite ordered trees, the nodes of which have outdegrees finite but without upper bounds on the number. The set of finite ordered trees is defined naturally if we use a subsidiary sort F (forest), which should be the sort of finite lists of trees. The following rules generate finite ordered trees t and forests f (see Figure 4):

$$\begin{array}{lcl} t & ::= & \mathtt{span}(f) \\ f & ::= & \mathtt{nil} \quad | \quad \mathtt{cons}(t, f) \end{array}$$

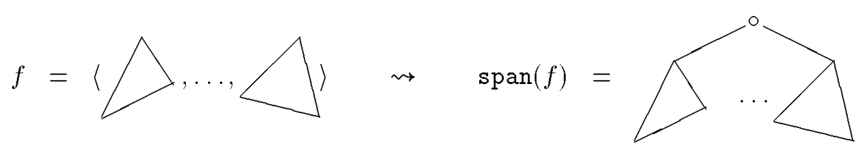


Figure 4

In other words, the algebra T of finite ordered trees is a solution of system of equations

$$\begin{aligned} T &= F \\ F &= 1 + T \cdot F. \end{aligned}$$

The algebra T is not definable in the sense of Definition 4.1 since it uses a subsidiary sort F and mutual recursion. In the form of μ-operators, the algebra T is written $\mu X. (\mu Y. 1 + XY)$ where two variables X and Y are in the scope of μY. Recall that the algebra in the sense of Definition 4.1 admits only one variable.

Allowing a finite number of sort variables, we can define more complicated structures and we need higher ordinals to reify them. We leave the extension to this direction to a forthcoming paper, and here are content with an analysis of the algebra T of finite ordered trees, which is important and the simplest (in a precise sense; the algebra T yields the least order type among those algebras requiring mutual recursion to define).

Finite trees in T are represented using a unary node `span`, a binary node `cons` and a 0-ary node `nil`. T should be different, however, from the algebra $\mu X. X^2 + X + 1$ that has the same family of nodes. The difference arises from the different notions of subterms. Suppose given a tree $t = t\langle t/0, \ldots, t/(n-1)\rangle$ (represented by `span`, `cons` and `nil`). For the notion of subtrees (subterms of sort T), there are few problems: every t/i is a subtree of t and every subtree of t/i is a subtree of t. The problem lies in the sort F of forests. Let $f = \langle t/0, \ldots, t/(n-1)\rangle$ be a forest. The term f is represented by `cons` and `nil` as

$$\texttt{cons}(t/0, \ldots \texttt{cons}(t/(n-1), \texttt{nil}) \ldots)$$

If we follow the idea in Section 4 that subterms are simply subexpressions, then $\langle t/1, \ldots, t/(n-1)\rangle$, $\langle t/2, \ldots, t/(n-1)\rangle$ etc. are sub-forests of f. There are, however, other subterms of sort F hidden in each t/i. If t/i is of the form $u\langle u/0, \ldots, u/(p-1)\rangle$, the forest $\langle u/0, \ldots, u/(p-1)\rangle$ is a subexpression of $\langle t/0, \ldots, t/i, \ldots, t/(n-1)\rangle$. But if we admitted this as subterms (recall that algebra embeddings are defined using subterms), then the embedding on the algebra T would be different with the tree embedding given in Definition 2.4. For example, consider two trees s and t in Figure 5. s does not embed

