Relational Limits in General Polymorphism

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Abstract

Parametric models of polymorphic lambda calculus have the structure of enriched categories with cotensors and ends in some generalized sense, and thus have many categorical data types induced by them. The ω-order minimum model is a parametric model.

1 Introduction

Higher order quantifier of polymorphic lambda calculus has several meanings. Two inventors of the calculus use different symbols. When Girard wrote $\bigwedge X.F(X)$ [10] ($\Pi X.F(X)$ in [12]), it corresponded to a higher order quantified formula $\forall X.F(X)$ via Curry-Howard isomorphism. When Reynolds wrote $\Delta X.F(X)$ [33], it was the type of polymorphism, especially of parametric polymorphism [34]. The third interpretation leaded by categorical semantics is that the quantified type, we write $\forall X.F(X)$, is a kind of limits. The notation $\Pi X.F(X)$ suggests that it might be regarded as a product of all $F(X)$ where X ranges over all types. That is to say, $\Pi X.F(X)$ is the collection of all sections (a section is a function sending a type X to a member of $F(X)$). For example, in per models (also called HEO [10]), $\Pi X.F(X)$ can be regarded as a product [25] that internally exists in a certain intuitionistic universe. But this aspect of quantified types is too extensive since they are allowed to contain sections that are gathers of programs having no interrelation, e.g., a family of binary functions that calculate $x + y$ if the type is `Int`, calculate $x - y$ if the type is `Real`, $\cdots$. Strachey called them ad hoc polymorphism, distinguished from parametric polymorphism. The latter is the name for families of programs such that the programs have strong uniformity, and relates to each other. Polymorphic lambda calculus is a mathematical formalization of the notion of parametric polymorphism.

Based on this consideration, the interpretation of $\forall X.F(X)$ may as well be more restrictive, collecting only those sections which have some structures modeling parametricity. For example, in some domain-theoretic models, [6], $\forall X.F(X)$ is the set of all continuous sections with respect to embedding-projections. Our choice is Reynolds parametricity [34]. It is a considerably strong kind of parametricity, because the uniformity it requires is the one with respect to all binary relations, not just with respect to embedding-projections.

Restricting $\forall X.F(X)$ to parametric polymorphism in Reynolds' sense, we obtain a desirable property. This time $\forall X.F(X)$ is not simply a product, but may be equalized much more. Then universal quantifier $\forall$ is regarded to be the limiting operator, $\underleftarrow{lim}$, in some extended sense, namely, the limit of cones which are but to a diagram of relations, not to a diagram of arrows as usual [13]. Compare it with Freyd adjoint functor theorem [27]. The theorem obtains an initial object by first making a product of a solution set and then collapsing it with an equalizer. Reynolds parametricity seems to have something to do with this collapsing procedure.

By the way logicians have long noticed that logical connectives are not independent. What we intend to say is not the distribution of negation in classical logic, as $\exists = \neg\forall\neg$. At the beginning of the twentieth century, Russell pointed out the disjunction $A \vee B$ is definable as

$$A \vee B = \forall X((A \Rightarrow X) \Rightarrow (B \Rightarrow X) \Rightarrow X)$$

where X ranges over propositions. Notably Prawitz proved [32] that, in intuitionistic second order logic, implication and universal quantification can define all other connectives. For example,

$$\begin{aligned} \exists X.F(X) &= \forall Y(\forall X(FX \Rightarrow Y) \Rightarrow Y) \\ A \wedge B &= \forall X((A \Rightarrow B \Rightarrow X) \Rightarrow X)\,. \end{aligned}$$

In [13] (also see [2]) we proved Reynolds parametricity is a necessary and sufficient condition for these logical representations to have natural categorical meanings. For example, $A \vee B$ above is a coproduct of A and B if it is parametric and the converse is true if the relations are restricted to the graphs of functions. Impredicativity of second order logic induces also inductive and coinductive types as initial and terminal fixed points. Although the first attempt of Reynolds to construct a parametric model in **Set** failed [35], now two models are known, that is, the parametric per model [2, 13] and the second order minimum model [29, 14].

The aim of this paper lies on the same line. In order to explain it, let us consider

$$\exists X.F(X) = \forall Y(\forall X(F(X) \Rightarrow Y) \Rightarrow Y)\,,$$

and ask the question what is the idea behind this. Since $\forall X(A \Rightarrow X)$ is regarded to be $\neg A$ intuitionistically, the representation of $\exists$ has similarity to $\neg\forall\neg$. But of course they are not equivalent in intuitionistic logic. It seems better to say that $\exists$ is a left adjoint of a substitution functor, following the traditional idea of hyperdoctrines.

In logic, existential quantifier is such that

$$\frac{A(X) \vdash B}{\exists X.A(X) \vdash B}$$

where X may occur in the upper sequent, but not in the lower sequent. Regarding $A(X)$ to be a function of X, and B in the upper sequent a constant function to B (write K_B), the rule above suggests that $\exists$ is a left adjoint of the functor that sends B to K_B. Since a left adjoint is a special case of right Kan extensions [27], we hope that $\exists X.F(X) = \forall Y(\forall X(FX \Rightarrow Y) \Rightarrow Y)$ may look as a right Kan extension, and indeed it does. The key is Kan theorem. We should consider a category $\mathbf{\Omega}^{\mathbf{\Omega}}$ ($\mathbf{\Omega}$ stands for the kind of type) such that such types with one parameter, as $F(X)$, are interpreted as its objects, and the homset $\mathbf{\Omega}^{\mathbf{\Omega}}(F, K_Y)$ is defined as $\forall X(FX \Rightarrow Y)$. But $\forall X(FX \Rightarrow Y)$ is a type, thus it is an object of $\mathbf{\Omega}$ rather than a hom'set'. So we arrive at an idea to use enriched category theory [19] where categories have hom-objects rather than homsets. In our case, the hom-objects should lie in $\mathbf{\Omega}$. Enriched category theory makes use of (co)tensor and (co)end. Our view is to regard $\forall$ and $\Rightarrow$ to be variations of end $\int_{\mathbf{\Omega}}$ and cotensor $\pitchfork_{\mathbf{\Omega}}$. Then $\exists X.F(X)$ can be written $\int_{Y\Omega} \mathbf{\Omega}^{\mathbf{\Omega}}(F, K_Y) \pitchfork_{\mathbf{\Omega}} Y$ which is a right Kan extension $\mathrm{Ran}_K 1_{\mathbf{\Omega}}$ by Kan theorem in enriched category theory [8], and is a left adjoint of K_{-}. Our aim is to investigate parametric models and pursue their categorical properties with the guide of enriched category theory.

The paper is formulated as follows. Section 2.1 gives a general framework, r-frame structure, to define the semantics of various polymorphic lambda calculi. Models of a system of polymorhic lambda calculus are r-frame structures admitting the interpretation of the system, as defined in Section 2.2. R-frame structures incorporate binary relations that allow us to associate parametricity to our models. Section 2.4 is a discussion how to provide enriched categorical structures with models. Our preference of enriched categories to ordinary categories is justified with the use of lambda calculus to denote the categorical structures.

In section 3.1, relational cotensor and relational ends are introduced, the latter of which may be regarded as an alternative presentation of parametricity. They yield right Kan extensions as in enriched category theory (Section 3.2) and thus give categorical meanings to some well-known logical expressions (Section 3.3). Likewise they are used to encode initial and terminal fixed points of endofunctors, as shown in Section 3.4. Putting these together, a parametric model yields a PL category (Section 3.5).

The ω-order minimum model presented in Section 4, is an example of parametric models that have all the categorical structures described in Section 3. This fact may be surprising, because the model is given no categorical structures a priori and even having direct products is not a matter of being straightforward.

2 Semantics of Polymorphic Lambda Calculus

2.1 R-frame Structure

An example of polymorphic program is

$$\texttt{head}_X : \texttt{List}(X) \Rightarrow X \,.$$

It is a function that returns the head of a finite list of type X. The program $\texttt{head}_X$ is *polymorphic* since it is defined for all type X. In addition, it is *parametric* in the sense of Strachey. Namely the definition of $\texttt{head}_X$ could be very uniform in the parameter X. It is not necessary to program (infinite number of) $\texttt{head}_X$ for X ranging over all types.

Polymorphic lambda calculus is the language that captures parametric polymorphism, and so polymorphic programs written in the language is highly uniform. Hence it is natural to think that semantics of polymorphic lambda calculus should contain somehow concepts of uniformity (= parametricity).

Reynolds' idea is to use binary relations [34]. Let r be a binary relation between the members of type X and those of type Y, that is, r is a subset of $X \times Y$. We can extend r to a binary relation between finite lists, $\texttt{List}(r) \subseteq \texttt{List}(X) \times \texttt{List}(Y)$ as $(\alpha, \beta) \in \texttt{List}(r)$ iff α and β have the same length, say $\alpha = \langle x_1, \ldots, x_n \rangle$ and $\beta = \langle y_1, \ldots, y_n \rangle$, and $(x_i, y_i) \in r$ for $i = 1, \ldots, n$. The parametricity of $\texttt{head}_X$ is then expressed in terms of binary relations. Supposed $(\alpha, \beta) \in \texttt{List}(r)$, their heads are related by r, i.e., $(\texttt{head}_X \alpha, \texttt{head}_Y \beta) \in r$. Reynolds abstraction theorem [34] asserts that programs written in polymorphic lambda calculus have this kind of stability with respect to binary relations. To state the theorem, the interpretation of types with parameters, such as $\texttt{List}(X)$, is so extended that they behave as maps on binary relations, as $\texttt{List}(r)$ in the example above.

For the purpose, we introduced [13] r-frames which are categories but without composition.

Definition 2.1 An *r-frame* $\mathbf{C}$ is a pair of two classes, the class of objects ($\mathrm{Obj}(\mathbf{C})$ or $|\mathbf{C}|$) and the class of relations ($\mathrm{Rel}(\mathbf{C})$), with three maps

$$\mathrm{Obj}(\mathbf{C}) \xrightarrow{id} \mathrm{Rel}(\mathbf{C}) \,, \quad \mathrm{Rel}(\mathbf{C}) \overset{d_0}{\underset{d_1}{\rightrightarrows}} \mathrm{Obj}(\mathbf{C})$$

such that $d_0 \circ id = 1$ and $d_1 \circ id = 1$. We write $r : A \multimap B$ for $d_0(r) = A$ and $d_1(r) = B$.

Morphisms of r-frames are similar to functors of categories except that composition does not matter.

Definition 2.2 An *r-frame morphism* F is a pair of functions of $\mathrm{Obj}(\mathbf{C}) \to \mathrm{Obj}(\mathbf{D})$ and of $\mathrm{Rel}(\mathbf{C}) \to \mathrm{Rel}(\mathbf{D})$, which are such that $F(r) : FA \multimap FB$ for $r : A \multimap B$ in $\mathrm{Rel}(\mathbf{C})$, and preserve id, that is, $F(id_A) = id_{FA}$.

In [13], r-frame morphisms are not required to preserve id. In this paper, however, only those preserving id appear. We note that relators [1] are the same as r-frame morphisms.

Definition 2.3 A *relation of r-frame morphisms*, $q : F \multimap G$ $(F, G : \mathbf{C} \rightrightarrows \mathbf{D})$, is a map from $\mathrm{Rel}(\mathbf{C})$ to $\mathrm{Rel}(\mathbf{D})$ such that $q(r) : FA \multimap GB$ for $r : A \multimap B$ in $\mathbf{C}$. An *identity relation* of r-frame morphisms, $id_F : F \multimap F$, is defined by $id_F = F$.

The last definition deviates from the analogy of category theory, that is, from the definition of natural transformations, and even from $\mathbf{C}_2$-transformation in relator theory [1]. So we should explain the motivation.

The category of small categories, **Cat**, is a cartesian closed category (shorthand ccc) where the exponentiation $\mathbf{D}^{\mathbf{C}}$ is a category of functors and natural transformations. Consider a binary functor $F : \mathbf{B} \times \mathbf{C} \to \mathbf{D}$. Its currification $\ulcorner F \urcorner : \mathbf{B} \to \mathbf{D}^{\mathbf{C}}$ is again a functor that sends an object B of the category $\mathbf{B}$ to a functor $F(B, -) \in |\mathbf{D}^{\mathbf{C}}|$. Also it must send an arrow $f : B \to B'$ of $\mathbf{B}$ to a natural transformation from $\mathbf{C}$ to $\mathbf{D}$, that is a map from $\mathrm{Obj}(\mathbf{C})$ to $\mathrm{Arr}(\mathbf{D})$. But $F(f, -)$ is rather a function from $\mathrm{Arr}(\mathbf{C})$ to $\mathrm{Arr}(\mathbf{D})$ as the arrow function of the functor F. The reason $F(f, -)$ is regarded to be a natural transformation is because F preserves composition. Namely $C \in |\mathbf{C}| \mapsto F(f, 1_C)$ is a natural transformation and $F(f, g)$ is determined by $F(f, 1_C) \circ F(1_B, g)$ where C is the codomain of g an arrow of $\mathbf{C}$.

In our case, composition is not involved. Therefore $F(r, -)$ should be a function from $\mathrm{Rel}(\mathbf{C})$ to $\mathrm{Rel}(\mathbf{D})$, rather than from $\mathrm{Obj}(\mathbf{C})$ to $\mathrm{Rel}(\mathbf{D})$. Hence to retain the ccc structure in the category of r-frames, relations of r-frame morphisms are defined as above.

We define the product of r-frames $\mathbf{A}$ and $\mathbf{B}$, $\mathbf{A} \times \mathbf{B}$, as $\mathrm{Obj}(\mathbf{A} \times \mathbf{B}) = \mathrm{Obj}(\mathbf{A}) \times \mathrm{Obj}(\mathbf{B})$ and $\mathrm{Rel}(\mathbf{A} \times \mathbf{B}) = \mathrm{Rel}(\mathbf{A}) \times \mathrm{Rel}(\mathbf{B})$ with obvious three maps, id, d_0, and d_1. We also define the product of 0 r-frame, i.e., a terminal object, $\mathbf{1}$, as the r-frame of one object and one (identity) relation, namely $\mathrm{Obj}(\mathbf{1}) = \{1\}$ and $\mathrm{Rel}(\mathbf{1}) = \{id_1 : 1 \multimap 1\}$.

In what follows, we provide the definition of r-frame structures that are general framework for defining models of various polymorphic lambda calculi, e.g. second order lambda calculus, and ω-order lambda calculus.

Definition 2.4 An *r-frame structure* is a category $\mathcal{R}f$ and its full subcategory $\mathcal{U}q$ satisfying the following conditions.

i) An object of $\mathcal{R}f$ is an r-frame and an arrow of $\mathcal{R}f$ is an r-frame morphism. $\mathcal{R}f$ has finite products, i.e., $\mathbf{1}$ and $\mathbf{A} \times \mathbf{B}$ for r-frames $\mathbf{A}$, $\mathbf{B}$ in $\mathcal{R}f$. Also $\mathcal{U}q$ has finite products.

ii) $\mathcal{R}f$ has a distinguished object $\mathbf{\Omega}$, and, for each r-frame $\mathbf{C}$ in the subcategory $\mathcal{U}q$, there is an exponentiation $\mathbf{\Omega}^{\mathbf{C}}$ such that the objects are r-frame morphisms from $\mathbf{C}$ to $\mathbf{\Omega}$ and the relations are their relations of r-frame morphisms. Here the exponentiation $\mathbf{\Omega}^{\mathbf{C}}$ means an object that induces an isomorphism $\mathcal{R}f(\mathbf{A} \times \mathbf{C}, \mathbf{\Omega}) \cong \mathcal{R}f(\mathbf{A}, \mathbf{\Omega}^{\mathbf{C}})$ natural in $\mathbf{A}$.

iii) There is a faithful r-frame morphism $D : \mathbf{\Omega} \to \mathbf{Rel}$ where $\mathbf{Rel}$ is the r-frame of all small sets and all set-theoretical binary relations (that is, $r : A \twoheadrightarrow B$ if $r \subseteq A \times B$). Here a faithful r-frame morphism means that the relation function of the r-frame morphism D is injective, analogously to faithful functors. We often identify the members of $\mathbf{\Omega}$ with their images of D, and write simply A for D_A and r for D_r.

iv) $\mathcal{R}f$ has r-frame morphisms

$$\begin{array}{ll} \Rightarrow : \mathbf{\Omega} \times \mathbf{\Omega} \to \mathbf{\Omega} & \text{and} \\ \forall^{\mathbf{C}} : \mathbf{\Omega}^{\mathbf{C}} \to \mathbf{\Omega} & \text{for each } \mathbf{C} \in |\mathcal{U}q|. \end{array}$$

Namely $\mathbf{C}$ is *universally quantifiable*. The symbol $\mathcal{U}q$ comes from this.

v) To $\Rightarrow$, *expansion functions* are associated:

$$\Phi^{\Rightarrow}_{A,B} : A \Rightarrow B \to (A \to B) \quad \text{for } A, B \in |\mathbf{\Omega}|$$

where $(A \to B)$ is the function space in the usual set-theoretical sense. (Here we write A for D_A, etc. So correctly, $\Phi^{\Rightarrow}_{A,B} : D_{A \Rightarrow B} \to (D_A \to D_B)$.) Also to $\forall^{\mathbf{C}}$, *expansion functions* are associated:

$$\Phi^{\forall^{\mathbf{C}}}_F : \forall^C F \to \Pi C \in |\mathbf{C}|.FC \quad \text{for } F \in |\mathbf{\Omega}^{\mathbf{C}}|$$

where $\Pi C \in |\mathbf{C}|.FC$ is the collection of $|\mathbf{C}|$-indexed families such that their components of index C are members of FC. We require $\Phi^{\Rightarrow}_{A,B}$ and $\Phi^{\forall^{\mathbf{C}}}_F$ are all one-to-one.

