

Olivier Cappé, Eric Moulines and Tobias Rydén

Inference in Hidden Markov Models

May 4, 2005

Springer
Berlin Heidelberg New York
Hong Kong London
Milan Paris Tokyo

Contents

Preface	V
1 Introduction	1
1.1 What Is a Hidden Markov Model?	1
1.2 Beyond Hidden Markov Models	4
1.3 Examples	6
1.3.1 Finite Hidden Markov Models.....	6
1.3.2 Normal Hidden Markov Models	13
1.3.3 Gaussian Linear State-Space Models	15
1.3.4 Conditionally Gaussian Linear State-Space Models	17
1.3.5 General (Continuous) State-Space HMMs	24
1.3.6 Switching Processes with Markov Regime.....	29
1.4 Left-to-Right and Ergodic Hidden Markov Models	33
2 Main Definitions and Notations.....	35
2.1 Markov Chains	35
2.1.1 Transition Kernels	35
2.1.2 Homogeneous Markov Chains	37
2.1.3 Non-homogeneous Markov Chains	40
2.2 Hidden Markov Models.....	42
2.2.1 Definitions and Notations	42
2.2.2 Conditional Independence in Hidden Markov Models ..	44
2.2.3 Hierarchical Hidden Markov Models	46
<hr/>	
Part I State Inference	
3 Filtering and Smoothing Recursions	51
3.1 Basic Notations and Definitions	53
3.1.1 Likelihood	53
3.1.2 Smoothing.....	54

3.1.3	The Forward-Backward Decomposition	56
3.1.4	Implicit Conditioning (Please Read This Section!)	58
3.2	Forward-Backward	59
3.2.1	The Forward-Backward Recursions	59
3.2.2	Filtering and Normalized Recursion	61
3.3	Markovian Decompositions	66
3.3.1	Forward Decomposition	66
3.3.2	Backward Decomposition	70
3.4	Complements	74
4	Advanced Topics in Smoothing	77
4.1	Recursive Computation of Smoothed Functionals	77
4.1.1	Fixed Point Smoothing	78
4.1.2	Recursive Smoothers for General Functionals	79
4.1.3	Comparison with Forward-Backward Smoothing	82
4.2	Filtering and Smoothing in More General Models	85
4.2.1	Smoothing in Markov-switching Models	86
4.2.2	Smoothing in Partially Observed Markov Chains	86
4.2.3	Marginal Smoothing in Hierarchical HMMs	87
4.3	Forgetting of the Initial Condition	89
4.3.1	Total Variation	90
4.3.2	Lipshitz Contraction for Transition Kernels	95
4.3.3	The Doeblin Condition and Uniform Ergodicity	97
4.3.4	Forgetting Properties	100
4.3.5	Uniform Forgetting Under Strong Mixing Conditions	105
4.3.6	Forgetting Under Alternative Conditions	110
5	Applications of Smoothing	121
5.1	Models with Finite State Space	121
5.1.1	Smoothing	122
5.1.2	Maximum <i>a Posteriori</i> Sequence Estimation	125
5.2	Gaussian Linear State-Space Models	127
5.2.1	Filtering and Backward Markovian Smoothing	127
5.2.2	Linear Prediction Interpretation	131
5.2.3	The Prediction and Filtering Recursions Revisited	137
5.2.4	Disturbance Smoothing	143
5.2.5	The Backward Recursion and the Two-Filter Formula	148
5.2.6	Application to Marginal Filtering and Smoothing in CGLSSMs	155
6	Monte Carlo Methods	161
6.1	Basic Monte Carlo Methods	161
6.1.1	Monte Carlo Integration	162
6.1.2	Monte Carlo Simulation for HMM State Inference	163
6.2	A Markov Chain Monte Carlo Primer	166

