Camera Placement in Integer Lattices

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CAMERA PLACEMENT IN INTEGER LATTICES[‡]

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Abstract

The camera placement problem concerns the placement of a fixed number of point-cameras on the integer lattice of d-tuples of integers in order to maximize their visibility. We give a caracterization of optimal configurations of size s less than 5^d and use it to compute in time $O(s \log s)$ an optimal *abstract* configuration under the assumption that the visibility of a configuration is computable in constant time.

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Contents

1 Introduction

Visibility and illumination problems are among the most appealing and intuitive research topics of combinatorial geometry. In many cases (though not all) their analysis requires nothing more than basic topics from geometry, number theory and graph theory and as such they are very well suited for a wide audience [3]. In recent years there has been particular emphasis on the algorithmic component of visibility problems in polygonal configurations and as such they have come to be studied under the area of "art gallery (watchman) problems". In turn this last area lies at the intersection of combinatorial and computational geometry [15].

Art gallery problems, theorems and algorithms are so named after the celebrated question first posed by V. Klee in 1973: "What is the minimum number of guards sufficient to cover the interior of an n-wall gallery?" The problem was solved soon thereafter first by Chvátal and subsequently also by Fisk. Since then art gallery problems have successfully emerged as a research area that stresses complexity and algorithmic aspects of visibility and illumination in configurations comprising "obstacles" and "guards". In fact by creating rather idealized situations the theory succeeds in abstracting the algorithmic essence of many visibility problems (like in partitioning theorems, mobile guard configurations, visibility graphs, etc.) thus significantly facilitating the study of their computational complexity.

In the present paper we focus on a particular class of art gallery problems, namely those visibility problems which concern configurations of points lying on the vertices of the integer lattice Λ. By this we assume that we have point obstacles (i.e. lattice points can block the view) and point guards (or cameras) which occupy the vertices of the lattice. We also assume that the cameras have "full visibility" (i.e. can survey the entire space) and see objects at any distance.

In particular we are interested in the following art gallery problem.

> *Given an integer s, determine a configuration S (of camera locations) contained in Λ and of cardinality s, such that the density of lattice points which are visible from at least one point of S is as large as possible.*

1.1 Some definitions

Before providing an outline of the main themes of investigation we remind the reader of some basic definitions and simple facts. By Λ we denote the d dimensional integer lattice consisting of d-tuples of integers and by Λ_n the set of lattice points in Λ whose coordinates have absolute value $\leq n/2$.* Very important for our subsequent optimization analysis is the notion of density of a set of lattice

*We suppress mention of the dimension d in our notation for Λ and Λ_n, but this will be always implicit in the context.

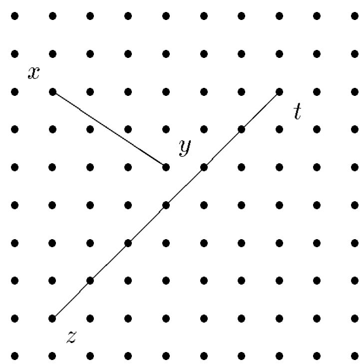

Figure 1: Points x and y are visible; points z and t are visible modulo p for $p \neq 2, 3$.

points. For any set $X \subseteq \Lambda$ of lattice points we define the density $D(X)$ of X as the limit (if it exists) of the ratio

$$D_n(X) = \frac{|X \cap \Lambda_n|}{|\Lambda_n|}$$

of the number of points in $X \cap \Lambda_n$ to the number of points in Λ_n as n tends to infinity. The upper and lower densities $\overline{D}(X), \underline{D}(X)$ are defined similarly by taking the lim sup, lim inf, respectively. It is easy to check that the density function is a finitely additive measure on those subsets of Λ which have density. In particular, we have

$$0 \leq D(X) \leq 1,$$
$$X \subseteq Y \Rightarrow D(X) \leq D(Y),$$
$$D(X \cup Y) + D(X \cap Y) = D(X) + D(Y).$$

Let $\mathcal{P} = \{2, 3, 5, \ldots\}$ be the set of prime numbers, and let p range over $\mathcal{P}$, and $\mathcal{Q}$ over subsets of $\mathcal{P}$. Two lattice points are called *visible modulo* p if they are distinct modulo p. Two lattice points are said *visible modulo* $\mathcal{Q}$ if they are visible modulo p for each prime $p \in \mathcal{Q}$. Two points visible modulo $\mathcal{P}$ are visible in the geometric sense, i.e. the open line segment joining them avoids all the lattice points (see Figure 1). For all $X \subseteq \Lambda$, X/p denotes the quotient set of X by the relation of equality modulo p.

1.2 Related literature

Interesting visibility problems have been studied on integer lattices [5, 8]. Of these we single out two which are relevant for our study.

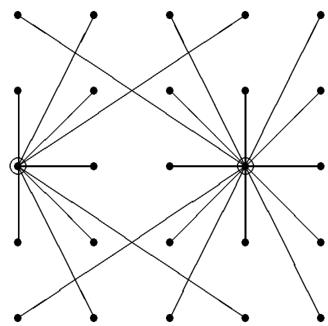

Figure 2: Two guards are enough to cover lattice of pairs of integers $\leq 5/2$.

Rumsey [22] shows that for any set S of lattice points, the density of the set of lattice points visible from each point of S is given by the infinite product

$$\prod_{p \in \mathcal{P}} \left(1 - \frac{|S/p|}{p^d}\right). \tag{1}$$

(In fact, Rumsey gives a characterization of the sets S for which the density formula (1) is true.) The above formula was previously obtained by G. Leujeune Dirichlet for the case $|S| = 1$ ("the probability that d integers chosen at random are relatively prime is $1/\zeta(d)$", where $\zeta(z) = \sum_{n \geq 1} n^{-z}$, $|z| > 1$, denotes the Riemann zeta function, [11, page 324]) and by Rearick [20, 21] for the case where $|S| = 2$ and the points of S are pairwise visible.

An interesting (and in general still open) art gallery problem was posed by Moser [14] in 1966: given a set P of points in the plane how many guards located at points of P are needed to see the unguarded points of P? Abbott [1] studies the case $P = \Lambda_n$ and shows that the minimum number $f(n)$ of guards which are necessary in order to see all the points of Λ_n (see Figure 2) verifies the inequalities

$$\frac{\ln n}{2 \ln \ln n} < f(n) < 4 \ln n.$$

The lower bound result follows by applying the Chinese remainder theorem and the Prime number theorem. For the upper bound Abbott constructs recursively a sequence $x_1, x_2, \ldots, x_k$ such that for each i, x_{i+1} is a point x in the set Λ_n for which the set-theoretic difference $V_n(x) \setminus (V_n(x_1) \cup \cdots \cup V_n(x_i))$, where $V_n(x)$ is the set of points of Λ_n visible from x, is of maximal size and shows that $k = O(\ln n)$ iterations of this procedure suffice in order to cover all the vertices of the lattice. His method however gives no "qualitative" information on the location of these points on the lattice. Nevertheless, he also shows using work of Erdös [4] that there exists a constant $\alpha > 0$ such that, for $d = 2$, every point of the lattice Λ_n is visible from the set $\{(1,0)\} \cup \{(0,j) \mid j = 0, 1, \ldots, k\}$, where $k = O(\ln^\alpha n)$. It is straightforward to see that his methods can easily be extended in order to yield similar results for the d-dimensional lattice Λ_n.

In some respects the camera placement problem can be thought of as a "qualitative" version of Abbott's problem. Despite the fact that Abbott's (and hence Moser's) question still remains open we expect that our investigations will also contribute to a better understanding of this problem.

At this point it is worth stressing one more time that we are considering point obstacles and point cameras. As a matter of fact, the density aspects of our problem would change were we to assume that the obstacles are discs of radius $\geq r$, say. According to a theorem of Pólya [18] no disc at a distance $\sqrt{1/r^2 - 1}$ from the origin can ever be visible from it. (This comes from Pólya's solution of the so-called "Pólya's orchard problem", i.e. "How thick must the trunks of the trees in a regularly spaced circular orchard grow if they are to block completely the view from the center?" [19][Chapter 5, Problem 239], [2].) See also [5][problem 13.2] for generalizations of Pólya's orchard problem.

Many other interesting visibility problems have been studied on integer lattices. The interested reader should consult [5, 8] for additional references.

1.3 Outline of the paper

Our paper is divided in the following parts. In section 2 we give a precise mathematical formulation of the camera placement problem and we introduce the notion of *abstract* configuration which captures the combinatorial part of the problem. In subsection 2.2 we show that a solution exists and we reformulate our problem into the following integer optimization problem:

$$\begin{array}{ll} \text{maximize} & u'(b_1) + u'(b_2) + \cdots + u'(b_m) \\ \text{subject to} & \begin{cases} (b_1, \ldots, b_m) = B(a_1, \ldots, a_m) \\ a_1 + \cdots + a_m = s, \qquad a_i \in \mathbb{N}, \end{cases} \end{array} \tag{2}$$

where u' is an absolutely monotone function, B is a linear operator and $m \in \mathbb{N}$; the three parameters u', B and m depend on s (see section 2 for the appropriate definitions of u',B and m).

This enables us to solve the problem, in section 2.3, for the case of $s \leq 3^d$ cameras. Subsequently in section 3 we make a deeper combinatorial analysis of our optimization technique and we provide a characterization of optimal configurations for all $s \leq 5^d$. This caracterization enables us to compute in time $O(s \log s)$ an *abstract* optimal configuration (i.e, a solution to (2)), assuming that the visibility function is computable in constant time. In section 4 we discuss open problems and related work.

2 Camera Placement Problem

The *camera placement problem* in multidimensional lattices is the following:

> Given an integer s, determine a configuration S of s lattice points (camera locations) such that the density of lattice points visible by at least one point of S is maximized.

More formally, we want to find conditions on the set S of possible camera locations so that the following quantity

$$u(S) := \sum_{E \subseteq S, E \neq \emptyset} (-1)^{|E|+1} \prod_{p \in \mathcal{P}} \left(1 - \frac{|E/p|}{p^d}\right), \tag{3}$$

which is obtained from the product formula (1) using the principle of inclusion/exclusion is maximized. The quantity $u(S)$ is called the *visibility* of the configuration. Configurations (if any) which, for a given s, attain the optimal density will be called *optimal*.

Clearly the visibility of a configuration depends only on the relations of visibility modulo p shared by its cameras as p ranges over $\mathcal{P}$. For example for $d = 2$ the visibility of a configuration of 4 pairwise visible cameras is given by

$$4 \prod_{p \in \mathcal{P}} \left(1 - \frac{1}{p^2}\right) - 6 \prod_{p \in \mathcal{P}} \left(1 - \frac{2}{p^2}\right) + 4 \prod_{p \in \mathcal{P}} \left(1 - \frac{3}{p^2}\right).$$

It follows that in order to control the variation of the visibilility function it will be more convenient to specify a configuration by its relations of visibility modulo p instead of the coordinates of its points. This leads to the notion of *abstract configuration*.

Definition 2.1 *An abstract configuration of size s is a family of equivalence relations*

$$(r_2, r_3, r_5, r_7, \ldots) = (r_p)_{p \in \mathcal{P}}$$

on the set $[s] = \{1, \ldots, s\}$ indexed by the set of prime numbers such that

$$i \, r_p \, j \qquad \text{if and only if} \qquad A_i - A_j \in p\,\Lambda,$$

for some ordered configuration $\{A_1, \ldots, A_s\}$ of s lattice points. In that case the configuration $\{A_1, \ldots, A_s\}$ is called a representative of the family $(r_p)_{p \in \mathcal{P}}$. Two ordered configurations which represent the same abstract configuration are called equivalent modulo $\mathcal{P}$.

In view of formula (3) it is obvious that two configurations which are equivalent modulo $\mathcal{P}$ have the same visibility.

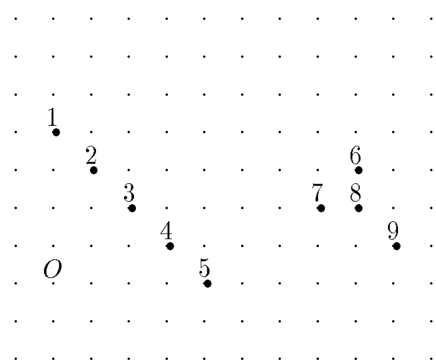

Figure 3: A 9-camera configuration.