For a relation $r : A \twoheadrightarrow B$ of $\mathbf{\Omega}$, we use the notation $r : a \mapsto b$ for $(a, b) \in r$ (precisely $(a, b) \in D_r$).

In general, the distinguished r-frame $\mathbf{\Omega}$ characterizes the r-frame structure. So we let $\mathbf{\Omega}$ be the representative of the structure and say as an "r-frame structure $\mathbf{\Omega}$".

iii) in the definition above means $\mathbf{\Omega}$ is such that an object A of $|\mathbf{\Omega}|$ can be considered to be a set and a relation $r : A \twoheadrightarrow B$ to be a subset $r \subseteq A \times B$. And v) shows that the meaning of $A \Rightarrow B$ is a set of functions from A to B after expanded by $\Phi^{\Rightarrow}_{A,B}$, and that the meaning of $\forall^{\mathbf{C}} F$ is a set of $|\mathbf{C}|$-indexed families.

It is convenient to also impose the following condition to r-frame structures. For each $A \in |\mathbf{A}|$, there is an r-frame morphism $K_A : \mathbf{1} \to \mathbf{A}$ such that $K_A(1) = A$ and

$K_A(id_1) = id_A$. K_A is sometimes written simply A. Furthermore $\forall^1 : \mathbf{\Omega}^1 \to \mathbf{\Omega}$ should be the canonical isomorphism $\mathbf{\Omega}^1 \xrightarrow{\sim} \mathbf{\Omega}$.

The hypothesis that $\Phi^{\Rightarrow}_{A,B}$ is one-to-one implies that for $A \Rightarrow B$ extensional equality holds. In contrast, at the level of r-frames and r-frame morphisms, there may be many arrows of $\mathcal{R}f$ that are the same in their extensionality, i.e., denote the same r-frame morphisms. However, for the models in this paper, extensionality holds even at this level.

The existence of $\forall^{\mathbf{C}}$ means that variables of $\mathbf{C}$ can be universally quantified, and therefore by controlling the subcategory $\mathcal{U}q$ we have systems of various logical strength, as discussed later. Our main focus is on the systems such that $\mathbf{\Omega}$ belongs to $\mathcal{U}q$, as models of impredicative logics.

We are interested in some extra conditions.

Definition 2.5 (Naturality Condition)

i) The r-frame morphism $\Rightarrow: \mathbf{\Omega} \times \mathbf{\Omega} \to \mathbf{\Omega}$ is *natural* provided, given relations $r : A \rightarrowtriangle A'$ and $s : B \rightarrowtriangle B'$ in $\mathbf{\Omega}$, the relation $r \Rightarrow s : (A \Rightarrow B) \rightarrowtriangle (A' \Rightarrow B')$ is defined as $r \Rightarrow s : f \mapsto f'$ iff $s : \Phi^{\Rightarrow}_{A,B}(f)(a) \mapsto \Phi^{\Rightarrow}_{A',B'}(f')(a')$ for all a and a' such that $r : a \mapsto a'$.

ii) The r-frame morphism $\forall^{\mathbf{C}}$ is *natural* provided, given $q : F \rightarrowtriangle F'$ in $\mathbf{\Omega}^{\mathbf{C}}$, the relation $\forall^{\mathbf{C}} q : \forall^{\mathbf{C}} F \rightarrowtriangle \forall^{\mathbf{C}} F'$ is defined as $\forall^{\mathbf{C}} q : f \mapsto f'$ iff $q(r) : \Phi^{\forall^{\mathbf{C}}}_{F}(f)(C) \mapsto \Phi^{\forall^{\mathbf{C}}}_{F'}(f')(C')$ for all $r : C \rightarrowtriangle C'$ in $\mathbf{C}$.

iii) An r-frame structure is *natural* if all of $\Rightarrow$ and $\forall^{\mathbf{C}}$ ($\mathbf{C}$ ranges over $\mathcal{U}q$) are natural.

Naturality condition, roughly, means that the relation functions of $\Rightarrow$ and $\forall^{\mathbf{C}}$ determine a logical relation [28].

Definition 2.6 $\mathbf{\Omega}$ has *enough relations* iff there is a one-to-one function $|\,|^{\mathbf{\Omega}}$ assigning to each $f \in A \Rightarrow B$ a relation $|f|^{\mathbf{\Omega}} : A \rightarrowtriangle B$, called the *graph* of f, in such a way that $|f|^{\mathbf{\Omega}} : a \mapsto b$ iff $b = \Phi^{\Rightarrow}_{A,B}(f)(a)$. (We often omit the superscript of $|\,|^{\mathbf{\Omega}}$.)

The models that appear in this paper all satisfy the naturality condition and have enough relations. An important model that does neither satisfy the naturality condition nor have enough relations is the coherence space model of Girard [11, 12] (see [13]). In this paper, however, we assume these two conditions are always satisfied, without mentioning it.

2.2 Polymorphic Lambda Calculus

We are interested in the r-frame structures that model polymorphic lambda calculi defined below, since the calculi are attractive in both logical and computational aspects.

Logically they are isomorphic to the natural deduction proof system [32] of higher order logics via Curry-Howard isomorphism [12]. Computationally the term rewriting of $\beta(\eta)$-conversion is Church-Rosser and strongly normalizing.

The syntax of a polymorphic lambda calculus consists of three layers, i.e., kinds, constructors, and terms, and of two sorts of judgements, i.e., constructor judgements and term judgements. In the following definition, we fix an r-frame structure $\mathbf{\Omega}$.

A *constructor judgement* has the form

$$X_1 : \mathbf{C}_1, \ldots, X_m : \mathbf{C}_m \vdash \sigma : \mathbf{D}$$

where $\mathbf{C}_1, \ldots, \mathbf{C}_m, \mathbf{D}$ are r-frames of $\mathcal{R}f$. Following the convention, we call them *kinds*. σ is called a *constructor* of kind $\mathbf{C}$. Constructors of a special kind $\mathbf{\Omega}$ are called *types*. $X_1, \ldots, X_m$ are *constructor variables* of kinds $\mathbf{C}_1, \ldots, \mathbf{C}_m$. We use Γ as an abbreviation of the sequence of constructor variables $X_1 : \mathbf{C}_1, \ldots, X_m : \mathbf{C}_m$, etc. The inference rules of constructor judgements are:

(C proj) $\qquad \Gamma \vdash X : \mathbf{C} \qquad (X : \mathbf{C}$ appears in $\Gamma)$

(C const) $\qquad \dfrac{\Gamma \vdash \sigma_1 : \mathbf{D}_1 \quad \cdots \quad \Gamma \vdash \sigma_k : \mathbf{D}_k}{\Gamma \vdash F(\sigma_1, \ldots, \sigma_k) : \mathbf{D}}$

$(F : \mathbf{D}_1 \times \cdots \times \mathbf{D}_k \to \mathbf{D}$ is an r-frame morphism of $\mathcal{R}f)$

(C lambda) $\qquad \dfrac{\Gamma, X : \mathbf{C} \vdash \sigma : \mathbf{D}}{\Gamma \vdash \lambda X^{\mathbf{C}} \sigma : \mathbf{D}^{\mathbf{C}}} \qquad$ (if $\mathbf{D}^{\mathbf{C}}$ exists in $\mathcal{R}f$)

The rule (C const) includes the following rules:

(C $\Rightarrow$) $\qquad \dfrac{\Gamma \vdash \sigma : \mathbf{\Omega} \quad \Gamma \vdash \tau : \mathbf{\Omega}}{\Gamma \vdash \sigma \Rightarrow \tau : \mathbf{\Omega}}$

(C $\forall$) $\qquad \dfrac{\Gamma \vdash \sigma : \mathbf{\Omega}^{\mathbf{C}}}{\Gamma \vdash \forall^{\mathbf{C}} \sigma : \mathbf{\Omega}}$

(C eval) $\qquad \dfrac{\Gamma \vdash \sigma : \mathbf{D}^{\mathbf{C}} \quad \Gamma \vdash \tau : \mathbf{C}}{\Gamma \vdash \epsilon_{\mathbf{C},\mathbf{D}}(\sigma, \tau) : \mathbf{D}}$

where $\epsilon_{\mathbf{C},\mathbf{D}}$ is the counit of adjunction (on $\mathbf{D}$) of product and exponentiation, namely, is an evaluation such that $\epsilon_{\mathbf{C},\mathbf{D}}(F, C) = F(C)$ for objects and $\epsilon_{\mathbf{C},\mathbf{D}}(q, r) = q(r)$ for relations. We write $\sigma\tau$ for $\epsilon_{\mathbf{C},\mathbf{D}}(\sigma, \tau)$. As for $\forall$, we sometimes use the notation $\forall X^{\mathbf{C}} \sigma$ for $\forall^{\mathbf{C}}(\lambda X^{\mathbf{C}} \sigma)$.

We should regard $\Gamma = X_1 : \mathbf{C}_1, \ldots, X_m : \mathbf{C}_m$ to be an ordered list. However we will follow a rough way of thinking. Exchanges of the order are handled implicitly when necessary. Otherwise, if one is interested only in the syntax, he may as well regard Γ to be a set.

The $\beta\eta$-conversion is defined as usual:

$$(\beta) \qquad (\lambda X^{\mathbf{C}}\sigma)\tau \triangleright \sigma[X := \tau]$$

$$(\eta) \qquad \lambda X^{\mathbf{C}}\sigma X \triangleright \sigma \qquad (X \text{ does not occur in } \sigma).$$

The least congruence relation including $\beta\eta$-conversion is called *$\beta\eta$-equality*.

An constructor judgement is interpreted as an r-frame morphism of $\mathcal{R}f$:

$$[\![X_1 : \mathbf{C}_1, \ldots, X_m : \mathbf{C}_m \vdash \sigma : \mathbf{D}]\!] : \mathbf{C}_1 \times \cdots \times \mathbf{C}_m \to \mathbf{D}\,.$$

The interpretation is defined in an obvious manner. $[\![\Gamma \vdash X : \mathbf{C}]\!]$ is a projection; $[\![\Gamma \vdash F(\sigma_1, \ldots, \sigma_k) : \mathbf{D}]\!]$ is a composition of $\langle [\![\Gamma \vdash \sigma_1 : \mathbf{D}_1]\!], \ldots, [\![\Gamma \vdash \sigma_k : \mathbf{D}_k]\!] \rangle$ followed by F; $[\![\Gamma \vdash \lambda X^{\mathbf{C}}\sigma : \mathbf{D}^{\mathbf{C}}]\!]$ is the currification of $[\![\Gamma, X : \mathbf{C} \vdash \sigma : \mathbf{D}]\!]$. The interpretation respects $\beta\eta$-equality. It is well-known that a ccc models a simply typed lambda calculus [21]. Although $\mathcal{R}f$ may have only restricted exponentiation, a similar argument shows the soundness of $\beta\eta$.

A *term judgement* is of the form

$$X_1 : \mathbf{C}_1, \ldots, X_m : \mathbf{C}_m \;\; ; \;\; x_1 : \sigma_1, \ldots, x_n : \sigma_n \vdash M : \tau$$

where $\sigma_1, \ldots, \sigma_n, \tau$ are types such that

$$\Gamma \vdash \sigma_1 : \mathbf{\Omega} \quad , \ldots , \quad \Gamma \vdash \sigma_n : \mathbf{\Omega} \quad , \quad \Gamma \vdash \tau : \mathbf{\Omega}$$

($\Gamma = X_1 : \mathbf{C}_1, \ldots, X_m : \mathbf{C}_m$) are constructor judgements obtained by the inference rules above. M is called a *term*. $x_1, \ldots, x_n$ are *term variables*. We use Θ for $x_1 : \sigma_1, \ldots, x_n : \sigma_n$. The inference rules of term judgements are

(t proj) $\qquad \Gamma\,;\,\Theta \vdash x : \sigma \quad (x : \sigma \text{ occurs in } \Theta)$

$$(\text{t} \Rightarrow\text{I}) \qquad \frac{\Gamma\;;\;\Theta, x : \sigma \vdash M : \tau}{\Gamma\,;\,\Theta \vdash \lambda x^{\sigma} M : \sigma \Rightarrow \tau}$$

$$(\text{t} \Rightarrow\text{E}) \qquad \frac{\Gamma\,;\,\Theta \vdash M : \sigma \Rightarrow \tau \quad \Gamma\,;\,\Theta \vdash N : \sigma}{\Gamma\,;\,\Theta \vdash MN : \tau}$$

$$(\text{t}\ \forall\text{I}) \qquad \frac{\Gamma, X : \mathbf{C}\;;\;\Theta \vdash M : \sigma}{\Gamma\,;\,\Theta \vdash \Lambda X^{\mathbf{C}} M : \forall X^{\mathbf{C}}\sigma}$$

if $\mathbf{C} \in |\mathcal{U}q|$ and Θ contains no free X

$$\text{(t }\forall\text{E)} \qquad \frac{\Gamma\,;\,\Theta \vdash M : \forall^{\mathbf{C}}\sigma \quad \Gamma \vdash \tau : \mathbf{C}}{\Gamma\,;\,\Theta \vdash M\{\tau\} : \sigma\tau}$$

The $\beta\eta$-conversion for terms is

$$(\beta) \qquad (\lambda x^{\sigma} M)N \triangleright M[x := N]$$
$$(\Lambda X^{\mathbf{C}} M)\{\tau\} \triangleright M[X := \tau]$$
$$(\eta) \qquad \lambda x^{\sigma} M x \triangleright M$$
$$\Lambda X^{\mathbf{C}} M\{X\} \triangleright M$$

where in two η rules x and X do not occur in M respectively. *$\beta\eta$-equality* is the least congruence relation including $\beta\eta$-conversion.

As is well-known [10, 12, 23], the $\beta(\eta)$-conversion have Church-Rosser and strong normalizing properties. Our models are for $\beta\eta$. But the strong normalizability of β turns out to be useful at one point later.

