Description Logics and Reasoning on Data
4: Reasoning in $\mathcal{EL}$

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Outline

Lightweight Description Logics

- Reasoning in $\mathcal{ALC}$ and all its extensions is ExpTime-hard
- ExpTime-hardness already holds for $\mathcal{FL}_0$, the $\mathcal{ALC}$ fragment without $\neg$, $\sqcup$ and $\exists$, whose concepts are built according to the following grammar: $C := \top \mid A \mid C \sqcap C \mid \forall R.C$
- Some applications require very large ontologies and/or data
  - SNOMED CT (medical ontology) $>$ 350 000 concepts
  - NCI (National Cancer Institute Thesaurus) $\approx$ 20 000 concepts
  - GO (Gene Ontology) $\approx$ 30 000 concepts
- Many of them do not require universal restrictions ($\forall R.C$) but rather existential restrictions ($\exists R.C$)
- Since the mid 2000's, increasing interest in lightweight DLs
  - reasoning in polynomial time
  - expressivity sufficient for many applications
  - allow for existential restrictions

Lightweight Description Logics

- Two main families of lightweight DLs
  - the $\mathcal{EL}$ family
    - designed to allow efficient reasoning with large ontologies
    - core of the OWL 2 EL profile
  - the DL-Lite family
    - designed for ontology-mediated query answering
    - core of the OWL 2 QL profile
    - cf. course on query rewriting

The $\mathcal{EL}$ Family

$\mathcal{EL}$ concepts are built according to the following grammar:

$$C := \top \mid A \mid C \sqcap C \mid \exists R.C$$

and an $\mathcal{EL}$ Tbox contains only concept inclusions $C_1 \sqsubseteq C_2$

- Fragment of $\mathcal{ALC}$ without $\neg$, $\sqcup$ and $\forall$
- Possible extensions that remain tractable
  - $\mathcal{EL}_\bot$: $\bot$ to express disjoint concepts
  - $\mathcal{EL}^{dr}$: domain and range restrictions
    - $\mathrm{dom}(R) \sqsubseteq C$ ($\equiv \exists R.\top \sqsubseteq C$, already in plain $\mathcal{EL}$)
    - $\mathrm{ran}(R) \sqsubseteq C$ ($\equiv \exists R^-.\top \sqsubseteq C$, not expressible in plain $\mathcal{EL}$)
  - $\mathcal{ELO}$: nominals $\{o\}$
  - (complex) role inclusions $R_1 \circ \cdots \circ R_n \sqsubseteq R_{n+1}$
    (includes transitivity (trans R) $\equiv R \circ R \sqsubseteq R$)
- OWL 2 EL profile includes all these extensions
- Adding any of the constructors $\neg$, $\sqcup$, $\forall$, R^- makes reasoning ExpTime-hard

Reasoning in $\mathcal{EL}$

Focus on plain $\mathcal{EL}$: the TBox contains concept inclusions $C_1 \sqsubseteq C_2$ with $C := \top \mid A \mid C \sqcap C \mid \exists R.C$

- Satisfiability is trivial
  - $\mathcal{I} = (\{e\}, \cdot^{\mathcal{I}})$, $a^{\mathcal{I}} = e$, $A^{\mathcal{I}} = \{e\}$, $R^{\mathcal{I}} = \{(e, e)\}$
- Subsumption/classification or instance checking are not!
  - cannot be reduced to satisfiability
  - focus on these reasoning tasks

Reasoning in $\mathcal{EL}$

Subsumption: Given an $\mathcal{EL}$ TBox $\mathcal{T}$ and two $\mathcal{EL}$ concepts C and D, decide whether $\mathcal{T} \models C \sqsubseteq D$

- We will assume that C and D are atomic concepts
  - if C, D are $\mathcal{EL}$ complex concepts,

$$\mathcal{T} \models C \sqsubseteq D \text{ iff } \mathcal{T} \cup \{A \sqsubseteq C, D \sqsubseteq B\} \models A \sqsubseteq B$$

    where A, B are fresh concept names

Classification: Given an $\mathcal{EL}$ TBox $\mathcal{T}$, find all atomic concepts A, B such that $\mathcal{T} \models A \sqsubseteq B$

Instance checking: Given an $\mathcal{EL}$ KB $\langle \mathcal{T}, \mathcal{A} \rangle$ and an $\mathcal{EL}$ concept C, decide for every individual a from $\mathcal{A}$ whether $\langle \mathcal{T}, \mathcal{A} \rangle \models C(a)$

- We will assume that C is an atomic concept
  - $\langle \mathcal{T}, \mathcal{A} \rangle \models C(a)$ iff $\langle \mathcal{T} \cup \{C \sqsubseteq A\}, \mathcal{A} \rangle \models A(a)$

Normal Form of $\mathcal{EL}$ TBoxes

An $\mathcal{EL}$ TBox is in normal form if it contains only concept inclusions of one of the following forms:

$$A \sqsubseteq B \quad A_1 \sqcap A_2 \sqsubseteq B \quad A \sqsubseteq \exists R.B \quad \exists R.A \sqsubseteq B$$

where A, A_1, A_2 and B are atomic concepts or $\top$

- For every $\mathcal{EL}$ TBox $\mathcal{T}$, we can construct in polynomial time $\mathcal{T}'$ in normal form (possibly using new concept names) such that
  - for every $C \sqsubseteq D$ which uses only concept names from $\mathcal{T}$, $\mathcal{T} \models C \sqsubseteq D$ iff $\mathcal{T}' \models C \sqsubseteq D$
  - for every ABox $\mathcal{A}$ and assertion α that uses atomic concepts from $\langle \mathcal{T}, \mathcal{A} \rangle$, $\langle \mathcal{T}, \mathcal{A} \rangle \models \alpha$ iff $\langle \mathcal{T}', \mathcal{A} \rangle \models \alpha$

We will assume that TBoxes are in normal form

Normalization of $\mathcal{EL}$ TBoxes

Normalization algorithm

Exhaustively apply the following normalization rules to $\mathcal{T}$

$$
\begin{array}{llcll}
\mathrm{NR}_0 & \hat{C} \sqsubseteq \hat{D} & \rightarrow & \hat{C} \sqsubseteq A, & A \sqsubseteq \hat{D} \\
\mathrm{NR}_\sqcap^{\ell,1} & C \sqcap \hat{D} \sqsubseteq B & \rightarrow & \hat{D} \sqsubseteq A, & C \sqcap A \sqsubseteq B \\
\mathrm{NR}_\sqcap^{\ell,2} & \hat{C} \sqcap D \sqsubseteq B & \rightarrow & \hat{C} \sqsubseteq A, & A \sqcap D \sqsubseteq B \\
\mathrm{NR}_\exists^{\ell} & \exists R.\hat{C} \sqsubseteq B & \rightarrow & \hat{C} \sqsubseteq A, & \exists R.A \sqsubseteq B \\
\mathrm{NR}_\exists^{r} & B \sqsubseteq \exists R.\hat{C} & \rightarrow & A \sqsubseteq \hat{C}, & B \sqsubseteq \exists R.A \\
\mathrm{NR}_\sqcap^{r} & B \sqsubseteq D \sqcap E & \rightarrow & B \sqsubseteq D, & B \sqsubseteq E
\end{array}
$$

where

- C, D, E are arbitrary $\mathcal{EL}$ concepts
- $\hat{C}, \hat{D}$ are $\mathcal{EL}$ concepts that are neither atomic concepts nor $\top$
- B is an atomic concept
- A is a fresh atomic concept