Example 2.1 *Figure 3 displays a 9-camera configuration* $\{A_1, \ldots, A_9\}$ *such that point* A_i *is positioned at the lattice vertex labeled by* i. *Assuming the origin is located at* $O = (0, 0)$ *the lattice points have the following coordinates:*

$$A_1 = (0,4), A_2 = (1,3), A_3 = (2,2), A_4 = (3,1), A_5 = (4,0),$$
$$A_6 = (8,3), A_7 = (7,2), A_8 = (8,2), A_9 = (9,1).$$

The corresponding equivalence relations $r_2, r_3, r_5, r_7, \ldots$ *are described by their equivalence classes as follows:*

r_2: $\{1,3,5,8\}, \{2,4,9\}, \{6\}, \{7\}$
r_3: $\{1,4,9\}, \{2\}, \{3,8\}, \{5\}, \{6\}, \{7\}$
r_5: $\{1\}, \{2\}, \{3,7\}, \{4\}, \{5\}, \{6\}, \{8\}, \{9\}$
r_7: $\{1\}, \{2,6\}, \{3\}, \{4\}, \{5\}, \{7\}, \{8\}, \{9\}$

and for r_p, $p \geq 11$, *all classes are singletons.*

The camera placement problem is then split into two subproblems.

> **Problem 1.** Give necessary and sufficient condition for an abstract configuration to be optimal.

> **Problem 2.** Compute a representative of a given abstract (optimal) configuration.

We will not consider the second problem in this article; the interested reader can consult [16] where it is shown that a representative of an optimal abstract configuration can be computed in expected time exponential in $s^{1/d}$ (for d fixed).

At this point and to guide our analysis it can be useful to make a conjecture about the solution. In view of formula (3) it is reasonable to believe that the cameras of an optimal configuration have to be evenly distributed in the classes of Λ/p as p ranges over the set of primes. This leads to the notion of *balanced* configuration.

Definition 2.2 *A configuration S is called balanced if*

$$\forall n \quad \textit{square free} \quad \forall c, c' \in \Lambda/n, \qquad ||S \cap c| - |S \cap c'|| \leq 1.$$

Conjecture 2.1 *An optimal configuration is necessarily balanced.*

Before we attempt to prove (or disprove!) this conjecture we must determine which family of equivalence relations have a representative; in particular we must determine if balanced configurations exist. We examine this question in the next section.

2.1 The realizability theorem

The paper of Rumsey [22] is concerned with the following problem.

> Given a set S of lattice points, let $V(S)$ be the set of lattice points which can see each of the points of S. Find the density of the set $V(S)$ (if it has one).

According to Rumsey [22] a set $X \subseteq \Lambda$ is called *periodic* of *period* the natural number m if $X = X + m\Lambda$; in other words X is the union of some classes of the quotient set of Λ by the relation of equality modulo m.

The following two results appear in [22].

Proposition 2.1 *A periodic set $X \subseteq \Lambda$ of period m has a density given by the ratio $|X/m|/|\Lambda/m|$. Moreover,*

$$|X/m| \left(\frac{\lfloor n/m \rfloor}{n} \right)^d \leq D_n(X) \leq |X/m| \left(\frac{\lceil n/m \rceil}{n} \right)^d . \qquad (4)$$

Proposition 2.2 *The density of a finite intersection of periodic sets whose periods are pairwise relatively prime is the product of the densities of the periodic sets.* □

These results suggest a posssible solution to Rumsey's problem. Recall that two points are visible modulo p if they are distinct modulo p and let $V_p(S)$ be the set of lattice points which can see modulo p each of the points of S. Rumsey [22] observes that

$$V(S) = \bigcap_{p \in \mathcal{P}} V_p(S)$$

and that $V_p(S)$ is periodic of period p. So it is tempting to assert [10] that the density of $V(S)$ is the product of the densities of the $V_p(S)$

$$D(V(S)) = \prod_{p\in\mathcal{P}} D(V_p(S)).$$

The main result of [22] is to give a necessary and sufficient condition so that the above formula is true. In particular this formula is true when S is finite.

What is this necessary and sufficient condition? It turns out that this necessary and sufficient condition can be reformulated in a much more versatile way when we replace the sequence $V_2(S), V_3(S), V_5(S), \ldots$ by an arbitrary sequence $X_1, X_2, X_3, \ldots$ of periodic sets of pairwise relatively prime periods. This reformulation is the following.

Theorem 2.1 *Let $(X_k)_{k\in\mathbb{N}}$ be a sequence of periodic sets such that X_k is periodic with respect to m_k. Assume that the m_k are pairwise relatively prime and put*

$$A(K) = \{\, x \in \Lambda \mid \exists k,\ x \in (X_1 \cap \ldots \cap X_{k-1} \cap \overline{X}_k),\ m_k > K\,|x|\,\}.$$

Then the following assertions are equivalent

i) $D(\bigcap_k X_k) = \prod_k D(X_k)$

ii) $\lim_{K\to\infty} \overline{D}(A(K)) = 0$.

Proof. We follow and adapt Rumsey's proof to work in the present framework.

Let $X = \bigcap_k X_k$; without loss of generality we assume that the sequence $m_1, m_2, \ldots$ is increasing.

First we prove that $i) \Rightarrow ii)$. Since the sequence m_k is increasing $A(K) \subseteq \left(\bigcap_{m_k\le K} X_k\right) \setminus X$. Hence we obtain that

$$\overline{D}(A(K)) \le \overline{D}\Bigg(\bigcap_{m_k\le K} X_k\Bigg) - \underline{D}(X) = \prod_{\rho_k\le K} D(X_k) - D(X).$$

Let now K tend to infinity to get the desired result.

We assume now $ii)$ and we prove $i)$. From $X = \bigcap_k X_k$ we get easily that

$$\overline{D}(X) \le \prod_k D(X_k). \tag{5}$$

In particular if the right-hand side of this inequality is zero then $\underline{D}(X) = D(X) = \overline{D}(X) = \prod_k D(X_k) = 0$ which proves (i). So without loss of generality we may assume that $\prod_k D(X_k) \neq 0$ or equivalently that $\sum_k D(\overline{X}_k) < \infty$. Let R be a natural number and choose n (the size of the box for computing the density) such that $Kn > R$. From $X = \bigcap_{m_k \leq R} X_k \setminus \bigcup_{m_k > R} F_k$ (together with the fact that the F_ks are disjoint) we get

$$D_n(X) = D_n\left(\bigcap_{m_k \leq R} X_k\right) - \sum_{R < m_k < Kn} D_n(F_k) - D_n\left(\bigcup_{m_k \geq Kn} F_k\right). \qquad (6)$$

The middle term in the right-hand side of (6) is estimated by using (4) as follows

$$D_n(F_k) \leq D_n(\overline{X}_k) \leq |\overline{X}_k/m_k| \left(\frac{\lceil n/m_k \rceil}{n}\right)^d.$$

Then we use $\rho_k \leq nK$ to bound $\lceil n/m_k \rceil / n$ by $(1+K)/m_k$. It follows that

$$D_n(F_k) \leq \left|\overline{X}_k/m_k\right| \frac{(1+K)^d}{m_k^d} = (1+K)^d D(\overline{X}_k).$$

The last term in (6) is majorized by $D_n(A(K))$ since

$$\left(\bigcup_{m_k \geq Kn} F_k\right) \bigcap \{\, x \in \Lambda \mid |x| \leq n \,\} \subseteq A(K).$$

Combining these majorizations we obtain

$$D_n(X) \geq D_n(\bigcap_{m_k \leq R} X_k) - (1+K)^d \sum_{R < m_k < nK} D(\overline{X}_k) - D_n(A(K)).$$

Now we let successively n, R and K go to infinity and we get

$$\begin{aligned} \underline{D}(X) &\geq \prod_{m_k \leq R} D(X_k) - (1+K)^d \sum_{R < m_k} D(\overline{X}_k) - \overline{D}(A(K)) \\ \underline{D}(X) &\geq \prod_k D(X_k) - \overline{D}(A(K)) \\ \underline{D}(X) &\geq \prod_k D(X_k). \end{aligned}$$

This last inequality combined with inequality (5) gives the desired result. □

Then Rumsey used the above theorem to prove that the set $V(S)$ admits a density when S is finite. We reformulate this result replacing the set $\mathcal{P}$ of prime numbers by a subset $\mathcal{Q}$ of $\mathcal{P}$.

Proposition 2.3 (Rumsey's Theorem) *Let S be a finite set of lattice points and let $\mathcal{Q}$ be a set of prime numbers. The set $V_{\mathcal{Q}}(S)$ of lattice points visible modulo $\mathcal{Q}$ from each point of S admits a density given by the infinite product*

$$\prod_{p \in \mathcal{Q}} \left(1 - \frac{|S/p|}{p^d}\right).$$

Furthermore $V_{\mathcal{Q}}(S)$ is nonempty if and only if $|S/p| < p^d$ for all $p \in \mathcal{Q}$.

Now we come to the existence question of a representative for a family of equivalence relations. First we generalize slightly the definition of abstract configuration.

Definition 2.3 *Let $\mathcal{Q}$ be a set of prime numbers. An abstract $\mathcal{Q}$-configuration of size s is a family $(r_p)_{p\in\mathcal{Q}}$ of equivalence relations on the set $[s]=\{1,\ldots,s\}$ indexed by the set $\mathcal{Q}$ such that*

$$i\, r_p\, j \qquad \text{if and only if} \qquad A_i - A_j \in p\,\Lambda,$$

for some ordered configuration $\{A_1,\ldots,A_s\}$ of s lattice points. In that case the configuration $\{A_1,\ldots,A_s\}$ is called a representative of the family $(r_p)_{p\in\mathcal{Q}}$. Two ordered configurations which represent the same abstract $\mathcal{Q}$-configuration are called equivalent modulo $\mathcal{Q}$.

Proposition 2.4 (Realizability Theorem) *Let $\mathcal{E}=(r_p)_{p\in\mathcal{Q}}$ be a family of equivalence relations on the set $[s]=\{1,\ldots,s\}$. A representative of $\mathcal{E}$ exists if and only if $|[s]/r_p|\le p^d$ for all prime $p\in Q$, and $|[s]/r_p|=s$, for p large enough. Furthermore the set of representatives of $\mathcal{E}$ admits a density given by the infinite product*

$$\prod_{p\in\mathcal{Q}} \frac{(p^d)_{|[s]/r_p|}}{p^{s\,d}},$$

where $(x)_y=x(x-1)\ldots(x-y+1)$ is the descent factorial.

Proof. In [12] the existence part of the Realizability Theorem was proved by a repeated application of Rumsey's Theorem. Here we deduce the existence from the expression of the density.

Let X_p be the set of s-tuples $(A_1,\ldots,A_s)$ of points of Λ such that $A_i-A_j\in p\,\Lambda$ if and only if $i\,r_p\,j$ for all i and j. Clearly the set, say X, of representatives of $\mathcal{E}$ is the intersection of the X_p as p ranges over $\mathcal{Q}$; furthermore we can easily verify that X_p is periodic with period p, and that its density is given by the ratio

$$\frac{(p^d)_{|[s]/r_p|}}{p^{s\,d}}.$$

Then we are in a position to apply Theorem 2.1.

First we observe that if $\prod_p D(X_p)=0$ then our theorem is proved since $D(X)\le\prod_p D(X_p)$. Hence without loss of generality we may assume that $\prod_p D(X_p)\neq 0$; in particular this condition implies that r_p is the identity relation for p large enough. Let now $A(K)$ be the set of lattice points $A=(A_1,\ldots,A_s)\in\Lambda^s$ such that $A_i-A_j\in p\,\Lambda$ and $p\ge K|A|>0$ for some i,j and $p\in\mathcal{Q}$. Then there exists a lattice point $U\in\Lambda$ such that $A_i-A_j=p\,U$. Hence it follows that

$$p\,|U| = |A_i-A_j|\le|A|\le p/K.$$

But applying this last inequality with $K = 2$ it follows that $U = 0$, i.e,

$$A_i = A_j.$$

The set $A(K)$ is then subdimensional, i.e, it is a subset of a finite number of hyperplanes, and consequently its density is null.