Interpretation of term judgements is to fulfill

$$[\![X_1 : \mathbf{C}_1, \ldots, X_m : \mathbf{C}_m\,;\, x_1 : \sigma_1, \ldots, x_n : \sigma_n \vdash M : \tau]\!]$$
$$\in \Pi \underline{C \in |\mathbf{C}|}([\![\Gamma \vdash \sigma_1 : \mathbf{\Omega}]\!](\underline{C}) \times \cdots \times [\![\Gamma \vdash \sigma_n : \mathbf{\Omega}]\!](\underline{C}) \to [\![\Gamma \vdash \tau : \mathbf{\Omega}]\!](\underline{C}))$$

(here $\Gamma = X_1 : \mathbf{C}_1, \ldots, X_n : \mathbf{C}_n$ and we use underline to denote a sequence), namely the interpretation is a $|\mathbf{C}|$-indexed family of n-ary function. Interpretation of term judgements has a subtle nature because it is not defined inductively, in contrast to that of constructor judgements. Rather it is provided in the form of conditions the interpretation should satisfy [4]:

(t proj) $\quad [\![\Gamma\,;\,\Theta \vdash x : \sigma]\!](\underline{C})(\underline{a}) = a$

(t $\Rightarrow$I) $\quad \Phi^{\Rightarrow}_{A,B}(\,[\![\Gamma\,;\,\Theta \vdash \lambda x^{\sigma} M : \sigma \Rightarrow \tau]\!](\underline{C})(\underline{a})\,)(a')$
$= [\![\Gamma\,;\,\Theta, x : \sigma \vdash M : \tau]\!](\underline{C})(\underline{a}, a')$
where $A = [\![\Gamma \vdash \sigma : \Omega]\!](\underline{C})$ and $B = [\![\Gamma \vdash \tau : \mathbf{\Omega}]\!](\underline{C})$

(t $\Rightarrow$E) $\quad [\![\Gamma\,;\,\Theta \vdash MN : \tau]\!](\underline{C})(\underline{a})$
$= \Phi^{\Rightarrow}_{A,B}(\,[\![\Gamma\,;\,\Theta \vdash M : \sigma \Rightarrow \tau]\!](\underline{C})(\underline{a})\,)(\,[\![\Gamma\,;\,\Theta \vdash N : \sigma]\!](\underline{C})(\underline{a})\,)$
where $A = [\![\Gamma \vdash \sigma : \mathbf{\Omega}]\!](\underline{C})$ and $B = [\![\Gamma \vdash \tau : \mathbf{\Omega}]\!](\underline{C})$

(t $\forall$I) $\quad \Phi^{\forall^{\mathbf{C}}}_{F}(\,[\![\Gamma\,;\,\Theta \vdash \Lambda X^{\mathbf{C}} M : \forall X^{\mathbf{C}}\sigma]\!](\underline{C})(\underline{a})\,)(C')$
$= [\![\Gamma, X : \mathbf{C}\,;\,\Theta \vdash M : \sigma]\!](\underline{C}, C')(\underline{a})$

where $F = [\![\Gamma \vdash \lambda X^{\mathbf{C}}\sigma : \mathbf{\Omega}^{\mathbf{C}}]\!](\underline{C})$

(t ∀E) $[\![\Gamma \,;\, \Theta \vdash M\{\tau\} : \sigma\tau]\!](\underline{C})(\underline{a})$

$$= \Phi_F^{\forall^{\mathbf{C}}}(\,[\![\Gamma \,;\, \Theta \vdash M : \forall^{\mathbf{C}}\sigma]\!](\underline{C})(\underline{a})\,)([\![\Gamma \vdash \tau : \mathbf{C}]\!](\underline{C}))$$

where $F = [\![\Gamma \vdash \sigma : \mathbf{\Omega}^{\mathbf{C}}]\!](\underline{C})$

By the familiar technique [4], we can show that interpretation of terms, if any, respects $\beta\eta$-equality.

Definition 2.7 An r-frame structure $\mathbf{\Omega}$ is a *parametric model* or simply a *model* if there is associated an interpretation $[\![\,]\!]$ satisfying the condition above.

We say, for example, a "model $\mathbf{\Omega}$" when $[\![\,]\!]$ is clear from the context.

Known from the name *parametric* model, our model implicates the notion of parametricity. The key is that the r-frame morphisms $\forall^{\mathbf{C}} : \mathbf{\Omega}^{\mathbf{C}} \to \mathbf{\Omega}$ must preserve id. All other than $\forall^{\mathbf{C}}$, i.e., evaluation $\epsilon_{\mathbf{C},\mathbf{D}} : \mathbf{D}^{\mathbf{C}} \times \mathbf{C} \to \mathbf{D}$, currification $\ulcorner\,\urcorner$ (for $F : \mathbf{A} \times \mathbf{C} \to \mathbf{D}$, $\ulcorner F \urcorner : \mathbf{A} \to \mathbf{D}^{\mathbf{C}}$) and $\Rightarrow$, prove to preserve id from the conditions imposed on the r-frame structure. In fact, $\epsilon_{\mathbf{C},\mathbf{D}}$ and $\ulcorner F \urcorner$ preserve id evidently (if F preserves id). The r-frame morphism $\Rightarrow$ preserves id by its naturality condition and extensionality. However the situation of $\forall^{\mathbf{C}}$ is more ad hoc. From its naturality condition, we know $\forall^{\mathbf{C}} id_F \subseteq id_{\forall^{\mathbf{C}} F}$, but the converse does not always hold. The converse inclusion $id_{\forall^{\mathbf{C}} F} \subseteq \forall^{\mathbf{C}} id_F$ means that for all $a \in \forall^{\mathbf{C}} F$ there holds $F(r) : \Phi_F^{\forall^{\mathbf{C}}}(a)(C) \mapsto\!\!\!\!\rightarrow \Phi_F^{\forall^{\mathbf{C}}}(a)(C')$ for all $r : C \multimap C'$ in $\mathbf{C}$ (or using interpretation $[\![\,]\!]$, it is written $F(r) : [\![a\{C\}]\!] \mapsto\!\!\!\!\rightarrow [\![a\{C'\}]\!]$; more precisely $[\![a\{C\}]\!]$ should be $[\![X\!:\!\mathbf{C} \;;\; x\!:\!\forall^{\mathbf{C}} F \;\vdash\; x\{X\} \;:\; FX]\!](C)(a)$). So, given any relation $r : C \multimap C'$, the components of a of index C (i.e., $a\{C\}$) and of index C' (i.e., $a\{C'\}$) are related by $F(r)$. In the example at the beginning of this section

$$\texttt{List}(r) \Rightarrow r : \texttt{head}_C \mapsto\!\!\!\!\rightarrow \texttt{head}_{C'}\,.$$

Therefore one may say $a \in \forall^{\mathbf{C}} F$ is *parametric* if $F(r) : [\![a\{C\}]\!] \mapsto\!\!\!\!\rightarrow [\![a\{C'\}]\!]$ for all $r : C \multimap C'$ in $\mathbf{C}$, or equivalently $\forall^{\mathbf{C}} id_F : a \mapsto\!\!\!\!\rightarrow a$. The requirement that $\forall^{\mathbf{C}}$ preserves id thus implicates parametricity of a polymorphic program $a \in \forall^{\mathbf{C}} F$.

The following theorem [34, 43, 13] asserts that programs coded by polymorphic lambda calculus are all parametric in the sense above. See [26] for a categorical treatment of the theorem.

Theorem 2.8 (Abstraction Theorem) *Assume $\mathbf{\Omega}$ is a natural model. Suppose given a term judgement $\Gamma \,;\, \Theta \vdash M : \tau$ (let Γ be $X_1\!:\!\mathbf{C}_1, \ldots, X_m\!:\!\mathbf{C}_m$ and Θ $x_1\!:\!\sigma_1, \ldots, x_n\!:\!\sigma_n$), and relations $r_i : C_i \multimap C_i'$ in $\mathbf{C}_i$ ($i = 1, \ldots, m$). Then, for any $[\![\Gamma \vdash \sigma_j]\!](\underline{r}) : a_j \mapsto\!\!\!\!\rightarrow a_j'$ ($j = 1, \ldots, n$),*

$$[\![\Gamma \vdash \tau]\!](\underline{r}) : [\![\Gamma \,;\, \Theta \vdash M : \tau]\!](\underline{C})(\underline{a}) \mapsto\!\!\!\!\rightarrow [\![\Gamma \,;\, \Theta \vdash M : \tau]\!](\underline{C'})(\underline{a'})\,.$$

Applying Abstraction Theorem to closed terms of the form $; \vdash M : \forall X^{\mathbf{C}}\sigma$ we obtain $\forall^{\mathbf{C}} id_{[\![X:\mathbf{C}\vdash\sigma]\!]} : [\![M]\!] \mapsto [\![M]\!]$, that is, parametricity of $[\![M]\!]$.

The logical strength of polymorphic lambda calculus depends on the full subcategory $\mathcal{U}q$ of universally quantifiable r-frames.

Definition 2.9 i) The set of kinds for *ω-order lambda calculus* (λ^ω in symbol) is given by the following:

$$\kappa ::= \mathbf{\Omega} \mid \kappa^\kappa .$$

A model $\mathbf{\Omega}$ is a *model of ω-order lambda calculus* if all the kinds of ω-order lambda calculus belong to $\mathcal{U}q$.

ii) *Second order lambda calculus* (λ^2 in symbol) should have a kind $\mathbf{\Omega}$. A model $\mathbf{\Omega}$ is a *model of second order lambda calculus* if $\mathbf{\Omega}$ is an r-frame of $\mathcal{U}q$.

iii) Let $\mathbf{G}_\Omega$ be an r-subframe of $\mathbf{\Omega}$ such that $|\mathbf{G}_\Omega| = |\mathbf{\Omega}|$ and $r : A \twoheadrightarrow B$ belongs to $\mathbf{G}_\Omega$ iff it is a graph of some $f \in A \Rightarrow B$, i.e., $r = |f|$. A model is a *model of second order lambda calculus with basic comprehension* if $\mathbf{G}_\Omega$ belongs to $\mathcal{U}q$ [13].

In this presentation, for example, third order free variables may occur even in second order lambda calculus, though they can never be universally quantified. They are parameters at meta level, and are convenient in some cases.

2.3 Parametric Per Model

An example of models of ω-order lambda calculus is the parametric per model defined as follows ([2, 13] for the cases of second order and of second order with basic comprehension). The model **Per** is also natural and has enough relations.

The distinguished r-frame $\mathbf{\Omega}$ is **Per**. Here the objects of **Per** are partial equivalence relations (shorthand, per) over the set of natural numbers $\mathbf{N}$ (in general, over a partial combinatory algebra). A per is identified with its subquotient set, written $\mathbf{N}/A$ where A is the per. The relations of **Per** are binary relations of the subquotient sets, namely, $r : A \twoheadrightarrow B$ means $r \subseteq \mathbf{N}/A \times \mathbf{N}/B$.

The category $\mathcal{R}f$ is the collection of all small r-frames and all their r-frame morphisms. $\mathcal{U}q$ is equal to $\mathcal{R}f$, and therefore **Per** is a model of ω-order lambda calculus. Exponentiation $\mathbf{D}^{\mathbf{C}}$ is the r-frame of all r-frame morphism from $\mathbf{C}$ to $\mathbf{D}$ and their relations.

For pers A and B, a per $A \Rightarrow B$ is defined as $e\ (A \Rightarrow B)\ e'$ (e, e' are natural numbers) iff $\varphi_e(a)\ B\ \varphi_{e'}(a')$ for any natural numbers a and a' such that $a\ A\ a'$, where φ_e is the e-th partial recursive function. For relations r and s of **Per**, a relation $r \Rightarrow s$ is defined so that $\Rightarrow$ satisfies the naturality condition.

A per $\forall^{\mathbf{C}} F$, given an r-frame morphism $F : \mathbf{C} \to \mathbf{\Omega}$, is defined as $e\ (\forall^{\mathbf{C}} F)\ e'$ iff $F(r) : [e]_{FC} \mapsto [e']_{FC'}$ for all $r : C \twoheadrightarrow C'$ in $\mathbf{C}$ where $[e]_{FC}$ is the equivalence class of e

modulo the per FC. For a relation of r-frame morphisms, $q : F \Rightarrow G$, a relation $\forall^{\mathbf{C}} q$ is defined so that $\forall^{\mathbf{C}}$ satisfies the naturality condition.

The expansion functions $\Phi^{\Rightarrow}_{A,B}$ is given by $\Phi^{\Rightarrow}_{A,B}([e]_{A\Rightarrow B})([a]_A) = [\varphi_e(a)]_B$, and $\Phi^{\forall^{\mathbf{C}}}_F$ by $\Phi^{\forall^{\mathbf{C}}}_F([e]_{\forall^{\mathbf{C}} F})(C) = [e]_{FC}$.

It is not difficult to define interpretation $[\![\,]\!]$, noticing the set of natural numbers models untyped lambda calculus. See [2, 13] for the detail and other properties. The papers deal with second order calculi, but the extension to the general case is immediate.

2.4 Models to Categories

The author feels sorry to non-experts of category theory (thus to himself), since we must use rather advanced issues that are not found, for example, in Mac Lane's book [27], that is, indexed categories [30] and enriched categories [19] (there is a brief description for the latter).

The reason these notions are necessary is evident. Polymorphic lambda calculus can define types with free constructor variables. Hence types are stratified by the kinds of the variables they contain, and to each stratum is associated a category in which the types of the stratum are interpreted. It is the indexed category theory that provides a language to deal with a collection of categories. The reason to stick to the enriched category theory is that we want to use polymorphic lambda calculus as an internal language to describe categorical constructions.

Indeed we need both issues at the same time. In an idealized situation, an indexed category $\mathcal{G}$ over a category $\mathcal{S}$ is a functor $\mathcal{G} : \mathcal{S}^{op} \to \mathbf{Cat}$. This is the case when the canonical isomorphisms [30] associated to the indexed category are identities. Although in general $\mathcal{G}$ should be a pseudo-functor, all the indexed categories that appear in this paper have the form of functors. Let $\mathbf{\Omega}$-$\mathbf{Cat}$ be the category of $\mathbf{\Omega}$(-enriched) categories and $\mathbf{\Omega}$-functors (see below). Then we coin an *indexed* $\mathbf{\Omega}$*-category over* $\mathcal{S}$ for a functor from $\mathcal{S}^{op}$ to $\mathbf{\Omega}$-$\mathbf{Cat}$.

Suppose given a model $\mathbf{\Omega}$. We are interested in $\mathbf{\Omega}$-enriched categories (or $\mathbf{\Omega}$-categories). In usual, $\mathbf{\Omega}$ should be a monoidal category [19]. So some comments are necessary, since $\mathbf{\Omega}$ is not even a category at this moment, though shortly $\mathbf{\Omega}$ is given the structure of a category and later the cartesian closed structure if $\mathbf{\Omega}$ has enough relations.

Hereafter we omit the symbol $[\![\,]\!]$ for simplicity and let lambda terms denote their interpretations.

Definition 2.10 Let $\mathbf{\Omega}$ be a model. An $\mathbf{\Omega}$*-category* $\mathbf{C}$ is given by the following data: a class of objects $|\mathbf{C}|$ and an object of $\mathbf{\Omega}$, $\mathbf{C}(A, B)$, for each $A, B \in |\mathbf{C}|$, together with $1^{\mathbf{C}}_A \in \mathbf{C}(A, A)$ for each $A \in |\mathbf{C}|$ and $comp^{\mathbf{C}}_{A,B,C} \in \mathbf{C}(A, B) \Rightarrow \mathbf{C}(B, C) \Rightarrow \mathbf{C}(A, C)$ for each $A, B, C \in |\mathbf{C}|$, subject to equations

$$\begin{aligned}
&\lambda f^{\mathbf{C}(A,B)} comp^{\mathbf{C}}_{A,B,B}(f)(1^{\mathbf{C}}_B) = \lambda f^{\mathbf{C}(A,B)} f \\
&\lambda f^{\mathbf{C}(A,B)} comp^{\mathbf{C}}_{A,A,B}(1^{\mathbf{C}}_A)(f) = \lambda f^{\mathbf{C}(A,B)} f \\
&\lambda f^{\mathbf{C}(A,B)} \lambda g^{\mathbf{C}(B,C)} \lambda h^{\mathbf{C}(C,D)} comp^{\mathbf{C}}_{A,C,D}(comp^{\mathbf{C}}_{A,B,C}(f)(g))(h) \\
&\quad = \lambda f^{\mathbf{C}(A,B)} \lambda g^{\mathbf{C}(B,C)} \lambda h^{\mathbf{C}(C,D)} comp^{\mathbf{C}}_{A,B,D}(f)(comp^{\mathbf{C}}_{B,C,D}(g)(h))
\end{aligned}$$

(ordinary equations of identity and associativity of composition).

$\mathbf{\Omega}$ is itself an $\mathbf{\Omega}$-category by $\mathbf{\Omega}(A, B) = A \Rightarrow B$, and $1^{\mathbf{\Omega}}_A = \lambda x^A x$, $comp^{\mathbf{\Omega}}_{A,B,C} = \lambda f^{A \Rightarrow B} \lambda g^{B \Rightarrow C} \lambda x^A g(fx)$.

