L-polytopes, Even Unimodular Lattices, and Perfect Lattices

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Abstract

It is shown here that every L-polytope of an even unimodular lattice does not generate the lattice.

It is given here the corrected formulation of a previous result of the author [3] on relations between extreme L-polytopes and perfect lattices. We prove here the following special case. If the square radius of the circumscribing sphere of an extreme L-polytope P of a lattice L is less than the minimal norm m of L, then the m-extension of P generates a perfect lattice.

1 Introduction

Recall some notions of integral lattices and L-polytopes. Details see in [1].

An *L-polytope* of a lattice L is the convex hull of all lattice points lying on an *empty sphere*. An empty sphere in a lattice L of dimension n is such a sphere that there is no lattice point inside the sphere, and the lattice points lying on the sphere affinely generate the n-dimensional space.

We call the radius of the empty sphere a radius of the inscribed L-polytope.

Let P be an L-polytope of a lattice L. We take the center of the empty sphere circumscribing the polytope P as the origin. Let $V = V(P)$ be the

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set of vertices of P, and let $v \in V(P)$ be the vector with endpoint in the vertex v. If r is the radius of P, then $v^2 = r^2$ for all $v \in V$.

Every point w of the lattice $L(P)$ affinely generated by vertices of the L-polytope P has the form

$$w = \sum_{v \in V} z_v v \text{ with } \sum_{v \in V} z_v = 1,\ z_v \in Z. \tag{1}$$

Clearly, $L(P) \subseteq L$. If $L(P) = L$, then the L-polytope P is called *generating*.

If P has dimension n, and there are $n+1$ affinely independent vectors $v_0, v_1, ..., v_n \in V$ such that any point $w \in L(P)$ is represented by the vectors, i.e. $w = \sum_0^n z_i v_i$, and $\sum_o^n z_i = 1$, then $\{v_0, v_1, ..., v_n\}$ is an affine basis of $L(P)$. In the case the L-polytope P is called *basic*. It is not proved that each L-polytope is basic. But examples of nonbasic L-polytopes are not known, and all L-polytopes of dimension ≤ 5 are basic.

If $v_0 \in V$, then the lattice $L(P)$ is *linearly* generated by the vectors $v - v_0$, $v \in V$. The vectors of the form $v - v'$ are called *lattice vectors*.

A minimal set of lattice vectors of a lattice L linearly generating L is called a *basis* of L. If dimension of L is n, then every basis of L contains n vectors.

Let $\mathcal{B} = \{b_1, ..., b_n\}$ be a basis of an n-dimensional lattice L. Then every lattice vector a of L is uniquely represented by the basis $\mathcal{B}$ as follows: $a = \sum_1^n z_i b_i$ with integral z_i.

A lattice L is called *integral* if the inner product (a_1, a_2) of any two lattice vectors $a_1, a_2 \in L$ is an integer.

For $a \in L$, the number $a^2 \equiv (a, a)$ is the *norm* of the vector a. If the norms of all vectors of an integral lattice are even, then the lattice is called *even*. So, an even lattice is integral.

Another way to represent a lattice L is to give a quadratic form representing norms of lattice vectors. If $a = \sum_1^n z_i b_i$, then

$$a^2 = \sum_{1 \leq i \leq j \leq n} b_{ij} z_i z_j = z^T B z,$$

where $B = (b_{ij}) = ((b_i, b_j))$ is the Gram matrix of the vectors b_i. The determinant of the matrix B, $detB$, does not depend on the basis B of the lattice L, and is denoted by $detL$.

A lattice L is called *perfect* if L is uniquely determined by vectors of minimal norm m. Let $\mathcal{M}(L) = \{a = \sum_1^n b_i z_i^a \in L : a^2 = m\}$ be the set of all minimal vectors of L. If L is perfect, then the system of linear equations

$$\sum_{1 \leq i \leq j \leq n} b_{ij} z_i^a z_j^a = m,\ a \in \mathcal{M}(L) \tag{2}$$

with the set of $n(n+1)/2$ unknowns b_{ij}, $1 \le i \le j \le n$, has rank $n(n+1)/2$.

A dual L^* of a lattice L is the set of all vectors c such that (c, a) is an integer for all $a \in L$. If L is integral, then $L \subseteq L^*$, and $detL^* = (detL)^{-1}$.

A lattice is called *unimodular* if $detL = 1$. An integral lattice is unimodular if and only if $L = L^*$.