To prove the existence part of the proposition notice that the product formula is positive if and only if $|[s]/r_p| \leq p^d$ for all prime $p \in Q$, and $|[s]/r_p| = s$, for p large enough. On the other hand these conditions must be satisfied since for any configuration S of s points we have $|S/p| \leq |\Lambda/p|$ which is equal to p^d and $|S/p| = |S|$ for p large enough since the coordinates of the points of S are bounded. □

Corollary 2.1 *Balanced configurations exist.*

Proof. To introduce balanced configurations of size s we argue as follows. Suppose we have indexed the classes of Λ/p with integers between 1 and p^d. So we can attach to each point A of Λ a finite sequence of integers, say $l(A)$, which represent the various classes of Λ/p at which A belongs as the prime number p ranges over the sequence $2, 3, 5, \ldots, p_r$ of primes less than $s^{1/d}$. Let i be the operator of pointwise incrementation, i.e. $i(x_1, x_2, \ldots, x_r) = (x_1 + 1, x_2 + 1, \ldots, x_r + 1)$ where the entry $x_i + 1$ is computed modulo p_i^d. Let 1 be the sequence $(1, 1, \ldots, 1)$. According to the Realizability Theorem, there exists a sequence of lattice points $A_1, \ldots, A_s$ such that *i)* A_k, A_l are visible modulo p for each prime $p \geq s^{1/d}$ and *ii)* $l(A_k) = i^{k-1}(1)$. Since the p_i^d are pairwise relatively prime, this configuration is clearly balanced. □

Remark 2.1 *We will use the Realizability Theorem in a sligthly stronger form. According to Theorem 2.1, if a representative of $\mathcal{E}$ exists then we can always find a representative belonging to some periodic set, assuming that this period is relatively prime to any element of $\mathcal{Q}$. For example we can impose on the A_i to verify the conditions*

$$A_i - B_i \in p\Lambda$$

where p ranges over a finite set of primes disjoint from $\mathcal{Q}$ and where the B_i are given lattice points. A similar remark applies to Rumsey's Theorem.

We end this section by introducing a few more notations and some technical points which will be useful in the sequel. Let

$$U_{\mathcal{Q}}(S) = \bigcup_{A \in S} V_{\mathcal{Q}}(\{A\}) \tag{7}$$

be the set of lattice points which are visible modulo $\mathcal{Q}$ from at least one point of S. From Rumsey's Theorem and the finite additivity of the density function it is clear that the set $U_{\mathcal{Q}}(S)$ admits a density. This density is called the $\mathcal{Q}$-*visibility* of the configuration S, and is denoted $u(\mathcal{Q}, S)$. According to the inclusion/exclusion principle,

$$u(\mathcal{Q}, S) = \sum_{E \subseteq S, E \neq \emptyset} (-1)^{|E|+1} \prod_{p \in \mathcal{Q}} \left(1 - \frac{|E/p|}{p^d}\right). \tag{8}$$

Note that two configurations which are equivalent modulo $\mathcal{Q}$ heve the same $\mathcal{Q}$-visibility.

Let now $S_1, \ldots, S_r$ and T be $r+1$ finite subsets of Λ. In our subsequent analysis we will encouter the set

$$U_{\mathcal{Q}}(S_1) \cap \cdots \cap U_{\mathcal{Q}}(S_r) \setminus U_{\mathcal{Q}}(T)$$

of lattice points which for each $i \leq r$ can see modulo $\mathcal{Q}$ at least one point of each set S_i and can not see modulo $\mathcal{Q}$ any of the points of T. This set admits a density, denoted by

$$u(\mathcal{Q}, S_1, S_2, \ldots, S_r; T).$$

It will be useful to relate this density to the following *difference operator.* For $A \subseteq \Lambda$ we define the operator Δ_A on the set of functions F from $\mathcal{P}(\Lambda)$ to $\mathbb{R}$ as follows

$$\Delta_A F(X) = F(A \cup X) - F(X). \tag{9}$$

Proposition 2.5 *The difference operator and the density function D verify*

1. $\Delta_A D(X) = D(A \setminus X)$
2. $\Delta_B \Delta_A D = -\Delta_{A \cap B} D$

Proof. The first part follows easily from the additivity of the density function and the set equality $X \cup Y = (X \setminus Y) \cup Y$. The second part follows from the following equations

$$\begin{aligned}
\Delta_B \Delta_A D(X) &= \Delta_A D(B \cup X) - \Delta_A D(X) \\
&= D(A \setminus (B \cup X)) - D(A \setminus X) \\
&= -[D(A \setminus X) - D(A \setminus (B \cup X))] \\
&= -[D((A \setminus X) \setminus (A \setminus (B \setminus X)))] \\
&= -D((A \cap B) \setminus X) \\
&= -\Delta_{A \cap B}(X). \square
\end{aligned}$$

By repeated application of part 2 of the previous proposition we get that

$$u(\mathcal{Q}, S_1, \ldots, S_r; T) = (-1)^{r+1} \Delta_{U_{\mathcal{Q}}(S_1)} \ldots \Delta_{U_{\mathcal{Q}}(S_r)} u(\mathcal{Q}, T). \tag{10}$$

Proposition 2.6 *Assume that $\mathcal{Q}$ is infinite. If $r^{1/d}$ is less than the minimal prime of $\mathcal{Q}$ then the set $U_{\mathcal{Q}}(S_1) \cap \cdots \cap U_{\mathcal{Q}}(S_r) \setminus U_{\mathcal{Q}}(T)$ admits a non-null density.*

Proof. Let A_i be a lattice point of S_i and for each lattice point B of T let q_B be a prime number of $\mathcal{Q}$ such that A_i and B are visible modulo q_B for all i. Let then $\mathcal{Q}_1 = \mathcal{Q} \setminus \{q_B \mid B \in T\}$, and let S be the set of A_i. Then the set, say V, of lattice points A such that

- $A - B \in q_B\,\Lambda$,
- and A is visible modulo $\mathcal{Q}_1$ from each of the A_i

is a subset of $U_{\mathcal{Q}}(S_1) \cap \cdots \cap U_{\mathcal{Q}}(S_r) \setminus U_{\mathcal{Q}}(T)$. But according to Rumsey's Theorem the density of V is non-null since, by hypothesis, $|S/p| \leq r < p^d$ for each prime $p \in \mathcal{Q}_1$. □

2.2 Reduction to an integer optimization problem.

The difficulty of the optimization problem previously stated is due not only to the way we specify and manipulate the locations of the cameras (this problem is now solved by the Realizability Theorem), but also on the formulation of $u(S)$ as an alternating sum in identity (3). In the sequel we will reformulate the problem as a non-linear integer optimization problem.

The key idea in overcoming the inherent complexity of optimizing $u(S)$ lies in an inductive formula for computing $u(S)$. We have the following theorem.

Theorem 2.2 (Reduction Theorem) *For any set $\mathcal{Q}$ of primes, any $p \in \mathcal{Q}$, and any configuration S,*

$$p^d\, u(\mathcal{Q}, S) = \sum_{c \in \Lambda/p} u(\mathcal{Q} \setminus \{p\}, S \setminus c)$$

Proof. Let c range over Λ/p. It is clear that a point of c is $\mathcal{Q}$-visible from a point of S if and only if it is $\mathcal{Q} \setminus \{p\}$-visible from a point of $S \setminus c$, i.e.

$$U_{\mathcal{Q}}(S) \cap c = U_{\mathcal{Q} \setminus \{p\}}(S \setminus c) \cap c.$$

According to Proposition 2.3 the set $U_{\mathcal{Q}}(S) \cap c$ admits a density given by $u(\mathcal{Q} \setminus \{p\}, S \setminus c)/p^d$. We use now the additivity of the density function to write

$$u(\mathcal{Q}, S) = \sum_{c \in \Lambda/p} D(U_{\mathcal{Q}}(S) \cap c)$$

which is, up to notations, the formula given in the theorem. □

A first application of the previous theorem is the following.

Theorem 2.3 (Finiteness Theorem) *A necessary condition for the optimality of a configuration S is that*

$$\forall p \in \mathcal{P} \;\; |S/p| = \min\{|S|, p^d\}.$$

In particular the cameras of an optimal configuration must be pairwise visible modulo p for all primes $p \geq |S|^{1/d}$.

Observe that this theorem proves our conjecture for all square free integer n divisible by a prime number $p \geq |S|^{1/d}$.

Proof. The inequality $|S/p| \leq \min\{|S|, p^d\}$ is always true even if S is not an optimal configuration. Now we assume that this inequality is strict for some $p \in \mathcal{P}$ and we construct a better configuration as follows. Let $c \in \Lambda/p$ be such that $S \cap c_1$ has at least two elements and split $S \cap c_1$ in two non-empty parts S_1, S_2. Since there exists a coset $c_2 \in \Lambda/p$ whose intersection with S is empty, the Realizability Theorem asserts that there exists a configuration S' in bijection with S such that

1. S and S' are equivalent modulo $\mathcal{P} \setminus \{p\}$
2. $S' \cap c_1 = S_1'$
3. $S' \cap c_2 = S_2'$
4. $S' \cap c = (S \cap c)'$ for all $c \neq c_1, c_2 \in \Lambda/p$

where E' stands for the image of E under the canonical bijection of S and S'. Let now $u'(.)$ stand for $u(\mathcal{P} \setminus \{p\}, .)$; since S and S' are equivalent modulo $\mathcal{P} \setminus \{p\}$ we have, for all $E \subseteq S$, the equality $u'(E) = u'(E')$; it follows, according to the reduction theorem, that

$$p^d\left(u(S') - u(S)\right) = u'(S \setminus S_1) + u'(S \setminus S_2) - u'(S \setminus (S_1 \cup S_2)) - u'(S).$$

But the right member of this equation is $-\Delta_{U_{\mathcal{P}'}(S_1)}\Delta_{U_{\mathcal{P}'}(S_2)}u'(S)$ where $\mathcal{P}' = \mathcal{P} \setminus \{p\}$ and Δ_A is the difference operator (9). According to equation (10) we can write

$$p^d(u(S') - u(S)) = u'(S_1, S_2; S \setminus S_1 \cup S_2)$$

which is, according to Proposition 2.6, positive. The proof of the theorem is complete. □

Example 2.2 *The configuration described in example 2.1 satisfies*

$$|S/2| = 4, |S/3| = 6, |S/5| = 8, |S/7| = 8.$$

By the Finiteness Theorem the corresponding configuration cannot be optimal.

An immediate consequence of the Finiteness Theorem is that a solution to the camera placement problem exists and that the number of solutions is finite modulo the relation of equivalence modulo $\mathcal{P}$. In some sense we can consider our problem as resolved: make an exhaustive search to determine an optimal abstract configuration and use remark 2.1 following the Realizability Theorem to find a representative of this abstract configuration. However the search space has size exponential in $s^{1+1/d}$ since, according to the Prime number Theorem,

m is of order $e^{s^{1/d}}$. Our goal is to reduce the size of the configuration space of candidates to optimality. We will achieve this goal when $s \leq 5^d$.

To each configuration S, let $\{p_1, \ldots, p_r\}$ be the sequence of prime numbers p such that $|S/p| \neq |S|$ and let $m = p_1 \ldots p_r$ be their product. We associate to S the family of integers (a_c) defined by

$$a_c = |S \cap c_1 \cap c_2 \cap \ldots \cap c_r|, \tag{11}$$

where the index $c = (c_1, \ldots, c_r)$ ranges over the set $\mathcal{C} := \Lambda/p_1 \times \cdots \times \Lambda/p_r$. The integer a_c is the number of cameras in the coset c of Λ/m. Conversely, the Realizability Theorem shows that given a family of numbers $(a_c)_{c \in \mathcal{C}}$ there exists a configuration S of $s = \sum_c a_c$ points such that $|S/p| = |S|$ for $p \neq p_i$ and to which the family (a_c) is associated by the above described procedure.

Example 2.3 *For the configuration of example 2.1 we have that* $r = 4$ *and* $p_1 = 2, p_2 = 3, p_3 = 5, p_4 = 7$.

Equipped with this new way of specifying a configuration of cameras we give now a new expression for the function $u(S)$ to be maximized. We introduce the *reduced density function*, defined on the subsets E of S by

$$u'(E) := u(\mathcal{P} \setminus \{p_1, \ldots, p_r\}, E), \tag{12}$$

and the family of *reduced configurations* $\mathcal{B}_c \subseteq S$ defined by

$$\mathcal{B}_c = S \setminus \bigcup_{i=1}^{r} c_i. \tag{13}$$

Then by a repeated application of the Reduction Theorem we get that the visibility of the configuration S is the mean of the $\mathcal{P} \setminus \{p_1, \ldots, p_r\}$-density of the $p_1^d \ldots p_r^d$ reduced configurations, i.e.

$$m^d\, u(S) = \sum_{c \in \mathcal{C}} u'(\mathcal{B}_c), \tag{14}$$

where $m^d = p_1^d \ldots p_r^d$. Before we give the properties of the reduced density function we recall that a real function $f(e)$ is called *absolutely monotone* if $(-1)^{n+1} \Delta^n f(e) > 0$ for all natural numbers $n \geq 1$, where Δ^n is the standard notation of the calculus of finite differences [9]

$$\Delta^1 f(x) = f(x+1) - f(x), \quad \Delta^{n+1} f = \Delta^1(\Delta^n f).$$

In particular an absolutely monotone function f is stricly increasing and stricly concave, i.e $f(e+1) - f(e)$ is strictly decreasing as a function of e.