Definition 2.11 An $\mathbf{\Omega}$-*functor* $F : \mathbf{C} \to \mathbf{D}$ is a pair of an object function $F : |\mathbf{C}| \to |\mathbf{D}|$ and the *strength* $\bar{F}_{A,B} \in \mathbf{C}(A, B) \Rightarrow \mathbf{D}(FA, FB)$ for each $A, B \in |\mathbf{C}|$. The strength $\bar{F}$ should preserve identity and composition:

$$\begin{aligned}
&\bar{F}_{A,A}(1^{\mathbf{C}}_A) = 1^{\mathbf{D}}_{FA} \\
&\lambda f^{\mathbf{C}(A,B)} \lambda g^{\mathbf{C}(B,C)} \bar{F}_{A,C}(comp^{\mathbf{C}}_{A,B,C}(f)(g)) \\
&\quad = \lambda f^{\mathbf{C}(A,B)} \lambda g^{\mathbf{C}(B,C)} comp^{\mathbf{D}}_{FA,FB,FC}(\bar{F}_{A,B}(f))(\bar{F}_{B,C}(g)).
\end{aligned}$$

Ordinary categories are **Set**-enriched categories. Hence the enriched category theory may be said to be the theory that replaces sets by objects of another base category. However some aspects of sets sneak in. For example, consider the enriched version of natural transformations. An $\mathbf{\Omega}$-*natural transformation* [19] $\nu : F \to G$ $(F, G : \mathbf{C} \rightrightarrows \mathbf{D})$ is a $|\mathbf{C}|$-indexed *family* of arrows $\nu_C \in FC \to GC$ with ordinary equations, where the family means a set. The consequence is that, in order to make a functor category a $\mathbf{C}$-enriched category, we need small completeness of the underlying category of $\mathbf{C}$ [7, 19]. For the issue of this paper, it is undesirable to impose completeness. In the parametric per model, **Per** is small, thus it cannot be small complete, though it is internally complete in the effective topos [16, 17] and also complete in the sense [19] of enriched category theory.

The situation is better if the $\mathbf{\Omega}$-category is also a universally quantifiable r-frame. Then we can define a *universal* $\mathbf{\Omega}$-*category* $\mathbf{C}$ to be an $\mathbf{\Omega}$-category with identity and composition in a universally quantified form:

$$\begin{aligned}
&id^{\mathbf{C}} \in \forall X^{\mathbf{C}} \mathbf{C}(X, X) \\
&comp^{\mathbf{C}} \in \forall X, Y, Z^{\mathbf{C}} \mathbf{C}(X, Y) \Rightarrow \mathbf{C}(Y, Z) \Rightarrow \mathbf{C}(X, Z)
\end{aligned}$$

with obvious equations. If $\mathbf{\Omega}$ belongs to $\mathcal{U}q$, then $\mathbf{\Omega}$ is a universal $\mathbf{\Omega}$-category by abstracting identity and composition with capital lambda. A *universal* $\mathbf{\Omega}$*-functor* $F : \mathbf{C} \to \mathbf{D}$ is an $\mathbf{\Omega}$-functor with the *universal strength*

$$\bar{F} \in \forall X, Y^{\mathbf{C}} \mathbf{C}(X, Y) \Rightarrow \mathbf{D}(FX, FY).$$

Let $F, G : \mathbf{C} \rightrightarrows \mathbf{D}$ be universal $\mathbf{\Omega}$-functors. We mean by a *universal* $\mathbf{\Omega}$*-natural transformation* $\alpha : F \to G$ a member of $\forall X^{\mathbf{C}} \mathbf{D}(FX, GX)$ such that

$$\begin{aligned} &\Lambda X, Y^{\mathbf{C}} \lambda f^{\mathbf{C}(X,Y)} comp^{\mathbf{D}}_{FX,FY,GY}(\bar{F}_{X,Y} f)(\alpha\{Y\}) \\ &\quad = \Lambda X, Y^{\mathbf{C}} \lambda f^{\mathbf{C}(X,Y)} comp^{\mathbf{D}}_{FX,GX,GY}(\alpha\{X\})(\bar{G}_{X,Y} f) \end{aligned}$$

In fact, if the $\mathbf{\Omega}$-categories and $\mathbf{\Omega}$-functors satisfy some conditions, then any member turns out to be a universal $\mathbf{\Omega}$-natural transformation, as proved later. In that case, we can put a hom-object of a functor category, $\mathbf{D}^{\mathbf{C}}(F, G)$, to be $\forall X^{\mathbf{C}} \mathbf{D}(FX, GX)$. This means that, in place of ends utilized in enriched category theory, we can use universal quantifier $\forall^{\mathbf{C}}$.

Now we construct an indexed $\mathbf{\Omega}$-category from a model $\mathbf{\Omega}$.

Definition 2.12 Suppose given a model $\mathbf{\Omega}$. Then an indexed $\mathbf{\Omega}$-category, also denoted by $\mathbf{\Omega}$, is defined as follows.

i) $\mathbf{\Omega}$ is indexed over $\mathcal{U}q$.

ii) For each $\mathbf{C} \in |\mathcal{U}q|$, the category of $\mathbf{C}$-indexed families (i.e., $\mathbf{\Omega}(\mathbf{C})$ for the functor $\mathbf{\Omega} : \mathcal{U}q \to \mathbf{\Omega}\text{-}\mathbf{Cat}$) is defined as $|\mathbf{\Omega}(\mathbf{C})| = |\mathbf{\Omega}^{\mathbf{C}}|$; and $\mathbf{\Omega}(\mathbf{C})(F, G) = \forall X^{\mathbf{C}}(FX \Rightarrow GX)$ together with

$$\begin{aligned} 1^{\mathbf{\Omega}^C}_F &= \Lambda X^{\mathbf{C}} \lambda x^{FX} x \\ comp^{\mathbf{\Omega}^C}_{F,G,H} &= \lambda f^{\forall X(FX \Rightarrow GX)} \lambda g^{\forall X(GX \Rightarrow HX)} \Lambda X^{\mathbf{C}} \lambda x^{FX} g\{X\}(f\{X\}x) \end{aligned}$$

We usually abuse $\mathbf{\Omega}^{\mathbf{C}}$ for $\mathbf{\Omega}(\mathbf{C})$.

iii) For an r-frame morphism $K : \mathbf{D} \to \mathbf{C}$ in $\mathcal{U}q$, an $\mathbf{\Omega}$-functor $\mathbf{\Omega}^K : \mathbf{\Omega}^{\mathbf{C}} \to \mathbf{\Omega}^{\mathbf{D}}$ is defined as follows. For F an object of $\mathbf{\Omega}^{\mathbf{C}}$, $\mathbf{\Omega}^K(F)$ is defined to be $F \circ K$. The strength $\bar{\mathbf{\Omega}}^K_{F,G} \in \forall X^{\mathbf{C}}(FX \Rightarrow GX) \Rightarrow \forall Y^{\mathbf{D}}(FK(Y) \Rightarrow GK(Y))$ is put $\lambda x^{\forall X(FX \Rightarrow GX)} \Lambda Y^{\mathbf{D}} x\{KY\}$.

It is easy to show that $\mathbf{\Omega}$ is an indexed $\mathbf{\Omega}$-category.

Remark 2.13 Consider the parametric per model **Per**. A universal **Per**-category $\mathbf{C}$ is regarded to be an internal category (see for example [18]) of the effective topos [15]. For $C, D \in \mathbf{C}$, put

$$[\![C = D]\!] = \begin{cases} \mathbf{N} & \text{if } C = D \\ \phi & \text{otherwise.} \end{cases}$$

And for $f, g \in \mathbf{C}(C, D)$,

$$[\![f = g]\!] = \begin{cases} f & \text{if } f = g \\ \phi & \text{otherwise.} \end{cases}$$

Here f and g are equivalence classes of the per $\mathbf{C}(C, D)$, so $f = g$ means they are the same equivalence class. The internalization of **Per** itself is found in [16]. Moreover a universal **Per**-functor $F : \mathbf{C} \to \mathbf{D}$ is an internal functor, and vice versa. Therefore any functor expressible in intuitionistic set theory turns out to be a universal **Per**-functor.

3 Relational Limits

In this section we study categorical properties of the indexed category $\mathbf{\Omega}$, especially those involving parametricity.

3.1 Relational Cotensor and Relational End

In the last section, we showed $\mathbf{\Omega}$ and $\mathbf{\Omega}^{\mathbf{C}}$ has enriched category structure. Limits in enriched categories are described with cotensor and end [8], or equivalently with indexed limits [19]. We introduce relational cotensor and relational end, and from them derive a lot of categorical constructions, among which some are well-known in category and enriched category literatures, as Kan theorem, and others are characteristic to the structure modeling polymorphic lambda calculus, as fixed points of universal $\mathbf{\Omega}$-endofunctors.

The following definition is an abstraction of properties polymorphic lambda terms should have, and will appear repeatedly below.

Definition 3.1 Let F be an r-frame morphism from $\mathbf{C}$ to $\mathbf{\Omega}$. A family $\{a_C \in FC \,|\, C \in |\mathbf{C}|\}$ is a *parametric family* iff the following two conditions are satisfied:

i) $\{a_C\}$ is parametric in the sense: for all relation $r : C \multimap D$ in the r-frame $\mathbf{C}$, it holds that $Fr : a_C \mapsto a_D$.

ii) For every r-frame morphism $G : \mathbf{E} \to \mathbf{C}$ where the r-frame $\mathbf{E}$ is universally quantifiable, we have $\Lambda X^{\mathbf{E}}.\, a_{GX}$ as an element of $\forall X^{\mathbf{E}}\, (F \circ G)(X)$.

Note that if $\{a_C\}$ is obtained from $a \in \forall X^{\mathbf{C}} F X$ by $a_C = a\{C\}$ then the family $\{a_C\}$ is a parametric family. The following additional condition to $\mathbf{\Omega}$-categories arises naturally, for our models are based on r-frames.

Definition 3.2 An r-frame of $\mathcal{R}f$, $\mathbf{C}$, has a *relational hom* iff $\mathcal{R}f$ has an r-frame morphism $\mathbf{C}(-,-) : \mathbf{C} \times \mathbf{C} \to \mathbf{\Omega}$ that makes $\mathbf{C}$ an $\mathbf{\Omega}$-category, and furthermore identity and composition

$$1^{\mathbf{C}}_A \in \mathbf{C}(A,A)$$
$$comp^{\mathbf{C}}_{A,B,C} \in \mathbf{C}(A,B) \Rightarrow \mathbf{C}(B,C) \Rightarrow \mathbf{C}(A,C)$$

are parametric families.

The superscripts are sometimes omitted. $comp^{\mathbf{C}}_{A,B,C}(f)(g)$ is also written $g \circ_{\mathbf{C}} f$ or $f;_{\mathbf{C}}\, g$.

Definition 3.3 An r-frame $\mathbf{C}$ with a relational hom has *graph relations* iff both

i) There is a one-to-one map $|\,|^{\mathbf{C}}$ assigning a relation of $\mathbf{C}$, $|f|^{\mathbf{C}} : C \rightharpoonup C'$, to each $f \in \mathbf{C}(C, C')$. We call $|f|^{\mathbf{C}}$ the *graph* of f; and

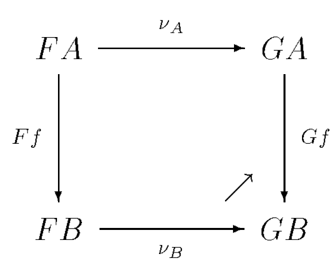
ii) $\mathbf{C}(-,-)$ and $|\,|^{\mathbf{C}}$ satisfy

$$\mathbf{C}(r, id_C); |\mathbf{C}(1_{D'}, f)| \subseteq \mathbf{C}(r, |f|^{\mathbf{C}}) \subseteq |\mathbf{C}(1_D, f)|; \mathbf{C}(r, id_{C'})$$
$$|\mathbf{C}(f, 1_D)|^{op}; \mathbf{C}(id_{C'}, r) \subseteq \mathbf{C}(|f|^{\mathbf{C}}, r) \subseteq \mathbf{C}(id_C, r); |\mathbf{C}(f, 1_{D'})|^{op}$$

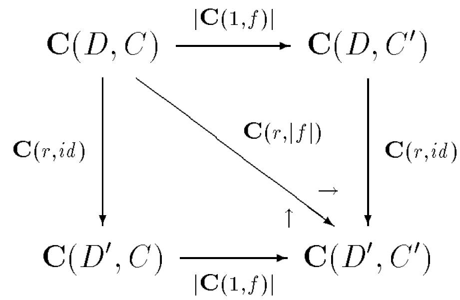
for any $f \in \mathbf{C}(C, C')$ and any $r : D \rightharpoonup D'$ in $\mathbf{C}$. (Notation: $r; s : A \rightharpoonup C$ (for $r : A \rightharpoonup B$ and $s : B \rightharpoonup C$ relations of $\mathbf{\Omega}$) is a diagramatic order composition of relations, i.e., $r; s : a \mapsto c$ iff there is a $b \in B$ such that $r : a \mapsto b$ and $s : b \mapsto c$ hold. $r^{op} : B \rightharpoonup A$ (for $r : A \rightharpoonup B$ a relation of $\mathbf{\Omega}$) is the opposite of r, i.e., $r^{op} : b \mapsto a$ iff $r : a \mapsto b$.)

The condition ii) includes the assumption that $\mathbf{\Omega}$ has the composition and the opposite displayed there.

To understand the inclusions of ii), the following consideration may do any help. The notion of lax natural transformation appears in dealing with relational algebras [5]. The difference from natural transformations, besides involving lax functors, is that the ordinary rectangle diagram



commutes only up to a 2-cell (not necessarily invertible), as exhibited in the diagram by the oblique arrow (this is a right lax natural transformation; a left lax natural transformation has a 2-cell in the converse direction [42]). The first expression of ii) may be pictured as



where all arrows should be read round-headed arrows that are not by a typographic reason, and the 2-cells, the small uparrow and the small rightarrow, are inclusions. This diagram asserts that $\mathbf{C}(r, |f|)$ is between two legs of the diagram of lax natural transformation. Therefore we call the condition ii) the *middle lax property* of $\mathbf{C}$. In the theory of relators [1], a notion similar to lax natural transformation appears.

The notion of graph relations is essential to develop categorical properties with relational limits, since we want to represent various diagrams by relations. A consequence of assuming for $\mathbf{C}$ to have graph relations is:

Proposition 3.4 *Suppose* $\mathbf{C}$ *has a relational hom with graph relations. For any* $f \in \mathbf{C}(C, C')$,

$$\begin{aligned} \mathbf{C}(id_D, |f|) &= |\mathbf{C}(1_D, f)| \\ \mathbf{C}(|f|, id_D) &= |\mathbf{C}(f, 1_D)|^{op} . \end{aligned}$$

Next we introduce relational cotensor and relational end.

Definition 3.5 Suppose $\mathbf{C}$ has a relational hom. A *relational cotensor* of $\mathbf{C}$ is an r-frame morphism of $\mathcal{R}f$, $\pitchfork_{\mathbf{C}}\colon \mathbf{\Omega} \times \mathbf{C} \to \mathbf{C}$, such that

$$A \Rightarrow \mathbf{C}(C, C') \cong \mathbf{C}(C, A \pitchfork_{\mathbf{C}} C')$$

where the isomorphism is given by parametric families. Namely there are two parametric families

$$\begin{aligned} i^{\pitchfork}_{A,C,C'} &\in (A \Rightarrow \mathbf{C}(C, C')) \Rightarrow \mathbf{C}(C, A \pitchfork_{\mathbf{C}} C') \\ {i^{\pitchfork}_{A,C,C'}}^{-1} &\in \mathbf{C}(C, A \pitchfork_{\mathbf{C}} C') \Rightarrow (A \Rightarrow \mathbf{C}(C, C')) \end{aligned}$$

such that ${i^{\pitchfork}_{A,C,C'}}^{-1} \circ_\Omega i^{\pitchfork}_{A,C,C'} = 1_{A \Rightarrow \mathbf{C}(C,C')}$ and $i^{\pitchfork}_{A,C,C'} \circ_\Omega {i^{\pitchfork}_{A,C,C'}}^{-1} = 1_{\mathbf{C}(C, A \pitchfork_{\mathbf{C}} C')}$ for any $A \in |\mathbf{\Omega}|$ and $C, C' \in |\mathbf{C}|$.

We say an r-frame $\mathbf{C}$ *has exponentiation w.r.t.* $\mathcal{U}q$ iff, for all $\mathbf{D}$ in $\mathcal{U}q$, there exists an exponentiation $\mathbf{C}^{\mathbf{D}}$. For example, $\mathbf{\Omega}$ and r-frames of the form $\mathbf{\Omega}^{\mathbf{C}}$ have exponentiation w.r.t. $\mathcal{U}q$ (for $\mathbf{\Omega}^{\mathbf{C}}$ put $(\mathbf{\Omega}^{\mathbf{C}})^{\mathbf{D}} = \mathbf{\Omega}^{\mathbf{D}\times\mathbf{C}}$). If $\mathbf{C}$ has exponentiation w.r.t. $\mathcal{U}q$, $\mathbf{C}^{\mathbf{D}}$ has a relational hom defined by $\mathbf{C}^{\mathbf{D}}(F,G) = \forall D^{\mathbf{D}}\mathbf{C}(FD, GD)$.