Normalization of $\mathcal{EL}$ TBoxes

Example

NR_0	$\hat{C} \sqsubseteq \hat{D}$	$\rightarrow$	$\hat{C} \sqsubseteq A,$	$A \sqsubseteq \hat{D}$
$\mathsf{NR}_{\sqcap}^{\ell,1}$	$C \sqcap \hat{D} \sqsubseteq B$	$\rightarrow$	$\hat{D} \sqsubseteq A,$	$C \sqcap A \sqsubseteq B$
$\mathsf{NR}_{\sqcap}^{\ell,2}$	$\hat{C} \sqcap D \sqsubseteq B$	$\rightarrow$	$\hat{C} \sqsubseteq A,$	$A \sqcap D \sqsubseteq B$
$\mathsf{NR}_{\exists}^{\ell}$	$\exists R.\hat{C} \sqsubseteq B$	$\rightarrow$	$\hat{C} \sqsubseteq A,$	$\exists R.A \sqsubseteq B$
$\mathsf{NR}_{\exists}^{r}$	$B \sqsubseteq \exists R.\hat{C}$	$\rightarrow$	$A \sqsubseteq \hat{C},$	$B \sqsubseteq \exists R.A$
$\mathsf{NR}_{\sqcap}^{r}$	$B \sqsubseteq D \sqcap E$	$\rightarrow$	$B \sqsubseteq D,$	$B \sqsubseteq E$

Normalize $\mathcal{T} = \{\exists R.C \sqcap D \sqsubseteq \exists S.\exists R.C\}$

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$\mathsf{NR}_\sqcap^{\ell,1}$	$C \sqcap \hat{D} \sqsubseteq B$	$\rightarrow$	$\hat{D} \sqsubseteq A,$	$C \sqcap A \sqsubseteq B$
$\mathsf{NR}_\sqcap^{\ell,2}$	$\hat{C} \sqcap D \sqsubseteq B$	$\rightarrow$	$\hat{C} \sqsubseteq A,$	$A \sqcap D \sqsubseteq B$
$\mathsf{NR}_\exists^{\ell}$	$\exists R.\hat{C} \sqsubseteq B$	$\rightarrow$	$\hat{C} \sqsubseteq A,$	$\exists R.A \sqsubseteq B$
$\mathsf{NR}_\exists^{r}$	$B \sqsubseteq \exists R.\hat{C}$	$\rightarrow$	$A \sqsubseteq \hat{C},$	$B \sqsubseteq \exists R.A$
$\mathsf{NR}_\sqcap^{r}$	$B \sqsubseteq D \sqcap E$	$\rightarrow$	$B \sqsubseteq D,$	$B \sqsubseteq E$

Normalize $\mathcal{T} = \{\exists R.C \sqcap D \sqsubseteq \exists S.\exists R.C\}$

$$\begin{array}{rcll} \exists R.C \sqcap D \sqsubseteq \exists S.\exists R.C & \rightarrow & \exists R.C \sqcap D \sqsubseteq A_1, \quad A_1 \sqsubseteq \exists S.\exists R.C & (\mathsf{NR}_0) \\ \exists R.C \sqcap D \sqsubseteq A_1 & \rightarrow & \exists R.C \sqsubseteq A_2, \quad A_2 \sqcap D \sqsubseteq A_1 & (\mathsf{NR}_\sqcap^{\ell,2}) \\ A_1 \sqsubseteq \exists S.\exists R.C & \rightarrow & A_1 \sqsubseteq \exists S.A_3, \quad A_3 \sqsubseteq \exists R.C & (\mathsf{NR}_\exists^{r}) \end{array}$$

Normalized TBox:

$\mathcal{T}' = \{\exists R.C \sqsubseteq A_2,\ A_2 \sqcap D \sqsubseteq A_1,\ A_1 \sqsubseteq \exists S.A_3,\ A_3 \sqsubseteq \exists R.C\}$

Normalization of $\mathcal{EL}$ TBoxes

Termination and complexity

For every input $\mathcal{EL}$ TBox $\mathcal{T}$, the normalization algorithm terminates in linear time w.r.t. the size of $\mathcal{T}$.

- Proof based on abnormality degree of $\mathcal{T}$
- Abnormal occurrence of a concept C within $\mathcal{T}$:
  - $C \sqsubseteq D$, where C, D are neither atomic concepts nor $\top$
  - C is neither an atomic concept nor $\top$, and is under a conjunction or an existential restriction
  - C is under a conjunction operator on the right hand side
- Abnormality degree of $\mathcal{T}$: number of abnormal occurrences
  - a TBox with abnormality degree 0 is in normal form
  - the abnormality degree is bounded by the size of $\mathcal{T}$
- Claim: Each rule decreases the abnormality degree of $\mathcal{T}$

Normalization of $\mathcal{EL}$ TBoxes

Termination and complexity – Proof of the claim

- If $\mathcal{T}'$ is obtained from $\mathcal{T}$ by applying NR_0
  - $\mathcal{T}' = \mathcal{T} \setminus \{\hat{C} \sqsubseteq \hat{D}\} \cup \{\hat{C} \sqsubseteq A,\ A \sqsubseteq \hat{D}\}$
  - decreases the abnormality degree by 1
    - removes abnormal occurrence $\hat{C} \sqsubseteq \hat{D}$ of $\hat{C}$
    - does not modify other abnormal occurrences

Normalization of $\mathcal{EL}$ TBoxes

Termination and complexity – Proof of the claim

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  - decreases the abnormality degree by 1
    - removes abnormal occurrence $\hat{C} \sqsubseteq \hat{D}$ of $\hat{C}$
    - does not modify other abnormal occurrences
- If $\mathcal{T}'$ is obtained from $\mathcal{T}$ by applying $\mathsf{NR}_{\sqcap}^{\ell,1}$
  - $\mathcal{T}' = \mathcal{T} \setminus \{C \sqcap \hat{D} \sqsubseteq B\} \cup \{\hat{D} \sqsubseteq A,\ C \sqcap A \sqsubseteq B\}$
  - decreases the abnormality degree by 1
    - removes abnormal occurrence $C \sqcap \hat{D}$ of $\hat{D}$
    - does not modify the number of other abnormal occurrences ($C \sqcap \hat{D}$ is an abnormal occurence of C iff $C \sqcap A$ is one)

Normalization of $\mathcal{EL}$ TBoxes

Termination and complexity – Proof of the claim

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  - $\mathcal{T}' = \mathcal{T} \setminus \{\hat{C} \sqsubseteq \hat{D}\} \cup \{\hat{C} \sqsubseteq A,\ A \sqsubseteq \hat{D}\}$
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  - decreases the abnormality degree by 1
    - removes abnormal occurrence $C \sqcap \hat{D}$ of $\hat{D}$
    - does not modify the number of other abnormal occurrences ($C \sqcap \hat{D}$ is an abnormal occurence of C iff $C \sqcap A$ is one)
- If $\mathcal{T}'$ is obtained from $\mathcal{T}$ by applying $\mathsf{NR}_{\exists}^{r}$
  - $\mathcal{T}' = \mathcal{T} \setminus \{B \sqsubseteq \exists R.\hat{C}\} \cup \{A \sqsubseteq \hat{C},\ B \sqsubseteq \exists R.A\}$
  - decreases the abnormality degree by 1
    - removes abnormal occurrence $\exists R.\hat{C}$ of $\hat{C}$
    - does not modify other abnormal occurrences