Theorem 2.4 (Optimization Theorem) *Let S be a configuration of s cameras and let u', and $(\mathcal{B}_c)$ be the corresponding reduced density function and family of reduced configurations associated to S. Then for $E \subseteq S$ the function $u'(E)$ depends only on the size $|E|$ of the set E. Let $u'(e) = u'(E)$, where $e = |E|$ and let $b_c = |\mathcal{B}_c|$. Then we can prove the following properties.*

1. *$u'(e)$ is absolutely monotone.*

2. $m^d u(S) = \sum_{c \in \mathcal{C}} u'(b_c)$

3. $b_c = \sum_{h(c,c')=r} a_{c'}$ *where the* Hamming distance $h(c, c')$ *is defined as the number of i such that $c_i \neq c'_i$.*

4. $\sum_{c \in \mathcal{C}} b_c = s \prod_{i=1}^{r} (p_i^d - 1)$

Proof. By hypothesis $|S/p| = |S|$, for all primes $p \notin \{p_1, \dots, p_r\}$. This implies that for such primes p any two cameras in S are pairwise visible modulo p. In particular for any set $E \subseteq S$, $|E/p| = |E|$. We conclude from equation (8) that the density function $u(\mathcal{P} \setminus \{p_1, \dots, p_r\}, E)$ depends only on the cardinality of the set $|E|$. More precisely we have

$$u'(E) = \sum_{k=1}^{e} (-1)^{k+1} \binom{e}{k} \prod_{p \neq p_1, \dots, p_r} \left(1 - \frac{k}{p^d}\right). \tag{15}$$

This proves the main assertion of the theorem regarding the function u'.

Next we proceed to the second part of the theorem. In view of the previous observations the function $u'(e)$ represents the $\mathcal{P} \setminus \{p_1, \dots, p_r\}$ density of a set of e cameras which are pairwise visible modulo p for each prime $p \in \mathcal{P} \setminus \{p_1, \dots, p_r\}$. Let $E \subseteq S$ and $A_1, \dots, A_n$ be n points of $S \setminus E$ such that $e = |E|$. According to equation (10) and Proposition 2.6 we have

$$(-1)^{n+1} \Delta^n u'(e) = u'(\{A_1\}, \dots, \{A_n\}; E) > 0$$

Parts 2. and 3. are trivial reformulations of equations (11), (13) and (14). Finally the last part follows from the following identities.

$$\begin{aligned} \sum_c b_c &= \sum_{c'} (a_{c'} |\{ c \mid h(c, c') = r \}|) \\ &= |\{ c \mid h(c, c') = r \}| \sum_{c'} a_{c'} \\ &= s \prod_{i=1}^{r} (p_i^d - 1). \end{aligned}$$

This completes the proof of the theorem. □

According to the Finiteness Theorem and the Optimization Theorem, our original optimization problem is reduced to the following integer optimization problem

$$\max \left\{ \sum_{I \in \mathcal{I}} u'(b_I) \mid b_I = \sum_{\substack{h(I,J)=r \\ J \in \mathcal{I}}} a_J, \quad \sum_{I \in \mathcal{I}} a_I = s, \quad a_I \in \mathbb{N} \right\} \tag{16}$$

where $\mathcal{I} = [1..2^d] \times \cdots \times [1..p_r^d]$ with $p_r < s^{1/d}$, h is the Hamming distance and u' is an absolutely monotone function.

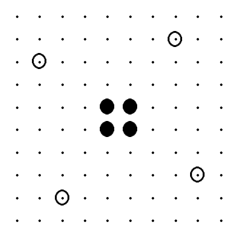

Figure 4: Two optimal 4 camera configurations in the plane.

2.3 Optimality for $s \leq 3^d$

Now it is possible to give characterizations of optimal configurations of size $\leq 3^d$.

Theorem 2.5 *A configuration of size $\leq 2^d$ is optimal if and only if its cameras are pairwise visible.*

Proof. Let S be an optimal configuration. From the Finiteness Theorem and our hypothesis $|S| \leq 2^d$, the cameras are pairwise visible modulo p for all prime numbers p. Consequently the cameras are pairwise visible. Conversely if the camers are pairwise visible, then their visibility is uniquely determined. □

Theorem 2.6 *A configuration S of size $\leq 3^d$ is optimal if and only if S is balanced.*

Proof. Let S be an optimal configuration. From the Finiteness Theorem and the hypothesis, $|S| \leq 3^d$, the cameras are pairwise visible modulo each prime $p \geq 3$. It remains to determine the visibility modulo 2. Our optimization problem is equivalent to the integer optimization problen

$$\max \left\{ \sum_{i=1}^{2^d} u'(b_i) \mid b_i = \sum_{k \neq i} a_k, \quad \sum_{i=1}^{2^d} a_i = s, \quad a_i \in \mathbb{N} \right\}$$

Here u' (the reduced density function) is an abolutely monotone function and a_i is interpreted as the number of cameras in the ith coset of $\Lambda/2$, provided we have numbered the 2^d cosets of $\Lambda/2$. In particular u' is strictly concave, i.e

$$\Delta u'(e) = u'(e+1) - u'(e)$$

is stricly decreasing. To show that the optimal solution is obtained when $|a_i - a_j| \leq 1$ we proceed as follows. Assume that for some i, j we have $a_i > a_j + 1$

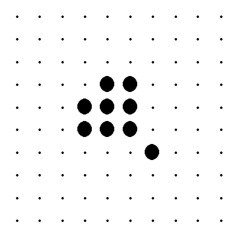

Figure 5: An optimal configuration of 9 cameras in the plane.

($\Leftrightarrow b_i < b_j - 1$). Then we claim that the objective function is increased by replacing a_i and a_j by $a_i - 1$ and $a_j + 1$. Indeed the variation of the objective function is

$$\Delta u'(b_i) - \Delta u'(b_j - 1)$$

which is > 0 since u' is stricly concave. To conclude the proof it remains to observe that the condition $|a_i - a_j| \leq 1$ determines the value of the objective function. $\square$

Example 2.4 *Figures 4 and 5 depict optimal configurations of 4 and 9 cameras in dimesion 2, respectively. However the configuration depicted in figure 3 is not optimal.*

3 Optimality for $s \leq 5^d$ Cameras

In the previous section we showed how to transform the original camera placement problem into a non-linear integer optimization problem and proved that optimal camera configurations of size $s \leq 3^d$ are precisely the balanced configuration (cf. Definition 2.2). In this section we extend our analysis even further in order to give optimality characterizations for the camera placement problem when the number of cameras is $s \leq 5^d$.

In the sequel it will be convenient to use the following notation. Let

$$L_1, \ldots, L_{3^d} \qquad \text{and} \qquad C_1, \ldots, C_{2^d}$$

be the 3^d classes of $\Lambda/3$ and 2^d classes of $\Lambda/2$, respectively. We use the abbreviations $l_i = |L_i \cap S|$, $c_j = |C_j \cap S|$, $a_{ij} = |L_i \cap C_j \cap S|$. Recall that our optimization problem has been reduced to the following problem

$$\begin{array}{ll} \text{maximize:} & \sum_{i,j} u'(b_{ij}) \\ \text{subject to:} & b_{ij} = \sum_{k \neq i, l \neq j} a_{kl}, \quad \sum_{i,j} a_{ij} = s, \quad a_{ij} \in \mathbb{N} \end{array}$$

and that optimal configurations are conjectured to be balanced. It turns out that this conjecture can not be proved or disproved using only the absolute monotony of the reduced density function. We refer the reader to [17] for this point. We prove a weaker result.

3.1 Proof of a weaker conjecture

As already proved the optimality of a configuration S of size $s \leq 5^d$, depends only on the relative sizes of the quantities l_i, c_j and a_{ij}. By taking advantage of our main optimization theorem we now give a partial description of the sizes of the equivalence classes in $\Lambda/2$, $\Lambda/3$ and $\Lambda/6$ when the configuration is optimal. Our main theorem is the following.

Theorem 3.1 *If S is an optimal configuration then*

$$|l_i - l_{i'}| \leq 1 \quad \textit{and} \quad |c_j - c_{j'}| \leq 1.$$

Furthermore there exists an integer $x \leq s$ such that, after any permutation of the indices which ensures that the sequences l_i and c_j are decreasing,

$$a_{ij} = \begin{cases} x \quad \textit{or} \quad x+1 & \textit{if } (i,j) > (i_0, j_0) \\ 0 \quad \textit{or} \quad 1 & \textit{otherwise} \end{cases}$$

where $i_0 = s \bmod 3^d$ and $j_0 = s \bmod 2^d$. □

Before we prove the theorem we introduce two transformations on the configurations which play a key role in the proof.

```
Exchange Procedure(L_i, L_i', S);
ε = 1;
for k in [1..2^d]
    do
    if a_i'k + a_ik is even then a_i'k := a_ik := (a_ik + a_i'k)/2
    else
        a_ik := (a_ik + a_i'k + ε)/2;
        a_i'k := (a_ik + a_i'k − ε)/2;
        ε := −ε;
    fi
    od
end;
```

Table 1: The exchange procedure.

3.1.1 The exchange and switch procedures

The Exchange Procedure

The first key transformation, called the *exchange* procedure, is depicted in Table 1. It takes as argument two classes, say L_i and $L_{i'}$, of $\Lambda/3$, and exchanges cameras between $L_i \cap C_k$ and $L_{i'} \cap C_k$ for $k = 1, \ldots, 2^d$ so that L_i and $L_{i'}$ are *balanced* i.e.,

$$|l_i - l_{i'}| \leq 1, \quad \text{and} \quad |a_{ik} - a_{i'k}| \leq 1. \tag{17}$$

The following lemma gives a sufficient condition for an exchange to be a visibility gain.

Lemma 3.1 *The exchange procedure, applied to L_i and $L_{i'}$, leaves b_{rk} and $b_{i'k}+b_{ik}$ unchanged for $r \neq i, i'$. Furthermore if the exchange procedure decreases the vector $(|b_{i'k} - b_{ik}|)_{k=1,\ldots,2^d}$, then the visibility of the configuration increases.*

Proof. The exchange procedure clearly leaves $a_{ik} + a_{i'k}$ and a_{rk} for $r \neq i, i'$ invariant; added to that, by definition, $b_{ij} = \sum_{k \neq i, l \neq j} a_{kl}$; therefore $b_{ik} + b_{i'k}$ and b_{rk} for $r \neq i, i'$ are also invariant.

The visibility variation, induced by the exchange procedure, is then, up to a positive factor, the sum for $k = 1, \ldots, 2^d$ of the variation of the quantity $u'(b_{ik}) + u'(b_{i'k})$; but, according to the strict concavity of the reduced density function u' and since $b_{ik}+b_{i'k}$ is constant, this last quantity increases if $|b_{ik}-b_{i'k}|$ decreases. □

Of course a similar definition of exchange procedure and a similar lemma hold for classes in $\Lambda/2$.

The Switch Procedure

The second key transformation, called the *switch* procedure, takes as argument two classes, say $L_i \cap C_j$ and $L_{i'} \cap C_{j'}$, of $\Lambda/6$, and increases by one the values of $a_{i'j}, a_{ij'}$, and decreases by one the values of $a_{ij}, a_{i'j'}$ (assuming that these quantities are positive). We can think of the switch as a transformation on 2×2 submatrices of the matrix of a_{ij}

$$\begin{array}{c} \\ i \\ i' \end{array} \begin{array}{c} \begin{array}{cc} j & j' \end{array} \\ \begin{pmatrix} a+1 & a' \\ a'' & a'''+1 \end{pmatrix} \end{array} \rightarrow \begin{pmatrix} a & a'+1 \\ a''+1 & a''' \end{pmatrix}. \tag{18}$$

The following lemma gives a sufficient condition for a switch to be a visibility gain.