Definition 3.6 Suppose $\mathbf{C}$ has a relational hom and has exponentiation w.r.t. $\mathcal{U}q$. Also suppose $\mathbf{D}$ belongs to $\mathcal{U}q$. A *relational end* of $\mathbf{D}$ in $\mathbf{C}$ is an r-frame morphism of $\mathcal{R}f$, $\int_{\mathbf{D}} : \mathbf{C}^{\mathbf{D}} \to \mathbf{C}$, such that there is an isomorphism given by parametric families

$$\mathbf{C}^{\mathbf{D}}(C, F(-)) \cong \mathbf{C}(C, \textstyle\int_{\mathbf{D}} F)\,,$$

namely there are two parametric families

$$\begin{aligned} i^{\int}_{C,F} &\in \mathbf{C}^{\mathbf{D}}(C, F(-)) \Rightarrow \mathbf{C}(C, \textstyle\int_{\mathbf{D}} F) \\ {i^{\int}_{C,F}}^{-1} &\in \mathbf{C}(C, \textstyle\int_{\mathbf{D}} F) \Rightarrow \mathbf{C}^{\mathbf{D}}(C, F(-)) \end{aligned}$$

which are mutually inverse for any C and F.

Observe that an enriched category which has both cotensor and (small) ends is called small complete [19]. So the requirement of existence of relational cotensor and relational end is very strong.

The behaviour of relational cotensor and relational end as $\mathbf{\Omega}$-functors are defined with the isomorphisms $i^{\pitchfork}$ and $i^{\int}$. For example, $A \pitchfork_{\mathbf{C}} f \in \mathbf{\Omega}(A \pitchfork_{\mathbf{C}} C, A \pitchfork_{\mathbf{C}} C')$ (for $f \in \mathbf{C}(C, C')$) is defined as

$$A \pitchfork_{\mathbf{C}} f = i^{\pitchfork}_{A, A\pitchfork C, C'}(\lambda x^A f \circ_{\mathbf{C}} ({i^{\pitchfork}_{A, A\pitchfork C, C}}^{-1}(1_{A\pitchfork C})x))\,.$$

If an r-frame has graph relations, the arrows are embedded into the relations by their graphs. Therefore we want the functor operating on arrows is compatible with the r-frame morphism operating on relations with respect to the graph:

Proposition 3.7 *i) Suppose* $\mathbf{C}$ *has a relational hom with graph relations, and has a relational cotensor. For* $f \in A' \Rightarrow A$ *and* $g \in \mathbf{C}(C, C')$*,*

$$\begin{aligned} \mathbf{C}(D, |f| \pitchfork C) &= |\mathbf{C}(D, f \pitchfork C)|^{op} \,:\, \mathbf{C}(D, A \pitchfork C) \twoheadrightarrow \mathbf{C}(D, A' \pitchfork C) \\ \mathbf{C}(D, A \pitchfork |g|) &= |\mathbf{C}(D, A \pitchfork g)| \,:\, \mathbf{C}(D, A \pitchfork C) \twoheadrightarrow \mathbf{C}(D, A \pitchfork C') \end{aligned}$$

for any $D \in |\mathbf{C}|$*.*

ii) Suppose $\mathbf{C}$ *and* $\mathbf{C}^{\mathbf{D}}$ *have relational homs with graph relations, and there is a relational end* $\int_{\mathbf{D}}$*. For* $h \in \mathbf{C}^{\mathbf{D}}(F, F')$*,*

$$\mathbf{C}(D, \textstyle\int_{\mathbf{D}} |h|) = |\mathbf{C}(D, \textstyle\int_{\mathbf{D}} h)| \,:\, \mathbf{C}(D, \textstyle\int_{\mathbf{D}} F) \twoheadrightarrow \mathbf{C}(D, \textstyle\int_{\mathbf{D}} F')$$

for any $D \in |\mathbf{C}|$*.*

(*Proof*) We prove $\mathbf{C}(id_D, A \pitchfork |g|) = |\mathbf{C}(D, A \pitchfork g)|$. First

$$\begin{array}{ccc}
\mathbf{C}(D, A \pitchfork C) & \xrightarrow{i^{\pitchfork -1}} & A \Rightarrow \mathbf{C}(D, C) \\
\Big\downarrow {\scriptstyle \mathbf{C}(1, A \pitchfork g)} & & \Big\downarrow {\scriptstyle A \Rightarrow \mathbf{C}(D, g)} \\
\mathbf{C}(D, A \pitchfork C') & \xrightarrow[i^{\pitchfork -1}]{} & A \Rightarrow \mathbf{C}(D, C')
\end{array}$$

commutes by parametricity of $i^{\pitchfork}_{A,C,D}$ in D. Thus $\mathbf{C}(id_D, A \pitchfork |g|) = |\mathbf{C}(1_D, A \pitchfork g)|$ iff $A \Rightarrow \mathbf{C}(id_D, |g|) = |A \Rightarrow \mathbf{C}(1_D, g)|$, by parametricity of $i^{\pitchfork}_{A,C,D}$ in C. The last equation is obtained from the hypothesis that $\mathbf{C}$ has graph relations (middle lax property). □

R-frames of the form $\mathbf{\Omega}^{\mathbf{C}}$ are examples of those which have relational homs, relational cotensors, and relational ends:

Proposition 3.8 *Suppose* $\mathbf{C}$ *and* $\mathbf{D}$ *be r-frames of* $\mathcal{U}q$.
i) $\mathbf{\Omega}^{\mathbf{C}}$ *has a relational hom.*
ii) $\mathbf{\Omega}^{\mathbf{C}}$ *has a relational cotensor.*
iii) A relational end of $\mathbf{D}$ *in* $\mathbf{\Omega}^{\mathbf{C}}$ *exists.*

(*Proof*) i) is discussed earlier. For ii) and iii), put

$$\begin{aligned} A \pitchfork_{\Omega^{\mathbf{C}}} F &= \lambda X^{\mathbf{C}}(A \Rightarrow FX) \\ \textstyle\int_{\mathbf{D}} F &= \lambda X^{\mathbf{C}} \forall D^{\mathbf{D}} FDX \,. \end{aligned}$$

The required isomorphisms are easily written in polymorphic lambda calculus and thus are parametric families. □

How about the condition of graph relations? This is the most subtle point of our formulation. Consider $f \in \mathbf{\Omega}(F, F')$. We must define its graph $|f| : F \rightharpoondown F'$, which is a relation of r-frame morphisms F and F', and thus the application of $r : C \rightharpoondown C'$ to $|f|$ should be such that $|f|(r) : FC \rightharpoondown FC'$. There are two natural ways to define the $|f|$, namely

$$\begin{aligned} |f|(r) &\stackrel{def}{=} |f\{C\}|; F'(r) \quad \text{and} \\ |f|(r) &\stackrel{def}{=} F(r); |f\{C'\}| \,. \end{aligned}$$

First of all, however, it is not sure that these $|f|$ can be defined (that is, compositions of the right hand side exist in $\mathbf{\Omega}$) and that $|f|$ belong to Rel($\mathbf{\Omega}^{\mathbf{C}}$). We must check it each time when defining a model. In our examples, i.e., the parametric per models and the ω-order minimum model, the condition evidently holds since $\mathbf{\Omega}$ includes all binary relations and $\mathbf{\Omega}^{\mathbf{C}}$ all relations of r-frame morphisms.

As a definition of $|f|$, we can adopt any of the above two, as far as it is well-defined. In fact, any one-to-one map $|\,|$ such that

$$F(r);|f\{C'\}| \subseteq |f|(r) \subseteq |f\{C\}|;F'(r)$$

should work as well. We decide to use $|f|(r) \stackrel{def}{=} |f\{C\}|;F'(r)$. It is one-to-one, since if $|f|(r) = |g|(r)$ for all relation r of $\mathbf{C}$, then $|f|(id_C) = |g|(id_C)$, i.e., $|f\{C\}| = |g\{C\}|$ for all object C of $\mathbf{C}$. Since $|\,|$ of $\mathbf{\Omega}$ and $\Phi_F^{\forall^{\mathbf{C}}}$ are injective, we can induce $f = g$.

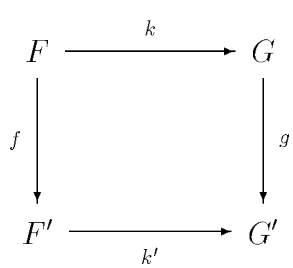
Proposition 3.9 *If there exists* $|\,|^{\mathbf{\Omega}^{\mathbf{C}}}$ *defined by* $|f|(r : C \multimap C') = |f\{C\}|;F'(r)$*, then* $\mathbf{\Omega}^{\mathbf{C}}$ *has graph relations.*

(*Proof*) As mentioned above, $|\,|$ is one-to-one. We must check the middle lax property

$$\mathbf{\Omega}^{\mathbf{C}}(q, id_F);|\mathbf{\Omega}^{\mathbf{C}}(1_{G'}, f)| \subseteq \mathbf{\Omega}^{\mathbf{C}}(q, |f|) \subseteq |\mathbf{\Omega}^{\mathbf{C}}(1_G, f)|;\mathbf{\Omega}^{\mathbf{C}}(q, id_{F'})$$
$$|\mathbf{\Omega}^{\mathbf{C}}(f, 1_G)|^{op};\mathbf{\Omega}^{\mathbf{C}}(id_{F'}, q) \subseteq \mathbf{\Omega}^{\mathbf{C}}(|f|, q) \subseteq \mathbf{\Omega}^{\mathbf{C}}(id_F, q);|\mathbf{\Omega}^{\mathbf{C}}(f, 1_{G'})|^{op}$$

for $f \in \mathbf{\Omega}^{\mathbf{C}}(F, F')$ and $q : G \multimap G'$ of $\mathbf{\Omega}^{\mathbf{C}}$. We left the detail to the reader. For example, the left inclusion of the first line is obtained from $F(r);|f\{C'\}| \subseteq |f\{C\}|;F'(r)$ where this inclusion is an immediate result of parametricity of f. Notice that the right inclusion of the first line is actually equality. $\Box$

Remark 3.10 The equality $\mathbf{\Omega}^{\mathbf{C}}(q, |f|) = |\mathbf{\Omega}^{\mathbf{C}}(1_G, f)|;\mathbf{\Omega}^{\mathbf{C}}(q, id_{F'})$ pointed out in the proof above implies that, for $f \in \mathbf{\Omega}^{\mathbf{C}}(F, F')$ and $g \in \mathbf{\Omega}^{\mathbf{C}}(G, G')$, $\mathbf{\Omega}^{\mathbf{C}}(|f|, |g|) : k \mapsto k'$ iff



commutes in $\mathbf{\Omega}^{\mathbf{C}}$. This equivalence is necessary to discuss fixed points of universal strong endofunctors later.

3.2 Right Kan Extension

In this section, we prove Kan theorem for right Kan extension. It is almost the reconstruction of the proof in enriched category theory [8].

Suppose given three r-frames $\mathbf{C}$, $\mathbf{D}$, and $\mathbf{E}$ where $\mathbf{D}$ and $\mathbf{E}$ are universally quantifiable. Also suppose that $\mathbf{C}$ has exponentiation w.r.t. $\mathcal{U}q$, and that $\mathbf{C}$, $\mathbf{E}$, $\mathbf{C}^{\mathbf{D}}$ and $\mathbf{C}^{\mathbf{E}}$ have relational homs with graph relations. Note that we can take $\mathbf{\Omega}^{\mathbf{C}}$ as $\mathbf{C}$ if it has graph relations.

Definition 3.11 $\mathbf{Us}(\mathbf{C}^{\mathbf{E}})$ is a full $\mathbf{\Omega}$-subcategory of $\mathbf{C}^{\mathbf{E}}$ such that F is an object of $\mathbf{Us}(\mathbf{C}^{\mathbf{E}})$ iff F has a universal strength

$$\bar{F} \in \forall E, E'^{\mathbf{E}}\mathbf{E}(E, E') \Rightarrow \mathbf{C}(FE, FE')$$

and satisfies

$$\mathbf{C}(r, id_{FE}); |\mathbf{C}(1_{D'}, \bar{F}g)| \subseteq \mathbf{C}(r, F|g|) \subseteq |\mathbf{C}(1_D, \bar{F}g)|; \mathbf{C}(r, id_{FE'})$$
$$|\mathbf{C}(\bar{F}g, 1_D)|^{op}; \mathbf{C}(id_{FE'}, r) \subseteq \mathbf{C}(F|g|, r) \subseteq \mathbf{C}(id_{FE}, r); |\mathbf{C}(\bar{F}g, 1_{D'})|^{op}$$

for all $g \in \mathbf{E}(E, E')$ and $r : D \rightharpoonup D'$ of $\mathbf{C}$.

We call the inclusions in the definition above *middle lax property* of F. Note that, since $\mathbf{Us}(\mathbf{C}^{\mathbf{E}})$ is full, $\mathbf{Us}(\mathbf{C}^{\mathbf{E}})(F, G) = \mathbf{C}^{\mathbf{E}}(F, G)$. The subcategory $\mathbf{Us}(\mathbf{C}^{\mathrm{E}})$ is meaningful since its morphisms are universal $\mathbf{\Omega}$-natural transformations:

Proposition 3.12 *Every $f \in \mathbf{Us}(\mathbf{C}^{\mathbf{E}})(F, G)$ gives a universal $\mathbf{\Omega}$-natural transformation from F to G.*

(*Proof*) For $g \in \mathbf{E}(E, E')$, it holds that $\mathbf{C}(F|g|, G|g|) : f\{E\} \mapsto f\{E'\}$, by parametricity of f. The middle lax property implies

$$\begin{aligned}\mathbf{C}(F|g|, G|g|) &\subseteq |\mathbf{C}(1_{FE}, \bar{G}g)|; \mathbf{C}(F|g|, id_{GE'}) \\ &= |\mathbf{C}(1_{FE}, \bar{G}g)|; |\mathbf{C}(\bar{F}g, 1_{GE'})|^{op}.\end{aligned}$$

Therefore

$$\begin{array}{ccc} FE & \xrightarrow{f\{E\}} & GE \\ {\scriptstyle Fg}\downarrow & & \downarrow{\scriptstyle Gg} \\ FE' & \xrightarrow[f\{E'\}]{} & GE' \end{array} \qquad \text{commutes.}$$

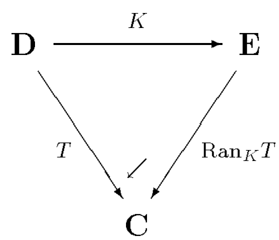
□

Definition 3.13 Let $\mathbf{C}$, $\mathbf{D}$, and $\mathbf{E}$ satisfy the conditions at the beginning of this section. Suppose given two r-frame morphisms $K : \mathbf{D} \to \mathbf{E}$ and $T : \mathbf{D} \to \mathbf{C}$, a *right Kan extension* of T along K is an object of $\mathbf{Us}(\mathbf{C}^{\mathrm{E}})$, denoted by $\mathrm{Ran}_K T$, such that there is an $\mathbf{\Omega}$-isomorphism

$$\mathbf{C}^{\mathbf{D}}(FK, T) \cong \mathbf{Us}(\mathbf{C}^{\mathbf{E}})(F, \mathrm{Ran}_K T)$$

$\mathbf{\Omega}$-natural in F.

The situation may be illustrated as



where the oblique small arrow is the counit $\epsilon \in \mathbf{C}^{\mathbf{D}}(\mathrm{Ran}_K T \circ K, T)$ obtained from the equivalence.

This definition differs with that of [8] in respect of the condition that, to disadvantage, the functors must have universal strengths rather than just strengths, and that, to advantage, K and T are not necessarily functorial. In fact, in the proof of Kan theorem below, we make no use that K and T are functors. An example where T is not functorial is provided later.