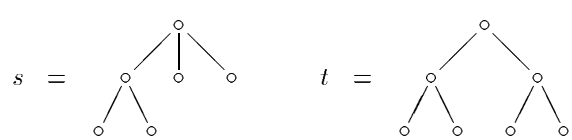


Figure 5

into t by tree embedding since t has no nodes of outdegree 3. If one admits all subexpressions of sort F as sub-forests, then s embeds into t, as in the following: Consider two forests $\langle\circ,\circ\rangle$ and $\langle\mathtt{span}\langle\circ,\circ\rangle\rangle$, where $\circ$ stands for $\mathtt{span}(\mathtt{nil})$. Since the first is a subexpression of the second, we have $\langle\circ,\circ\rangle \trianglelefteq_F \langle\mathtt{span}\langle\circ,\circ\rangle\rangle$ by the projection rule of algebra embeddings. Then one can derive $\langle\mathtt{span}\langle\circ,\circ\rangle,\circ,\circ\rangle \trianglelefteq_F \langle\mathtt{span}\langle\circ,\circ\rangle,\mathtt{span}\langle\circ,\circ\rangle\rangle$ and then $\mathtt{span}\langle\mathtt{span}\langle\circ,\circ\rangle,\circ,\circ\rangle \trianglelefteq_T \mathtt{span}\langle\mathtt{span}\langle\circ,\circ\rangle,\mathtt{span}\langle\circ,\circ\rangle\rangle$ by congruence rules. The last means $s \trianglelefteq_T t$ and thus the embedding on T is undesirably not equal to the tree embedding.

This dilemma is settled simply by rejecting that $\langle\circ,\circ\rangle$ is a subterm of $\langle\mathtt{span}\langle\circ,\circ\rangle\rangle$. In general, if $f = \langle t/0,\ldots,t/n-1\rangle$ is a term of sort F, then the subterms of f having sort F are only $\langle t/1,\ldots,t/n-1\rangle$, $\langle t/2,\ldots,t/n-1\rangle$ etc., and never look inside of the already formed tree t/i. In the form of inference rules, a is a subterm of b (where a,b are terms of sort T or F) if and only if the relation $a \subseteq b$ is derived by the following rules:

$$a \subseteq a \qquad \text{for every term } a$$

$$\frac{a \subseteq f}{a \subseteq \mathtt{cons}(t,f)} \qquad t:T \text{ and } f:F$$

$$\frac{a \subseteq f}{a \subseteq \mathtt{span}(f)} \qquad f:F$$

Note that there is no rule inferring from $a \subseteq \mathtt{cons}(t,f)$ from $a \subseteq t$. Then the embeddings associated to the algebras T and F are defined in the same way as in Definition 4.7. One has only to read $a'^{\circ} \subseteq a$ the relation derived by the above rules. It is easy to see that $\trianglelefteq_T$ is the tree embedding on finite ordered trees and $\trianglelefteq_F$ is the Higman embedding induced from the tree embedding.

We give a reification of the tree embedding by the ordinal $\varphi(\Omega^\omega,0)$. Recall that this ordinal is the supremum of the order types of all (single recursive) algebras. We associate to each bad sequence of finite ordered trees a descending sequence of algebras. The reification technique given in Section 5 works completely as well. To $F[X]=\mu Y.\,1+XY$ and $f=\gamma_F(a):F[T]$ where $a:1+TF$, an algebra $F^\vee_f[X]$ is associated by $F^\vee_f[X]=\mu Y.\,(1+XY)^\vee_a$ where all $X^\vee_s$ is replaced by $T^\vee_s$ and all $Y^\vee_g$ by $F^\vee_g[X]$. Moreover to $T=\mu X.\,F[X]$ and $t=\gamma_T(f)$, an algebra $T^\vee_t$ is associated by $T^\vee_t=\mu X.\,F^\vee_f[X]$ where each $X^\vee_s$ is replaced by $T^\vee_s$.

7.1 Example

Let $t:T$ be $\mathtt{span}\langle t/0,\ldots,t/n-1\rangle$.

$$\begin{aligned} T^\vee_t &= \mu X.\,F^\vee_{\langle t/0,\ldots,t/n-1\rangle}[X] \\ &= \mu X.\,(\mu Y.\,1+T^\vee_{t/0}Y+X\cdot F^\vee_{\langle t/1,\ldots,t/n-1\rangle}[X]) \\ &\cong \mu X.\,(T^\vee_{t/0})^*(1+X\cdot F^\vee_{\langle t/1,\ldots,t/n-1\rangle}[X]) \\ &\;\vdots \end{aligned}$$

$$\cong \quad \mu X.\,(T^{\vee}_{t/0})^* + X(T^{\vee}_{t/0})^*(T^{\vee}_{t/1})^* + \cdots + X^{n-1}(T^{\vee}_{t/0})^* \cdots (T^{\vee}_{t/n-1})^*$$