Normalization of $\mathcal{EL}$ TBoxes

Termination and complexity – Proof of the claim

- If $\mathcal{T}'$ is obtained from $\mathcal{T}$ by applying NR_0
  - $\mathcal{T}' = \mathcal{T} \setminus \{\hat{C} \sqsubseteq \hat{D}\} \cup \{\hat{C} \sqsubseteq A,\ A \sqsubseteq \hat{D}\}$
  - decreases the abnormality degree by 1
    - removes abnormal occurrence $\hat{C} \sqsubseteq \hat{D}$ of $\hat{C}$
    - does not modify other abnormal occurrences
- If $\mathcal{T}'$ is obtained from $\mathcal{T}$ by applying $\mathrm{NR}_\sqcap^{\ell,1}$
  - $\mathcal{T}' = \mathcal{T} \setminus \{C \sqcap \hat{D} \sqsubseteq B\} \cup \{\hat{D} \sqsubseteq A,\ C \sqcap A \sqsubseteq B\}$
  - decreases the abnormality degree by 1
    - removes abnormal occurrence $C \sqcap \hat{D}$ of $\hat{D}$
    - does not modify the number of other abnormal occurrences ($C \sqcap \hat{D}$ is an abnormal occurence of C iff $C \sqcap A$ is one)
- If $\mathcal{T}'$ is obtained from $\mathcal{T}$ by applying $\mathrm{NR}_\exists^r$
  - $\mathcal{T}' = \mathcal{T} \setminus \{B \sqsubseteq \exists R.\hat{C}\} \cup \{A \sqsubseteq \hat{C},\ B \sqsubseteq \exists R.A\}$
  - decreases the abnormality degree by 1
    - removes abnormal occurrence $\exists R.\hat{C}$ of $\hat{C}$
    - does not modify other abnormal occurrences
- $\mathrm{NR}_\sqcap^{\ell,2}$, $\mathrm{NR}_\exists^\ell$, $\mathrm{NR}_\sqcap^r$: left as practice

Conservative Extensions

$\mathcal{T}_2$ is a conservative extension of $\mathcal{T}_1$ if:

- the signature of $\mathcal{T}_1$ is included in the signature of $\mathcal{T}_2$
- every model of $\mathcal{T}_2$ is a model of $\mathcal{T}_1$
- for every model $\mathcal{I}_1$ of $\mathcal{T}_1$, there exists a model $\mathcal{I}_2$ of $\mathcal{T}_2$ with:
  - $\Delta^{\mathcal{I}_1} = \Delta^{\mathcal{I}_2}$
  - $A^{\mathcal{I}_1} = A^{\mathcal{I}_2}$ for every atomic concept in the signature of $\mathcal{T}_1$
  - $R^{\mathcal{I}_1} = R^{\mathcal{I}_2}$ for every role in the signature of $\mathcal{T}_1$

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  - $A^{\mathcal{I}_1} = A^{\mathcal{I}_2}$ for every atomic concept in the signature of $\mathcal{T}_1$
  - $R^{\mathcal{I}_1} = R^{\mathcal{I}_2}$ for every role in the signature of $\mathcal{T}_1$

Properties of conservative extensions

- Transitivity: If $\mathcal{T}_2$ is a conservative extension of $\mathcal{T}_1$, and $\mathcal{T}_3$ is a conservative extension of $\mathcal{T}_2$, then $\mathcal{T}_3$ is a conservative extension of $\mathcal{T}_1$
- If $\mathcal{T}_2$ is a conservative extension of $\mathcal{T}_1$
  - if C and D are concepts containing only concept and role names from $\mathcal{T}_1$, then it holds that $\mathcal{T}_1 \models C \sqsubseteq D$ if and only if $\mathcal{T}_2 \models C \sqsubseteq D$
  - for every ABox $\mathcal{A}$ and assertion α that use only atomic concepts and roles from $\mathcal{T}_1$, $\langle \mathcal{T}_1, \mathcal{A} \rangle \models \alpha$ iff $\langle \mathcal{T}_2, \mathcal{A} \rangle \models \alpha$

Normalization of $\mathcal{EL}$ TBoxes

Soundness and completeness

- $\mathcal{T}$ and $\mathcal{T}'$ need not be equivalent due to the introduction of new atomic concepts by the normalization rules
- Claim: $\mathcal{T}'$ is a conservative extension of $\mathcal{T}$

Show that if $\mathcal{T}_2$ is obtained from $\mathcal{T}_1$ by applying one of the normalization rules, then $\mathcal{T}_2$ is a conservative extension of $\mathcal{T}_1$. The claim follows by transitivity.

- If $\mathcal{T}_2$ is obtained from $\mathcal{T}_1$ by applying NR_0
  - $\mathcal{T}_2 = \mathcal{T}_1 \setminus \{\hat{C} \sqsubseteq \hat{D}\} \cup \{\hat{C} \sqsubseteq A,\ A \sqsubseteq \hat{D}\}$
  - every model of $\mathcal{T}_2$ is a model of $\mathcal{T}_1$
  - for every model $\mathcal{I}_1$ of $\mathcal{T}_1$, define $\mathcal{I}_2$
    - $\Delta^{\mathcal{I}_2} = \Delta^{\mathcal{I}_1}$, $R^{\mathcal{I}_2} = R^{\mathcal{I}_1}$ for every role
    - $B^{\mathcal{I}_2} = B^{\mathcal{I}_1}$ for every atomic concept different from A
    - $A^{\mathcal{I}_2} = \hat{C}^{\mathcal{I}_1}$
  - $\mathcal{I}_2 \models \mathcal{T}_2$
- Other rules left as practice

Compact Canonical Model

- To decide entailment of an axiom or assertion in DL, we normally need to consider all the models of the KB
- In $\mathcal{EL}$, for every KB $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$, there exists a finite model $\mathcal{C}_{\mathcal{K}}$ which can be used to check whether an assertion or an inclusion between two atomic concepts is entailed
- $\mathcal{C}_{\mathcal{K}}$ is the compact canonical model of $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$

Compact Canonical Model

Construction of $\mathcal{C}_\mathcal{K}$

Let $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$ with $\mathcal{T}$ an $\mathcal{EL}$ TBox in normal form

- Start with $\mathcal{I}_0$ defined by

$$\begin{aligned}
\Delta^{\mathcal{I}_0} &= \{a \mid a \text{ individual from } \mathcal{A}\} \cup \{e_A \mid A \text{ atomic concept}\} \cup \{e_\top\} \\
A^{\mathcal{I}_0} &= \{a \mid A(a) \in \mathcal{A}\} \cup \{e_A\} \\
R^{\mathcal{I}_0} &= \{(a,b) \mid R(a,b) \in \mathcal{A}\} \\
a^{\mathcal{I}_0} &= a \text{ for every individual from } \mathcal{A}
\end{aligned}$$

- $\mathcal{I}_{n+1}$ is obtained from $\mathcal{I}_n$ by applying one of the following rules (note that C can be an atomic concept A, $A_1 \sqcap A_2$ or $\exists R.A$)

R_1 : if $C \sqsubseteq B \in \mathcal{T}, x \in C^{\mathcal{I}_n}$ and $x \notin B^{\mathcal{I}_n}$, then $B^{\mathcal{I}_{n+1}} = B^{\mathcal{I}_n} \cup \{x\}$