Theorem 3.14 (Kan Theorem) *Suppose* $\mathbf{C}$*,* $\mathbf{D}$*, and* $\mathbf{E}$ *fulfill the conditions at the beginning of section 3.2, and* $K : \mathbf{D} \to \mathbf{E}$ *and* $T : \mathbf{D} \to \mathbf{C}$ *are given. Also suppose a relational cotensor* $\pitchfork_{\mathbf{C}}: \mathbf{\Omega} \times \mathbf{C} \to \mathbf{C}$ *and a relational end* $\int_{\mathbf{D}} : \mathbf{C}^{\mathbf{D}} \to \mathbf{C}$ *exist. Then*

$$\mathrm{Ran}_K T \overset{def}{=} \int_{X^{\mathrm{D}}} \mathbf{E}(-, KX) \pitchfork_{\mathbf{C}} TX$$

is a right Kan extension of T *along* K *(notation:* $\int_{X^{\mathrm{D}}} FX = \int_{\mathbf{D}} F$*).*

The proof is similar to those in [27, 8].
(*Proof*) First we prove

$$\forall E^{\mathbf{E}}(\mathbf{E}(E, KD) \Rightarrow \mathbf{C}(FE, TD)) \underset{j_D}{\cong} \mathbf{C}(FKD, TD).$$

The isomorphism j is defined by parametric families

$$j_D = \lambda x^{\forall E(\mathbf{E}(E,KD)\Rightarrow\mathbf{C}(FE,KD))} x\{KD\} 1_{KD}$$
$$j_D^{-1} = \lambda y^{\mathbf{C}(FKD,TD)} \Lambda E^{\mathbf{E}} \lambda g^{\mathbf{E}(E,KD)} \bar{F}_{E,KD}(g);_{\mathbf{C}}\, y\,.$$

$j_D \circ j_D^{-1} = 1$ is immediate. As for $j_D^{-1} \circ j_D = 1$, use the commutative diagram

$$\begin{array}{ccc}
\mathbf{E}(KD,KD) & \xrightarrow{x_{KD}} & \mathbf{C}(FKD,TD) \\
{\scriptstyle \mathbf{E}(g,1)}\downarrow & & \downarrow{\scriptstyle \mathbf{C}(Fg,1)} \\
\mathbf{E}(E,KD) & \xrightarrow[x_E]{} & \mathbf{C}(FE,TD)
\end{array}$$

for $x \in \forall E^{\mathbf{E}}(\mathbf{E}(E,KD)\Rightarrow\mathbf{C}(FE,KD))$, which diagram is obtained from parametricity of x and the middle lax property of F.

Then

$$\begin{aligned}
\forall E^{\mathbf{E}}\mathbf{C}(FE, \mathrm{Ran}_K T(E)) &\cong \forall E^{\mathbf{E}}\forall D^{\mathbf{D}}(\mathbf{E}(E,KD)\Rightarrow\mathbf{C}(FE,TD)) \\
&\cong \forall D^{\mathbf{D}}\forall E^{\mathbf{E}}(\mathbf{E}(E,KD)\Rightarrow\mathbf{C}(FE,TD)) \\
&\cong \forall D^{\mathbf{D}}\mathbf{C}(FKD,TD)\,.
\end{aligned}$$

The isomorphisms are all written in polymorphic lambda calculus together with isomorphisms of relational cotensor and relational end, so are parametric in F. Then, by the middle lax property of $\mathbf{C}^{\mathbf{D}}$ and $\mathbf{C}^{\mathbf{E}}$, we have $\mathbf{\Omega}$-naturality. $\square$

3.3 Right Kan Extension to Others

From right Kan extension, we obtain many categorical constructions. First of all, a left adjoint is a special case of a right Kan extension, as is well-known. As usual, we say $\mathrm{Ran}_K T \in |\mathbf{Us}(\mathbf{C}^{\mathbf{E}})|$ is *preserved* by a universal $\mathbf{\Omega}$-functor $G : C \to C'$ if $G \circ \mathrm{Ran}_K T$ with induced counit $G\epsilon$ is a right Kan extension of $G \circ T$ along K.

Theorem 3.15 *Suppose* $\mathbf{C}$ *and* $\mathbf{D}$ *of* $\mathcal{U}q$ *have relational homs with graph relations and have exponentiation w.r.t.* $\mathcal{U}q$*. Also suppose* $G : \mathbf{D} \to \mathbf{C}$ *is a universal* $\mathbf{\Omega}$*-functor with middle lax property. Then* $F = \mathrm{Ran}_G 1_{\mathbf{D}}$ *is a left adjoint of* G *iff* F *is preserved by* G*.*

The isomorphism $\mathbf{D}(FC, D) \cong \mathbf{C}(C, GD)$ is given by parametric families and is $\mathbf{\Omega}$-natural. This theorem opens a way to define tensor and coend.

Corollary 3.16 *A* tensor*,* $\otimes_{\mathbf{C}} : \mathbf{\Omega} \times \mathbf{C} \to \mathbf{C}$*, is defined by* $A \otimes_{\mathbf{C}} C = \mathrm{Ran}_{\mathbf{C}(C,-)} 1_{\mathbf{C}}(A)$ *if the right hand side exists. Namely,*

$$A \otimes_{\mathbf{C}} C = \int_{X^{\mathbf{C}}} (A \Rightarrow \mathbf{C}(C, X)) \pitchfork_{\mathbf{C}} X$$

where an isomorphism

$$\mathbf{C}(A \otimes_{\mathbf{C}} C, D) \cong A \Rightarrow \mathbf{C}(C, D)$$

is $\mathbf{\Omega}$*-natural in* A*,* C*, and* D*.*

In particular, when $\mathbf{C}$ is $\mathbf{\Omega}$, the tensor $A \otimes_{\mathbf{\Omega}} C \cong \forall X^{\mathbf{\Omega}}((A \Rightarrow C \Rightarrow X) \Rightarrow X)$ is a cartesian product $A \times C$ [13].

Corollary 3.17 *A* coend *of* $\mathbf{D}$ *in* $\mathbf{C}$*,* $\int^{\mathbf{D}} : \mathbf{C}^{\mathbf{D}} \to \mathbf{C}$*, is defined by* $\int^{\mathbf{D}} = \mathrm{Ran}_{I_{\mathbf{D}}} 1_{\mathbf{C}}$ *if the right hand side exists (here* $I_{\mathbf{D}} : \mathbf{C} \to \mathbf{C}^{\mathbf{D}}$ *is an embedding* $I_{\mathbf{D}}(C) = \lambda D^{\mathbf{D}} C$*). Namely*

$$\int^{\mathbf{D}} F = \int_{X^{\mathbf{C}}} (\forall Y^{\mathbf{D}} \mathbf{C}(FY, X)) \pitchfork_{\mathbf{C}} X$$

where an isomorphism

$$\mathbf{C}(\textstyle\int^{\mathbf{D}} F, C) \cong \mathbf{C}^{\mathbf{D}}(F, I_{\mathbf{D}} C)$$

is $\mathbf{\Omega}$*-natural in* F *and* C*.*

In particular, if $\mathbf{C}$ is $\mathbf{\Omega}$, we have a well-known encoding of existential quantifier [32],

$$\exists^{\mathbf{D}} F = \forall X^{\mathbf{\Omega}} (\forall Y^{\mathbf{D}} (FY \Rightarrow X) \Rightarrow X) .$$

The construction of coend may be compared with the following situation. Suppose $\mathbf{C}$ is a small complete small category in some (intuitionistic) universe (see [16, 17] for the existence). As pointed out in [16], the completeness implies cocompleteness, since a colimit is expressed as a limit of all cones. The representation of $\int^{\mathbf{D}}$ above means a limit of the relational cones from F to X, regarding $\forall Y^{\mathbf{D}} \mathbf{C}(FY, X)$ to be the collections of relational cones.

In the theorem above, a left adjoint is obtained by letting T be an identity. Instead, putting K to be an identity, we have *Yoneda lemma*:

Proposition 3.18 *Let* $F : \mathbf{D} \to \mathbf{C}$ *be an* $\mathbf{\Omega}$*-functor with middle lax property. Then* $F \cong \mathrm{Ran}_{1_{\mathbf{D}}} F$*, that is,*

$$F \cong \int_{X^{\mathbf{D}}} \mathbf{D}(-, X) \pitchfork_{\mathbf{C}} FX .$$

Left Kan extension is an example of constructions we can obtain from tensor and coend.

Definition 3.19 Suppose $\mathbf{C}$, $\mathbf{D}$, and $\mathbf{E}$ are r-frames satisfying the condition at the beginning of section 3.2. Given two r-frame morphisms $K : \mathbf{D} \to \mathbf{E}$ and $T : \mathbf{D} \to \mathbf{C}$, a *left Kan extension* is an object of $\mathbf{Us}(\mathbf{C}^{\mathbf{E}})$, denoted by $\mathrm{Lan}_K T$, such that there is an $\mathbf{\Omega}$-isomorphism

$$\mathbf{Us}(\mathbf{C}^{\mathbf{E}})(\mathrm{Lan}_K T, F) \cong \mathbf{C}^{\mathbf{D}}(T, FK)$$

$\mathbf{\Omega}$-natural in F.

Theorem 3.20 *Suppose* $\mathbf{C}$, $\mathbf{D}$, *and* $\mathbf{E}$ *are r-frames satisfying the condition at the beginning of the last section. For* $K : \mathbf{D} \to \mathbf{E}$ *and* $T : \mathbf{D} \to \mathbf{C}$,

$$\mathrm{Lan}_K T = \int^{X^{\mathbf{D}}} \mathbf{E}(KX, -) \otimes_{\mathbf{C}} TX$$

if the right hand side exists.

The proof is similar to that of Theorem 3.14.

An interesting property of Kan extensions is that even if T is not functorial, its Kan extension is a universally strong functor. As an application of this property, we show a coproduct of $\mathbf{\Omega}$ is defined as a right Kan extension. Note that colimits are usually represented by left Kan extensions [27].

In [36] Russell provided two definitions of disjunction from what he called the indefinables. The one is $A \vee B = (A \Rightarrow B) \Rightarrow B$, and the other is $\forall X((A \Rightarrow X) \Rightarrow (B \Rightarrow X) \Rightarrow X)$ where X ranges over all propositions. His intent is in classical logic. But the latter representation is meaningful in intuitionistic logic. Indeed Prawitz showed [32] the representation works in intuitionistic second order logic. In parametric models of second order lambda calculus, which corresponds to intuitionistic second order logic, $A \vee B = \forall X((A \Rightarrow X) \Rightarrow (B \Rightarrow X) \Rightarrow X)$ is a coproduct of A and B [13]. As for the first representation $A \vee B = (A \Rightarrow B) \Rightarrow B$, it seems not to be appropriate for intuitionistic logic. However it still has a connection with the latter representation. Let $T : \mathbf{\Omega} \to \mathbf{\Omega}$ be $(A \Rightarrow -) \Rightarrow -$. Since the hole $(-)$ appears both positively and negatively, it is not a functor. However we can apply the construction of Yoneda lemma, though the isomorphism of the lemma fails. Anyway we have a universal strong functor $\mathrm{Ran}_{1_\Omega} T : \mathbf{\Omega} \to \mathbf{\Omega}$. By Kan theorem,

$$\mathrm{Ran}_{1_\Omega} T(B) = \forall X^{\mathbf{\Omega}}((B \Rightarrow X) \Rightarrow (A \Rightarrow X) \Rightarrow X)$$

which is the second representation of $B \vee A$. This consideration leads us to the following theorem, which is proved without Kan theorem.

Theorem 3.21 *Let* $T : \mathbf{\Omega} \to \mathbf{\Omega}$ *be* $(A \Rightarrow -) \Rightarrow -$. *Then* $\mathrm{Ran}_{1_\Omega} T$ *is a coproduct* $(-) + A$.

(*Proof*) We should define (put $R = \mathrm{Ran}_{1_\Omega} T$)

$$\begin{array}{ll} \iota \in \mathbf{\Omega}^{\mathbf{\Omega}}(1_{\mathbf{\Omega}}, R) & \text{i.e., } \forall X^{\mathbf{\Omega}}(X \Rightarrow X + A) \\ \iota' \in \mathbf{\Omega}^{\mathbf{\Omega}}(A, R) & \text{i.e., } \forall X^{\mathbf{\Omega}}(A \Rightarrow X + A) \end{array}$$

$$\frac{f \in \mathbf{\Omega}(B, C) \quad g \in \mathbf{\Omega}(A, C)}{[f, g] \in \mathbf{\Omega}(RB, C)}$$

satisfying equations

$$\begin{array}{ll} [f, g] \circ \iota_B = f & (f \in B \Rightarrow C) \\ [f, g] \circ \iota'_B = g & (g \in A \Rightarrow C) \\ [\iota_B k, \iota'_B k] = k & (k \in RB \Rightarrow C)\,. \end{array}$$

By definition of right Kan extension, there is an isomorphism $\mathbf{\Omega}^{\mathbf{\Omega}}(F, T) \underset{j_F}{\cong} \mathbf{\Omega}^{\mathbf{\Omega}}(F, R)$ $\mathbf{\Omega}$-natural in F. Put

$$\begin{aligned} \iota &= j_{1_\Omega}(\Lambda X^{\mathbf{\Omega}} \lambda x^X \lambda z^{A \Rightarrow X} x) \\ \iota' &= j_A(\Lambda X^{\mathbf{\Omega}} \lambda a^A \lambda z^{A \Rightarrow X} za) \\ [f, g] &= \lambda y^{RB}(j^{-1}_{RB \times (B \Rightarrow -)}(\Lambda X^{\mathbf{\Omega}} \lambda p^{RB \times (B \Rightarrow X)} \bar{R}(\pi' p)(\pi p))) \{C\} \langle y, f \rangle g \end{aligned}$$

(π and π' are the first and second projections and $\langle\, ,\, \rangle$ is the pairing). The equations of coproduct follows from the naturality of j_F. □

3.4 Fixed Points

The categorical data types given so far are basic ones, and we can do almost nothing in real programming without inductive and coinductive data types. For example, the data type of natural numbers is defined inductively from 0 and a successor. In polymorphic lambda calculus, the type of natural numbers is represented as

$$\mathtt{Nat} = \forall X^{\mathbf{\Omega}}(X \Rightarrow (X \Rightarrow X) \Rightarrow X)\,.$$

This representation appears to be very essential, since we can extract programs in the form of polymorphic lambda calculus from proofs of validity of the programs in intuitionistic higher order logic [10, 12, 24, 40]. However there is still something unnatural. For example, the following two terms programmable in the language of polymorphic lambda calculus

$$\lambda x^{\mathtt{Nat}} \lambda y^{\mathtt{Nat}} x + y \quad \text{and} \quad \lambda x^{\mathtt{Nat}} \lambda y^{\mathtt{Nat}} y + x$$

are different (i.e., not $\beta\eta$-equal) in the syntax. So it would be worth to examine the meaning of $\mathtt{Nat} = \forall X^{\mathbf{\Omega}}(X \Rightarrow (X \Rightarrow X) \Rightarrow X)$ and other inductive and coinductive types in a semantical framework. In [3], Böhm and Berarducci proved that the closed terms of type $\mathtt{Nat}$ just correspond bijectively to natural numbers, and likewise for other algebras without generators. In this paper, as in [2, 13], the problem is examined using categorical language.

In category theory, inductive and coinductive types are described with F-algebras [22]. For $F : \mathbf{C} \to \mathbf{C}$ an endofunctor, let F-**Alg** denote the category of F-algebras, namely, the category whose object is an arrow $f : FA \to A$ of $\mathbf{C}$ and whose arrow $k : (FA \xrightarrow{f} A) \to (FB \xrightarrow{g} B)$ is $k : A \to B$ of $\mathbf{C}$ rendering the diagram

$$\begin{array}{ccc} FA & \xrightarrow{Fk} & FB \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[k]{} & B \end{array}$$

commutative. The category of F-coalgebras, $(1_{\mathbf{C}}, F)$, is defined dually.

Definition 3.22 An *initial fixed point* of $F : \mathbf{C} \to \mathbf{C}$ is an initial object (if any) of F-**Alg**. A *terminal fixed point* of $F : \mathbf{C} \to \mathbf{C}$ is a terminal object of F-**Coalg**.

We denote an initial fixed point by μF and a terminal fixed point by νF.

Proposition 3.23 *An initial fixed point μF is a fixed point in the sense $F(\mu F) \cong \mu F$. Likewise $F(\nu F) \cong \nu F$.*

See [20, 38] for the proof. μF corresponds to an inductive type and νF to a coinductive type.

In this section we give representations of initial and terminal fixed points by (co)tensor and (co)end. For that, we need a condition in addition to the middle lax property. As pointed out at 3.10, in the r-frame $\mathbf{\Omega}^{\mathbf{D}}$,

$$\mathbf{\Omega}^{\mathbf{D}}(|f|, |g|) : k \mapsto k' \quad \text{iff} \quad \begin{array}{ccc} F & \xrightarrow{k} & G \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ F' & \xrightarrow[k']{} & G' \end{array} .$$

We impose this condition on $\mathbf{C}$ for which an endofunctor is considered. Namely,

Condition 3.24 For $f \in \mathbf{C}(A, A')$ and $g \in \mathbf{C}(B, B')$,

$$\mathbf{C}(|f|, |g|) : k \mapsto k' \quad \text{iff} \quad \begin{array}{ccc} A & \xrightarrow{k} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ A' & \xrightarrow[k']{} & B' \end{array} .$$

Theorem 3.25 *Let* $\mathbf{C}$ *satisfy Condition 3.24, and* $F : \mathbf{C} \to \mathbf{C}$ *a universal* $\mathbf{\Omega}$*-functor with middle lax property.*

i) $\mu F = \int_{X^{\mathbf{C}}} \mathbf{C}(FX, X) \pitchfork_{\mathbf{C}} X$ *is, if it exists, an initial fixed point of* F.

ii) $\nu F = \int^{X^{\mathbf{C}}} \mathbf{C}(X, FX) \otimes_{\mathbf{C}} X$ *is, if it exists, a terminal fixed point of* F.

