Hypercube Embedding of Distances with Few Values

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Abstract

A distance d on a finite set V is hypercube embeddable if it can be isometrically embedded into the hypercube $\{0,1\}^m$ for some $m \geq 1$, i.e., if the elements $i \in V$ can be labeled by sets A_i in such a way that $d(i,j) = |A_i \triangle A_j|$ for all $i, j \in V$. Testing whether a distance d is hypercube embeddable is an NP-complete problem, even if d is assumed to take its values in $\{2,3,4,6\}$ ([Chv80]). On the other hand, the hypercube embeddability problem is polynomial when restricted to the class of distances with values in $\{1,2,3\}$ ([Avi90]). Let x, y be positive integers such that exactly two of $x, y, x+y$ are odd. We show that, for fixed x, y, the hypercube embeddability problem remains polynomial for the class of distances with values in $\{x, y, x+y\}$.

1 Introduction

Let $V := \{1, \ldots, n\}$ be a finite set. A function $d : V \times V \longrightarrow \mathbb{R}_+$ is a **distance** on V if d is symmetric, i.e., $d(i,j) = d(j,i)$ for all $i, j \in V$, and $d(i,i) = 0$ for all $i \in V$. A distance d is a **semimetric** if it satisfies the following **triangle inequality**

$$d(j,k) \leq d(i,j) + d(i,k) \tag{1}$$

for all distinct $i, j, k \in V$. If $d(i,j) = 0$ holds only for $i = j$, then d is a **metric**.

A distance d is said to be **hypercube embeddable** if d can be isometrically embedded into some hypercube $\{0,1\}^m$ for some $m \geq 1$. Equivalently, d is hypercube embeddable if one can label the points $i \in V$ by sets A_i in such a way that

$$d(i,j) = |A_i \triangle A_j| \text{ for all } i, j \in V. \tag{2}$$

Any labeling of the points of V by sets A_i satisfying (2) is called an h-**labeling** of d.

It is easy to see that every hypercube embeddable distance is a semimetric. Also, a semimetric is hypercube embeddable if and only if the metric formed by contracting the points at zero distance to single points is hypercube embeddable. Hence, we can assume to deal with integral distances taking nonzero values on distinct pairs of points.

The following problem is called the **hypercube embeddability problem.**

Given a distance d on n points, test whether d is hypercube embeddable.

This problem has been long studied and has many applications, in particular, for binary adressing in telecommunication networks (see [GP71]). When restricted to the class (ii) below of path metrics of graphs, this is the problem of embedding graphs isometrically into the hypercube. It can be checked in polynomial time whether a given graph is an isometric subgraph of a hypercube ([Djo73]). On the other hand, it is NP-complete to decide whether a given graph is a subgraph (not necessarily isometric) of a hypercube ([APP85, KVC86]); in fact, it is NP-hard to compute the minimum dimension of a hypercube in which a tree can be embedded as a subgraph ([WC90]). The (isometric) hypercube embeddability problem is NP-complete for general distances; actually, it remains NP-complete when restricted to the following class (i) of distances ([Chv80]):

(i) distances having one point at distance 3 from all other points with the distances between those points belonging to $\{2, 4, 6\}$.

The hypercube embeddability problem is also NP-complete for the class of distances with values in $\{2, 4, 6, 8\}$ ([Avi93]). However, several classes of metrics are known for which the hypercube embeddability problem is polynomial. This is the case, in particular, for the following classes of distances:
(ii) path metrics of graphs ([Djo73]),
(iii) distances with values 1,2 ([AD80]),
(iv) distances with values 2,4 and with one point at distance 2 from all other points ([Chv80]),
(v) distances with values 1,2,3 ([Avi90]),
(vi) distances d on a set V such that $d(i, j) = 2$ for all $(i, j) \in (S \times S) \cup (T \times T)$, for some partition of V as $V = S \cup T$ ([DL91a]).

Note that the metrics from $(ii) - (vi)$, either have some inner structure (they arise from graphs in (ii)), or have some restriction on the values that they can take (it is easily observed that a metric from (vi) takes at most four distinct values).

In this paper, we consider the following classes (vii) and $(viii)$ of distances:
(vii) distances with values x, y, where x, y are positive integers such that exactly one of them is odd,

$(viii)$ distances with values $x, y, x+y$, where x, y are positive integers such that exactly two of $x, y, x+y$ are odd.
Hence, the class (vii) contains (iii) as subcase $x = 1,\ y = 2$, and the class $(viii)$ contains (v) as subcase $x = 1,\ y = 2$.

Our main result is that, for fixed x, y, the hypercube embeddability problem is polynomial over each of the classes (vii) and $(viii)$. We refer to the results from Propositions 3.4, 3.5, 4.2, 5.1, 6.1, and 6.2, for the corresponding characterizations of hypercube embeddability.

However, we have no characterization for the classes of hypercube embeddable distances that take two or three values, all of them even. Indeed, our technique of proof relies strongly on the existence of some odd values, and on the fact that their number is small (less than or equal to 2). For instance, the complexity of the hypercube embeddability problem for the class of distances with values in $\{2, 4, 6\}$ is not known. We show in Figure 1 an example of a distance d with values in $\{2, 4, 6\}$, which is hypercube embedable (an h-labeling is shown in Figure 1), but such that the distance obtained by adding a new point at distance 3 from the other points is not hypercube embeddable (to see it, check that d admits no h-labeling where all labels are sets of cardinality 3). Hence, the NP-completeness result for the class (i) has no immediate implication for the complexity of the hypercube embeddability problem for $\{2, 4, 6\}$-valued distances.

Still, the class (iv), which consists of distances with values 2,4, can be tested in polynomial time ([Chv80]). We recall below the proof, which relies on a connection between hypercube embeddable distances and intersection patterns.

Figure 1

A symmetric function $a : V \times V \longrightarrow$ is called an **intersection pattern** if there exist n sets $A_1, \ldots, A_n$ such that

$$a(i,j) = |A_i \cap A_j| \tag{3}$$

for all $i, j \in V$.
Let d be a distance on $V \cup \{n+1\}$ and let a denote the symmetric function on V defined by

$$\begin{cases} a(i,i) & = d(i,n+1) & \text{for all } i \in V, \\ a(i,j) & = (d(i,n+1) + d(j,n+1) - d(i,j))/2 & \text{for all } i \neq j \in V. \end{cases} \tag{4}$$

Then, one can verify that d is hypercube embeddable if and only if a is an intersection pattern ([Dez73a]). (Note that one can use the same sets $A_1, \ldots, A_n$ for the labeling of d and the representation of a as (3) if one chooses $A_{n+1} = \emptyset$.)