Note that $T^{\vee}_t$ is a single recursive algebra. □

We can show that there is an embedding-reflecting injection $T_t \rightarrowtail T^{\vee}_t$ for all $t : T$ as in Lemma 5.5. Hence to each bad sequence $\langle t/0, t/1, \ldots \rangle$ of terms of sort T we can associate a descending sequence $T^{\vee}_{\langle t/0 \rangle}, T^{\vee}_{\langle t/0, t/1 \rangle}, \ldots$ of algebras. Therefore well-orderedness of the relation $<$ on the class of algebras, which is equivalent to well-orderedness of $\varphi(\Omega^\omega, 0)$, shows that the tree embedding is well-partial-ordered.

Conversely well-partial-orderedness of $\trianglelefteq_T$ implies well-orderedness of $\varphi(\Omega^\omega, 0)$. To see this, note that $\varphi(\Omega^\omega, 0)$ is the supremum of $\varphi(\Omega^n, 0)$ $(n < \omega)$ and $\varphi(\Omega^n, 0)$ is the order type of well-partial-order on the algebra $\mu X.\, X^{n+1} + 1$ of $(n+1)$-ary trees for $n \geq 2$. There is an embedding-reflecting map from $\mu X.\, X^{n+1} + 1$ to T as easily seen. Hence $\mathit{Wpo}(T)$ implies $\mathit{Wpo}(\mu X.\, X^{n+1} + 1)$ for all n, which in turn implies well-orderedness of $\varphi(\Omega^\omega, 0)$.

7.2 Theorem

Well-partial-orderedness of the tree embedding on finite ordered trees is equivalent to well-orderedness of the ordinal $\varphi(\Omega^\omega, 0)$. □

This theorem is first proved by Rathjen and Weiermann [29] for non-ordered trees. We do not claim our method is essentially different with theirs. We include our method here hoping that the the reader might feel the argument is simpler and having an intention to provide a guide to more general arguments for mutual recursion.

We also give a brief description for another result on finite trees, which already appeared in the literature but can be simplified by our method. In [18], Gupta proved that well-partial-orderedness of the tree minor relation is equivalent to well-orderedness of the ordinal ϵ_0. The method in [18] is to associate regular expressions to trees and ordinals to regular expressions so that bad sequences with respect to the tree minor yield descending sequences of ordinals less than ϵ_0.

A finite non-ordered tree is a *minor* of another if and only if the former is obtained from the latter by several applications of edge deletion and edge contraction. We write $s \trianglelefteq_m t$ if s is a tree minor of t (equal to the relation $\triangleleft_{m^r}$ in [18]). This is the graph minor relation applied on rooted trees but respecting the orientation on the edges (rooted trees may be regarded as oriented graphs where the orientations on the edges are from roots to leaves).

In the above, we modified the notion of subterms to obtain the tree embedding as the embedding on the algebra T. In Figure 5, we observed that if the notion of subterms is not modified then s embeds into t undesirably. However the reader might notice that s is a minor of t by contracting the rightmost edge from the root of t (s, t are ordered trees here). In fact, if we add the omitted rule of subterms

$$\frac{a \subseteq t}{a \subseteq \ \mathtt{cons}(t, f)} \qquad t : T \text{ and } f : F$$

then a finite ordered tree s embeds into t by the algebra embedding $\trianglelefteq_T$ if and only if s is obtained from t by several applications of two rewriting rules in Figure 6. Observe that the rule (i) is the combination of a single application of edge contraction and several applications of edge deletion, and the rule (ii) is a consequence of several applications of edge deletions. So fixing in some canonical way the order on the sets of immediate successors of all nodes in order to transform non-ordered trees to ordered trees, we easily see $s \trianglelefteq_T t$ (with the added rule above for subterms) implies $s \trianglelefteq_m t$. Therefore $Wpo(\trianglelefteq_m)$ implies $Wpo(\trianglelefteq_T)$.