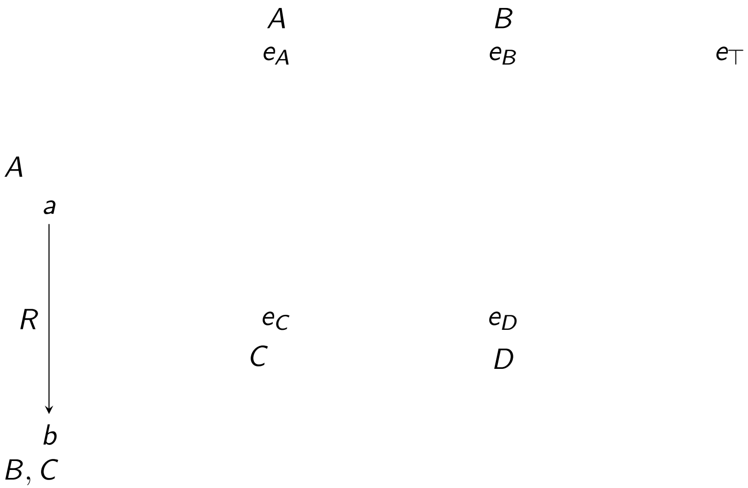
R_2 : if $A \sqsubseteq \exists R.B \in \mathcal{T}, x \in A^{\mathcal{I}_n}$ and $(x, e_B) \notin R^{\mathcal{I}_n}$, then $R^{\mathcal{I}_{n+1}} = R^{\mathcal{I}_n} \cup \{(x, e_B)\}$

- When we reach $\mathcal{I}_k$ such that no more rules apply, set $\mathcal{C}_\mathcal{K} = \mathcal{I}_k$

Compact Canonical Model

Example

$$\mathcal{T} = \{A \sqsubseteq \exists R.B,\ \exists R.C \sqsubseteq D,\ A \sqcap D \sqsubseteq C,\ C \sqsubseteq \exists R.C\}$$
$$\mathcal{A} = \{A(a),\ R(a,b),\ B(b),\ C(b)\}$$



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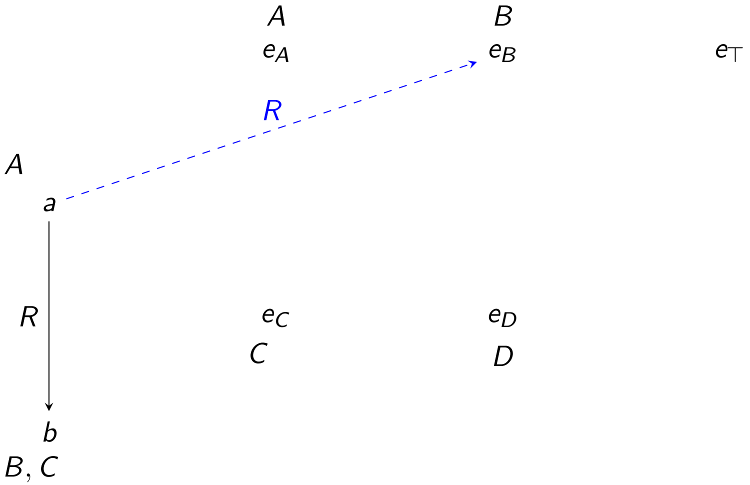

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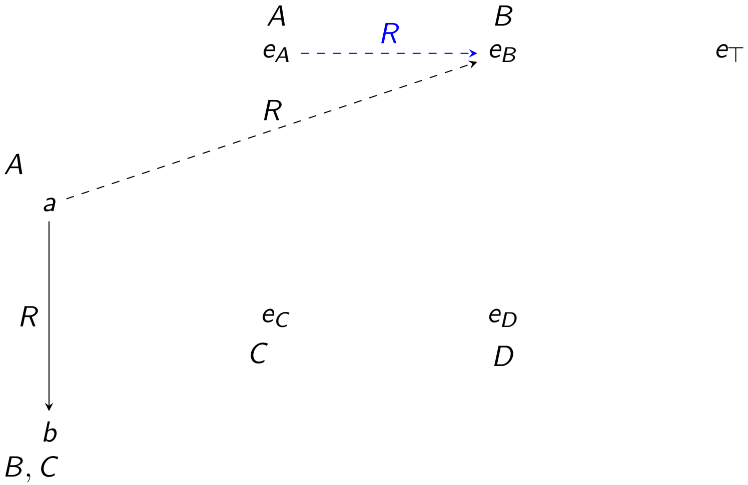

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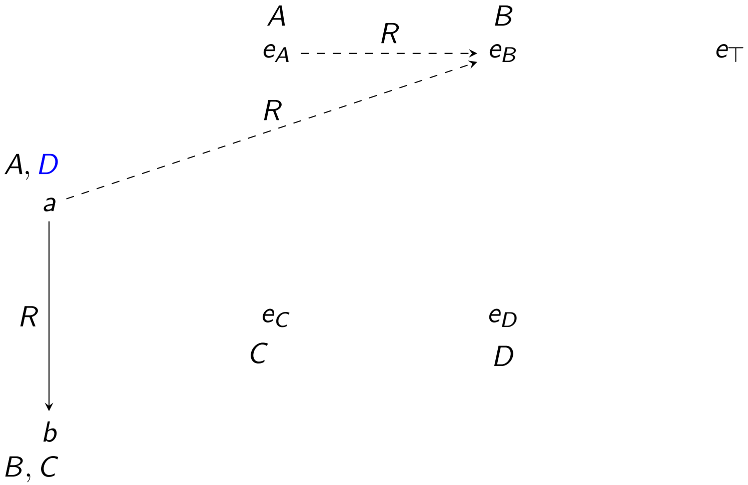

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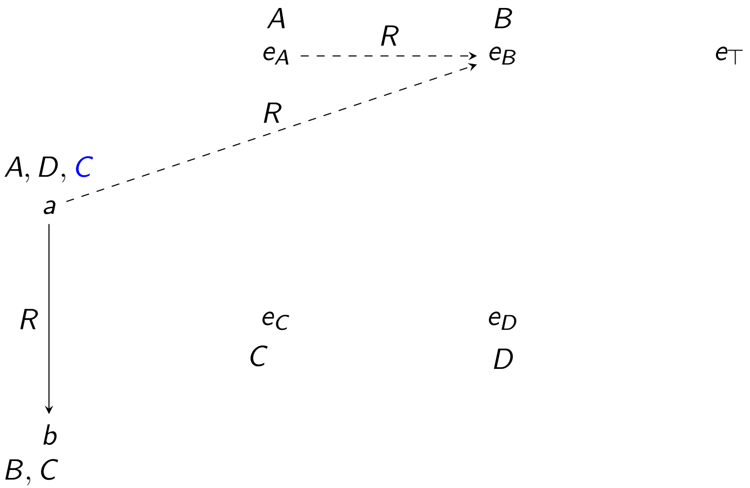

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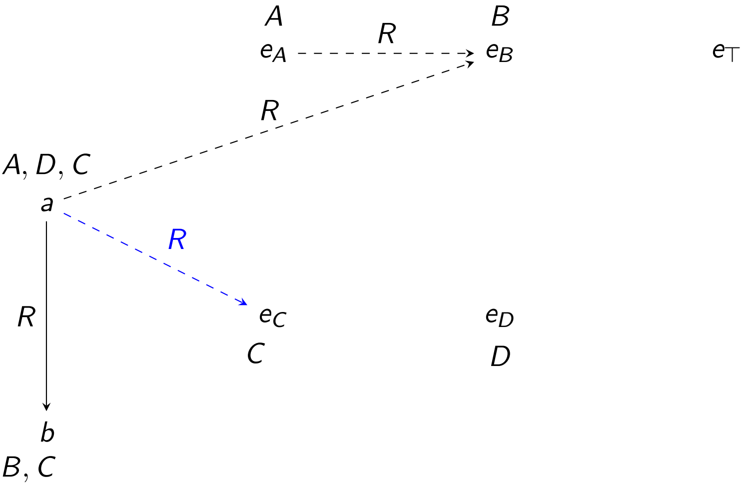