Lemma 3.2 *The switch procedure, applied to $L_i \cap C_j$ and $L_{i'} \cap C_{j'}$, increases by one $b_{i'j}$ and $b_{ij'}$, and decreases by one b_{ij} and $b_{i'j'}$, while the other entries b_{rk} are unchanged. Furthermore if the vector $(|b_{i'j} - b_{ij}|, |b_{i'j'} - b_{ij'}|)$ decreases, then the visibility of the configuration increases.*

Proof. Similar to the proof of Lemma 3.1. □

3.1.2 Balancing the classes.

The following four identities are not difficult to prove from the above definitions and will be useful in the sequel.

$$\begin{aligned} b_{ik} &= s - l_i - c_k + a_{ik} && (19) \\ b_{i'k} - b_{ik} &= (l_i - l_{i'}) - (a_{ik} - a_{i'k}) && (20) \\ \sum_k (a_{ik} - a_{i'k}) &= l_i - l_{i'} && (21) \\ \sum_k (b_{i'k} - b_{ik}) &= (2^d - 1)(l_i - l_{i'}) && (22) \end{aligned}$$

We are now in position to commence with the proof of the main theorem. We break up the proof into several lemmas.

We begin by examining the exchange procedure.

Lemma 3.3 *Assume that $l_i - l_{i'}$ is even. If S is an optimal configuration, then L_i and $L_{i'}$ are balanced.*

Proof. First we note that the exchange procedure leaves invariant the parity of $b_{ik} - b_{i'k}$. It follows that the vector $(|b_{ik} - b_{i'k}|)_{k=1,\ldots,3^d}$ is bounded below by the 0-1 vector $(b_{ik} - b_{i'k} \bmod 2)_{k=1,\ldots,3^d}$. But this last vector is precisely obtained by the exchange procedure when $l_i - l_{i'}$ is even. According to Lemma 3.1 it follows that $|b_{ik} - b_{i'k}| = b_{ik} - b_{i'k} \bmod 2$; therefore L_i and $L_{i'}$ are balanced according to (22) and (20). □

Lemma 3.4 *Let L_i and $L_{i'}$ two equivalence classes of $\Lambda/3$ such that $l_i - l_{i'} = \Delta l > 0$ is odd. If S is an optimal configuration, then for every k we have*

$$b_{i'k} - b_{ik} \geq 0 \ \ (\Leftrightarrow \Delta l \geq a_{ik} - a_{i'k}).$$

Furthermore the number of k such that $b_{i'k} - b_{ik} = 0$ is at least one more than the number of k such that $b_{i'k} - b_{ik}$ is an even non-null number, with equality if only if L_i and $L_{i'}$ are balanced.

Proof. We use the equivalent form "if, for some k, $b_{i'k} - b_{ik} \leq -1$ then S is not optimal" and prove this assertion directly. According to Equation (22) one has $b_{i'k'} - b_{ik'} > \Delta l$ for some k'; furthermore $a_{ik} > 0$ and $a_{i'k'} > 0$ (cf. Equation (20)). Now a switch between the classes $L_i \cap C_k$ and $L_{i'} \cap C_{k'}$ decreases $|b_{i'k'} - b_{ik'}|$ by two, and decreases $|b_{i'k} - b_{ik}|$ by 0 or 2; according to Lemma 3.2 this means that S is not optimal.

Now we prove the second part of the lemma. Let $\beta(k) = b_{i'k} - b_{ik}$. Assume, without loss of generality, that $\beta(k)$ is null for $k = 1, \ldots, n_1$, non-null and even for $k = n_1 + 1, \ldots, n_1 + n_2 = n$, and odd for $k = n + 1, \ldots, 3^d$. We prove that $n_1 \geq n_2 + 1$. From (21) and $l_i - l_i'$ odd, n is odd. Let $n = 2n' + 1$. Now the exchange procedure leads to $l_i - l_{i'} = 1$ and to $\beta(k) = 1$ for $k = n + 1, \ldots, 3^d$, and $\beta(k) = 0$ for $n' + 1$ values of $k \in \{1, \ldots, n\}$ and $\beta(k) = 2$ for the remaining values of k; the important point is that we can choose at our convenience the $n' + 1$ values of $b_{i'k} - b_{ik}$ null among the n possible ones. Let us choose the $n' + 1$ first values of k i.e., $k = 1, \ldots, n' + 1$. According to Lemma 3.1 and the optimality of S, $n_1 \leq n' + 1$ and consequently $n_1 \geq n_2 + 1$ since $n_1 + n_2 = 2n' + 1$.

It remains to show that the equality is obtained only for balanced classes. Assume $n_1 = n_2 + 1$, then $b_{i'k} - b_{ik} = 0, 1$ or 2, and from (22),(20) we get $\Delta l = 1$ and $a_{ik} - a_{i'k} = -1, 0$ or 1. □

From Lemma 3.3 we obtain a partition of $\Lambda/3$ and $\Lambda/2$ in two parts of balanced classes. Without loss of generality, there exist an $0 \leq i_0 < 3^d$ and a $0 \leq j_0 < 2^d$ such that

$$\begin{array}{lllllll} l_1 & = & \cdots & = & l_{i_0} & = & l + \Delta l \\ l_{i_0+1} & = & \cdots & = & l_{3^d} & = & l \\ c_1 & = & \cdots & = & c_{j_0} & = & c + \Delta c \\ c_{j_0+1} & = & \cdots & = & c_{2^d} & = & c \end{array} \tag{23}$$

where Δl and Δc are odd. Furthermore, if $l_i = l_{i'}$ then for every k, $|a_{ik} - a_{i'k}| \leq 1$, and if $c_j = c_{j'}$ then for every k, $|a_{kj} - a_{kj'}| \leq 1$. In fact we have a stronger result. Let A, B, C and D be the four blocks

$$\begin{array}{r} A = \{\,(i,j) \mid i \leq i_0, j \leq j_0\,\} \\ B = \{\,(i,j) \mid i_0 < i, j \leq j_0\,\} \\ C = \{\,(i,j) \mid i \leq i_0, j_0 < j\,\} \\ D = \{\,(i,j) \mid i_0 < i, j_0 < j\,\}. \end{array} \tag{24}$$

Lemma 3.5 *In each block A,B,C and D we have* $|a_{ij} - a_{i'j'}| \leq 1$. ($\Leftrightarrow |b_{ij} - b_{i'j'}| \leq 1$)

Proof. We prove the contrapositive. Assume there exist $(i, j), (i', j')$ in the same block such that $a_{ij} = a$, $a_{ij'} = a_{i'j} = a+1$ and $a_{i'j'} = a+2$. Since $l_i = l_{i'}$ there exists a j'' such that $a_{ij''} = a' + 1$ and $a_{i'j''} = a'$. Now it can be verified that the switch procedure, applied to $L_i \cap C_{j''}$ and $L_{i'} \cap C_j$, leaves invariant the visibility of the configuration. But since $a_{i'j'}$ and $a_{i'j}$ differs now by 2, we can apply the exchange procedure to C_j and $C_{j'}$ in order to increase the visibility of the configuration. □

3.1.3 Refining the balancing

In the sequel we write $\max A$ for $\max\{ a_{ij} \mid (i, j) \in A \}$, and $\min A$ for $\min\{ a_{ij} \mid (i, j) \in A \}$. Similar notations are used for B, C, D.

Lemma 3.6 *Assume that A and B are nonempty. There exist* i, i', k *such that* $a_{ik} = \max A$ *and* $a_{i'k} = \min B$.

A similar result holds for the pairs $(A, C), (C, D)$ *and* (B, D) *as well as the transposed pairs* $(B, A), (C, A), (D, C)$ *and* $(D.B)$.

Proof. Let i, i', k, k' such that $a_{ik} = \max A$ and $a_{i'k'} = \min B$. We claim that $\min B = a_{i''k}$ or $\max A = a_{i''k'}$ for some i'', which will prove our lemma. Assume the contrary; then $a_{i''k} = \max B > \min B$ for all $i'' > i_0$ and $a_{i''k'} = \min A < \max A$ for all $i'' \leq i_0$; consequently $c_k - c_{k'} = \sum_j (a_{jk} - a_{jk'}) \geq 2$. But $c_k - c_{k'} = 0$; contradiction. □

Before we exploit the previous lemmas we examine the case where one of the integers i_0 and j_0 is null.

Lemma 3.7 *If either* i_0 *or* j_0 *is null then the configuration is balanced.*

Proof. Assume first that $i_0 = j_0 = 0$; then $A = B = C = \emptyset$, and the configuration is balanced according to lemma 3.5.

Assume now that $i_0 \neq 0$ and $j_0 = 0$ (the case $i_0 = 0, j_0 \neq 0$ is similar). Let $i \leq i_0 < i'$ and k_1 such that $a_{ik_1} = \max C$ and $a_{i'k_1} = \min D$ (cf. Lemma 3.6) and let $e = b_{i'k_1} - b_{ik_1} = \Delta l - (\max C - \min D)$. From $j_0 = 0$ we get $A = B = \emptyset$, and consequently $b_{i'k} - b_{ik} \in \{e, e+1, e+2\}$ for $k = 1, \ldots, 3^d$. But according to Lemma 3.4 $b_{i'k} - b_{ik} \geq 0$ and vanishes at least once; therefore $e = 0$, and $b_{i'k} - b_{ik} \in \{0, 1, 2\}$, which according to (22) gives $\Delta l = 1$ and $\max C - \min D = 1$. □

In the sequel we suppose that both i_0 and j_0 are non-null.

Lemma 3.8 $\Delta l = \max(\max A - \min B, \max C - \min D))$.

Proof. It is an obvious consequence of Lemmas 3.4 and 3.6. □

Lemma 3.9 *One of the three following cases holds.*

1. $\Delta l = \max A - \min B = \max C - \min D = 1$

2. $\Delta l = \max A - \min B$ *and* $\max C \leq \min D$

3. $\Delta l = \max C - \min D$ *and* $\max A \leq \min B$

Proof. If $\Delta l = 1$, it is simply a reformulation of Lemma 3.8.

Assume now that $\Delta l = \max A - \min B \geq 3$ (the case $\Delta l = \max C - \min D$ is similar) and let $e = \max C - \min D$. Let $i \leq i_0 < i'$ and $k_1 \leq j_0$ such that $a_{ik_1} = \max A$ and $a_{i'k_1} = \min B$ (cf. lemma 3.6). We put $\beta(k) = b_{i'k} - b_{ik}$ for $k \neq k_1$ and denote by n_1 and n_2, respectively, the number of $k \neq k_1$ such that $\beta(k) = 0$ and the number of k such that $\beta(k)$ is even and non-null. From

1. $n_1 > n_2$, (cf. lemma 3.4)

2. the expected value of $\beta(k)$ for $k \neq k_1$ is Δl, (cf. equation (22));

3. $\beta(k) \in \{ 0, 1, 2, \Delta l - e, \Delta l - e + 1, \Delta l - e + 2 \}$

we clearly get $e = \max C - \min D < 1$. □

Let us denote $(P_1), (P_2), (P_3)$, in this order, the three cases of our previous lemma, and let $(Q_1), (Q_2), (Q_3)$ be their counterpart for Δc; $(Q_1), (Q_2)$ and (Q_3) are simply obtained from $(P_1), (P_2)$ and (P_3) by replacing Δl by Δc and by permuting B and C; to be more explicit

$$(P_1) \begin{cases} \Delta l = 1 \\ \max A = 1 + \min B \\ \max C = 1 + \min D \end{cases} \qquad (Q_1) \begin{cases} \Delta c = 1 \\ \max A = 1 + \min C \\ \max B = 1 + \min D \end{cases}$$

$$(P_2) \begin{cases} \Delta l = \max A - \min B \\ \min D \geq \max C \end{cases} \qquad (Q_2) \begin{cases} \Delta c = \max A - \min C \\ \min D \geq \max B \end{cases}$$

$$(P_3) \begin{cases} \Delta l = \max C - \min D \\ \min B \geq \max A \end{cases} \qquad (Q_3) \begin{cases} \Delta c = \max B - \min D \\ \min C \geq \max A \end{cases}$$

Lemma 3.10 *If (P_1) and (Q_1) hold then the configuration is balanced.*

Proof. In view of (P_1) and (Q_1) it suffices to prove that $\max B = 1 + \min B$ ($\Leftrightarrow \max C = 1 + \min C$). Assume on the contrary that $\max B = \min B$ ($\Leftrightarrow \max C = \min C$) and let x be the common value of the entries a_{ij} in the blocks B and C; according to (P_1) and (Q_1) there exists an entry $a_{ij} = x + 1$ in block A and an entry $a_{i'j'} = x - 1$ in block D; but then $\Delta l = \sum_k a_{ik} - a_{i',k} \geq 2$; contradiction. □