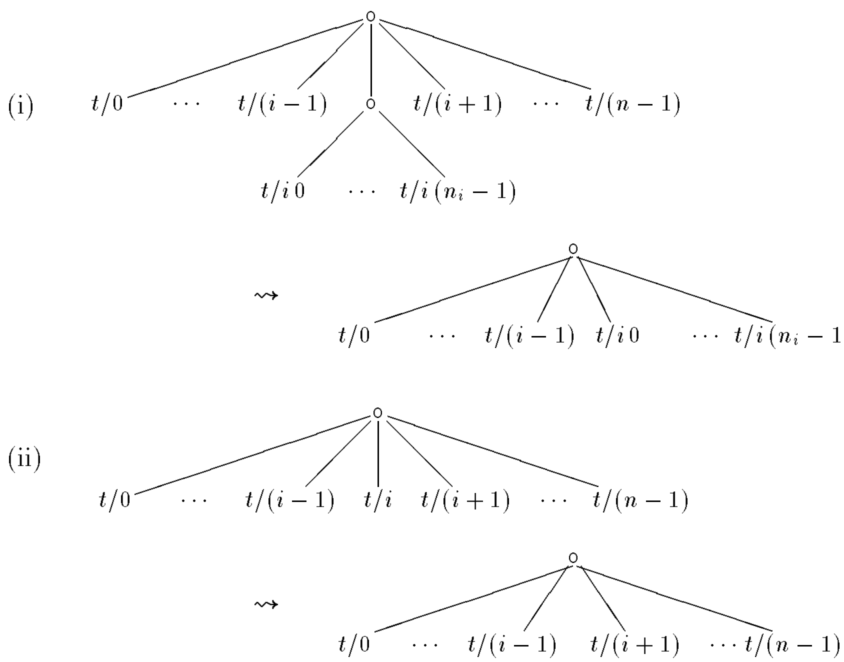


Figure 6

On the other hand, the rules in Figure 6 correspond exactly the algebra embedding on $B = \mu X. X^2 + 1$ of binary trees by the transformation $s \mapsto s^\flat$ from B to T in Figure 7. Hence $Wpo(\trianglelefteq_B)$ implies $Wpo(\trianglelefteq_T)$ and so in turn $Wpo(\trianglelefteq_m)$. Since Theorem 5.11 shows the algebra B is reified by the ordinal ϵ_0, we finally have the proof that well-orderedness of ϵ_0 implies well-partial-orderedness of the tree minor relation. The converse is easier. Therefore we have the following theorem first proved by Gupta [18].

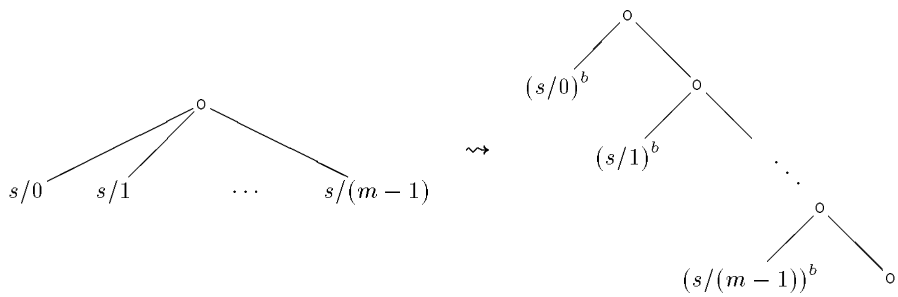


Figure 7

7.3 Theorem

Well-partial-orderedness of the tree minor is equivalent to well-orderedness of the ordinal ϵ_0 $\square$

So it is independent from system ACA_0 that the set of finite non-ordered trees is well-partial-ordered with respect to the tree minor relation. In addition, since the argument is formalizable in intuitionistic logic, we have a constructive proof of the fact that the tree minor is a well-partial-ordered, from a constructive proof of well-orderedness of ϵ_0 (See [18]).

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