Let d be a distance from the class (iv) and let a be its transform by (4). Then, $a_{ii} = 2$ for all $i \in V$ and $a_{ij} = 0, 1$ for all distinct $i, j \in V$. It is easy to see that such a is an intersection pattern if and only if the graph on V whose edges are the pairs ij for which $a(i,j) = 1$ is a line-graph. As line-graphs can be recognized in polynomial time ([Bei70]), the class (iii) can be tested in polynomial time for hypercube embeddability.

In the same way, testing whether a symmetric function a with $a(i,i) = 3$ for all $i \in V$ is an intersection pattern, is an NP-complete problem or, equivalently, the hypercube embeddability problem is NP-complete for the class (i) of distances ([Chv80]).

2 Preliminaries

We group in this section several observations on hypercube embeddings.

Let d be a distance on $V = \{1, \ldots, n\}$. If $A_1, \ldots, A_n$ is an h-labeling of d, then $d(i,j) + d(i,k) + d(j,k) = 2(|A_i| + |A_j| + |A_k| - |A_i \cap A_j| - |A_i \cap A_k| - |A_j \cap A_k|)$ holds for all $i, j, k \in V$. This imples the following result.

CLAIM 2.1 ([Dez61]) *If d is hypercube embeddable, then d satisfies the following* **even condition***:*

$$d(i,j) + d(i,k) + d(j,k) \textit{ is an even integer for all } i, j, k \in V. \tag{5}$$

CLAIM 2.2 *If $A_1, \ldots, A_n$ is an h-labeling of d and, for some distinct $i, j, k \in V$, $d(j,k) = d(i,j) + d(i,k)$, then $A_j \cap A_k \subseteq A_i \subseteq A_j \cup A_k$.*

Clearly, if $A_1, \ldots, A_n$ is an h-labeling of d and A is an arbitrary set, then $A_1 \triangle A, \ldots, A_n \triangle A$ is also an h-labeling of d. Also, permuting the elements of $\bigcup_{1 \leq i \leq n} A_i$ yields another h-labeling of d. Two h-labelings of d are said to be **equivalent** if they differ by the above operations. If d has a unique (up to equivalence) h-labeling, then d is said to be **rigid**.

Suppose we are trying to construct an h-labeling of d. We can always supose that one of the points of V, say the point n, is labeled by $\emptyset$. Then, d is hypercube embeddable if and only if we can find $n-1$ sets $A_1, \ldots, A_{n-1}$ such that

$$\begin{cases} |A_i| & = d(i,n) & \text{for all } 1 \leq i \leq n-1, \\ |A_i \cap A_j| & = (d(i,n) + d(j,n) - d(i,j))/2 & \text{for all } 1 \leq i < j \leq n-1. \end{cases}$$

Therefore, we are looking for a collection of $n-1$ sets satisfying some conditions on the cardinalities of the sets and of their pairwise intersections.

The following set families will play an important role here. Let $\mathcal{A}$ be a family of subsets of $V = \{1, \ldots, n\}$. Then, $\mathcal{A}$ is called a $(h, k; n)$**-intersecting system** if $|A| = h$ for all $A \in \mathcal{A}$ and $|A \cap A'| = k$ for all distinct $A, A' \in \mathcal{A}$. Moreover, if there exists a set K of cardinality k such that $A \cap A' = K$ for all distinct $A, A' \in \mathcal{A}$, then $\mathcal{A}$ is called a Δ**-system** with **kernel** K and **parameters** $(h, k; n)$. Clearly, $|\mathcal{A}| \leq \frac{n-k}{h-k}$ holds for such a Δ-system. When we do not want to specify the size n of the groundset, we speak of a (h, k)-intersecting system or of a Δ-system with parameters (h, k). Let $f(h, k; n)$ denote the maximum cardinality of a $(h, k; n)$-intersecting system.

It is known that every (h, k)-intersecting system $\mathcal{A}$ whose cardinality $|\mathcal{A}|$ is large with respect to h (namely, $|\mathcal{A}| \geq h^2 - h + 2$) is necessarily a Δ-system ([Dez74]). In particular, if $\mathcal{A}$ is a $(2t, t)$-intersecting system with $|\mathcal{A}| \geq t^2 + t + 2$, then $\mathcal{A}$ is a Δ-system ([Dez73b]).

Let $2t\mathbb{1}_n$ denote the distance on V defined by $d(i, j) = 2t$ for all $i \neq j \in V$. If we label one point of V by $\emptyset$, then the h-labelings of $2t\mathbb{1}_n$ are exactly the $(2t, t)$-intersecting systems of cardinality $n-1$. Therefore, the above result can be reformulated as follows. For more information on the variety of hypercube embeddings of the equidistant metric $2t\mathbb{1}_n$, see [DL93].

CLAIM 2.3 ([Dez73b]) *If $n \geq t^2 + t + 3$, then the distance $2t\mathbb{1}_n$ is rigid, i.e., its unique h-labeling (up to equivalence) consists of n pairwise disjoint sets each of cardinality t; this labeling is called the* **star-labeling**.

A natural weakening of the notion of hypercube embeddability is that of ℓ_1-embeddability. A distance d on V is said to be ℓ_1**-embeddable** if there exist n vectors $u_i \in \mathbb{R}^m$ $(m \geq 1)$ for $i \in V$ such that $d(i, j) = \| u_i - v_j \|_1$, where $\| u \|_1 = \sum_{1 \leq h \leq m} |u_h|$ for $u \in \mathbb{R}^m$. Testing whether a given distance on n points is ℓ_1-embeddable is also NP-complete ([Kar85]). It is interesting to note that hypercube and ℓ_1-embeddable distances can be expressed in terms of cut semimetrics.

Given a subset $S \subseteq V$, the **cut semimetric** d_S is defined by $d_S(i,j) = 1$ if $(i,j) \in S \times T$ and $d(i,j) = 0$ if $(i,j) \in S^2 \cup T^2$. Given a distance d on V, then d is ℓ_1-embeddable (resp. hypercube embeddable) if and only if $d = \sum_{S \subseteq V} \lambda_S d_S$ for some nonnegative (resp. nonnegative integer) scalars λ_S ([AD82]). Therefore, for d integral, d is ℓ_1-embeddable if and only if λd is hypercube embeddable for some scalar λ.

Hence, the set of ℓ_1-embeddable distances on n points forms a cone. This cone is called the cut cone; it has been extensively studied in the literature (see, e.g., [DL92] and references there). Each of the valid inequalities for this cone yields a necessary condition for hypercube embeddability; many valid inequalities for the cut cone are known (see, e.g., [DL91b] for a survey). In particular, all ℓ_1-embeddable distances satisfy the hypermetric inequalities defined below (introduced in [Dez61]).