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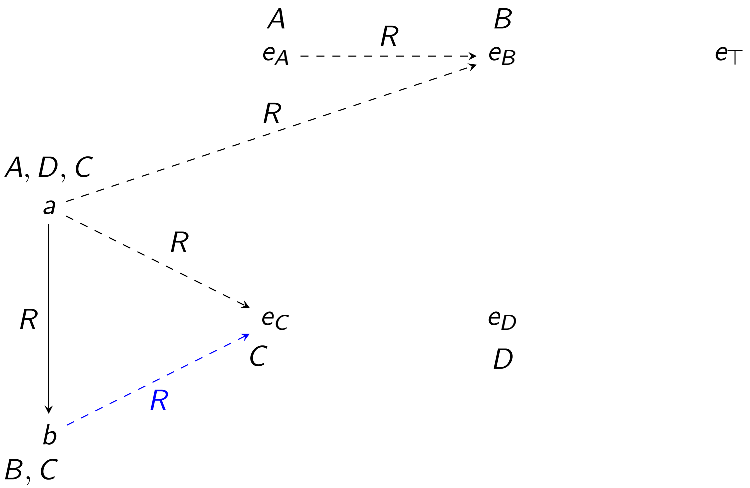

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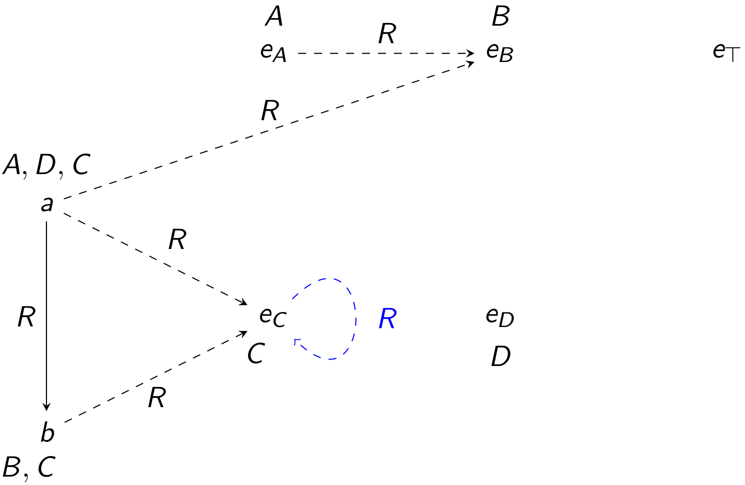

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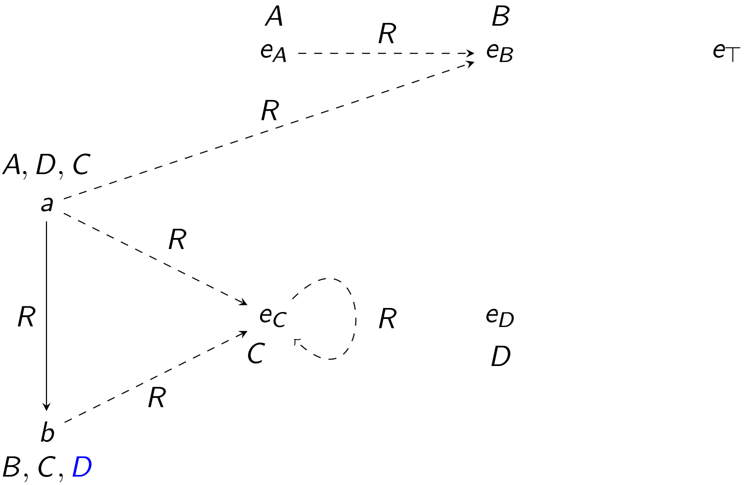

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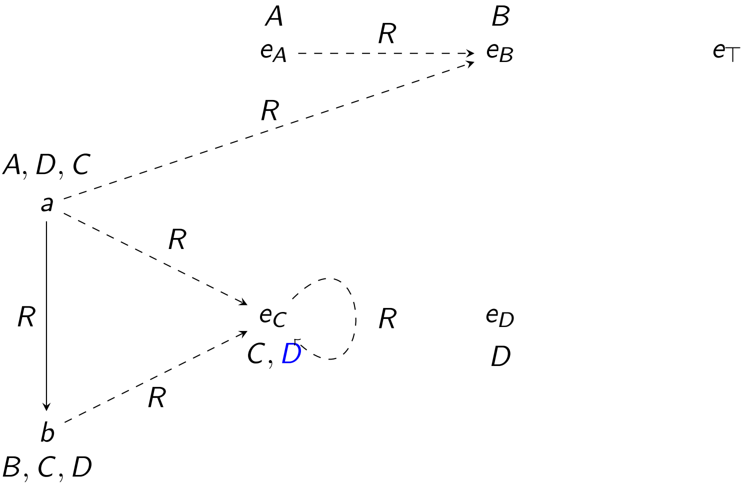

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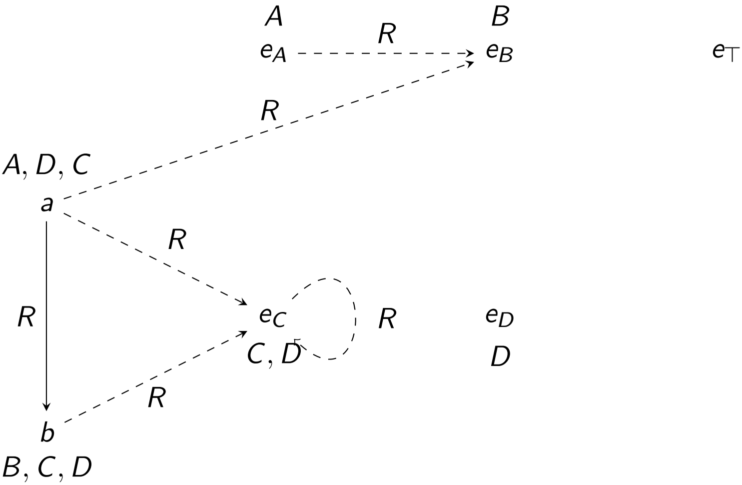

Compact Canonical Model

Properties of $\mathcal{C}_\mathcal{K}$

- $\mathcal{C}_\mathcal{K}$ can be constructed in polynomial time
  - $\Delta^{\mathcal{C}_\mathcal{K}}$ is linear in the size of $\mathcal{K}$
  - each rule application adds an element or pair of elements of $\Delta^{\mathcal{C}_\mathcal{K}}$ to the interpretation of an atomic concept or role from $\mathcal{K}$