Lemma 3.11 *If (P_1) (resp. (Q_1)) and (Q_2) or (Q_3) (resp. (P_2) or (P_3)) hold then the configuration is balanced.*

Proof. We examine the case $(P_1), (Q_2)$ (the other cases are similar). From $\Delta c \geq 1$ and (Q_2) we get that $\max A - \min C > \max B - \min D$ which combined with (P_1) gives $\max C - \min C > \max B - \min B$; hence it follows that $\max C = 1 + \min C$ and $\max B = \min B$. But then $\Delta c = 1 + \max B - \min D \leq 1$, which implies that $\Delta c = 1$ and that $\max B = \min D$. □

Lemma 3.12 *(P_2) and (Q_3) (resp. (P_3) and (Q_2)) are not compatible.*

Proof. Indeed from (P_2) and (Q_3) we deduce that $\Delta l + \Delta c = \max A - \min D + \max B - \min B \leq \min C - \max C + \max B - \min B \leq 1$ which is incompatible with the fact that Δl and Δc are odd. □

Lemma 3.13 *If (P_2) and (Q_2) hold and if $\min D = \min C$ (or $\min D = \min B$) then the configuration is balanced.*

Proof. We assume that $\min D = \min C$ (the case $\min D = \min B$ is similar). Let $i \leq i_0 < i'$ and $k_1 \leq j_0$ such that $a_{ik_1} = \max A$ and $a_{i'k_1} = \min B$ (cf. lemma 3.6). We put $\beta(k) = b_{i'k} - b_{ik}$ for $k \neq k_1$ and denote by n_1 and n_2, respectively, the number of $k \neq k_1$ such that $\beta(k) = 0$ and the number of k such that $\beta(k)$ is even and non-null. From

1. $n_1 > n_2$, (cf. lemma 3.4)
2. the expected value of $\beta(k)$ for $k \neq k_1$ is Δl, (cf. equation (22));
3. $\beta(k) \in \{ 0, 1, 2, \Delta l, \Delta l + 1 \}$

we get that $\Delta l = 1$. According to (P_2) and (Q_2) one has $\max A = 1 + \min B = \Delta c + \min C = \Delta c + \min D \geq \Delta c + \max B$; it follows that $\Delta c = 1$ and $\min B = \max B = \min D = \min C$; the configuration is then balanced. □

Lemma 3.14 *If (P_3) and (Q_3) hold and if $\min B = \min A$ (or $\min C = \min A$) then the configuration is balanced.*

Proof. Assume that $\min B = \min A$ (the case $\min C = \min A$ is similar); as in the previous lemma we can deduce that $\Delta l = 1$. According to (P_3) and (Q_3) one has $\min D = \max C - 1 = \max B - \Delta c$ and $\min C \geq \min B = \min A = \max A$; then $\Delta c = \max B - \max C + 1 \leq 2$. It follows that $\Delta c = 1$ and $\max B = \max C = 1 + \min D$. □

The above analysis has shown that an optimal configuration is either balanced or it satisfies one of the two following conditions.

$$
(S_1) \begin{cases} \Delta l = \max A - \min B \\ \Delta c = \max A - \min C \\ \min D > \min C \\ \min D > \min B \end{cases} \qquad (S_2) \begin{cases} \Delta l = \max C - \min D \\ \Delta c = \max B - \min D \\ \min B > \min A \\ \min C > \min A \end{cases}
$$

Lemma 3.15 *If* (S_2) *holds then the configuration is balanced.*

Proof. First we introduce some notations. Let $i \leq i_0 < i'$ such that $a_{ik} - a_{i'k}$ equal $\min A - \max B$ for some k, and similarly let $j \leq j_0 < j'$ such that $a_{kj} - a_{kj'}$ equal $\min A - \max C$ for some k (cf. Lemma 3.6).

Set $\omega(k) = a_{ik} - a_{i'k}$ and let $k_1 > j_0$ such that $\omega(k_1) = \Delta l$ (cf. Lemma 3.4). We denote by ω_1 the expected value of $-\omega(k)$ for $k \leq j_0$ and by ω_2 the expected value of $\omega(k)$ for $k \neq k_1 > j_0$. The equation (21), $\sum_k (a_{ik} - a_{i'k}) = l_i - l_{i'}$, may then be rewritten

$$(2^d/2 + y)\omega_1 = (2^d/2 - y - 1)\omega_2 \tag{25}$$

where $y = j_0 - 2^d/2$. Furthermore it is clear that

$$\begin{cases} \max C - \min D - 1 \leq \omega_2 \leq \max C - \min D = \Delta l \\ \min B - \max A \leq \omega_1 \leq \max B - \min A \end{cases} \tag{26}$$

We note also that ω_1 and ω_2 are non-null; indeed from our hypothesis (S_2) $\min B - \max A \geq 0$ and $\max B - \min A \geq 1$; but $\omega_1 \geq \min B - \max A$ and $-\omega(k) = \max B - \min A$ for some k; consequently $\omega_1 \neq 0$ and $\omega_2 \neq 0$ follows from (25) and $2^d/2 + y = j_0 \neq 0$.

Similarly we set $\tau(k) = a_{kj} - a_{kj'}$, and introduce τ_1 the expected value of $-\tau(k)$ for $k \leq i_0$ and τ_2 the expected value of $\tau(k)$ for $k \neq k_2 > i_0$ with $\tau(k_2) = \Delta c$. Related to equations (25) and (26) are then

$$(3^d/2 + x)\tau_1 = (3^d/2 - x - 1)\tau_2 \tag{27}$$

where $x = i_0 - 3^d/2$, and

$$\begin{cases} \max B - \min D - 1 \leq \tau_2 \leq \max B - \min D = \Delta c \\ \min C - \max A \leq \tau_1 \leq \max C - \min A \end{cases} \tag{28}$$

At last τ_1 and τ_2 are non-null.

Recall that Δl and Δc are odd. The proof is divided in 3 parts: first we assume Δl and $\Delta c \geq 3$, then $\Delta l \geq \Delta c = 1$ and at last $\Delta c > \Delta l = 1$.

Case 1. The quantities Δl and Δc are ≥ 3. From Δl and $\Delta c \geq 3$, we get $\min C$ and $\min B \geq 2$; therefore the number $\sum_{B \cup C} a_{ij}$ is bounded below by $2\#B \cup C$ which is equal to $6^d - 4xy$; but this number is, by hypothesis, less than or equal to 5^d; consequently

$$4xy \geq 6^d - 5^d. \tag{29}$$

It follows that that x and y are both positive or both negative.

Case 1.1 The quantities x and y are negative. If x and y are negative then $x \leq -1/2$ and $y \leq -1$ since $x + 1/2$ and y are integers; it follows, according to (25) and (27), that $\tau_2 \le \tau_1$ and $\omega_2 < \omega_1$. Using the inequalities (26) and (28) we have then

$$\begin{array}{l} \max B - \min D - 1 \le \max C - \min A \\ \max C - \min D - 1 < \max B - \min A \end{array} \tag{30}$$

from which we deduce that $|\max B - \max C| \le \min D - \min A + 1$. But $\min D = 0$ since $\#C \cup D = 3^d(2^d/2 - y) \ge 3^d(2^d/2+1) \ge 5^d$ and $\max B - \max C = \Delta l - \Delta c$ is even; consequently $\max B = \max C = \Delta l = \Delta c$ and $\min A = 0$.

Now the ratios $\omega_2/\omega_1 = (2^d/2 + y)/(2^d/2 - y - 1)$ and $\tau_2/\tau_1 = (3^d/2 + x)/(3^d/2 - x - 1)$ are bounded below by $(\Delta l - 1)/\Delta l$ whose minimal value $2/3$ is obtained for $\Delta l = 3$. Simple manipulations give then

$$-5x \le \frac{3^d}{2} + 2 \quad \text{and} \quad -5y \le \frac{2^d}{2} + 2 \tag{31}$$

from which we deduce that $4xy \le 4(2^d/2+2)(3^d/2+2)/25$ which is incompatible with (29).

Case 1.2 The quantities x and y are positive. If x and y are positive, according to (25) and (27), one has $\tau_2 < \tau_1$ and $\omega_1 < \omega_2$. Using the inequalities (26) and (28) we have then

$$\begin{array}{l} \min C - \max A < \max B - \min D \\ \min B - \max A < \max C - \min D \end{array} \tag{32}$$

from which we deduce that $|\max C - \max B| < \max A - \min D + 1$. But $\min A = 0$ since $\#B \cup A = 3^d(2^d/2 + y) \ge 3^d(2^d + 1) > 5^d$ and $\max B - \max C = \Delta l - \Delta c$ is even; consequently $\min D \le \max A \le 1$ and $\Delta l = \Delta c$.

Now the ratios $\omega_2/\omega_1 = (2^d/2 + y)/(2^d/2 - y - 1)$ and $\tau_2/\tau_1 = (3^d/2 + x)/(3^d/2 - x - 1)$ are bounded above by $\Delta l/(\Delta l - 2)$ whose maximal value is 3 obtained for $\Delta l = 3$. Simple manipulations give then

$$4x \le 3^d - 3 \quad \text{and} \quad 4y \le 2^d - 3 \tag{33}$$

and consequently $4xy \le (2^d - 3)(3^d - 3)/4$ which is incompatible with (29).

Case 2. We suppose $\Delta c \ge \Delta l = 1$.

Case 2.1 We suppose $\min D = 0$. If $\min D = 0$ then, according to (S_2), $\max C = \min C = 1$, $\min A = 0$, and $\max B = \Delta c$. It follows that $\beta(k) = b_{kj'} - b_{kj}$ belongs to $\{0, 1, 2, \Delta c, \Delta c+1\}$, and consequently that $\Delta c = \max B = 1$. The configuration is then balanced.

Case 2.2 We suppose $\min D \ne 0$. From $5^d \ge \sum_{C \cup D} a_{ij} \ge \#C \cup D = 3^d(2^d/2 - y)$ we deduce that $y \ge 0$ and consequently that $\omega_1 < \omega_2$. It follows, according to (26), that $\min B - \max A < \max C - \min D = 1$; and consequently that $\min B = \max A$. But $\min A = 0$ since $\sum_{A \cup B \cup C \cup D} a_{ij} \le 5^d$; hence $\min B =$

$\max A = 1$ and $\max B = \Delta c + \min D \leq 2$; then $\Delta c = 1$ and $\min D = 1$. Futhermore $\max C = \max B = 2$.

We claim that there exist $(i,j) \in A$ and $(i',j') \in D$ such that

$$a_{ij} = a_{i'j'} = 1 \qquad \text{and} \qquad a_{i'j} = a_{ij'} = 2.$$

Indeed let (i,j) such that $a_{ij} = \max A = 1$ and fix $j'' > j_0$; since $b_{kj} - b_{kj''}$ vanishes for some k (cf. Lemma 3.4) it follows that $a_{kj} = \max B$ for some k, say i'. Now according to Lemma 3.4 $b_{i'k} - b_{ik}$ vanish for some $k > j_0$, say j', i.e., $a_{ij'} = \max C = 2$ and $a_{i'j'} = \min D = 1$. But the switch

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

increases the visibility of the configuration; indeed the corresponding transformation on the b_{ij} matrix is

$$\begin{pmatrix} b+1 & b+3 \\ b+3 & b+3 \end{pmatrix} \rightarrow \begin{pmatrix} b+2 & b+2 \\ b+2 & b+4 \end{pmatrix}$$

where $b = s - l - 1 - c - 1$, and the visibility variation is $\Delta^3 u'(b+1)$ which is positive.

Case 3. We suppose $\Delta l > \Delta c = 1$.

Case 3.1 We suppose $\min D = 0$**.** As in **case 2.1** we get $\Delta l = 1$; contradiction.

Case 3.2 We suppose $\min D \neq 0$**.** From $\Delta l \geq 3$ we get $\min C \geq 3$. and from $5^d \geq \sum_{B \cup C \cup D} a_{ij} \geq \# B \cup D + 3\# C \geq 2^d(3^d/2 - x) + 3(3^d/2 + x)$ we get that $x \leq 1/2$ and consequently $\tau_1 < \tau_2$ and we can finish the proof as in **Case 2.2**. □

Lemma 3.16 *If* (S_1) *holds then* $\Delta l = \Delta c = 1$.