The hypermetric condition. If d is hypercube embeddable, then

$$\sum_{i,j \in V} b_i b_j d(i,j) \leq 0 \tag{6}$$

for all integers b_i, $i \in V$, such that $\sum_{i \in V} b_i = 1$. The inequality (6) with b_i integer for $i \in V$ and $\sum_{i \in V} b_i = 1$ is called a **hypermetric inequality**. A distance d is said to be **hypermetric** if it satisfies all hypermetric inequalities. If $|b_i| = 1$ for all $i \in V$ and $\sum_{i \in V} |b_i| = k$, then the inequality (6) is called a k**-gonal** inequality. For instance, the 3-gonal inequalities are the triangle inequalities (1) and the 5-gonal inequalities are of the form

$$d(i_1,i_2) + d(i_1,i_3) + d(i_2,i_3) + d(i_4,i_5) - \sum_{\substack{h=1,2,3 \\ k=4,5}} d(i_h,i_k) \leq 0$$

for $i_1, i_2, i_3, i_4, i_5 \in V$.

We consider here the following classes of distances, where a, b are positive integers.
(a) d takes the two values $2a$,b (b odd);
(b) d takes the three values $a, b, a+b$ (a, b odd);
(c) d takes the three values $2a, b, b+2a$ (b odd).
We show that, for fixed a, b, each of these classes can be tested in polynomial time for hypercube embeddability.

Our proof goes as follows. Suppose that d is a distance on V, $|V| = n$, from one of the classes $(a) - (c)$.

If n is bounded by a function of a and b, then one can test directly whether d is hypercube embeddable, for instance, by brute force enumeration.

Suppose now that n is large with respect to a and b, namely, $n \geq 2n(a,b)-1$, where $n(a,b) = a^2+a+3$ for the cases $(a),(c)$, and $n(a,b) = (\frac{a+b}{2})^2+\frac{a+b}{2}+3$ for the case (b). Then, we are able to characterize the hypercube embeddable distances on $n \geq 2n(a,b)-1$ points by a set of conditions which can be checked in time polynomial in n; see Proposition 3.4 for the case (a), Proposition 4.2 for the case (b), and Propositions 5.1 and 6.1 for the case (c). We use as basic tools the following facts.

By the even condition (5), the set of pairs ij for which $d(i,j)$ is odd is a complete bipartite graph, i.e., V is partitioned into $V = S \cup T$ with, for instance, $|S| \geq |T|$, $d(i,j)$ even for $(i,j) \in S^2 \cup T^2$, and $d(i,j)$ odd if $(i,j) \in S \times T$.

By Claim 2.3, the projection of d on $S \times S$ is rigid since $|S| \geq n(a,b)$. This forces the points of S to be labeled by the star-labeling (or an equivalent of it) in any h-labeling of d.

In the cases $(b),(c)$, one of the values taken by d is the sum of the other two. This will give us more information on a possible h-labeling of d, by applying Claim 2.2.

For the case (c), we must distinguish two subcases, depending whether $b < 2a$ or $2a < b$. These two cases have a quite different behaviour, as we shall see from the proof. The subcase $b < 2a$ contains the instance $a = b = 1$, which was considered in [Avi90]; actualy, it can be treated in essentially the same way as the special instance $a = b = 1$.

3 Distances with values $2a, b$ (b odd)

Let a, b be two positive integers with b odd. Let d be a distance on V which takes the two values $2a, b$. We can assume that $b \geq a$ (else, d is not a semimetric). If d is hypercube embeddable, then V is partitioned into $V = S \cup T$ with, for instance, $|S| \geq |T|$, $d(i,j) = 2a$ if $(i,j) \in S^2 \cup T^2$, and $d(i,j) = b$ if $(i,j) \in S \times T$ (by (5)). Suppose that we label a node $j_0 \in T$ by $\emptyset$.

Then, an h-labeling of d exists if and only if there exist two set families $\mathcal{A}$ and $\mathcal{B}$ with $|\mathcal{A}| = |S|$, $|\mathcal{B}| = |T|-1$, and satisfying

$$\begin{cases} \mathcal{A} \text{ is a } (b, b-a)\text{-intersecting system,} \\ \mathcal{B} \text{ is a } (2a, a)\text{-intersecting system,} \\ |A \cap B| = a \text{ for all } A \in \mathcal{A}, B \in \mathcal{B}. \end{cases} \tag{7}$$

(Indeed, label the points of S by the members of $\mathcal{A}$ and the points of $T \setminus \{j_0\}$ by the members of $\mathcal{B}$.)

Note that, if $\mathcal{B} = \emptyset$ and $\mathcal{A}$ is a Δ-system with parameters $(b, b-a)$, then they satisfy relation (7). In other words, d is hypercube embeddable if $|T| = 1 \leq |S|$. This is the distance shown in Figure 2.

The distance from Figure 3 (i.e., the case $|S| = |T| = 2$) is also hypercube embeddable. (Indeed, label the two nodes of T by $\emptyset$ and $A \cup A'$, and the two nodes of S by $A_0 \cup A$ and $A_0 \cup A'$, where A_0, A, A' are disjoint sets of respective cardinalities $b-a, a, a$.)

Figure 2 Figure 3 Figure 4

CLAIM 3.1 *Let $\mathcal{A}, \mathcal{B}$ be set families satisfying (7). If $\mathcal{B} \neq \emptyset$ and $\mathcal{A}$ is a Δ-system, then $|\mathcal{A}| \leq \frac{a}{2a-b} + 1$.*

PROOF. Let A_0, $|A_0| = b-a$, denote the kernel of the Δ-system $\mathcal{A}$. Let $B \in \mathcal{B}$ and set $\alpha = |B \cap A_0| \leq b-a$. Then, $|B \cap (A \setminus A_0)| = a - \alpha$ for all $A \in \mathcal{A}$. Hence, $2a = |B| \geq \alpha + |\mathcal{A}|(a-\alpha) = a|\mathcal{A}| - \alpha(|\mathcal{A}|-1) \geq a|\mathcal{A}| - (b-a)(|\mathcal{A}|-1) = (2a-b)|\mathcal{A}| + b - a$, which implies that $|\mathcal{A}| \leq \frac{3a-b}{2a-b} = \frac{a}{2a-b} + 1$. ∎

CLAIM 3.2 *(i) If $b \geq 2a$, then d is hypercube embeddable.*
(ii) If $b < 2a$ and $2 \leq |T| \leq |S| \leq \frac{a}{2a-b} + 1$, then d is hypercube embeddable.
(iii) If $b < 2a$ and d is hypercube embeddable, then $\min(|T|, |S|-1) \leq \lfloor \frac{b}{2a-b} \rfloor$ (else, d violates a $(2\lfloor \frac{b}{2a-b} \rfloor + 3)$-gonal inequality).