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- $\mathcal{C}_\mathcal{K}$ is a model of $\mathcal{K}$
  - $\mathcal{I}_0 \models \mathcal{A}$ so $\mathcal{C}_\mathcal{K} \models \mathcal{A}$
  - for every $C \sqsubseteq B \in \mathcal{T}$, $C^{\mathcal{C}_\mathcal{K}} \subseteq B^{\mathcal{C}_\mathcal{K}}$ (otherwise R_1 would apply)
  - for every $A \sqsubseteq \exists R.B \in \mathcal{T}$ and $x \in A^{\mathcal{C}_\mathcal{K}}$, $(x, e_B) \in R^{\mathcal{C}_\mathcal{K}}$ (otherwise R_2 would apply), and since $e_B \in B^{\mathcal{C}_\mathcal{K}}$, $x \in \exists R.B^{\mathcal{C}_\mathcal{K}}$
  - hence $\mathcal{C}_\mathcal{K} \models \mathcal{T}$

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  - hence $\mathcal{C}_\mathcal{K} \models \mathcal{T}$
- for every concept inclusion between atomic concepts $A \sqsubseteq B$, $\mathcal{K} \models A \sqsubseteq B$ iff $\mathcal{C}_\mathcal{K} \models B(e_A)$
  - if $\mathcal{K} \models A \sqsubseteq B$, $\mathcal{C}_\mathcal{K} \models A \sqsubseteq B$ so since $e_A \in A^{\mathcal{C}_\mathcal{K}}$, $\mathcal{C}_\mathcal{K} \models B(e_A)$
  - Claim 1: if $\mathcal{C}_\mathcal{K} \models B(e_A)$, then $\mathcal{K} \models A \sqsubseteq B$

Compact Canonical Model

Properties of $\mathcal{C}_\mathcal{K}$

- $\mathcal{C}_\mathcal{K}$ can be constructed in polynomial time
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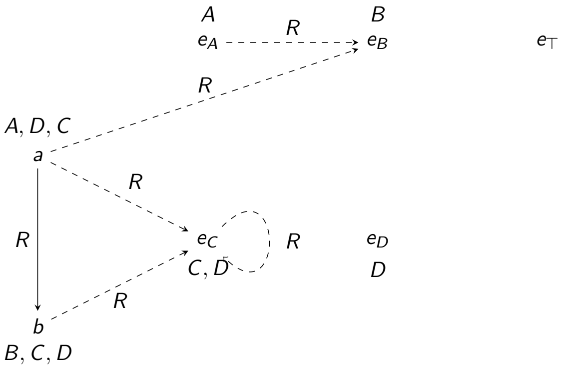
- for every concept inclusion between atomic concepts $A \sqsubseteq B$, $\mathcal{K} \models A \sqsubseteq B$ iff $\mathcal{C}_\mathcal{K} \models B(e_A)$
  - if $\mathcal{K} \models A \sqsubseteq B$, $\mathcal{C}_\mathcal{K} \models A \sqsubseteq B$ so since $e_A \in A^{\mathcal{C}_\mathcal{K}}$, $\mathcal{C}_\mathcal{K} \models B(e_A)$
  - Claim 1: if $\mathcal{C}_\mathcal{K} \models B(e_A)$, then $\mathcal{K} \models A \sqsubseteq B$
- for every assertion α, $\mathcal{K} \models \alpha$ iff $\mathcal{C}_\mathcal{K} \models \alpha$
  - if $\mathcal{K} \models \alpha$, $\mathcal{C}_\mathcal{K} \models \alpha$
  - $\mathcal{C}_\mathcal{K} \models R(a, b)$ with a, b individuals implies $R(a, b) \in \mathcal{A}$
  - Claim 2: if $\mathcal{C}_\mathcal{K} \models A(a)$ with a individual, then $\mathcal{K} \models A(a)$

Compact Canonical Model

Example

$$\mathcal{T} = \{A \sqsubseteq \exists R.B,\ \exists R.C \sqsubseteq D,\ A \sqcap D \sqsubseteq C,\ C \sqsubseteq \exists R.C\}$$
$$\mathcal{A} = \{A(a),\ R(a,b),\ B(b),\ C(b)\}$$



$$\mathcal{C}_\mathcal{K} \models C(a) \Rightarrow \mathcal{K} \models C(a) \qquad \mathcal{C}_\mathcal{K} \models D(a) \Rightarrow \mathcal{K} \models D(a)$$
$$\mathcal{C}_\mathcal{K} \models D(b) \Rightarrow \mathcal{K} \models D(b) \qquad \mathcal{C}_\mathcal{K} \models D(e_C) \Rightarrow \mathcal{K} \models C \sqsubseteq D$$

Compact Canonical Model

Properties of $\mathcal{C}_{\mathcal{K}}$ – Proof of Claim 1

For all atomic concepts A, B, $\mathcal{C}_{\mathcal{K}} \models B(e_A)$ implies $\mathcal{K} \models A \sqsubseteq B$

Proof by induction on n such that $e_A \in B^{\mathcal{I}_n}$

- Base case: $e_A \in B^{\mathcal{I}_0}$ implies that $B = A$ and $\mathcal{K} \models A \sqsubseteq A$
- Induction hypothesis (IH): For every atomic concepts A and B, $e_A \in B^{\mathcal{I}_n}$ implies $\mathcal{K} \models A \sqsubseteq B$

Compact Canonical Model

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- Induction hypothesis (IH): For every atomic concepts A and B, $e_A \in B^{\mathcal{I}_n}$ implies $\mathcal{K} \models A \sqsubseteq B$
- Induction step: Assume that $e_A \in B^{\mathcal{I}_{n+1}}$
  - If $e_A \in B^{\mathcal{I}_n}$, $\mathcal{K} \models A \sqsubseteq B$ by IH
  - If $e_A \notin B^{\mathcal{I}_n}$, e_A has been added to $B^{\mathcal{I}_{n+1}}$ by applying rule R_1: there exists $C \sqsubseteq B \in \mathcal{T}$ such that $e_A \in C^{\mathcal{I}_n}$

Compact Canonical Model

Properties of $\mathcal{C}_{\mathcal{K}}$ – Proof of Claim 1

For all atomic concepts A, B, $\mathcal{C}_{\mathcal{K}} \models B(e_A)$ implies $\mathcal{K} \models A \sqsubseteq B$

Proof by induction on n such that $e_A \in B^{\mathcal{I}_n}$

- Base case: $e_A \in B^{\mathcal{I}_0}$ implies that $B = A$ and $\mathcal{K} \models A \sqsubseteq A$
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  - If $e_A \in B^{\mathcal{I}_n}$, $\mathcal{K} \models A \sqsubseteq B$ by IH
  - If $e_A \notin B^{\mathcal{I}_n}$, e_A has been added to $B^{\mathcal{I}_{n+1}}$ by applying rule R_1: there exists $C \sqsubseteq B \in \mathcal{T}$ such that $e_A \in C^{\mathcal{I}_n}$
    - case C atomic concept: $\mathcal{K} \models A \sqsubseteq C$ (by IH). It is then easy to check that $\mathcal{K} \models A \sqsubseteq B$
    - case $C = A_1 \sqcap A_2$: $e_A \in A_1^{\mathcal{I}_n}$ and $e_A \in A_2^{\mathcal{I}_n}$ so $\mathcal{K} \models A \sqsubseteq A_1$ and $\mathcal{K} \models A \sqsubseteq A_2$ (by IH). Since $A_1 \sqcap A_2 \sqsubseteq B \in \mathcal{T}$, it is then easy to check that $\mathcal{K} \models A \sqsubseteq B$
    - case $C = \exists R.D$: there exists $e_X \in D^{\mathcal{I}_n}$ s.t. $(e_A, e_X) \in R^{\mathcal{I}_n}$. $(e_A, e_X) \in R^{\mathcal{I}_n}$ has been added by rule R_2 so $E \sqsubseteq \exists R.X \in \mathcal{T}$ and $e_A \in E^{\mathcal{I}_n}$. $\mathcal{K} \models X \sqsubseteq D$ and $\mathcal{K} \models A \sqsubseteq E$ (by IH). Since $\mathcal{K} \models A \sqsubseteq E$, $\mathcal{K} \models E \sqsubseteq \exists R.X$, $\mathcal{K} \models X \sqsubseteq D$ and $\mathcal{K} \models \exists R.D \sqsubseteq B$, it is easy to check that $\mathcal{K} \models A \sqsubseteq B$

