A Geometric Proof of the Enumeration Formula for Sturmian Words

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Abstract

The number of factors of length m of Sturmian words is known to be $1+\sum_1^m (m-i+1)\phi(i)$. We give a geometric proof of this formula, based on duality and on Euler's relation for planar graphs.

Résumé

Le nombre de facteurs de longueur m des mots de Sturm est connu pour être donné par la formule $1+\sum_1^m (m-i+1)\phi(i)$. Nous donnons une preuve géométrique de ce resultat basée sur la dualité et la relation d'Euler pour les graphes planaires.

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1 Introduction

The Sturmian word with real parameters $\alpha, \beta \in [0, 1[$ is defined as the 0-1 bi-infinite sequence

$$(\lfloor \alpha(n+1) + \beta \rfloor - \lfloor \alpha n + \beta \rfloor)_{n \in \mathbb{Z}}. \tag{1}$$

Sturmian words have a long history. A clear exposition of early work by J. Bernoulli, Christoffel, and A. A. Markov is given in the book by Venkov [19]. The term "sturmian" has been used by Hedlund and Morse in their development of symbolic dynamics [7, 8, 9]. There is a large literature about properties of these sequences (see for example Series [17], Fraenkel *et al.* [6], Stolarsky [18]). From a combinatorial point of view, they have been considered by Rauzy [13, 14, 15], Brown [2], Ito, Yasutomi [10] in particular in relation with iterated morphisms, and by Séébold [16], Mignosi [11]. Sturmian words appear in ergodic theory [12], in computer graphics [1], and in cristallography. Duluc and Gouyou-Beauchamps [4] considered the set of all finite words that are factors of some Sturmian word. They proved that the complement, say S, of this set is a context-free language, and they conjectured that S is inherently ambiguous. To show this, they in fact conjectured Theorem 1 below. Since the generating series of these numbers is transcendental, the Chomsky-Schützenberger theorem would prove inherent ambiguity (see Flajolet [5] for a systematic exposition).

The set S admits a nice combinatorial description due to Coven and Hedlund [3], and Luc Boasson (personal communication) uses this description to prove directly that S is inherently ambiguous by applying Ogden's Lemma.

Mignosi [11] proved the following result.

Theorem 1 *The number of factors of length m of Sturmian words is given by the sum*

$$1 + \sum_{1}^{m} (m - i + 1)\phi(i), \tag{2}$$

where ϕ is the Euler function, i.e., $\phi(n)$ is the number of natural integers less than n and coprime to n.

Mignosi's proof is based on a delicate analysis of the structure of Sturmian words. The aim of the present note is to give another, and we think simpler proof, based on a completely different argument. Since each Sturmian word is defined by a line, it is natural to consider the geometric dual of the plane. The dual of any set of lattice points is an arrangement of lines. It appears that the factors of fixed length of Sturmian words are in bijection with the faces of some specific arrangement restricted to the unit square.

We exploit Euler's relation on planar graphs for gaining an expression for the number of faces which, by a simple counting argument, gives the desired result. In order to keep the proof elementary, we considered duality on the plane rather than on the torus which might be more natural.

The next section contains a brief account of duality; then Euler's formula is used to give the proof.

2 Duality

We call a factor of length m of a Sturmian word, a *Sturmian m-factor.* It is easily verified that the sequence

$$(\lfloor\alpha(n+1)+\beta\rfloor - \lfloor\alpha n+\beta\rfloor)_{0\leq n<m} \tag{3}$$

ranges over all Sturmian m-factors when α and β range over $[0,1[$. We say that the Sturmian m-factor (3) is *defined* by the straight line ℓ with equation $y=\alpha x+\beta$.

Let (O,x,y) be a coordinate system for the Eucidean plane $\mathcal{P}$. Let $\mathcal{H}$ be the set of straight lines with finite slope and let $\mathcal{L}$ be the set of straight lines with equation $y=\alpha x+\beta$ with α and β in $[0,1[$. Given a line $\ell\in\mathcal{H}$ with equation $y=\alpha x+\beta$ we denote by ℓ^+ the half-plane $y\leq\alpha x+\beta$. Let

$$H_m=\{\,(x,y)\mid x,y\in\mathbb{N},\quad 0\leq x\leq m\,\}. \tag{4}$$

We prove now the following "geometric" result.