PROOF. For $(i), (ii)$, we show how to construct some families $\mathcal{A}$, $\mathcal{B}$ satisfying (7) and with $|\mathcal{A}| = |S|$, $|\mathcal{B}| = |T| - 1$. In both cases, we take for $\mathcal{A}$ a Δ-system with parameters $(b, b-a)$ and kernel A_0, $|A_0| = b - a$.
In case (i), as $b \geq 2a$, we can find a subset B_0 of A_0 with $|B_0| = a$. Then, we take for $\mathcal{B}$ a Δ-system with parameters $(2a, a)$ and kernel B_0 such that $(A \setminus A_0) \cap (B \setminus B_0) = \emptyset$ for all $A \in \mathcal{A}$, $B \in \mathcal{B}$.
In case (ii), we have that $a \geq (s-1)(2a-b)$ (setting $s := |S|$), i.e., for each $A \in \mathcal{A}$, we can find $s-1$ disjoint subsets $A^{(1)}, \ldots, A^{(s-1)}$, of $A \setminus A_0$, each of cardinality $2a-b$. Note that $x := b - a + s(2a-b) \leq 2a$. Let $X^{(1)}, \ldots, X^{(s-1)}$ be disjoint sets of cardinality $2a - x$, disjoint from $\bigcup_{A \in \mathcal{A}} A$. Given $A_1 \in \mathcal{A}$, we set $\mathcal{B} := \{B^{(1)}, \ldots, B^{(s-1)}\}$, where, for $1 \leq j \leq s-1$, $B^{(j)} = A_0 \cup A_1^{(1)} \cup \bigcup_{A \in \mathcal{A} \setminus \{A_1\}} A^{(j)} \cup X^{(j)}$. Then, $\mathcal{A}$, $\mathcal{B}$ satisfy (7).

(iii) Set $k := \min(|T|, |S|-1)$. Suppose, for contradiction, that $k \geq \lfloor \frac{b}{2a-b} \rfloor + 1$. Set $b_i = 1$ for $k+1$ points of S, $b_i = -1$ for k points of T, and $b_i = 0$ for the remaining points of V. Then, $\sum_{i,j \in V} b_i b_j d(i,j) = 2k(k(2a-b)-b) > 0$. Hence, d violates a $(2\lfloor \frac{b}{2a-b} \rfloor + 3)$-gonal inequality, contradicting the fact that d satisfies the hypermetric condition. ∎

CLAIM 3.3 *Suppose $|S| \geq a^2 + a + 3$ and $|T| \geq 2$. Then, d is hypercube embeddable if and only if $b \geq 2a$.*

PROOF. If $b \geq 2a$, then d is hypercube embeddable by Claim 3.2 (i). Conversely, if d is hypercube embeddable then, by Claim 2.3, the collection $\mathcal{A}$ is a Δ-system. Hence, $|\mathcal{A}| \leq \frac{a}{2a-b} + 1$, by Claim 3.1, contradicting the assumption $|S| = |\mathcal{A}| \geq a^2 + a + 3$. ∎

So, we have the following results.

PROPOSITION 3.4 *Let $a \leq b$ be positive integers with b odd. Let d be a distance on n points taking the values $2a$ and b. Suppose that d satisfies the even condition (5). If $b \geq 2a$, then d is hypercube embeddable. In particular, if $n \geq 2a^2 + 2a + 5$, then d is hypercube embedable if and only if $b \geq 2a$ or d is the distance from Figure 2.*

PROPOSITION 3.5 *Let $a \leq b$ be positive integers with b odd and $b < \frac{4}{3}a$. Let d be a distance taking the values $2a$ and b. Suppose that d satisfies the even condition (5). The following assertions are equivalent.*
(i) d is hypercube embeddable.
(ii) d satisfies the 5-gonal inequality (i.e., d does not contain as substructure the distance from Figure 4).
(iii) d is one of the distances from Figures 2 and 3.

Proposition 3.5 follows from Claim 3.2 (iii) after noting that $\lfloor \frac{b}{2a-b} \rfloor = 1$ if $b < \frac{4}{3}a$.

4 Distances with values $a, b, a+b$ (a,b odd)

Let a, b be positive odd integers with $a < b$. Let d be a distance on V taking the values $a, b, a+b$. If d is hypercube embeddable, then $V = S \cup T$, $S \cap T = \emptyset$, with $d(i,j) = a+b$ for $(i,j) \in S^2 \cup T^2$ and $d(i,j) \in \{a,b\}$ for $(i,j) \in S \times T$. Moreover, the pairs ij with $d(i,j) = a$ form a matching, by the triangle inequality (1).

CLAIM 4.1 *If there are at least two pairs at distance a, then $|S| = |T| = 2$ (else, d would violate the 5-gonal inequality).*

PROOF. Let $i, i' \in S$, $j, j' \in T$ such that $d(i,j) = d(i',j') = a$ and suppose, for contradiction, that there exist another point k. Set $b_i = b_{i'} = b_k = 1$, $b_j = b_{j'} = -1$, and $b_h = 0$ for the remaining points. Then, $\sum_{i,j \in V} b_i b_j d(i,j) = 4a > 0$, contradicting the fact that d satisfies the 5-gonal inequalities. ∎

From now on, we can suppose that there is exactly one pair (i_0, j_0) at distance a, where $i_0 \in S$, $j_0 \in T$.

We can suppose that j_0 is labeled by $\emptyset$ and, then, i_0 should be labeled by a set A_0 of cardinality a. Therefore, an h-labeling of d exists if and only if there exist two $(b, \frac{b-a}{2})$-intersecting systems $\mathcal{A}$ and $\mathcal{B}$ disjoint from A_0 such that $|\mathcal{A}| = |S| - 1$, $|\mathcal{B}| = |T| - 1$, and $|A \cap B| = \frac{a+b}{2}$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$. (Label the points of $S \setminus \{i_o\}$ by the members of $\mathcal{A}$ and the points of $T \setminus \{j_0\}$ by $A_0 \cup B$ where $B \in \mathcal{B}$.)

Such set families $\mathcal{A}$ and $\mathcal{B}$ can be constructed, for instance, if $|S| = |T| = 2$, or $|T| = 1 \leq |S|$, implying that d is hypercube embeddable in these cases.