Compact Canonical Model

Properties of $\mathcal{C}_\mathcal{K}$ – Proof of Claim 2

For every concept assertion $A(a)$, if $\mathcal{C}_\mathcal{K} \models A(a)$, then $\mathcal{K} \models A(a)$

Proof by induction on n such that $a \in A^{\mathcal{I}_n}$

- Base case: $a \in A^{\mathcal{I}_0}$ implies $A(a) \in \mathcal{A}$
- Induction hypothesis (IH): For every atomic concept A and individual a, $a \in A^{\mathcal{I}_n}$ implies $\mathcal{K} \models A(a)$

Compact Canonical Model

For every concept assertion $A(a)$, if $\mathcal{C}_\mathcal{K} \models A(a)$, then $\mathcal{K} \models A(a)$

Proof by induction on n such that $a \in A^{\mathcal{I}_n}$

- Base case: $a \in A^{\mathcal{I}_0}$ implies $A(a) \in \mathcal{A}$
- Induction hypothesis (IH): For every atomic concept A and individual a, $a \in A^{\mathcal{I}_n}$ implies $\mathcal{K} \models A(a)$
- Induction step: Assume that $a \in A^{\mathcal{I}_{n+1}}$
  - If $a \in A^{\mathcal{I}_n}$, $\mathcal{K} \models A(a)$ by IH
  - If $a \notin A^{\mathcal{I}_n}$, a has been added to $A^{\mathcal{I}_{n+1}}$ by applying rule R_1: there exists $C \sqsubseteq A \in \mathcal{T}$ such that $a \in C^{\mathcal{I}_n}$

Compact Canonical Model

Properties of $\mathcal{C}_\mathcal{K}$ – Proof of Claim 2

For every concept assertion $A(a)$, if $\mathcal{C}_\mathcal{K} \models A(a)$, then $\mathcal{K} \models A(a)$

Proof by induction on n such that $a \in A^{\mathcal{I}_n}$

- Base case: $a \in A^{\mathcal{I}_0}$ implies $A(a) \in \mathcal{A}$
- Induction hypothesis (IH): For every atomic concept A and individual a, $a \in A^{\mathcal{I}_n}$ implies $\mathcal{K} \models A(a)$
- Induction step: Assume that $a \in A^{\mathcal{I}_{n+1}}$
  - If $a \in A^{\mathcal{I}_n}$, $\mathcal{K} \models A(a)$ by IH
  - If $a \notin A^{\mathcal{I}_n}$, a has been added to $A^{\mathcal{I}_{n+1}}$ by applying rule R_1: there exists $C \sqsubseteq A \in \mathcal{T}$ such that $a \in C^{\mathcal{I}_n}$
    - case C atomic concept: $\mathcal{K} \models C(a)$ (by IH). It is then easy to check that $\mathcal{K} \models A(a)$
    - case $C = A_1 \sqcap A_2$: $\mathcal{K} \models A_1(a)$ and $\mathcal{K} \models A_2(a)$ (by IH). Since $A_1 \sqcap A_2 \sqsubseteq A \in \mathcal{T}$, it is then easy to check that $\mathcal{K} \models A(a)$
    - case $C = \exists R.D$: there exists $x \in D^{\mathcal{I}_n}$ s.t. $(a, x) \in R^{\mathcal{I}_n}$
      – if x is an individual, $R(a, x) \in \mathcal{A}$ and $\mathcal{K} \models D(x)$ (by IH) so since $\exists R.D \sqsubseteq A \in \mathcal{T}$, it is easy to check that $\mathcal{K} \models A(a)$
      – if $x = e_X$, $E \sqsubseteq \exists R.X \in \mathcal{T}$ and $a \in E^{\mathcal{I}_n}$ so $\mathcal{K} \models E(a)$ (by IH). By Claim 1, $\mathcal{K} \models X \sqsubseteq D$. It is then easy to check that $\mathcal{K} \models A(a)$

Exercise

Build the compact canonical model of $\langle \mathcal{T}, \mathcal{A} \rangle$ and use it to classify $\mathcal{T}$ and find all assertions entailed by $\langle \mathcal{T}, \mathcal{A} \rangle$

$$
\begin{aligned}
\mathcal{T} = \{ & A \sqcap B \sqsubseteq D, \quad B \sqcap D \sqsubseteq C, \quad \exists S.D \sqsubseteq D, \\
& C \sqsubseteq \exists R.A, \quad C \sqsubseteq \exists R.B, \quad B \sqsubseteq \exists S.D \} \\
\mathcal{A} = \{ & A(a), \quad B(a), \quad S(a, b), \quad D(b) \}
\end{aligned}
$$

Classification Algorithm

Given a TBox $\mathcal{T}$ in normal form, complete $\mathcal{T}$ using saturation rules

$$\mathsf{CR}_1^{\mathcal{T}} \ \frac{}{A \sqsubseteq A} \qquad \mathsf{CR}_2^{\mathcal{T}} \ \frac{}{A \sqsubseteq \top} \qquad \mathsf{CR}_3^{\mathcal{T}} \ \frac{A_1 \sqsubseteq B \quad B \sqsubseteq A_2}{A_1 \sqsubseteq A_2}$$

$$\mathsf{CR}_4^{\mathcal{T}} \ \frac{A \sqsubseteq A_1 \quad A \sqsubseteq A_2 \quad A_1 \sqcap A_2 \sqsubseteq B}{A \sqsubseteq B} \qquad \mathsf{CR}_5^{\mathcal{T}} \ \frac{A \sqsubseteq \exists R.A_1 \quad A_1 \sqsubseteq B_1 \quad \exists R.B_1 \sqsubseteq B}{A \sqsubseteq B}$$

- Instantiated rule: obtained by replacing A, A_1, A_2, B, B_1 by atomic concepts or $\top$ and R by atomic role
- Instantiated rule with premises $\alpha_1, \ldots, \alpha_n$ and conclusion β is applicable if $\{\alpha_1, \ldots, \alpha_n\} \subseteq \mathcal{T}$ and $\beta \notin \mathcal{T}$.
  - premises: axioms above the line
  - conclusion: axiom below the line

Applying the rule adds β to $\mathcal{T}$

Classification Algorithm

$$\text{CR}_1^{\mathcal{T}}\ \frac{}{A \sqsubseteq A} \qquad \text{CR}_2^{\mathcal{T}}\ \frac{}{A \sqsubseteq \top} \qquad \text{CR}_3^{\mathcal{T}}\ \frac{A_1 \sqsubseteq B \quad B \sqsubseteq A_2}{A_1 \sqsubseteq A_2}$$

$$\text{CR}_4^{\mathcal{T}}\ \frac{A \sqsubseteq A_1 \quad A \sqsubseteq A_2 \quad A_1 \sqcap A_2 \sqsubseteq B}{A \sqsubseteq B} \qquad \text{CR}_5^{\mathcal{T}}\ \frac{A \sqsubseteq \exists R.A_1 \quad A_1 \sqsubseteq B_1 \quad \exists R.B_1 \sqsubseteq B}{A \sqsubseteq B}$$