An even unimodular lattice exists only for dimension which is a multiple of 8. There are one 8-dimensional even unimodular lattice, the root lattice E_8, two lattices of dimension 16, and 24 lattices of dimension 24, one of which is the famous Leech lattice. There are more than 8.10^7 even unimodular lattices of dimension 32.

The distance space (V, d) (where $d(u, v) = (u - v)^2$ is the squared Euclidean distance between vertices of an L-polytope P) is *hypermetric*, i.e. it satisfies all hypermetric inequalities

$$\sum_{u,v \in V} z_u z_v d(u, v) \le 0 \text{ for all integral } z_v \text{ such that } \sum_{v \in V} z_v = 1.$$

The inequality is an expansion of the condition $w^2 \ge r^2$ for w given by (1).

A *t-extension* of the distance space (V, d) is the distance space$(V \cup \{w\}, d)$ such that $d(w, v) = t$ for all $v \in V$. If $V = V(P)$, we say that we have a t-extension of P.

Let P be a basic L-polytope with the affine basis $\{v_0, v_1, ..., v_n\}$. Every vertex $v \in V(P)$ is represented in the basis as follows: $v = \sum_0^n z_i^v v_i$ with integral z_i^v and $\sum_0^n z_i^v = 1$. The L-polytope P is called *extreme* if the system of linear equations

$$\sum_{0 \le i < j \le n} d(v_i, v_j) z_i^v z_j^v = 0, \; v \in V(P) \tag{3}$$

with the set of $n(n+1)/2$ unknowns $d(v_i, v_j)$, $0 \le i < j \le n$, has rank $n(n+1)/2 - 1$.

An extreme L-polytope generates in a sence extreme rays of the cone Hyp_n of all hypermetrics on n points (for details, see [1]). Hyp_n has nontrivial extreme rays (=extreme rays distinct from cut rays) only for $n \ge 7$. Since L-polytopes of dimension $n \le 5$ are basic, and extreme rays of Hyp_n for $n \le 6$ are trivial (Hyp_n coincides with the cut cone for $n \le 6$), extreme L-polytopes exist for dimension ≥ 6, only.

In low dimensions, there are known only two nontrivial extreme L-polytopes:

a) the 6-dimensional asymmetric Schläfli polytope with 27 vertices, and

b) the 7-dimensional symmetric Gosset polytope with 56 vertices.

The extremality of the Schläfli and the Gosset polytopes is proved in [1], see also [2].

2 L-polytopes of an even unimodular lattice

In this section we prove the following

Theorem 2.1. *Every L-polytope of an even unimodular lattice is not generating.*

The theorem is a corollary of the following

Lemma 2.2. *Let P be a generating L-polytope of an even lattice L. Then the center of the polytope P belongs to the dual lattice L^*.*

Proof. Let the origin be in the center of P. Let V be the set of vertices of P, and let $v_0 \in V$. By definition, $v^2 = r^2$ for all $v \in V$ where r is the radius of P.

Since L is even, the norm of the lattice vector $v - v_0$,

$$(v - v_0)^2 = v^2 - 2(v, v_0) + v_0^2 = 2(r^2 - (v, v_0)),$$

is an even integer for every $v \in V$. In other words

$$r^2 - (v, v_0) = \text{ an integer for all } v \in V.$$

Consider the vector v_0, and show that the vector belongs to the dual lattice L^*. In fact, we have

$$-(v_0, v - v_0) = v_0^2 - (v, v_0) = r^2 - (v, v_0) = \text{ an integer.}$$

Since P is generating, the vectors $v - v_0$, for $v \in V$, linearly generate the lattice L. Hence (v_0, a) is an integer for every $a \in L$, i.e. $v_0 \in L^*$. Since endpoints of v_0 are points of L, and the vector v_0 belongs to L^*, the center of P, which is one of the endpoints of the vector v_0, belongs to L^*. $\square$

3 Extreme L-polytopes and perfect lattices

It is given in [3] a theorem which is slightly not correct. I give here the correct formulation.

Theorem 3.1. (Proposition 10 of [3]) *Let (V, d) be an extreme hypermetric space generating a lattice L_0. Let r be radius of the circumscribing* *sphere, and $r^2 \leq t$. Let w be an extension point of an t-extension of (V, d). Then the lattice L affinely generated by V and w is t-perfect if there are $v, v' \in V$ such that $d(v, v') = t$.*

If t of Theorem 3.1 is minimal norm of the lattice L, then L is perfect. But we cannot demand in the formulation of Theorem 3.1 that t is minimal norm of L.