Proposition 1 *Two lines ℓ and ℓ' of $\mathcal{L}$ define the same Sturmian m-factor if and only if*

$$\ell^+\cap H_m=\ell'^+\cap H_m.$$

Proof. The set $\ell^+\cap H_m$ is clearly defined by the lattice points $(n,\lfloor\alpha n+\beta\rfloor)$ for $n=0,\ldots,m$, and consequently by the Sturmian m-factor defined by ℓ since $\beta\in[0,1[$. □

To exploit the previous proposition it will be convenient to represent a line by a point. At this end we use the duality transform $x\in\mathcal{H}\cup\mathcal{P}\mapsto x^*\in\mathcal{H}\cup\mathcal{P}$. Duality maps the line $\ell\in\mathcal{H}$ with equation $y=\alpha x+\beta$ to the point $\ell^*\in\mathcal{P}$ with coordinates $(\alpha,-\beta)$ and the point p with coordinates (α,β) to

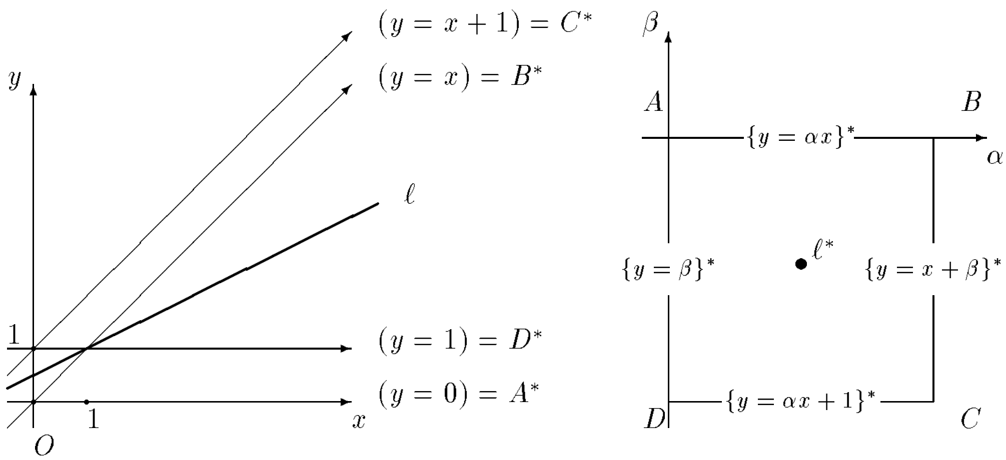

Figure 1: The duality transform of $\mathcal{L}$.

the line p^* with equation $y = \alpha x - \beta$. It can be easily verify that the duality transform is an involution, and that it preserves the incidence relation i.e.,

$$p \in \ell \Leftrightarrow \ell^* \in p^*, \qquad p \in \ell^+ \Leftrightarrow \ell^* \in (p^*)^+. \tag{5}$$

The set $\mathcal{L}$ is mapped (see Figure 1), by the duality transform, onto the square

$$\mathcal{C} = \{ (\alpha, \beta) \mid 0 \leq \alpha, -\beta < 1 \}. \tag{6}$$

The extreme points $A(0, 0)$, $B(1, 0)$, $C(-1, 1)$, $D(0, -1)$ of the topological closure $\overline{\mathcal{C}}$ of $\mathcal{C}$ are, respectively, the dual images of the lines $y = 0, y = x, y = x + 1$, $y = 1$; its edges $]A, B[,]B, C[,]C, D[,]D, A[$ are, respectively, the dual images of the lines $\{y = \alpha x\}, \{y = x + \beta\}, \{y = \alpha x + 1\}$ and $\{y = \beta\}$, and its interior is the dual image of the set $\{y = \alpha x + \beta\}$ with $0 < \alpha, \beta < 1$.