This is obvious for the case $|T| = |S| = 2$. If $T = \{j_0\}$, then we can take for $\mathcal{A}$ a Δ-system with parameters $(b, \frac{b-a}{2})$.

But, in general, we do not know about the existence of the families $\mathcal{A}$ and $\mathcal{B}$. Note that, if $|T| \geq 2$, i.e., $\mathcal{B} \neq \emptyset$, then we can take for $\mathcal{A}$ a Δ-system only if $|\mathcal{A}| = |S| - 1 \leq \frac{b}{a}$. Indeed, let A_1, $|A_1| = \frac{b-a}{2}$, denote the kernel of $\mathcal{A}$ and let $B \in \mathcal{B}$, then $|B \cap (A \setminus A_1)| \geq a$ for all $A \in \mathcal{A}$, implying that $b = |B| \geq a|\mathcal{A}|$, i.e., $|\mathcal{A}| \leq \frac{b}{a}$.

Let us assume that d is a distance on $n \geq 2(\frac{a+b}{2})^2 + a + b + 5$ and $|S| \geq |T|$. Then, $|S| \geq (\frac{a+b}{2})^2 + \frac{a+b}{2} + 3$ and, thus, by Claim 2.3, the points of S must be labeled by the star-labeling in any h-labeling of d. It is easy to see that this implies that $\mathcal{A}$ is a Δ-sytem. Therefore, if $|T| \geq 2$, then $|S| \leq \frac{b}{a} + 1$, contradicting the above assumption on $|S|$. Hence, $|T| = 1$ in which case we saw above that d is indeed hypercube embeddable.

So, we have shown the following result.

PROPOSITION 4.2 *Let $a < b$ be odd integers and let d be a distance on $n \geq 2(\frac{a+b}{2})^2 + a + b + 5$ points taking the values $a, b, a+b$. Suppose that d satisfies the even condition* (5). *Then, d is hypercube embeddable if and only if d is the distance from Figure* 5.

Figure 5

In the general case, hypercube embeddability depends on the existence of two $(b, \frac{b-a}{2})$-intersecting systems $\mathcal{A}$ and $\mathcal{B}$ satisfying the "transversality" condition $|A \cap B| = \frac{b+a}{2}$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Using the hypermetric condition, we can give an upper bound on $\min(|\mathcal{B}|-1, |\mathcal{A}|-2)$ (if $|\mathcal{B}| \leq |\mathcal{A}|$).

CLAIM 4.3 *Suppose that* $|T| \leq |S|$ *and* $\min(|T|, |S|-1) \geq 2$. *Then,* $\min(|T|, |S|-1) \leq \lfloor \frac{b}{a} \rfloor - 1$ *(else, d violates the* $(2\lfloor \frac{b}{a} \rfloor + 1)$*-gonal inequality).*

PROOF. Set $k := \min(|T|, |S|-1)$ and suppose, for contradiction, that $k \geq \lfloor \frac{b}{a} \rfloor$. Set $b_i = 1$ for $k+1$ points of S including i_0, $b_i = -1$ for k points of T including j_0, and $b_i = 0$ for the remaining points. Then, $\sum_{i,j \in V} b_i b_j d(i,j) = 2(k-1)(a(k+1)-b) > 0$. This contradicts the fact that d satisfies the $(2\lfloor \frac{b}{a} \rfloor + 1)$-gonal inequality. ∎

In particular, if $b < 2a$, then $\min(|T|-1, |S|) = 1$, in which case d is indeed hypercube embeddable. Therefore,

PROPOSITION 4.4 *Let a, b be odd integers such that $a < b < 2a$. Let d be a distance taking the values $a, b, a+b$. Then, d is hypercube embeddable if and only if d is one of the distances from Figures* 5 *and* 6.

Figure 6

5 Distances with values $2a, b, 2a+b$ (b odd, $2a < b$)

Let a, b be positive integers such that b is odd and $2a < b$. Let d be a distance on n points that takes the values $2a, b$ and $2a+b$. We assume that d satisfies the even condition (5), i.e., V is partitioned into $V = S \cup T$ with $d(i,j) = 2a$ for $(i,j) \in S^2 \cup T^2$, $d(i,j) \in \{b, b+2a\}$ for $(i,j) \in S \times T$, and $|T| \leq |S|$.

Set $I = \{j \in T : d(i,j) = b + 2a \text{ for all } i \in S\}$, $U = \{j \in T : d(i,j) = b \text{ for all } i \in S\}$, and $M = T \setminus I \cup U$. For $j \in T$, set $N_b(j) = \{i \in S : d(i,j) = b\}$ and call $|N_b(j)|$ the **valency** of j. Two distinct elements $j, j' \in M$ are said to be **twins** (resp. **pseudotwins**, **symmetric**) if $N_b(j) = N_b(j')$ (resp. $|N_b(j) \triangle N_b(j')| = 1$, $|N_b(j) \setminus N_b(j')| = |N_b(j') \setminus N_b(j)| = 1$). A subset $M' \subseteq M$ is a **twin class** (resp. a **symmetric class**) if any two distinct elements of M' are twins (resp. symmetric).

We recall that $f(2a, a; a+b)$ denotes the maximum cardinality of a $(2a, a; a+b)$-intersecting system.

We show the following result.

Proposition 5.1 *With the notation above, suppose d is a distance on $n \geq 2a^2 + 2a + 5$ points. Then, d is hypercube embeddable if and only if (i) or (ii) holds.*
(i) $M = \emptyset$ and $|U| \leq \frac{b}{a}$ if $|I| \geq 2$, $|U| \leq f(2a, a; a+b)$ if $|I| = 1$.
(ii) $M = T$ and any two elements of T are twins, pseudotwins or symmetric. Moreover,
- either $|N_b(j)| = v$ for all $j \in T$ for some $1 \leq v \leq \frac{b}{a} + 1$ and T is a twin class or a symmetric class,
- or $|N_b(j)| \in \{v, v+1\}$ for all $j \in T$ for some $1 \leq v \leq \frac{b}{a}$. Set $T' = \{j \in T : |N_b(j)| = v\}$ and $T'' = T \setminus T'$. Then, either $|T'| = 1$, T'' is a symmetric class, or T'' is a twin class with $|T''| \leq \frac{b}{a} - v + 1$); or T' is a twin class with $|T'| \geq 2$ and T'' is a symmetric class; or T' is a symmetric class with $|T'| = 2$ and T'' is a twin class with $|T''| \leq \frac{b}{a} - v + 1$.

Consider the distance d from Figure 7. It is interesting to note that, if $|S| \geq a^2 + a + 3$, then d is hypercube embeddable if and only if $|U| \leq f(2a, a; a+b)$, i.e., there exists a $(2a, a; a+b)$-intersecting system of cardinality $|U|$.