Proof. First we introduce some notations. We fix $i \leq i_0 < i'$ (resp. $j \leq j_0 < j'$), set $\omega(k) = a_{ik} - a_{i'k}$ (resp. $\tau(k) = a_{kj} - a_{kj'}$) and let $k_1 \leq j_0$ (resp. $k_2 \leq i_0$) such that $\omega(k_1) = \Delta l$ (resp. $\tau(k_2) = \Delta c$). We denote by ω_1 (resp. τ_1) the expected value of $\omega(k)$ (resp. $\tau(k)$) for $k \neq k_1 \leq j_0$ (resp. $k \neq k_2 \leq i_0$), and denote by ω_2 (resp. τ_2) the expected value of $-\omega(k)$ (resp. $-\tau(k)$) for $k > j_0$ (resp. $k > i_0$). According to equation (21) one has

$$(2^d/2 + y - 1)\omega_1 = (2^d/2 - y)\omega_2 \tag{34}$$

$$(3^d/2 + x - 1)\tau_1 = (3^d/2 - x)\tau_2. \tag{35}$$

where $y = j_0 - 2^d/2$ and $x = i_0 - 3^d/2$. Furthermore its is clear that

$$\left\{ \begin{array}{c} \max A - \min B - 1 \leq \omega_1 \leq \max A - \min B = \Delta l \\ \min D - \max C \leq \omega_2 \leq \max D - \min C \end{array} \right. \tag{36}$$

$$\left\{ \begin{array}{c} \max A - \min C - 1 \leq \tau_1 \leq \max A - \min C = \Delta c \\ \min D - \max B \leq \tau_2 \leq \max D - \min B \end{array} \right. \tag{37}$$

The proof is devided in two parts: first we assume $\Delta l \neq \Delta c$, then $\Delta l = \Delta c \geq 3$.

Case 1. We suppose $\Delta l \neq \Delta c$. We assume $\Delta l > \Delta c$ (the case $\Delta c > \Delta l$ is similar). Since Δl and Δc are odd one has $\min C - \min B = \Delta l - \Delta c \geq 2$; and consequently $\min D > \min C \geq 2$; furthermore $\min A \geq 2$ since $\Delta l \geq 3$. We claim that $y > 0$ and $x < 0$; indeed $\sum_{A \cup C} a_{ij}$ and $\sum_{C \cup D} a_{ij}$ are bounded above by 5^d and bounded below by $2\,2^d(3^d/2 + x)$ and $2\,3^d(2^d/2 - y)$, respectively. It follows, according to (34) and (35) that $\omega_1 \leq \omega_2$ and that $\tau_2 \leq \tau_1$, and consequently, according to (36) and (37), one has $\Delta l - 1 \leq \max D - \min C$ and $\min D - \max B \leq \Delta c$; combining these two last inequalities we get $4 \leq 2\,(\Delta l - \Delta c) \leq \max D - \min D + \max B - \min B + 1 \leq 3$; contradiction.

Case 2. We suppose $\Delta l = \Delta c \geq 3$. We observe that $\max A \geq 3$, and $\min D > \min B = \min C$; consequently $\min B = \min C = 0$ since the number of cameras is less than 5^d.

Now we prove that $\max A$ appears at least 9 times in block A; indeed according to Lemma 3.4 and since L_k and $L_{k'}$ for $k \leq i_0 < k'$ are not balanced $\beta(l) = b_{k'l} - b_{kl}$ vanishes at least 3 times, and consequently the entry $\max A$ appears at least 3 times in each "row" L_k for $k \leq i_0$; similarly $\max A$ appears at least 3 times in each "column" C_l for $l \leq j_0$; this forces i_0 and j_0 to be ≥ 3 and consequently proves our claim.

We claim now that $xy < 0$; indeed $\sum_{A \cup D} a_{ij}$ is bounded above by 5^d and bounbed below by $18 + \#A \cup D = 18 + 6^d/2 + 2xy > 5^d + 2xy$.

Let

$$r_1 = (3^d/2 + x - 1)(3^d/2 - x) = \tau_2/\tau_1, \tag{38}$$

and

$$r_2 = (2^d/2 + y - 1)/(2^d/2 - y) = \omega_2/\omega_1. \tag{39}$$

According to (34),(35),(36),(37) the ratios r_1 and r_2 are bounded below by $m = (\min D - 1)/\max A$ and bounded above by $M = \max D/(\max A - 1)$. But $xy < 0$ implies that $r_1 r_2 > 1$, and consequently $m < 1 < M$ which is equivalent to $\min D = \max A - 1$ or $\min D = \max A$.

Assume that $\min D = \max A - 1$. Then

$$1/3 \leq r_1, r_2 \leq 3/2$$

and it can be verified that $\sum_{A\cup D} a_{ij}$ is bounded below by

$$9 + 2\,\frac{2^d+3}{4}\,\frac{3^d+1}{2} + 2\,\frac{2^d-1}{2}\,\frac{2\,3^d-2}{5}$$

(or the same expression after permutation of 2^d and 3^d) In each case we can verify that this number is greater than 5^d.

Similarly assume that $\min D = \max A$. Then

$$\frac{2}{3} \leq r_1, r_2 \leq 2$$

and it can be verified that $\sum_{A\cup D} a_{ij}$ is bounded below by

$$9 + 2\,\frac{2\,2^d+3}{5}\,\frac{3^d+1}{2} + 3\,\frac{2\,2^d-1}{3}\,\frac{3^d-2}{3}$$

(or the same expression after permutation of 2^d and 3^d) In each case we can verify that this number is greater than 5^d. □

Proof of Theorem 3.1. From the above analysis an optimal configuration is either balanced or satisfies $\Delta l = \Delta c = 1$ and $\max A = 1+\min B = 1+\min C < 1+\min D$. This proves the theorem taking $x = \min D$ and using $\sum_{i,j} a_{ij} \leq 5^d$.□

3.2 The optimal configurations

Let us call $\mathcal{C}(x)$ the set of configurations of size s which satisfy the conditions of Theorem 3.1. According to $\sum_i l_i = \sum_j c_j = \sum_{i,j} a_{ij} = s$ these conditions are equivalent to

$$\begin{array}{lllllll} l_1 & = & \cdots & = & l_{i_0} & = & l+1 \\ l_{i_0+1} & = & \cdots & = & l_{3^d} & = & l \\ c_1 & = & \cdots & = & c_{j_0} & = & c+1 \\ c_{j_0+1} & = & \cdots & = & c_{2^d} & = & c \end{array} \tag{40}$$

and

$$a_{ij} = \left\{ \begin{array}{cccc} x & \text{or} & x+1 & \text{if } (i,j) > (i_0, j_0) \\ 0 & \text{or} & 1 & \text{otherwise} \end{array} \right. \tag{41}$$

where

$$\begin{array}{lllllll} l & = & \lfloor \frac{s}{3^d} \rfloor & & i_0 & = & s \bmod 3^d \\ c & = & \lfloor \frac{s}{2^d} \rfloor & & j_0 & = & s \bmod 2^d. \end{array} \tag{42}$$

It will be convenient to use the notation i'_0 and j'_0 for $2^d - i_0$ and $3^d - j_0$, respectively.

Clearly balanced configurations are in $\mathcal{C}(0)$ and optimal configurations are in $\mathcal{C}(x)$ for some $x \leq s$. The union of the $\mathcal{C}(x)$ is denoted $\mathcal{C}$.

Let $\lambda(x)$ be defined by the recurrence relation

$$\left\{ \begin{array}{lcl} \lambda(0) & = & \Delta^3 u'(b) \\ \lambda(x+1) & = & \lambda(x) + \Delta^2 u'(b+x+2) \end{array} \right. \tag{43}$$

where $b = s - l - c - 2$, and let

$$\begin{array}{lcl} M(s) & = & \min\{\, i'_0 l, j'_0 c, i_0 j_0 - j_0(c+1) + i'_0 l \,\} \\ m(s) & = & \max\{\, 0, i'_0 l - j_0(c+1), i'_0 l - i'_0 j_0, j'_0 c - i_0 j'_0 \,\} \\ x(s) & = & \left\lceil \frac{M(s)}{i'_0 j'_0} \right\rceil - 1. \end{array} \tag{44}$$

We are now in position to give the main result of this section.

Theorem 3.2 *Optimal configuration belong to* $\mathcal{C}(x_1)$ *where*

$$x_1 = \max\{\, x \in [0, x(s)] \mid \lambda(x) \geq 0 \,\}$$

and are characterized by

$$\sum_{i > i_0, j > j_0} a_{ij} \quad \left\{ \begin{array}{l} \textit{maximal if } \lambda(x_1) > 0 \\ \textit{any value if } \lambda(x_1) = 0 \end{array} \right.$$

Furthermore for each $x \in [0, x(s)]$ *an abstract configuration of* $\mathcal{C}(x)$ *such that* $\sum_{i > i_0, j > j_0} a_{ij}$ *is maximal can be computed in time* $O(s \log s)$ *and space* $O(s)$.

Before we give the proof of this theorem we introduce a few more notations.

Given a configuration S of $\mathcal{C}$, let

$$\begin{array}{lclclcl} n_A & = & \sum_{i \leq i_0, j \leq j_0} a_{ij} & \quad & n_B & = & \sum_{i > i_0, j \leq j_0} a_{ij} \\ n_C & = & \sum_{i \leq i_0, j > j_0} a_{ij} & \quad & n_D & = & \sum_{i > i_0, j > j_0} a_{ij}; \end{array} \tag{45}$$

we use also the notations $n(S)$ for n_D and $x(S)$ (resp. $x'(S)$) for the minimal (resp. maximal) x such that $S \in \mathcal{C}(x)$. Clearly $x(S), x'(S)$ and $n(S)$ are related by

$$x(S) = \left\lceil \frac{n(S)}{i'_0 j'_0} \right\rceil - 1 \qquad x'(S) = \left\lfloor \frac{n(S)}{i'_0 j'_0} \right\rfloor. \tag{46}$$

Straightforward calculations give then

$$\begin{array}{lcl} 6^d u(S) & = & n_1 u'(b) + n_2 u'(b+1) + n_3 u'(b+2) \\ & & + n_4 u'(b+x+2) + n_5 u'(b+x+3) \end{array} \tag{47}$$

where $b = s - l - c - 2$, $x = x'(S)$ and

$$\begin{array}{lcl} n_1 & = & i_0 j_0 - n_A \\ n_2 & = & n_A + i'_0 j_0 - n_B + i_0 j'_0 - n_C \\ n_3 & = & n_B + n_C \\ n_4 & = & i'_0 j'_0 - n_D \bmod i'_0 j'_0 \\ n_5 & = & n_D \bmod i'_0 j'_0. \end{array} \tag{48}$$

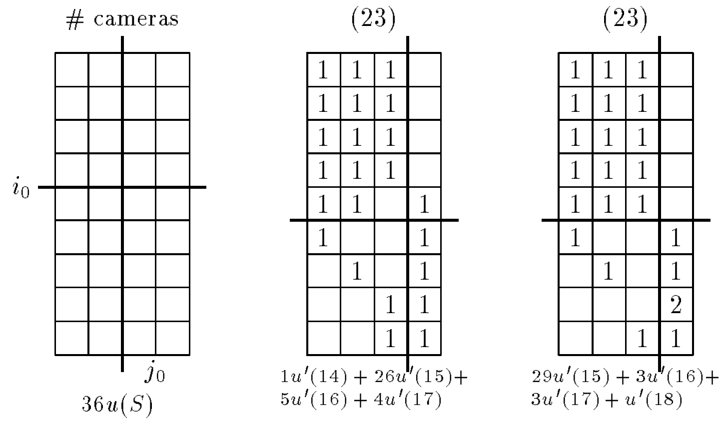

Figure 6: The two candidates to optimality of the 23-cameras placement problem.

In particular the visibility of the configuration S depends only on the value of $n(S)$. For example two abstract configurations of 23 cameras in dimension 2 which belong to $\mathcal{C}(0)$ and $\mathcal{C}(1)$ respectively are represented by their a_{ij}'s matrix in Figure 6.

We recall that the switch operator picks a camera from $L_{i'} \cap C_j$ and puts it in $L_i \cap C_j$ and similarly from $L_i \cap C_{j'}$ to $L_{i'} \cap C_{j'}$; we restrict our attention to the ones which leave the configuration in $\mathcal{C}(x)$, i.e, we assume that $a_{ij} = 0$, $a_{i'j'} = x$ or 0 and $a_{ij'} = a_{i'j} = 1$ (in that case we will say that the switch is *applicable* to the configuration). There are two kinds of such transformations. Firstly $i, i' \leq i_0$ (or $i, i' > i_0$ or $j, j' \leq j_0$ or $j, j' > j_0$); in that case n_A, n_B, n_C and n_D remain constant; such a transformation will be called a *u-switch*. Secondly $i \leq i_0 < i'$ and $j \leq j_0 < j'$; in that case we must have $a_{ij} = 0, a_{i'j} = a_{ij'} = 1$ and $a_{i'j'} = x$ and after the transformation we have $a_{ij} = 1, a_{i'j} = a_{ij'} = 0$ and $a_{i'j'} = x + 1$; such a transformation will be called an $x+1$-*switch*. An $x+1$-switch increases the values of n_A and n_D by one and decreases the values of n_B and n_C by one.