We now define the *arrangement* $\mathcal{A}_m$ of the square $\overline{\mathcal{C}}$ induced by the dual lines in H_m^*: there are $m(m+1)/2$ lines in H_m^* which intersect the interior of $\mathcal{C}$ and yield a segment; these $m(m+1)/2$ lines are precisely the duals of the lattice points (x, y) such that $1 \leq y \leq x \leq m$. We add, to this set of segments, the four line segments enclosing the square $\overline{\mathcal{C}}$. By definition the arrangement $\mathcal{A}_m$ is the cell decomposition of the square $\overline{\mathcal{C}}$ whose vertices, edges and faces are respectively the intersection points of all these segments, the maximal connected components of the union of the segments minus the

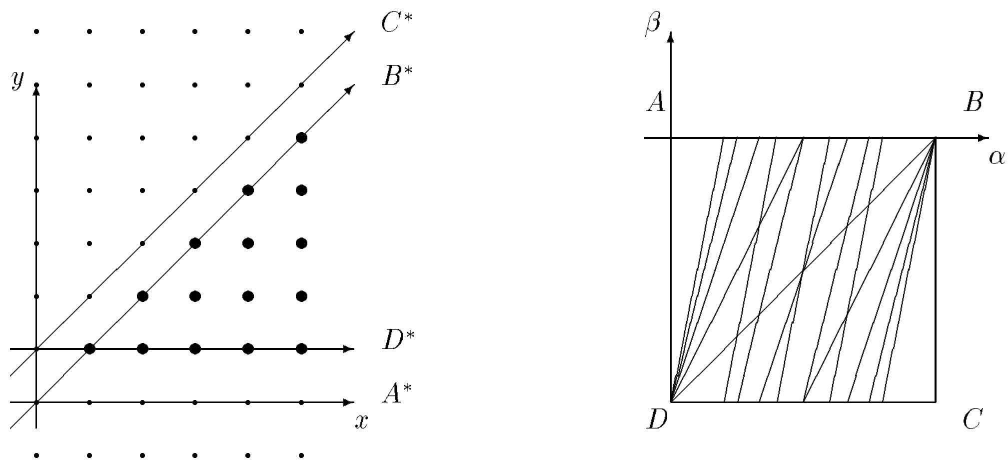

Figure 2: The arrangement A_5

vertices, and the maximal connected components of the square $\overline{\mathcal{C}}$ minus the segments. The arrangementt $\mathcal{A}_5$ is depicted in Figure 2: there are 19 segments, 29 vertices, 52 edges and 24 faces. Observe that in our definition we don't count the external face.

We define the *upper enveloppe* of a set X of the plane to be the set of points (x, y) of the topological closure of X such that $(x + \epsilon, y - \epsilon) \in X$ for some positive ϵ. The relation between $\mathcal{A}_m$ and the Sturmian m-factors is given in the following proposition (See Figure 3).

Proposition 2 *Two lines ℓ, ℓ' of $\mathcal{L}$ define the same Sturmian m-factor if and only if ℓ^* and ℓ'^* belong to the upper envelope of the same face of the arrangement $\mathcal{A}_m$.*

Proof. This is the dual formulation of the previous proposition. □

The following corollary gives the first step in the proof of the enumeration formula.

Corollary 1 *The number of Sturmian m-factors is equal to the number of faces of the arrangement $\mathcal{A}_m$.* □