Figure 7

In the remaining of this section, we give the proof of Proposition 5.1. As $n \geq 2n^2 + 2a + 5$ and $|S| \geq |T|$, the points of S should be labeled by the star-labeling (or an equivalent of it) in any h-labeling of d. So, we can suppose that the points $i \in S$ are labeled by sets A_i, where the A_i's are pairwise disjoint sets of cardinality a. Set $A = \bigcup_{i \in S} A_i$.

The next Claim 5.2 establishes Proposition 5.1 (i).

Claim 5.2 *Assume that $M = \emptyset$, i.e., $I, U \neq \emptyset$. Then, d is hypercube embeddable if and only if $|U| \leq \frac{b}{a}$ if $|I| \geq 2$, and $|U| \leq f(2a, a; a+b)$ if $|I| = 1$.*

PROOF. We show that the following assertions (a) – (c) are equivalent.
(a) d is hypercube embeddable,
(b) the projection of d on $\{i, i'\} \cup T$, where i, i' are distinct elements of S, is hypercube embeddable,
(c) $|U| \leq \frac{b}{a}$ if $|I| \geq 2$, and $|U| \leq f(2a, a; a+b)$ if $|I| = 1$.
$(a) \Longrightarrow (b)$ holds obviously. We show $(c) \Longrightarrow (a)$. Let X be a set of cardinality $a+b$ disjoint from A. Suppose first that $|I| = 1$. By assumption, we can find a $(2a, a; a+b)$-intersecting system $\mathcal{B}$ on X with cardinality $|U|$. Label the element of I by X and the elements $j \in U$ by the sets $B \triangle X$ for $B \in \mathcal{B}$. This gives an h-labeling of d. Suppose now that $|I| \geq 2$. As $|U| \leq \frac{b}{a}$, we can construct a Δ-system $\mathcal{B}$ on X with parameters $(2a, b; a+b)$ and $|\mathcal{B}| = |U|$. Let B_1 denote the kernel of $\mathcal{B}$, let $j_0 \in I$, and let C_j, $j \in I \setminus \{j_0\}$, be pairwise disjoint sets of cardinality a disjoint from X. Label the elements of U by the sets $B \triangle X$ for $B \in \mathcal{B}$, label j_0 by X and the elements $j \in I \setminus \{j_0\}$ by the sets $X \triangle (B_1 \cup C_j)$. This gives an h-labeling of d.

$(b) \Longrightarrow (c)$ Consider an h-labeling of the projection of d on $\{i, i'\} \cup T$ in which a given element $j_0 \in I$ is labeled by $\emptyset$. Denote by A, A', C_j $(j \in I \setminus \{j_0\})$, and B_k $(k \in U)$ the sets labeling i, i', $j \in I \setminus \{j_0\}$, $k \in U$, respectively. Then, $|A| = |A'| = b + 2a$, $|A \cap A'| = b + a$, and both $\mathcal{B} = \{B_k : k \in U\}$ and $\mathcal{C} = \{C_j : j \in I \setminus \{j_0\}\}$ are $(2a, a)$-intersecting systems. Moreover, $b = |B_k \triangle A|$, implying that $B_k \subseteq A$. Hence, $B_k \subseteq A \cap A'$ for all $k \in U$, i.e., $\mathcal{B}$ is a $(2a, a; a+b)$-intersecting system, which implies that $|\mathcal{B}| = |U| \leq f(2a, a; a+b)$. Suppose that $|I| \geq 2$ and let $j \in I \setminus \{j_0\}$. Then, $b + 2a = |C_j \triangle A|$, i.e., $|A \cap C_j| = a$. Also, $2a = |B_k \triangle C_j|$, i.e., $B_k \cap C_j| = a$. Therefore, $B_k \cap C_j = A \cap C_j \subseteq B_k$ for all $k \in U$. This shows that $\mathcal{B}$ is a Δ-system, implying that $|\mathcal{B}| = |U| \leq \frac{b}{a}$. ∎

From now on, we assume that $M \neq \emptyset$.

CLAIM 5.3 *Let $j \in M$. Then, $v := |N_b(j)| \leq \frac{b}{a}+1$ and j should be labeled by $\bigcup_{i \in N_b(j)} A_i \cup X_j$, where X_j is a set disjoint from A with $|X_j| = b - va + a$. We call X_j the* **residual label** *of j.*

PROOF. Denote by B the set labeling a given point $j \in M$. Let $i, i' \in S$ such that $d(i,j) = b$ and $d(i',j) = b + 2a$. Then, $b = |B \triangle A_i|$, i.e., $|B| = b - a + 2|B \cap A_i| \leq b + a$ since $|B \cap A_i| \leq a$. Also, $b + 2a = |B \triangle A_{i'}|$, i.e., $|B| = b + a + 2|B \cap A_{i'}| \geq b + a$. Hence, $|B| = b + a$, $A_i \subseteq B$ for all $i \in N_b(j)$, and $A_{i'} \cap B = \emptyset$ for all $i \in S \setminus N_b(j)$. Therefore, $B = \bigcup_{i \in N_b(j)} A_i \cup X_j$, where $|X_j| = b - va + a$, implying that $v \leq \frac{b}{a} + 1$. ∎

CLAIM 5.4 *Let j, j' be distinct elements of M. Then, either $N_b(j) = N_b(j')$ (j, j' are twins), or $N_b(j) \subset N_b(j')$ with $|N_b(j') \setminus N_b(j)| = 1$ (j, j' are pseudotwins), or $|N_b(j) \setminus N_b(j')| = |N_b(j') \setminus N_b(j)| = 1$ (j, j' are symmetric).*

PROOF. Suppose that $N_b(j) \subset N_b(j')$ and j, j' have respective valencies v, v' with $v < v'$. By Claim 5.3, j, j' have residual labels $X_j, X_{j'}$, with $|X_j| = b - va + a$, $|X_{j'}| = b - v'a + a$. Then, $2a = d(j, j') = |X_j \triangle X_{j'}| + (v' - v)a$, implying that $|X_j \cap X_{j'}| = b - va$. Hence, $|X_{j'} \setminus X_j| = a(v - v' + 1) \geq 0$, which implies that $v' = v + 1$, i.e., j, j' are pseudotwins. Suppose now that $N_b(j) \not\subset N_b(j')$ and $N_b(j') \not\subset N_b(j)$. Suppose that we can find distinct elements $i, i' \in N_b(j) \setminus N_b(j')$ and let $i'' \in N_b(j') \setminus N_b(j)$. Set $b_i = b_{i'} = b_{j'} = 1$, $b_{i''} = b_j = -1$, and $b_h = 0$ for the remaining elements. Then, $\sum_{h,h' \in V} b_h b_{h'} d(h, h') = 4a > 0$. Hence, d violates the 5-gonal inequality, contradicting the hypermetric condition. This shows that $|N_b(j) \triangle N_b(j')| = 2$, i.e., j, j' are symmetric. ∎

Note that any two symmetric elements $j, j' \in M$ should receive the same residual label, i.e., $X_j = X_{j'}$. Hence, the relation "j, j' are symmetric" is an equivalence relation on M.