(*Proof*) Let us prove ii). We must define

$$\delta \in \mathbf{C}(\nu F, F(\nu F))$$

$$\frac{f \in \mathbf{C}(X, FX)}{f^{\nu} \in \mathbf{C}(X, \nu F)}.$$

There is an isomorphism natural in C

$$\begin{aligned} \mathbf{C}(\nu F, C) &\cong \forall X^{\mathbf{C}} \mathbf{C}(\mathbf{C}(X, FX) \otimes_{\mathbf{C}} X, C) \\ &\cong \forall X^{\mathbf{C}} (\mathbf{C}(X, FX) \Rightarrow \mathbf{C}(X, C)) \end{aligned}$$

which we denote by i_C. Define

$$\begin{aligned} f^{\nu} &= i_{\nu F}(1_{\nu F})\{X\} f \\ \delta &= i^{-1}_{F(\nu F)}(\Lambda X^{\mathbf{C}} \lambda f^{\mathbf{C}(X, FX)} \bar{F}(f^{\nu}) \circ_{\mathbf{C}} f) \end{aligned}$$

We can prove $\delta \circ_{\mathbf{C}} f^{\nu} = \bar{F}(f^{\nu}) \circ_{\mathbf{C}} f$ by naturality of i. Let us verify

$$\text{if} \quad \begin{array}{ccc} A & \xrightarrow{k} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ FA & \xrightarrow[\bar{F}k]{} & FB \end{array} \quad \text{then} \quad \begin{array}{ccc} A & \xrightarrow{k} & B \\ & {\scriptstyle f^{\nu}}\searrow \quad \swarrow{\scriptstyle g^{\nu}} & \\ & \nu F & \end{array} .$$

By Condition 3.24 and the middle lax property of F, the left rectangle diagram implies $\mathbf{C}(|k|, F|k|) : f \mapsto g$. Parametricity of $i_{\nu F}(1_{\nu F})$ then implies $\mathbf{C}(|k|, \nu F) : f^{\nu} \mapsto g^{\nu}$, i.e., $f^{\nu} = g^{\nu} \circ_{\mathbf{C}} k$ the right triangle diagram. In particular, applying to $\delta \circ_{\mathbf{C}} f^{\nu} = \bar{F}(f^{\nu}) \circ_{\mathbf{C}} f$, we obtain $f^{\nu} = \delta^{\nu} \circ_{\mathbf{C}} f^{\nu}$. Since $\delta^{\nu} \circ_{\mathbf{C}} f^{\nu} = i_{\nu F}(\delta^{\nu})\{X\} f$ by naturality of i, there holds $f^{\nu} = i_{\nu F}(\delta^{\nu})\{X\} f$, then $\delta^{\nu} = 1_{\nu F}$ abstracting by f and X. Therefore if the above rectangle diagram holds for $g = \delta$, then $f^{\nu} = \delta^{\nu} \circ_{\mathbf{C}} k = k$. It shows that $f^{\nu} : (A \xrightarrow{f} FA) \to (\nu F \xrightarrow{\delta} F(\nu F))$ is unique, i.e., νF is a terminal fixed point of F. $\Box$

In case of $\mathtt{Nat} = \forall X^{\mathbf{\Omega}}(X \Rightarrow (X \Rightarrow X) \Rightarrow X)$,

$$\begin{aligned} \mathtt{Nat} &\cong \forall X^{\mathbf{\Omega}}((\top \Rightarrow X) \times (X \Rightarrow X) \Rightarrow X) \\ &\cong \forall X^{\mathbf{\Omega}}(((\top + X) \Rightarrow X) \Rightarrow X) \\ &\cong \mu(\top + (-)). \end{aligned}$$

In category theory, a natural numbers object is an initial object of the category of diagrams $(\top \xrightarrow{z} X \xrightarrow{s} X)$ [18, 21]. We easily see $\mu(\top + (-))$ is a natural numbers object.

3.5 PL category

Seely [37] defined a general framework of categorical models of (ω-order) lambda calculus, called a PL category (see [31] for the second order fragment).

Definition 3.26 A *PL category* $(\mathcal{G}, \mathcal{S})$ is given by the following data.

i) A category $\mathcal{S}$ has a distinguished object $\mathbf{\Omega}$, products, and exponentiation of the form $\mathbf{\Omega}^{\mathbf{C}}$.

ii) $\mathcal{G}$ is an indexed category over $\mathcal{S}$ such that

- a) $\mathcal{S}^{op} \to \mathbf{Cat} \to \mathbf{Set}$ is given by $\mathcal{S}(-, \mathbf{\Omega})$ where $\mathcal{S} \to \mathbf{Cat}$ is $\mathcal{G}$ and $\mathbf{Cat} \to \mathbf{Set}$ is an obvious forgetful functor.
- b) For $\mathbf{C} \in |\mathcal{S}|$, the category of $\mathbf{C}$-indexed families, $\mathcal{G}(\mathbf{C})$, is a ccc; for $K : \mathbf{D} \to \mathbf{C}$ in $\mathcal{S}$, the substitution functor $\mathcal{G}(K) : \mathcal{G}(\mathbf{C}) \to \mathcal{G}(\mathbf{D})$ is a ccc functor.
- c) Let $\mathcal{G}^{\mathbf{C}}$ be an indexed category defined as $\mathcal{G}^{\mathbf{C}}(\mathbf{D}) = \mathcal{G}(\mathbf{C} \times \mathbf{D})$, and $\kappa_{\mathbf{C}} : \mathcal{G} \to \mathcal{G}^{\mathbf{C}}$ an indexed functor defined as $\kappa_{\mathbf{C}}(\mathbf{D}) = \mathcal{G}(\pi' : \mathbf{C} \times \mathbf{D} \to \mathbf{D})$. Then there are two indexed functors $\Sigma_{\mathbf{C}}, \Pi_{\mathbf{C}} : \mathcal{G}^{\mathbf{C}} \rightrightarrows \mathcal{G}$ such that $\Sigma_{\mathbf{C}} \dashv \kappa_{\mathbf{C}} \dashv \Pi_{\mathbf{C}}$ (indexed adjoint).

We show a parametric model $\mathbf{\Omega}$ of ω-order lambda calculus gives a PL category, if all $\mathbf{\Omega}^{\mathbf{C}}$ has graph relations.

Let $\mathcal{S}$ be $\mathcal{U}q$. Evidently i) is satisfied. As for ii)a), we put the category of $\mathbf{C}$-indexed families to be $\mathbf{\Omega}^{\mathbf{C}}$. Although $|\mathbf{\Omega}^{\mathbf{C}}|$ differs with $\mathcal{U}q(\mathbf{C}, \mathbf{\Omega})$, in any way, they are isomorphic.

In order to prove ii)b), we must define exponentiation and finite products in $\mathbf{\Omega}^{\mathbf{C}}$, which are preserved by substitution functors. Exponentiation is defined as

$$G^F = \lambda X^{\mathbf{C}}(FX \Rightarrow GX)\,.$$

Cartesian product is defined in two alternative ways:

$$\begin{aligned} F \times G &= \lambda X^{\mathbf{C}} \forall Y^{\mathbf{\Omega}^{\mathrm{C}}}((FX \Rightarrow GX \Rightarrow YX) \Rightarrow YX) \quad \text{or} \\ F \times G &= \lambda X^{\mathbf{C}} \forall Y^{\mathbf{\Omega}}((FX \Rightarrow GX \Rightarrow Y) \Rightarrow Y)\,. \end{aligned}$$

To prove the substitution functor $\mathbf{\Omega}^K$ is a ccc functor, that is, preserves exponentiation and product, the latter definition of product is convenient (moreover, for the former, $\mathbf{\Omega}^{\mathbf{C}}$ must belong to $\mathcal{U}q$). Indeed

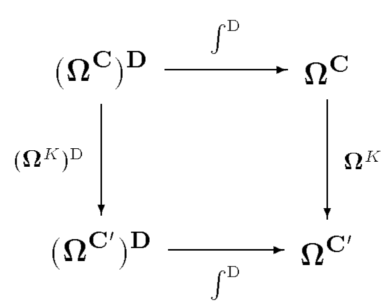
$$\begin{aligned} \mathbf{\Omega}^K(F \times G) &= \lambda D^{\mathbf{D}} \forall Y^{\mathbf{\Omega}}((FK(D) \Rightarrow GK(D) \Rightarrow Y) \Rightarrow Y) \\ &= FK \times GK \\ &= \mathbf{\Omega}^K F \times \mathbf{\Omega}^K G\,. \end{aligned}$$

Then, since a product is unique up to isomorphism, even the former definition works. Exponentiation is clearly preserved by $\mathbf{\Omega}^K$.

ii)c) is proved by showing

$$\textstyle\int^{\mathbf{D}} \dashv \kappa_{\mathbf{D}} \dashv \int_{\mathbf{D}} \qquad \text{(indexed adjoint).}$$

We must prove the end $\int_{\mathbf{D}}$ and the coend $\int^{\mathbf{D}}$ are indexed functors, and natural transformations of the adjunctions are indexed natural transformations. For example, for $K : \mathbf{C}' \to \mathbf{C}$,



must commute up to isomorphism.

We have no direct proof of this. First prove a new definition

$$\textstyle\int'^{\mathbf{D}} F = \lambda C^{\mathbf{C}} \forall Y^{\mathbf{\Omega}}(\forall D^{\mathbf{D}}(FDC \Rightarrow Y) \Rightarrow Y)$$

is also a coend of F (the old definition is $\int^{\mathbf{D}} F = \lambda C^{\mathbf{C}} \forall Y^{\mathbf{\Omega}^{\mathbf{C}}} (\forall D^{\mathbf{D}} (FDC \Rightarrow YC) \Rightarrow YC))$. It is easily showed that, for $\int'^{\mathbf{D}}$, the rectangle diagram above commutes. Then, from uniqueness of adjunction up to isomorphism, the diagram commutes even for $\int^{\mathbf{D}}$ up to isomorphism, and therefore the coend $\int^{\mathbf{D}}$ is an indexed functor. The proof that the end $\int_{\mathbf{D}}$ is an indexed functor is immediate. We leave the remainder to the reader.

Hence we have proved

Theorem 3.27 *Suppose* $\mathbf{\Omega}$ *is a model of* ω*-order lambda calculus such that all* $\mathbf{\Omega}^{\mathbf{C}}$ *has graph relations. Then it determines a PL category by the data above.*

In fact, the PL category has more properties; every $\mathbf{\Omega}^{\mathbf{C}}$ has initial and terminal fixed points of universal strong endofunctors with middle lax property; every r-frame morphism T to $\mathbf{\Omega}^{\mathbf{C}}$ has right and left Kan extensions along any r-frame morphism K.

Since the parametric per model **Per** is a model of ω-order lambda calculus, and have all relations and all relations of r-frame morphisms, we have a corollary:

Corollary 3.28 *The parametric per model* **Per** *provides a PL category.*

4 ω-order Minimum Model

In this section we develop the ω-order minimum model that is a straightforward extension of the second order minimum model [29]. The ω-order minimum model is a model of ω-order lambda calculus such that all $\mathbf{\Omega}^{\mathbf{C}}$ has graph relations. So all the categorical constructions described in the earlier sections exist in the model. For a similar discussion of the second order minimum model, see [14].

There are some terms such that they should have the same meaning from natural and intuitive aspects, but are still different, as $\lambda x^{\mathtt{Nat}} \lambda y^{\mathtt{Nat}} x + y$ and $\lambda x^{\mathtt{Nat}} \lambda y^{\mathtt{Nat}} y + x$. A model provides a denotation that suggests which terms should be the same and which should not. So each model gives an equivalence relation of (closed) terms, called theories. Among them there is a maximum one that equates terms as much as possible so that any more equations cause inconsistency. The theory is called the ω-order maximum consistent theory and the ω-order minimum model is obtained by collecting the equivalence classes of the theory.

Definition 4.1 A *pretheory* E is an equivalence relation of *closed* terms of polymorphic lambda calculus λ^{ω} such that

i) if $M\ E\ N$, then M and N have the same *closed* type.

ii) if $M =_{\beta} N$ then $M\ E\ N$.

iii) if $M\ E\ M'$ of type $\sigma \Rightarrow \tau$ and $N\ E\ N'$ of type σ then $(MN)\ E\ (M'N')$.

iv) if $M\ E\ M'$ of type $\forall^{\mathbf{C}} \sigma$ then $(M\{C\})\ E\ (M'\{C\})$ for all C, an object of $\mathbf{C}$.

The condition ii) means that E respects β-equality, and iii)iv) mean that E is congruent w.r.t. applications. Note that our pretheory does not necessarily respect η-equality.

Each model $\mathbf{\Omega}$ determines a pretheory $E_{\mathbf{\Omega}}$ as

$$M\ E_{\mathbf{\Omega}}\ N \qquad \text{iff} \qquad [\![M]\!] = [\![N]\!]\,.$$

A pretheory obtained from a model is called a *theory*.

Regarding pretheories as a subset of $Ct \times Ct$ where Ct is the set of closed terms of closed types, we have a partial order of pretheories by inclusion. By a usual argument, the partial order turns out to be a complete lattice, where the maximum element is the pretheory (indeed a theory) equating any terms of the same type. However it is not the maximum consistent theory that we want to have, because it is inconsistent in the following sense. Let `Bool` be the type $\forall X^{\mathbf{\Omega}}(X \Rightarrow X \Rightarrow X)$, which is regarded to be the (two-valued) set of truth values. `Bool` has only two β-normal closed terms, `true` $= \Lambda X^{\mathbf{\Omega}}\lambda x^X \lambda y^X x$ and `false` $= \Lambda X^{\mathbf{\Omega}}\lambda x^X \lambda y^X y$. If a pretheory equates `true` and `false`, we say the pretheory is *inconsistent*, otherwise *consistent*. The maximum theory is thus inconsistent, and actually it is the unique inconsistent pretheory. In fact, if `true` E `false` then $M\ E\ M'$ for any terms M and M' of the same type.

The maximum consistent theory is defined using the type `Bool`. One may regard two closed terms to be equivalent insofar as they are not separated by some predicates (i.e., functions to `Bool`). This idea is formalized as follows.

Definition 4.2 A pretheory $=_m$, to be called the *ω-order maximum consistent theory*, is defined as, given closed terms M and N of closed type σ, $M =_m N$ iff $KM =_\beta KN$ for any closed term K of type $\sigma \Rightarrow$ `Bool`.

Note that in the right hand side of "iff" the equality is $=_\beta$ ($=_{\beta\eta}$ results in the same), not $=_m$.

We must check that $=_m$ is actually a "maximum", "consistent" "pretheory", as well as that it is a "theory". The first three are easy.

Proposition 4.3 *$=_m$ is a pretheory.*

(*Proof*) We prove iii) of Definition 4.1. i)ii) are easy and iv) is similar. Let K be a closed term of type $\tau \Rightarrow$ `Bool`. Then we have

$$\lambda x^\sigma K(Mx) : \sigma \Rightarrow \texttt{Bool}$$
$$\lambda y^{\sigma \Rightarrow \tau} K(yN') : (\sigma \Rightarrow \tau) \Rightarrow \texttt{Bool}.$$

By $N =_m N'$ and $M =_m M'$ (hypotheses), $K(MN) =_\beta K(MN')$ and $K(MN') =_\beta K(M'N')$. Hence $K(MN) =_\beta K(M'N')$, i.e., $MN =_m M'N'$. □

Proposition 4.4 $=_m$ *is consistent.*

(*Proof*) Immediate. $\square$

Proposition 4.5 $=_m$ *is maximum in the sublattice of consistent pretheories.*

(*Proof*) Note that, in the type `Bool`, any consistent pretheory E coincides with $=_m$, since `true` and `false` are only two β-normal closed terms of `Bool`. We prove $E \subseteq =_m$. For $M\ E\ N$ of type σ, there follows $(KM)\ E\ (KN)$ for any $K : \sigma \Rightarrow$ `Bool`, thus $KM =_\beta KN$. Hence $M =_m N$. $\square$

The hardest part is to prove $=_m$ is a theory. It is necessary to construct a model $\mathbf{\Omega}_m$, that is defined as follows.