We need the following lemma.

Lemma 3.17 *With the above notations we have*

1. *if* $n(S) < M(s)$ *(resp.* $n(S) > m(s)$*) then there exists a sequence of at most four u-switches followed by an (resp. an inverse)* $x'(S)+1$*-switch applicable to* S*;*

2. $\{\, n(S) \mid S \in \mathcal{C} \,\} = [m(s), M(s)]$ *and* $\{\, x(S) \mid S \in \mathcal{C} \,\} = [0, x(s)]$

3. $\lambda(0) > 0$ *and* $\lambda(x)$ *is stricly decreasing.*

4. $\lambda(x)$ *is the variation of visibility induced by an* $x+1$*-switch.*

Proof. Let $\mu(x)$ be the variation of visibility induced by an $x+1$-switch. From equation (47) a simple calculation gives

$$\mu(x) = 3\,u'(b+1) + u'(b+x+3) - u'(b) - 2\,u'(b+2) - u'(b+x+2). \quad (49)$$

Claims (3) and (4) are consequence of the following expressions of $\mu(0)$ and $\Delta\mu(x)$

$$\begin{aligned} \mu(0) &= 3\,u'(b+1) + u'(b+3) - u'(b) - 3\,u'(b+2) \\ &= \Delta^3 u'(b) \end{aligned}$$

$$\begin{aligned} \mu(x) - \mu(x+1) &= 2\,u'(b+x+3) - u'(b+x+4) - u'(b+x+2) \\ &= -\Delta^2 u'(b+x+2) \end{aligned}$$

which are easily obtained from (49) and which are stricly positive because of the absolute monotonicity of the reduced density function.

To prove claims (1) and (2) we use the notations introduced above. Obviously the 4-tuple (n_A, n_B, n_C, n_D) verifies the relations

$$\begin{array}{rclcrcl} n_A + n_B &=& j_0(c+1) & \qquad & n_A + n_C &=& i_0(l+1) \\ n_B + n_D &=& i'_0 l & & n_C + n_D &=& j'_0 c. \end{array} \quad (50)$$

and

$$\begin{array}{ll} 0 \le n_A \le i_0 j_0 & \qquad 0 \le n_D \\ 0 \le n_B \le i'_0 j_0 & \qquad 0 \le n_C \le i_0 j'_0 \end{array} \quad (51)$$

Elementary algebra shows that n_D, subject to the only constraints (50) and (51) (i.e. we do not care what n_D is for some configuration S), is maximal if and only if $n_A = i_0 j_0$ or $n_B n_C = 0$, and n_D is minimal if and only if $n_A n_D = 0$ or $n_B = i'_0 j_0$ or $n_C = i_0 j'_0$. Simple calculation gives then that $m(s)$ and $M(s)$ are respectively the minimal and maximal value of n_D. Clearly $m(s)$ and $M(s)$ are respectively a lower and an upper bound on the minimal and maximal value of $n(S)$ as S ranges over $\mathcal{C}$; consequently claim (2) is consequence of claim (1).

Suppose now that $n(S) < M(s)$. Then there exist $i \le i_0 < i'$ and $j \le j_0 < j'$ such that $a_{ij'} = a_{i'j} = 1$. Now we show that in at most four u-switches we can transform the configuration in such a way that $a_{ij} = 0$ and $a_{i'j'} = x$, hence we can apply a $x+1$-switch to increase by one the value of n_D. We explain the procedure for a_{ij} (for $a_{i'j'}$ it is similar). There are three cases illustrated in Figure 7).

1. Firstly, $a_{ij} = 0$; then we do nothing.

2. Secondly, $a_{ij} = 1$ but there exists $j'' \le j_0$ such that $a_{ij''} = 0$; then since $c_j = c''_j$ there exists i'' such that $a_{i''j} = 0$ and $a_{i''j''} = 1$ and we can apply a u-switch to the configuration to get $a_{ij} = 0$. Similarly we can do the same work if $a_{ij} = 1$ and there exists $i'' \le i_0$ such that $a_{i''j} = 0$.

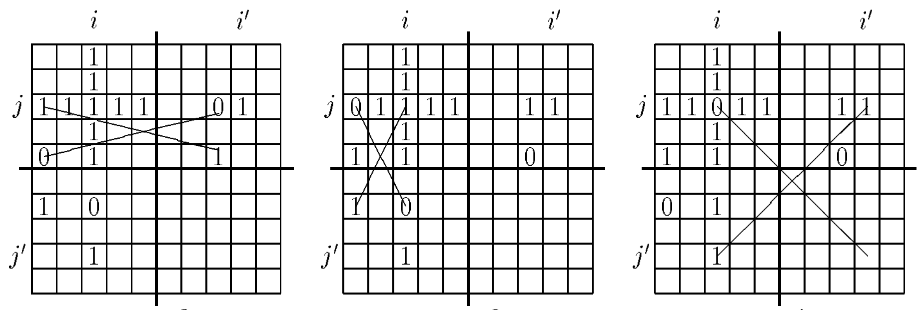

Figure 7: Illustration of Lemma 3.17.

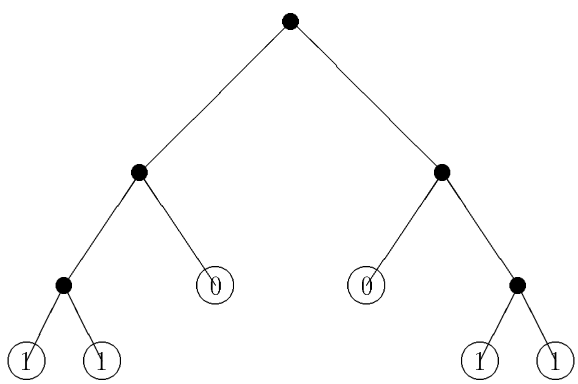

Figure 8: representation of the word 11000011

3. Thirdly, $a_{il} = a_{kj} = 1$ for all $l \leq j_0$ and all $k \leq i_0$; then since $n(S) = M(s)$ is not satisfied there exist an $i'' \leq i_0$ and a $j'' \leq j_0$ such that $a_{i''j''} = 0$; but since $l_i = l''_i$ there exists $j''' > j_0$ such that $a_{i''j'''} = 1$ and $a_{ij'''} = 0$ and we can apply a u-switch to the configuration to get $a_{ij''} = 0$ and we are in the situation of the second case, so we can apply a new u-switch to get $a_{ij} = 0$.

The case $n(S) > m(s)$ is similar. These considerations prove claim (1). □

Proof of Theorem 3.2. The first part is consequence of the above lemma; indeed starting from a balanced configuration S (i.e., $x(S) = 0$) and applying successively the sequence of switches which increases $n(S)$ by one until $n(S)$ reaches its maximal value $M(s)$ (claims (1), (2)) we get configurations of increasing visibility while the variation of visibility $\lambda(x)$ is positive (i.e $x(S) \leq x_1$) and then configurations of decreasing visibility since the function $\lambda(x)$ is decreasing (claim (3)).

For the last part, the idea is to start from a balanced configuration and to apply a sequence of at most 5 switches to increase the value of n_D by one. The proof of the existence of this sequence of switches is constructive and it remains to give its complexity analysis.

It will be useful to think of the abstract representation of a configuration of $\mathcal{C}(x)$ as a complete bipartite graph between the 3^d classes $L_1, \ldots, L_{3^d}$ of $\Lambda/3$ and the 2^d classes $C_1, \ldots, C_{2^d}$ of $\Lambda/2$, where the edge connecting L_i and C_j is labeled with the value $0, 1, x$ or $x+1$ of a_{ij}. The set of edges emanating from vertex L_i (respectively, C_j) would appear as a word of length 2^d (respectively, 3^d) in the alphabet $\{0, 1, x, x+1\}$. Such a word is represented by the binary tree obtained from the complete binary tree whose leaves are labeled with the letters of the word and by replacing every subtree whose leaves are labeled by 0 with a leave of label 0 (see Figure 8). The size of such a tree is O(number of edges labeled with $1, x$ or $x+1$). The size of the entire structure is $O(s)$ (here we assume that $s \geq 3^d$). With ad hoc information on the internal nodes of these trees one can see that it is possible to make search and modify in logarithmic time the labels of a given tree, or also, always in logarithmic time, an edge of given label. It is easy to see that with this "manipulation primitive" it is possible to execute a sequence of ad hoc switches permitting an increase of n_D by one. It remains to explain how to construct a balanced configuration. For this one uses the balanced configuration described in Corollary 2.1. The structure of the bipartite graph is initialized with labels 0 and then changes to 1 the value of the labels of s edges between L_i and C_j, with $(i,j) = (1,1), (2,2), \ldots, (s,s)$, where the last coordinates are calculated modulo $(3^d, 2^d)$. □

A natural question arises then. For which values of s is the number of candidates to optimality exactly 1 or equivalently for which value of s is $x(s) = 0$? For example for $d = 2$ we have $x(s) = 0$ for all s but 23 for which $x(23) = 1$. For $d = 3$ we have $x(79) = x(102) = x(125) = 1$ and $x(103) = 2$ and $x(s) = 0$ for the other values of s. In fact one can easily show from the expression of $x(s)$ given in (44) that the ratio of s for which $x(s) = 0$ is more than $(1-(5/6)^d)$. In that case Theorem 3.2 asserts that the optimal configurations are precisely the balanced configurations for which $\sum_{i>i_0, j>j_0} a_{ij}$ is maximal, and that an abstract optimal configuration is computable in $O(s \log s)$ time. Concerning the case $x(s) \neq 0$, things are less satisfactory since to determine an optimal configuration requires to compute the sign of $\lambda(x)$. While a numerical computation of $\lambda(x)$ is efficiently possible (see [23]), we leave the determination of its sign in full generality as an open problem. However it is possible to prove (see [17]) that the sign of $\lambda(x)$ is independant of the absolute monotony of the reduced density function.

4 Conclusion and Suggestions for Further Work

In this article we defined and analyzed the camera placement problem in complete integer lattices. We have reduced the combinatorial part of the problem to an instance of a general integer optimization problem involving absolute monotone function and solved efficiently the problem when the number of camera is less than 5^d in d dimensions. Can our analysis be improved to solve the case of $s > 5^d$ cameras? Our discussion for $s \leq 5^d$ indicates that our conjecture (optimal configuration are balanced) might be too strong since it depends on properties of the visibility function which are independant of its absolute monotony. However we have shown that an optimal configuration is close (in the sense that it is possible to "move" to the optimal one by a series of simple transformations, called switches) to a balanced configuration; furthermore we have generated a small number of "candidates" to optimality. The key procedure in accomplishing this task is an exchange procedure which "calibrates" the classes of Λ/p, for $p = 2, 3$. However, does this procedure generalize to an arbitrary number of cameras? At this point it is reasonable to conjecture that an optimal configuration of size s is close to a balanced configuration up to a polynomial number of simple transformations.

The analysis of the camera placement problem proposed in this paper raises several interesting questions which may become the object of further study.

The first type of questions concerns extensions to systems of points other than the integer lattice Λ: for example the vertices of any tiling system [6, 7] or any point system you can imagine. A priori this is a nontrivial question. In general it requires the study of the following subproblems: (1) Give a number theoretic characterization of the visibility relation for points of the given tiling system; (2) Extend Rumsey's theorem; in particular, it is necessary to determine the density of the visibility sets $V(S)$ in arbitrary tiling systems; (3) Investigate combinatorial optimization techniques in order to construct optimal configurations. We refer the reader to our survey paper [13] where it is shown that our optimization methodology is fairly general and will be useful even in some of these more general cases.

The second type of questions concern the non-linear optimization problem to which our camera placement problem has been reduced. This problem is interesting in is own right and it will be interesting to investigate it in more details. For example, in Chapter 3, an instance of this problem has been solved in linear time up to a logarithmic factor (counting for 1 the evaluation cost of the function u'). To prove this result we used the fact that the parameter s is, roughly speaking, smaller than the number of integer variables a_i to be found. Is it an artifact of our proof? For additional questions and considerations we refer the reader to [13, 16].

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