CLAIM 5.5 *If $M \neq \emptyset$, then $I = U = \emptyset$.*

PROOF. Let $j \in M$ with valency v and let $k \in I \cup U$ be labeled by a set C. Then, $2a = d(j, k) = |C \triangle (\bigcup_{i \in N_b(j)} A_i \cup X_j)| \geq \sum_{i \in N_b(j)} |C \triangle A_i| \geq vb > 2av$, since $d(i, k) = |A_i \triangle C| \in \{b, b + 2a\}$ and $b > 2a$. We obtain a contradiction. Hence, $I = U = \emptyset$. ∎

By Claim 5.5, $T = M$ and, by Claim 5.4, either all elements $j \in T$ have the same valency v ($1 \leq v \leq \frac{b}{a} + 1$), or they have valency v or $v + 1$ ($1 \leq v \leq \frac{b}{a}$). Set $T' = \{j \in T : |N_b(j)| = v\}$ and $T'' = T \setminus T'$. Then, each of T', T'' is a symmetric class or a twin class.

Suppose first that all elements of T have the same valency v. Then, d is hypercube embeddable. Indeed, if T is a twin class, we can choose for the residual labels X_j, $j \in T$, the members of a Δ-system with parameters $(b - va + a, b - va)$. If T is a symmetric class, take all the residual labels of $j \in T$ equal to a given set X of cardinality $b - va + a$. This provides in both cases an h-labeling of d.

Suppose now that the elements of T have two possible valencies v and $v + 1$.

CLAIM 5.6 *(i) If $|T'| = 1$, then d is hypercube embeddable if and only if T'' is a symmetric class, or T'' is a twin class with $|T''| \leq \frac{b}{a} - v + 1$.*
(ii) If $|T'| \geq 2$ and T' is a twin class, then d is hypercube embeddable if and only if T'' is a symmetric class.
(iii) If $|T'| \geq 2$ and T' is a symmetric class, then d is hypercube embeddable if and only if $|T'| = 2$ and T'' is a twin class with $|T''| \leq \frac{b}{a} - v + 1$.

PROOF. (i) Let $T' = \{j_0\}$. If T'' is a symmetric class, then d is hyeprcube embeddable. Indeed, let X be a set disjoint from A with $|X| = b-va+a$ and let $Y \subseteq X$ with $|Y| = b-va$. Take X as residual label for j_0 and Y as residual label for all the elements of T''. Suppose now that T'' is a twin class. Let X denote the residual label of j_0 and let X_k denote the residual label of $k \in T''$. Then, $X_k \subseteq X$, $|X_k| = b - va$, $|X| = b - va + a$, $|X_k \triangle X_{k'}| = 2a$ for $k \neq k' \in T''$. Set $Y_k = X \setminus X_k$ for $k \in T''$. Then, the Y_k's are pairwise disjoint subsets of X of cardinality a. This implies that $|T''| \leq \frac{b}{a} - v + 1$. Conversely, if T'' is a twin class with $|T''| \leq \frac{b}{a} - v + 1$, then d is hypercube embeddable (take the residual labels in the way indicated above).

(ii) Since T' is a twin class the residual labels X_j, $j \in T'$, form a $(b - va + a, b - va)$-intersecting system. Let $k \in T''$ with residual label X_k and $j \in T'$. Then, $2a = d(j,k) = |X_j \triangle X_k| + a$, from which we deduce that $X_k \subseteq X_j$. As $|T'| \geq 2$, we have that $X_k = X_j \cap X_{j'}$ for $j \neq j' \in T'$. Hence, T'' is a symmetric class. Then, d is indeed hypercube embeddable. Indeed, let X be a set disjoint from A with $|X| = b - va$. Take the residual labels of the elements of T'' all equal to X and take for the residual labels of the elements of T' a Δ-system with kernel X and parameters $(b - va + a, b - va)$.

(iii) Since T' is a symmetric class, all the elements of T' have the same residual label X, $|X| = b - va + a$. For $j \neq j' \in T'$, $k \in T''$, we have that $N_b(k) = N_b(j) \cup N_b(j')$, which implies that $|T'| = 2$ and that T'' is a twin class. If $|T''| \geq 2$, then all residual labels X_k, $k \in T''$, are contained in X, from which we deduce as in case (i) that $|T''| \leq \frac{b}{a} - v + 1$. Conversely, if $|T'| = 2$, $|T''| \leq \frac{b}{a} - v + 1$, T' is a symmetric class, and T'' is a twin class, then d is hypercube embeddable. Indeed, let X be the residual label of both elements of T' and let $X_k = X \setminus Y_k$ be the residual labels of $k \in T''$, where the Y_k's are disjoint subsets of X of cardinality a. ∎

This concludes the proof of Proposition 5.1.

6 Distances with values $b, 2a, b + 2a$ (b odd, $b < 2a$)

Let a, b be positive integers with b odd and $b < 2a$. Let d be a distance on V, $|V| = n$, that takes the values $b, 2a$ and $b + 2a$. We show the following result.

PROPOSITION 6.1 *Let a, b be positive integers with b odd and $b < 2a$. Let d be a distance on $n \geq 2a^2 + 2a + 5$ points that takes the values $b, 2a, b + 2a$. Suppose that d satisfies the even condition* (5) *and the triangle inequality* (1). *Then, the following assertions are equivalent.*
(i) d is hypercube embeddable.

(ii) d does not contain as substructure any of the distances from Figures 8-15.
In particular, if $b < a$, then d is always hypercube embeddable. (In Figures 8-15, a plain edge represents distance $2a + b$, a dotted edge distance b and no edge means distance $2a$.)

Figure 9

Figure 10

Figure 8

Figure 11

Figure 12

Figure 13

Figure 14

Figure 15

PROOF. For the implication $(i) \Longrightarrow (ii)$, we check that none of the distances from Figures 8-15 is hypercube embeddable.