If t is minimal norm of the lattice $L(P) = L_0$, and $t > r^2$, then one can prove that t is minimal norm of the lattice L, too. Below, we reformulate Theorem 3.1 for t equal to minimal norm of $L(P)$ as Theorem 3.2. The given proof is in essential the same as the proof of Theorem 3.1 given in [3].

If t is minimal norm of the lattice $L(P)$, and $t = r^2$, then we cannot prove that t is minimal norm of the lattice L. In the case, we demand in Theorem 3.3 that t is minimal norm of the lattice L (not of $L(P)$). But now, there is no necessity in condition that t is a distance between two points of V. In fact, in many cases t is less than minimal norm of $L(P)$.

Theorem 3.2. *Let P be an extreme L-polytope generating a lattice $L(P)$ of minimal norm m, and there are $v, v' \in V(P)$ with $d(v, v') = m$. If the squared radius of P is less than m, then the m-extension of P generates a perfect lattice.*

Proof. Let w be the extension point of P. Let $\{v_0, v_1, ..., v_n\}$ be an affine basis of $L(P)$, $v_i \in V(P)$. Let r be radius of P. Recall that the origin is in the center of the sphere.

Since $r^2 < m$, the point w does not lie in the n-dimensional space spanned by P.

If we set $b_i = v_i - w$, $0 \leq i \leq n$, then $\{b_0, b_1, ..., b_n\}$ is a basis of the (n+1)-dimensional lattice L generated by w and V.

The lattice L is composed of n-dimensional layers isomorphic to $L(P)$.

The minimal norm of L is m. In fact, if there is a vector $a_0 \in L$ of norm less than m, then its endpoints lie in different (and neighbouring) layers. W.l.o.g. we can suppose that w is one of endpoints of a_0, and the other endpoint lies in the space spanned by $L(P)$. Since $a_0^2 < m$, the endpoint of a_0 lies inside the sphere circumscribing P. This contradicts to that P is an L-polytope of the lattice $L(P)$, and $L(P)$ is a layer of L.

Let $v = \sum_0^n z_i^v v_i$, $\sum_0^n z_i^v = 1$, be the representation of a point $v \in V$ in the affine basis $\{v_0, ..., v_n\}$. Then the lattice vector $a_v = v - w$ of the lattice L has the following representation in the basis $\{b_0, ..., b_n\}$

$$a_v = \sum_0^n z_i^v b_i.$$

Note that

$$d(v_i, v_j) = (v_i - v_j)^2 = ((v_i - w) - (v_j - w))^2 = (b_i - b_j) = b_i^2 + b_j^2 - 2(b_i, b_j).$$

Since $d(v_i, w) = b_i^2 = m$, we obtain that $d(v_i, v_j)$ and the coefficients b_{ij} of the Gram matrix B are related as follows

$$d(v_i, v_j) = 2(m - b_{ij}),\ 0 \leq i \leq j \leq n. \tag{4}$$

Since $d(v_i, v_i) = 0$, we can use the equations (3) with $i = j$ in the sum. Substituting (4) in such transformed (3) and using the equality $\sum_0^n z_i^v = 1$, we obtain

$$\sum_{0 \leq i \leq j \leq n} z_i^v z_j^v b_{ij} = m. \tag{5}$$

Note that this equality is the expansion of the condition that each vector a_v, $v \in V$, having the representation $a_v = \sum z_i^v b_i$ has the norm m. For $v = v_i$, the corresponding equation of (5) is the equalitiy $b_i^2 = m$.

If the polytope P is extreme, then the system of equations (3) determines uniquely (up to positive multiple) the distances $d(v_i, v_j)$.

The equality $d(v, v') = m$ determines the multiple. In fact, the vector $v - v'$ is a lattice vector of the lattice L. Hence it has representation $v - v' = \sum_0^n z_i b_i$ in the basis $\{b_0, ..., b_n\}$. Since the vector $v - v'$ has minimal norm, we obtain an additional equation of the type (5):

$$\sum_{0 \leq i \leq j \leq n} z_i z_j b_{ij} = m. \tag{6}$$

This implies that the system of equalities (5) and (6) determines uniquely the matrix (b_{ij}), i.e. the lattice L is perfect. □

Recall that the *contact polytope* of a lattice is the convex hull of endpoints of all minimal vectors of the lattice.