6.2.1	The Accept-Reject Algorithm	166
6.2.2	Markov Chain Monte Carlo	170
6.2.3	Metropolis-Hastings	171
6.2.4	Hybrid Algorithms	179
6.2.5	Gibbs Sampling	180
6.2.6	Stopping an MCMC Algorithm	185
6.3	Applications to Hidden Markov Models	186
6.3.1	Generic Sampling Strategies	186
6.3.2	Gibbs Sampling in CGLSSMs	194
7	Sequential Monte Carlo Methods	209
7.1	Importance Sampling and Resampling	210
7.1.1	Importance Sampling	210
7.1.2	Sampling Importance Resampling	211
7.2	Sequential Importance Sampling	214
7.2.1	Sequential Implementation for HMMs	214
7.2.2	Choice of the Instrumental Kernel	218
7.3	Sequential Importance Sampling with Resampling	231
7.3.1	Weight Degeneracy	231
7.3.2	Resampling	236
7.4	Complements	242
7.4.1	Implementation of Multinomial Resampling	242
7.4.2	Alternatives to Multinomial Resampling	244
8	Advanced Topics in Sequential Monte Carlo	251
8.1	Alternatives to SISR	251
8.1.1	I.I.D. Sampling	253
8.1.2	Two-Stage Sampling	256
8.1.3	Interpretation with Auxiliary Variables	260
8.1.4	Auxiliary Accept-Reject Sampling	261
8.1.5	Markov Chain Monte Carlo Auxiliary Sampling	263
8.2	Sequential Monte Carlo in Hierarchical HMMs	264
8.2.1	Sequential Importance Sampling and Global Sampling	265
8.2.2	Optimal Sampling	267
8.2.3	Application to CGLSSMs	274
8.3	Particle Approximation of Smoothing Functionals	278
9	Analysis of Sequential Monte Carlo Methods	287
9.1	Importance Sampling	287
9.1.1	Unnormalized Importance Sampling	287
9.1.2	Deviation Inequalities	291
9.1.3	Self-normalized Importance Sampling Estimator	293
9.2	Sampling Importance Resampling	295
9.2.1	The Algorithm	295
9.2.2	Definitions and Notations	297

XIV Contents

9.2.3	Weighting and Resampling	300
9.2.4	Application to the Single-Stage SIR Algorithm	307
9.3	Single-Step Analysis of SMC Methods	311
9.3.1	Mutation Step	311
9.3.2	Description of Algorithms	315
9.3.3	Analysis of the Mutation/Selection Algorithm	319
9.3.4	Analysis of the Selection/Mutation Algorithm	320
9.4	Sequential Monte Carlo Methods	321
9.4.1	SISR	321
9.4.2	I.I.D. Sampling	324
9.5	Complements	333
9.5.1	Weak Limits Theorems for Triangular Array	333
9.5.2	Bibliographic Notes	342

Part II Parameter Inference

10	Maximum Likelihood Inference, Part I: Optimization Through Exact Smoothing	347
10.1	Likelihood Optimization in Incomplete Data Models	347
10.1.1	Problem Statement and Notations	348
10.1.2	The Expectation-Maximization Algorithm	349
10.1.3	Gradient-based Methods	353
10.1.4	Pros and Cons of Gradient-based Methods	358
10.2	Application to HMMs	359
10.2.1	Hidden Markov Models as Missing Data Models	359
10.2.2	EM in HMMs	360
10.2.3	Computing Derivatives	362
10.2.4	Connection with the Sensitivity Equation Approach ..	364
10.3	The Example of Normal Hidden Markov Models	367
10.3.1	EM Parameter Update Formulas	367
10.3.2	Estimation of the Initial Distribution	370
10.3.3	Recursive Implementation of E-Step	371
10.3.4	Computation of the Score and Observed Information ..	374
10.4	The Example of Gaussian Linear State-Space Models	384
10.4.1	The Intermediate Quantity of EM	385
10.4.2	Recursive Implementation	387
10.5	Complements	389
10.5.1	Global Convergence of the EM Algorithm	389
10.5.2	Rate of Convergence of EM	392
10.5.3	Generalized EM Algorithms	393
10.5.4	Bibliographic Notes	394