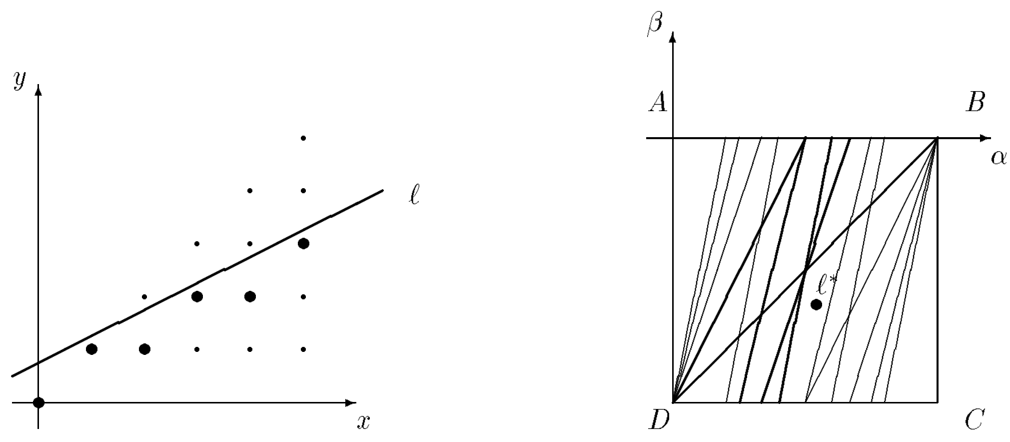

Figure 3: A line and its corresponding dual face

3 Proof of the formula

For any line $\ell \in \mathcal{H}$, let $c(\ell) = \mathrm{Card}(\ell \cap H_m)$, and let $\mathcal{L}'$ be the lines in $\mathcal{L}$ with positive slope and such that $c(\ell) \geq 2$. We start by proving the following formula.

Lemma 1 *The number f of faces of $\mathcal{A}_m$ is*

$$f = 1 + m + \sum_{\ell \in \mathcal{L}'} (c(\ell) - 1) \tag{7}$$

Proof. Let e, v be respectively the number of edges and vertices of the arrangements $\mathcal{A}_m$. According to Euler's relation for planar graphs

$$f - e + v = 1. \tag{8}$$

Remember that we don't count the external face. Let V be the set of vertices of the arrangement $\mathcal{A}_m$. As in any graph, the number e of edges satisfies the relation

$$2e = \sum_{s \in V} deg(s) \tag{9}$$

where $deg(s)$ is, as usual, the number of edges incident to s. We partition the set vertices V into

$$V = \{A, B, C, D\} \cup V_0 \cup V_1 \cup W \tag{10}$$

where V_0 and V_1 are the set of vertices lying on $]A, B[$ and on $]C, D[$ respectively, and where W is the set of vertices in the interior of $\mathcal{C}$.

It can be easily verified that $deg(A) = deg(C) = 2$ and $deg(B) = deg(D) = m + 2$; furthermore according to (5)

$$deg(s) = \begin{cases} 2\,c(s^*) & \text{if } s \in W \\ c(s^*) + 1 & \text{if } s \in V_0 \cup V_1; \end{cases} \tag{11}$$

Observe also that $(\alpha, 0) \mapsto (\alpha, -1)$ is a degree preserving bijection from V_0 to V_1; consequently

$$\sum_{s \in V_0 \cup V_1} deg(s) = 2 \sum_{s \in V_0} deg(s). \tag{12}$$

From (8) and (9), we get

$$\begin{aligned} f &= 1 + e - v \\ &= 1 + \sum_{s \in V} (deg(s)/2 - 1) \\ &= 1 + m + 2 \sum_{s \in V_0} (deg(s)/2 - 1) + \sum_{s \in W} (deg(s)/2 - 1) \end{aligned}$$

The last equality follows from (12). In view of (11)

$$f = 1 + m + \sum_{s \in V_0 \cup W} c(s^*) - 1. \tag{13}$$

Since $V_0 \cup W$ and $\mathcal{L}'$ are dual sets the formula follows. □

Proof of the theorem. In view of Lemma 1 it suffices to evaluate the sum

$$\sum_{\ell \in \mathcal{L}'} c(\ell) - 1. \tag{14}$$

Assume that the slope of the line ℓ is q/p with p and q coprime. Then $c(\ell) - 1$ is the number of points $(x, y) \in H_m$ such that $(x + p, y + q)$ also belongs to H_m, and such that the line defined by (x, y) and $(x + p, y + q)$ is precisely the line ℓ. Hence, the sum (14) is equal to the number of 4-tuples (x, y, p, q) of integers such that

1. $0 \le x \le m$ and $0 \le x + p \le m$
2. $1 \le q < p$ and p and q coprime

3. the ordinate of the intersection point of the line trough (x, y) with slope q/p and the y-axis belong to $[0, 1[$.

Straightforward calculations show that the last condition is equivalent to $0 \leq y - qx/p < 1$ which shows that $y = \lceil qx/p \rceil$ is uniquely determinated by the triples (x, p, q). It follows that the number of 4-tuples (x, y, p, q) is the number of triples (x, p, q) such that conditions 1 and 2 above hold. This number is equal to $\sum_{p=2}^{m}(m - p + 1)\phi(p)$. This proves the formula. $\square$

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