We note that the L-polytope P can be obtained from the lattice L affinely generated by w and V by the construction described in [1], Lemma 7.2, i.e. by a section of the contact polytope of a lattice. Hence we can reformulate Theorem 3.2 as follows.

Theorem 3.2'. *Let an extreme n-dimensional L-polytope P is a section of the contact polytope of a (n+1)-dimensional lattice L by a hyperplane not going through the center of the contact polytope. Then the lattice L is perfect.* □

Since extreme L-polytopes exist only for dimension not less than 6, the perfect lattices which can be obtained as in Theorem 3.2 have dimension not less than 7. In general, using Table 1 of [4] of all known 7-dimensional perfect lattices one can obtain all extreme polytopes which are a section of the contact polytope of one of the 7-dimensional perfect lattices. But no effective algorithm of finding such a section is known. I can only say that among sections orthogonal to a minimal vector there is only one giving an extreme L-polytope. This is the case of the lattice E_7^*. The corresponding extreme L-polytope is the Schläfli polytope having 27 vertices.

Another examples of perfect lattices obtained as an extension are the root lattice E_8, the lattice Λ'_{16} of [1] and the Leech lattice. The corresponding extreme polytopes are the Gosset polytope 3_{21}, the polytope P^{16} related to the Barnes-Wall lattice and the polytope P_{23} related to the Leech lattice. The polytopes are described in [1].

Theorem 3.3. *Let P be an extreme L-polytope of radius r. Let its set of vertices V and the center of P generate a lattice L with minimal norm r^2. Then L is perfect.*

Proof. By supposition $V \subseteq \mathcal{M}(L)$, where $\mathcal{M}(L)$ is the set of all minimal vectors of L.

If $\{v_0, v_1, ..., v_n\}$ is an affine basis of P, then the system (3) determines uniquely up to a multiple λ the distances $d(v_i, v_j)$. We have

$$d(v_i, v_j) = (v_i - v_j)^2 = 2(r^2 - (v_i, v_j)) = 2r^2(1 - \phi_{ij})$$

where ϕ_{ij} is the angle between the vectors v_i and v_j. Hence if we fix the radius r, we fix the multiple λ.

Since $\{v_0, v_1, ..., v_n\}$ is an affine basis, there is only one linear dependency between the vectors $v_i, 1 \leq i \leq n$. Hence we can suppose that $v_1, ..., v_n$ are linearly independent. Let W be the matrix with the elements (v_i, v_j), $1 \leq i \leq j \leq n$. Recall that, for fixed r, the elements of W are uniquely determined by the system (3). The system is equivalent the system of all equations

$$v^2 = r^2, \text{ i.e. } \sum_{0 \leq i \leq j \leq n} z_i^v z_j^v (v_i, v_j) = r^2, \; v \in V. \tag{7}$$

Let $\{b_1, ..., b_n\}$ be a basis of L, and let $v_i = \sum_{i=1}^n x_i^k b_k$. Let X be the matrix whose i-th row is $\{x_i^k : 1 \leq k \leq n\}$, and let B be the Gram matrix of the vectors b_i. We have

$$W = XBX^T. \tag{8}$$

Since the matrix X is not singular, we obtain

$$B = X^{-1}W(X^T)^{-1}.$$

Since the matrix W is uniquely determined by the system (3) and by the value of r, the Gram matrix B is determined uniquely, too. We see that if we substitute (8) in (7), we obtain a subsystem of the system (2) uniquely determining b_{ij}. $\square$

There are 7 perfect lattices in dimension 6. There are 33 known perfect lattices in dimension 7. (See [4]).

I know only the following examples of L-polytopes of dimension 6 and 7 satisfying the conditions of Theorem 3.3.

The asymmetric Schläfli polytope is an extreme L-polytope of the root lattice E_6. The polytope with its center generates the dual lattice E_6^* which is perfect. The contact polytope of E_6^* is the diplo-Schläfli polytope, vertices of which are vertices of the Scläfli polytope and its antipodes (the name is taken from [5]).

The symmetric Gosset polytope is an extreme L-polytope of the root lattice E_7 and the contact polytope of the perfect lattice E_7^* (see [5]).

Unfortnately Theorems 3.2 and 3.3 cannot be reversed. In the first case, not all perfect lattices provide extreme L-polytopes. In the second case, the contact polytope not of each perfect lattice is an L-polytope. But if it is an L-polytope, it is not always extreme.

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