- Definition of an r-frame $\mathbf{\Omega}_m$ and an r-frame morphism $D : \mathbf{\Omega}_m \to \mathbf{Rel}$:
  - A is an object of $\mathbf{\Omega}_m$ iff $A = [\sigma]$ is a $\beta\eta$-equivalence class of a closed type σ. $D_{[\sigma]}$ is the set of equivalence classes $[M]$ of closed terms M of type σ modulo $=_m$. Namely $D_{[\sigma]} = Ct(\sigma)/ =_m$ where $Ct(\sigma)$ is the collection of closed terms of type σ. For simplicity let σ denote the equivalence class $[\sigma]$.
  - $r : \sigma \multimap \tau$ is a relation of $\mathbf{\Omega}_m$ iff r is a subset of $D_\sigma \times D_\tau$. D_r is simply r. That is to say, the relation class of $\mathbf{\Omega}_m$ is the collection of all binary relations.
- Definition of $\mathcal{R}f$ and $\mathcal{U}q$:
  - An object $\mathbf{A}$ of $\mathcal{R}f$ is of the form $\kappa_1 \times \cdots \times \kappa_n$ $(n \geq 0)$ where κ_i is a kind of λ^ω, in which $\mathbf{\Omega}$ is taken to be $\mathbf{\Omega}_m$. So typical examples are $\mathbf{\Omega}_m \times \mathbf{\Omega}_m^{\mathbf{\Omega}_m}$, $\mathbf{\Omega}_m^{\mathbf{\Omega}_m^{\mathbf{\Omega}_m}}$, etc. Products with $\mathbf{1}$, e.g. $\mathbf{1} \times \mathbf{A}$, are identified with $\mathbf{A}$.
  - $F : \kappa_1 \times \cdots \times \kappa_m \to \kappa'_1 \times \cdots \times \kappa'_n$ is an arrow of $\mathcal{R}f$ iff F is an n-tuple of constructor judgements

$$\Gamma \vdash \sigma_1 : \kappa'_1 \quad , \ldots , \quad \Gamma \vdash \sigma_n : \kappa'_n$$

    where Γ is of length m. Precisely, σ should be a $\beta\eta$-equivalence class, and the differences of names of free and/or bound variables are ignored. The meaning of F as an r-frame morphism is given by substitution, i.e., for objects,

$$\begin{aligned}&\langle \Gamma \vdash \sigma_1 : \kappa'_1, \ldots, \Gamma \vdash \sigma_n : \kappa'_n \rangle(\langle \tau_1, \ldots, \tau_n \rangle) \\ &\quad = \langle \sigma_1[\Gamma := \underline{\tau}], \ldots, \sigma_n[\Gamma := \underline{\tau}] \rangle\,;\end{aligned}$$

and for relations, the meaning as a relation function is determined by the condition imposed on models (including the naturality condition), since all $\Gamma \vdash \sigma_i : \kappa'_i$ are constructed from $\Rightarrow$, $\forall^{\mathbf{C}}$, $\epsilon_{\mathbf{C},\mathbf{D}}$, and $\lambda Y^{\mathbf{C}}$.

 – $\mathbf{\Omega}^{\mathbf{C}}_m$ is an r-frame where the objects are closed constructors σ such that $\vdash \sigma : \mathbf{\Omega}^{\mathbf{C}}$, and the relations are any relations of r-frame morphisms.

- $\Rightarrow: \mathbf{\Omega}_m \times \mathbf{\Omega}_m \to \mathbf{\Omega}_m$ is defined as $\Rightarrow(\langle\sigma,\tau\rangle) = \sigma \Rightarrow \tau$ for objects; for relations, $\Rightarrow(\langle r,s\rangle)$ is determined by the naturality condition of $\Rightarrow$.

- $\forall^{\mathbf{C}} : \mathbf{\Omega}^{\mathbf{C}}_m \to \mathbf{\Omega}_m$ is defined as $\forall^{\mathbf{C}}(\sigma) = \forall^{\mathbf{C}}\sigma$ for objects; for relations, $\forall^{\mathbf{C}} q$ is determined by the naturality condition of $\forall^{\mathbf{C}}$. We must check $\forall^{\mathbf{C}}$ preserves *id* (see below).

- $\Phi^{\Rightarrow}_{\sigma,\tau} : (\sigma \Rightarrow \tau) \to (\sigma \to \tau)$ is such that $\Phi^{\Rightarrow}_{\sigma,\tau}([M])$ is a function sending $[N]$ to $[MN]$ And $\Phi^{\forall^{\mathbf{C}}}_{\sigma} : \forall^{\mathbf{C}}\sigma \to \Pi C \in |\mathbf{C}|.\sigma[X := C]$ is such that $\Phi^{\forall^{\mathbf{C}}}_{\sigma}([M])$ is a function sending C to $[M\{C\}]$. We must give a proof that $\Phi^{\Rightarrow}_{\sigma,\tau}$ and $\Phi^{\forall^{\mathbf{C}}}_{\sigma}$ are all one-to-one.

- Interpretation $[\![\,]\!]$ is almost obvious. For a constructor judgement, $[\![\Gamma \vdash \sigma : \mathbf{D}]\!]$ is simply $\Gamma \vdash \sigma : \mathbf{D}$. As for a term judgement, $[\![\Gamma; \Theta \vdash M : \tau]\!]$ is a function sending $C_i \in |\mathbf{C}_i|$, $a_j \in [\![\Gamma \vdash \sigma_j : \mathbf{\Omega}]\!](\underline{C})$ to $[M[\Gamma := \underline{C}][\Theta := \underline{a}]]$.

Next we verify all $\Phi^{\Rightarrow}_{\sigma,\tau}$ and $\Phi^{\forall^{\mathbf{C}}}_{\sigma}$ are all one-to-one. It suffices to prove that $=_m$ satisfies the ω-rule.

Theorem 4.6 *$=_m$ satisfies the ω-rule:*

i) Let M be a closed term of function type $\sigma \Rightarrow \tau$. If $MN =_m M'N$ for all closed N of type σ, then $M =_m M'$.

ii) Let M be a closed term of universal type $\forall^{\mathbf{C}}\sigma$. If $M\{\tau\} =_m M'\{\tau\}$ for all closed constructor τ of kind $\mathbf{C}$, then $M =_m M'$.

To prove this theorem, we need the following definition.

Definition 4.7 A term M is in *long β-normal form* iff all of

i) M is written $l_1 \cdots l_m.N$ where l_i is a first order binding $\lambda x_i^{\sigma_i}$ or a constructor binding $\Lambda X_i^{\mathbf{C}_i}$ according to the type of M.

ii) Let τ be the type of N and assume τ is in $\beta\eta$-normal form. We require the outermost connective of τ is neither $\Rightarrow$ nor $\forall^{\mathbf{D}}$ (namely either τ begins with ϵ or is a constructor variable itself).

iii) N is in β-normal form. Hence it is of the form $xN_1 \cdots N_n$ where x is a term variable and N_i is a term or a constructor.

The definition of long β-normal form differs with that of long $\beta\eta$-normal form in respect of iii) where N is simply in β-normal form.

Proposition 4.8 *Every term has a unique long β-normal form.*

(*Proof*) Given a term M, it is evident that there is $l_1 \cdots l_m.N$ satisfying i) and ii), by η-expansion. By induction on the type of M, we can prove the form of the bindings $l_1 \cdots l_m$ satisfying i) and ii) is uniquely determined. As for iii), the β-normal form of N is unique. □

Since the proof of the ω-rule is a little complicated, we first explain the background idea.

The claim is that, to prove i), if $M =_m M'$ of type $\sigma \Rightarrow \tau$ then for any $K : (\sigma \Rightarrow \tau) \Rightarrow \texttt{Bool}$, there holds $KM =_\beta KM'$ or equivalently $KM =_m KM'$. Assume

$$K =_{\beta\eta} \lambda x^{\sigma \Rightarrow \tau} x L_1 \cdots L_n \, .$$

If $L_1, \ldots, L_n$ are closed, then $KM =_{\beta\eta} ML_1 \cdots L_n$ and likewise for M'. Since $MN =_m M'N$ for all closed $N : \sigma$ by hypothesis, we obtain $ML_1 \cdots L_n =_m M'L_1 \cdots L_n$. Hence $KM =_m KM'$, as required. In general, however, x may occur in $L_1, \ldots, L_n$. It makes the argument difficult. In this case we replace only the head x of $xL_1 \cdots L_n$ by M, leaving the other x's in L_i as they are. Then we reduce the obtained $ML_1 \cdots L_n$ by β-reduction to a β-normal form, which should begin with x again. Therefore we can argue by induction on the length of β-reduction.

Furthermore K may not be written in the above form. But it has a long β-normal form and is handled using the fact that `Bool` has only two members.

(*Proof*) (of Theorem 4.6) We treat (i) and (ii) at the same time. The claim is that, for any closed $K : \rho \Rightarrow \texttt{Bool}$, it holds that $KM =_{\beta\eta} KM'$ where $\rho = \sigma \Rightarrow \tau$ or $= \forall Y^{\mathbf{C}}.\sigma$. W.l.o.g. we assume K is in long β-normal form

$$K = \lambda x^\rho \Lambda X^{\mathbf{\Omega}} \lambda y^X \lambda z^X h L_1 \cdots L_n$$

where h is a term variable and L_i is a term or a constructor according to the type of h. Since K is closed, h is x (then $n \geq 1$), y or z (then $n = 0$). Define $\mathrm{mln}(N)$ to be the maximal length of β-reductions from N. The proof is by induction on $\mathrm{mln}(KM)$ where $K : \rho \Rightarrow \texttt{Bool}$ is in long β-normal form.

(BASE) $\mathrm{mln}(KM) = 1$. This is the case that $h = y$ or $= z$. Then evidently $KM =_{\beta\eta} KM'$.

(INDUCTION STEP) $\mathrm{mln}(KM) > 1$. This is the case that $h = x$. Put

$$\overline{K} = \lambda x^\rho \Lambda X^{\mathbf{\Omega}} \lambda y^X \lambda z^X M L_1 \cdots L_n$$

and let K' be the long β-normal form of $\overline{K}$, written

$$K' = \lambda x^{\rho} \Lambda X^{\mathbf{\Omega}} \lambda y^{X} \lambda z^{X} h' L'_1 \cdots L'_{n'} .$$

K' is obtained from $\overline{K}$ by $M L_1 \cdots L_n \rhd_{\beta} h' L'_1 \cdots L'_{n'}$. Now we infer $\mathrm{mln}(KM) > \mathrm{mln}(K'M)$. Indeed if $h' = y$ or $= z$, then the inequality is immediate. If $h' = x$, any β-reductions from $K'M$ are such that

$$K'M \rhd_{\beta} M(L'_1[x := M]) \cdots (L'_{n'}[x := M]) \rhd_{\beta} \cdots .$$

Then there is a sequence of β-reductions from KM such that

$$\begin{array}{lll} KM & \rhd_{\beta} & \Lambda X^{\mathbf{\Omega}} \lambda y^{X} \lambda z^{X} M(L_1[x := M]) \cdots (L_n[x := M]) \\ & \rhd_{\beta}^{\geq 1} & \Lambda X^{\mathbf{\Omega}} \lambda y^{X} \lambda z^{X} M(L'_1[x := M]) \cdots (L'_{n'}[x := M]) \\ & \rhd_{\beta} & \cdots \end{array}$$

where the reductions at $\rhd_{\beta}^{\geq 1}$ are the substitution of $M L_1 \cdots L_n \rhd_{\beta}^{\geq 1} x L'_1 \cdots L'_{n'}$. Hence even in this case $\mathrm{mln}(KM) > \mathrm{mln}(K'M)$ holds. By induction hypothesis, $K'M =_{\beta\eta} K'M'$. So

$$KM =_{\beta\eta} \overline{K}M =_{\beta\eta} K'M =_{\beta\eta} K'M' =_{\beta\eta} \overline{K}M' .$$

Let θ be a substitution $[X := \mathtt{Bool}; x := M', y := \mathtt{true}, z := \mathtt{false}]$, and a closed constructor ω the type of $L_1\theta$ if ρ is $\sigma \Rightarrow \tau$ (then L_1 is a term), or $L_1\theta$ itself if ρ is $\forall Y^{\mathbf{C}}.\sigma$ (then L_1 is a constructor). Note that L_1 exists since $n \geq 1$. We put $J = \lambda v.v(L_2\theta) \cdots (L_n\theta)$ where the type of v is τ if ρ is $\sigma \Rightarrow \tau$, or $\sigma[Y := \omega]$ if ρ is $\forall Y^{\mathbf{C}}\sigma$. Then ω is a closed constructor and J is a closed term of type $\tau \Rightarrow \mathtt{Bool}$ or $\sigma[Y := \omega] \Rightarrow \mathtt{Bool}$. By definition of J,

$$\left\{ \begin{array}{l} J(M(L_1\theta)) =_{\beta\eta} \overline{K}M'\{\mathtt{Bool}\}(\mathtt{true})(\mathtt{false}) \\ J(M'(L_1\theta)) =_{\beta\eta} KM'\{\mathtt{Bool}\}(\mathtt{true})(\mathtt{false}) \end{array} \right.$$

both closed terms of type `Bool`. Note that the only closed terms of type `Bool` are `true` and `false` and for $b = \mathtt{true}$ or $= \mathtt{false}$ it holds that $b =_{\beta\eta} b\{\mathtt{Bool}\}(\mathtt{true})(\mathtt{false})$. Hence

$$\left\{ \begin{array}{l} J(M(L_1\theta)) =_{\beta\eta} \overline{K}M' \\ J(M'(L_1\theta)) =_{\beta\eta} KM' . \end{array} \right.$$

By the hypothesis of the ω rule for M and M', it follows that $M(L_1\theta) =_m M'(L_1\theta)$ thus $J(M(L_1\theta)) =_{\beta\eta} J(M'(L_1\theta))$. Therefore $\overline{K}M' =_{\beta\eta} KM'$ holds. □

A semantical proof for second order calculus is found in [29]

Corollary 4.9 $\Phi^{\Rightarrow}_{\sigma,\tau}$ *and* $\Phi^{\forall^{\mathbf{C}}}_{\sigma}$ *are all one-to-one.*

Another thing we have to do is to prove the parametricity of the r-frame structure $\mathbf{\Omega}_m$:

Theorem 4.10 *Every* $\forall^{\mathbf{C}} : \mathbf{\Omega}_m^{\mathbf{C}} \to \mathbf{\Omega}_m$ *preserves id.*

(*Proof*) Note that all element of $A \in |\mathbf{\Omega}_m|$ is an equivalence class of a closed term. The claim is to show $\forall^{\mathbf{C}}(id_\sigma) : [M] \mapsto [M]$ for any $\sigma \in |\mathbf{\Omega}_m^{\mathbf{C}}|$ (i.e., $\vdash \sigma : \mathbf{\Omega}^{\mathbf{C}}$) and any $M \in [\![\forall^{\mathbf{C}}\sigma]\!]$ (i.e., $; \vdash M : \forall^{\mathbf{C}}\sigma$). Abstraction theorem (Theorem 2.8) implies

$$\forall^{\mathbf{C}}(id_\sigma) : [\![M]\!] \mapsto [\![M]\!] .$$

Notice that $[\![M]\!] = [M]$. □

Therefore we have proved the next theorem.

Theorem 4.11 $\mathbf{\Omega}_m$ *is a model.*

Now we can call $=_m$ the ω-order maximum consistent theory.

$\mathbf{\Omega}_m$ and $\mathbf{\Omega}_m^{\mathbf{C}}$ have graph relations, since they have all relations and all relations of r-frame morphisms. Therefore $\mathbf{\Omega}_m$ fulfills all the conditions necessary for the categorical constructions in the earlier sections. In particular,

Theorem 4.12 *The indexed category* $\mathbf{\Omega}_m$ *is a PL category. Moreover every* $\mathbf{\Omega}_m^{\mathbf{C}}$ *has initial and terminal fixed points of universal strong functors.*

Note that ω-order lambda calculus does not contain the primitives of products and existential quantifier as opposed to the calculus in [37]. Nevertheless the ω-order minimum model has products and existential quantifier (as a coend).

Remark 4.13 As mentioned earlier, it causes no problem that some pair of arrows $\mathbf{C} \rightrightarrows \mathbf{D}$ of $\mathcal{R}f$ denotes the same r-frame morphism. However, in the model $\mathbf{\Omega}_m$, all arrows of $\mathcal{R}f$ are distinct with each other. To prove it, extend the result of Statman for simply typed lambda calculus [39, 41].

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