Indeed, the distances from Figures 8-14 violate the hypermetric condition (thus, they are not ℓ_1-embeddable). The numbers assigned to the nodes in Figures 8-14 indicate a choice of integers b_i's for which the hypermetric inequality (6) is violated. For instance, for the distance from Figure 8, we have that $\sum_{i,j\in V} b_i b_j d(i,j) = 4a(2a(2a-b)-b) \geq 4a > 0$ since $2a - b \geq 1$.

The distance from Figure 15 is not hypercube embeddable by Corollary 3.3.

We now show the implication $(ii) \Longrightarrow (i)$. As d satisfies the even condition, V is partitioned into $S \cup T$ with $|S| \geq |T|$, $d(i,j) = 2a$ for $(i,j) \in S^2 \cup T^2$, $d(i,j) \in \{b, b+2a\}$ for $(i,j) \in S \times T$. Set $s = |S|$. For $j \in T$, set $N_b(j) = \{i \in S : d(i,j) = b\}$. For $v \in \{0, 1, 2, \ldots, s-1, s\}$, set $T_v = \{j \in T : |N_b(j)| = v\}$. We group below several observations on the sets T_v.

(i) $T_{s-1} = \emptyset$ (since d does not contain the configuration from Figure 8).
(ii) $|T_s| \leq 1$ (since d does not contain the configuration from Figure 15).
(iii) All T_v are empty except maybe T_0, T_1, T_2, T_s (indeed, $|N_b(j)| \leq 2$ or $|N_b(j)| \geq s-1$ for all $j \in T$, since d does not contain the substructure from Figure 9).
(iv) At most one of T_0 and T_2 is not empty (since d does not contain the substructure from figure 10).
(v) If $|T_1| \geq 2$, then
$(v1)$ either all $N_b(j)$, $j \in T_1$, are equal,
$(v2)$ or all $N_b(j)$, $j \in T_1$, are distinct
(since d does not contain the substructure from Figure 11).
(vi) If $j \neq j' \in T_2$, then $|N_b(j) \cap N_b(j')| = 1$ (use Figures 11 and 12).
(vii) If $j \in T_1$ and $j' \in T_2$, then $N_b(j) \cap N_b(j') \neq \emptyset$ (by Figure 11).
$(viii)$ If $b < a$, then $T_2 = T_s = \emptyset$ (by the triangle inequality).

We now show how to construct an h-labeling of d. Let A_i, $i \in S$, be disjoint sets of cardinality a. Set $A = \cup_{i\in S} A_i$. Label the elements of S by the A_i's.

Suppose first that $b < a$. Then, by $(viii)$, $d(i_1, j_1) = \ldots = d(i_r, j_r) = b$ for some $i_1, \ldots, i_r \in S$, $j_1, \ldots, j_r \in T$, $1 \leq r \leq |T|$. Let X, B_j $(j \in T \setminus \{j_1, \ldots, j_r\})$, be disjoints sets, disjoint from A, and with $|X| = b$, $|B_j| = a$. Label $j_1, \ldots, j_r$ by $A_{i_1} \cup X, \ldots, A_{i_r} \cup X$, respectively, and $j \in T \setminus \{j_1, \ldots, j_r\}$ by $X \cup B_j$. This gives an h-labeling of d.

We now suppose that $b \geq a$. Let X be a set disjoint from A with $|X| = b - a$.
- If $T_s \neq \emptyset$, $T_s = \{x\}$ (by (i)), then label x by X.
- Label each element $j \in T_2$ by $\bigcup_{i\in N_b(j)} A_i \cup X$ (this gives already an h-labeling of the projection of d on $S \cup T_s \cup T_2$ (by (vi))).
- Suppose that all $N_b(j)$, $j \in T_1$, are equal to, say, $\{i_0\}$, as in $(v1)$. Let Y_j, $j \in T_1$, be disjoint sets, disjoint from A and X, and with cardinality a. Label $j \in T_1$ by $A_{i_0} \cup X \cup Y_j$. If all $N_b(j)$, $j \in T_1$, are distinct, as in $(v2)$, then label $j \in T_1$ by $\bigcup_{i\in N_b(j)} A_i \cup X \cup Y$, where Y is a set disjoint from A and X with $|Y| = a$.
(In both cases, we have obtained an h-labeling of the projection of d on $S \cup T_s \cup T_2 \cup T_1$

(by (vii)).)
- Suppose that $T_0 \neq \emptyset$. Then, $T_2 = \emptyset$ by (iv). Let Z_k, $k \in T_0$, be disjoint sets, disjoint from all the sets constructed so far, with cardinality a.
If we are in case $(v1)$, then $|T_1| \leq 1$ or ($|T_1| \leq 2$ and $|T_0| = 1$). (Indeed, if $|T_1|, |T_2| \geq 2$, then d contains the substructure from Figure 14 and, if $|T_1| \geq 3$, $|T_0| = 1$, then we have the substructure from Figure 13.) If $|T_1| = 1$, $T_1 = \{j\}$, label $k \in T_0$ by $X \cup Y_j \cup Z_k$. If $|T_1| = 2$, $T_1 = \{j, j'\}$, then label the unique element $k \in T_0$ by $X \cup Y_j \cup Y_{j'}$.
Else, we are in case $(v2)$. Then, label $k \in T_0$ by $X \cup Y \cup Z_k$.
In both cases, we have constructed an h-labeling of d. ∎

Observe that the exclusion of the distance from Figure 15 is used only for showing that $|T_s| \leq 1$, i.e., that at most one point is at distance b from all points of S. Consider the distance d_s on $s+2$ points which has the same configuration as in Figure 15 but with s nodes on the top level instead of a^2+a+3. Let $s_2(a,b)$ denote the largest integer s such that d_s is hypercube embeddable. Then, Proposition 6.1 remains valid if we exclude the distance $d_{s_2(a,b)+1}$ instead of excluding the distance d_{a^2+a+3} from Figure 15. Note that $2 \leq \frac{a}{2a-b} + 1 \leq s_2(a,b) \leq a^2+a+2$, with $s_2(a,b) = 2$ if $b < \frac{4}{3}a$ (use Claim 3.2). So, we have the following result.

Proposition 6.2 *Let a, b be positive integers with b odd and $b < \frac{4}{3}a$. Let d be a distance on $n \geq 2a^2 + 2a + 5$ points that takes the values $b, 2a, b+2a$. Suppose that d satisfies the even condition* (5) *and the triangle inequality* (1). *Then, the following assertions are equivalent.*
(i) d is hypercube embeddable.
(ii) d is ℓ_1-embeddable.
(iii) d is hypermetric.
(iv) d does not contain as substructure any of the distances from Figures 4 *and* 8-14.

Note that Proposition 6.2 is a direct extension of the result given in [Avi90] for the subcase $a = b = 1$.

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