11 Maximum Likelihood Inference, Part II:	
Monte Carlo Optimization	397
11.1 Methods and Algorithms	398
11.1.1 Monte Carlo EM	398
11.1.2 Simulation Schedules	403
11.1.3 Gradient-based Algorithms	408
11.1.4 Interlude: Stochastic Approximation and the Robbins-Monro Approach	411
11.1.5 Stochastic Gradient Algorithms	412
11.1.6 Stochastic Approximation EM	414
11.1.7 Stochastic EM	416
11.2 Analysis of the MCEM Algorithm	419
11.2.1 Convergence of Perturbed Dynamical Systems	420
11.2.2 Convergence of the MCEM Algorithm	423
11.2.3 Rate of Convergence of MCEM	426
11.3 Analysis of Stochastic Approximation Algorithms.....	429
11.3.1 Basic Results for Stochastic Approximation Algorithms	429
11.3.2 Convergence of the Stochastic Gradient Algorithm.....	431
11.3.3 Rate of Convergence of the Stochastic Gradient Algorithm	432
11.3.4 Convergence of the SAEM Algorithm	433
11.4 Complements	435
12 Statistical Properties of the Maximum Likelihood Estimator	441
12.1 A Primer on MLE Asymptotics	442
12.2 Stationary Approximations	443
12.3 Consistency	446
12.3.1 Construction of the Stationary Conditional Log-likelihood	446
12.3.2 The Contrast Function and Its Properties	448
12.4 Identifiability	450
12.4.1 Equivalence of Parameters	451
12.4.2 Identifiability of Mixture Densities.....	454
12.4.3 Application of Mixture Identifiability to Hidden Markov Models	455
12.5 Asymptotic Normality of the Score and Convergence of the Observed Information	457
12.5.1 The Score Function and Invoking the Fisher Identity ..	457
12.5.2 Construction of the Stationary Conditional Score	459
12.5.3 Weak Convergence of the Normalized Score	464
12.5.4 Convergence of the Normalized Observed Information ..	465
12.5.5 Asymptotics of the Maximum Likelihood Estimator ..	465
12.6 Applications to Likelihood-based Tests	466

XVI Contents

12.7 Complements	468
13 Fully Bayesian Approaches	471
13.1 Parameter Estimation	471
13.1.1 Bayesian Inference	471
13.1.2 Prior Distributions for HMMs	475
13.1.3 Non-identifiability and Label Switching	478
13.1.4 MCMC Methods for Bayesian Inference	481
13.2 Reversible Jump Methods	488
13.2.1 Variable Dimension Models	488
13.2.2 Green's Reversible Jump Algorithm	490
13.2.3 Alternative Sampler Designs	498
13.2.4 Alternatives to Reversible Jump MCMC	500
13.3 Multiple Imputations Methods and Maximum <i>a Posteriori</i> ..	501
13.3.1 Simulated Annealing	502
13.3.2 The SAME Algorithm	503

Part III Background and Complements

14 Elements of Markov Chain Theory	513
14.1 Chains on Countable State Spaces	513
14.1.1 Irreducibility	513
14.1.2 Recurrence and Transience	514
14.1.3 Invariant Measures and Stationarity	517
14.1.4 Ergodicity	519
14.2 Chains on General State Spaces	520
14.2.1 Irreducibility	521
14.2.2 Recurrence and Transience	523
14.2.3 Invariant Measures and Stationarity	534
14.2.4 Ergodicity	541
14.2.5 Geometric Ergodicity and Foster-Lyapunov Conditions ..	548
14.2.6 Limit Theorems	552
14.3 Applications to Hidden Markov Models	556
14.3.1 Phi-irreducibility	557
14.3.2 Atoms and Small Sets	558
14.3.3 Recurrence and Positive Recurrence	560
15 An Information-Theoretic Perspective on Order Estimation	565
15.1 Model Order Identification: What Is It About?	566
15.2 Order Estimation in Perspective	567
15.3 Order Estimation and Composite Hypothesis Testing	569
15.4 Code-based Identification	571
15.4.1 Definitions	571

15.4.2	Information Divergence Rates	574
15.5	MDL Order Estimators in Bayesian Settings	576
15.6	Strongly Consistent Penalized Maximum Likelihood Estimators for HMM Order Estimation	577
15.7	Efficiency Issues	580
15.7.1	Variations on Stein's Lemma	581
15.7.2	Achieving Optimal Error Exponents	584
15.8	Consistency of the BIC Estimator in the Markov Order Estimation Problem	587
15.8.1	Some Martingale Tools	589
15.8.2	The Martingale Approach	591
15.8.3	The Union Bound Meets Martingale Inequalities	592
15.9	Complements	600

Part IV Appendices

A	Conditioning	605
A.1	Probability and Topology Terminology and Notation	605
A.2	Conditional Expectation	606
A.3	Conditional Distribution	611
A.4	Conditional Independence	614
B	Linear Prediction	617
B.1	Hilbert Spaces	617
B.2	The Projection Theorem	619
C	Notations	621
C.1	Mathematical	621
C.2	Probability	622
C.3	Hidden Markov Models	622
C.4	Sequential Monte Carlo	624
References	625
Index	645