Classify $\mathcal{T}$: find all atomic concepts A, B such that $\mathcal{T} \models A \sqsubseteq B$

- Exhaustively apply instantiated saturation rules to $\mathcal{T}$
  - the resulting TBox sat($\mathcal{T}$) is called the saturated TBox
- For every atomic concepts A and B, return that $\mathcal{T} \models A \sqsubseteq B$ iff $A \sqsubseteq B \in \text{sat}(\mathcal{T})$

Lemma

All exhaustive sequences of rule applications lead to a unique saturated TBox

Classification Algorithm

Example

$$\mathcal{T} = \{A \sqsubseteq D, \quad C \sqcap D \sqsubseteq B, \quad D \sqsubseteq \exists R.D,$$
$$\exists R.E \sqsubseteq C, \quad \exists R.\top \sqsubseteq E\}$$

Classification Algorithm

Example

$$\mathcal{T} = \{A \sqsubseteq D, \quad C \sqcap D \sqsubseteq B, \quad D \sqsubseteq \exists R.D,$$
$$\exists R.E \sqsubseteq C, \quad \exists R.\top \sqsubseteq E\}$$

$$\frac{}{A \sqsubseteq A} \qquad \frac{}{B \sqsubseteq B} \qquad \frac{}{C \sqsubseteq C} \qquad \frac{}{D \sqsubseteq D} \qquad \frac{}{E \sqsubseteq E}$$

$$\frac{}{A \sqsubseteq \top} \qquad \frac{}{B \sqsubseteq \top} \qquad \frac{}{C \sqsubseteq \top} \qquad \frac{}{D \sqsubseteq \top} \qquad \frac{}{E \sqsubseteq \top}$$

Classification Algorithm

Example

$$\mathcal{T} = \{A \sqsubseteq D, \quad C \sqcap D \sqsubseteq B, \quad D \sqsubseteq \exists R.D, \quad \exists R.E \sqsubseteq C, \quad \exists R.\top \sqsubseteq E\}$$

$$\frac{}{A \sqsubseteq A} \qquad \frac{}{B \sqsubseteq B} \qquad \frac{}{C \sqsubseteq C} \qquad \frac{}{D \sqsubseteq D} \qquad \frac{}{E \sqsubseteq E}$$

$$\frac{}{A \sqsubseteq \top} \qquad \frac{}{B \sqsubseteq \top} \qquad \frac{}{C \sqsubseteq \top} \qquad \frac{}{D \sqsubseteq \top} \qquad \frac{}{E \sqsubseteq \top}$$

$$\frac{D \sqsubseteq \exists R.D \quad D \sqsubseteq \top \quad \exists R.\top \sqsubseteq E}{D \sqsubseteq E} \qquad \frac{D \sqsubseteq \exists R.D \quad D \sqsubseteq E \quad \exists R.E \sqsubseteq C}{D \sqsubseteq C}$$

Classification Algorithm

Example

$$\mathcal{T} = \{A \sqsubseteq D, \quad C \sqcap D \sqsubseteq B, \quad D \sqsubseteq \exists R.D,$$
$$\exists R.E \sqsubseteq C, \quad \exists R.\top \sqsubseteq E\}$$

$$\frac{}{A \sqsubseteq A} \quad \frac{}{B \sqsubseteq B} \quad \frac{}{C \sqsubseteq C} \quad \frac{}{D \sqsubseteq D} \quad \frac{}{E \sqsubseteq E}$$

$$\frac{}{A \sqsubseteq \top} \quad \frac{}{B \sqsubseteq \top} \quad \frac{}{C \sqsubseteq \top} \quad \frac{}{D \sqsubseteq \top} \quad \frac{}{E \sqsubseteq \top}$$

$$\frac{D \sqsubseteq \exists R.D \quad D \sqsubseteq \top \quad \exists R.\top \sqsubseteq E}{D \sqsubseteq E} \qquad \frac{D \sqsubseteq \exists R.D \quad D \sqsubseteq E \quad \exists R.E \sqsubseteq C}{D \sqsubseteq C}$$

$$\frac{D \sqsubseteq D \quad D \sqsubseteq C \quad D \sqcap C \sqsubseteq B}{D \sqsubseteq B}$$

Classification Algorithm

Example

$$\mathcal{T} = \{A \sqsubseteq D, \quad C \sqcap D \sqsubseteq B, \quad D \sqsubseteq \exists R.D, \\ \exists R.E \sqsubseteq C, \quad \exists R.\top \sqsubseteq E\}$$

$$\frac{}{A \sqsubseteq A} \qquad \frac{}{B \sqsubseteq B} \qquad \frac{}{C \sqsubseteq C} \qquad \frac{}{D \sqsubseteq D} \qquad \frac{}{E \sqsubseteq E}$$

$$\frac{}{A \sqsubseteq \top} \qquad \frac{}{B \sqsubseteq \top} \qquad \frac{}{C \sqsubseteq \top} \qquad \frac{}{D \sqsubseteq \top} \qquad \frac{}{E \sqsubseteq \top}$$

$$\frac{D \sqsubseteq \exists R.D \quad D \sqsubseteq \top \quad \exists R.\top \sqsubseteq E}{D \sqsubseteq E} \qquad \frac{D \sqsubseteq \exists R.D \quad D \sqsubseteq E \quad \exists R.E \sqsubseteq C}{D \sqsubseteq C}$$

$$\frac{D \sqsubseteq D \quad D \sqsubseteq C \quad D \sqcap C \sqsubseteq B}{D \sqsubseteq B}$$

$$\frac{A \sqsubseteq D \quad D \sqsubseteq E}{A \sqsubseteq E} \qquad \frac{A \sqsubseteq D \quad D \sqsubseteq C}{A \sqsubseteq C} \qquad \frac{A \sqsubseteq D \quad D \sqsubseteq B}{A \sqsubseteq B}$$

Classification Algorithm

Termination and complexity

Classification algorithm runs in polynomial time w.r.t. the size of $\mathcal{T}$

- Each rule application adds a concept inclusion of the form $A \sqsubseteq B$ with A and B atomic concepts from $\mathcal{T}$ or $\top$
- The number of such concept inclusions is quadratic in the number of atomic concepts that occur in $\mathcal{T}$

Classification Algorithm

Soundness

$$\text{CR}_1^{\mathcal{T}}\ \frac{}{A \sqsubseteq A} \qquad \text{CR}_2^{\mathcal{T}}\ \frac{}{A \sqsubseteq \top} \qquad \text{CR}_3^{\mathcal{T}}\ \frac{A_1 \sqsubseteq B \quad B \sqsubseteq A_2}{A_1 \sqsubseteq A_2}$$

$$\text{CR}_4^{\mathcal{T}}\ \frac{A \sqsubseteq A_1 \quad A \sqsubseteq A_2 \quad A_1 \sqcap A_2 \sqsubseteq B}{A \sqsubseteq B} \qquad \text{CR}_5^{\mathcal{T}}\ \frac{A \sqsubseteq \exists R.A_1 \quad A_1 \sqsubseteq B_1 \quad \exists R.B_1 \sqsubseteq B}{A \sqsubseteq B}$$

If $A \sqsubseteq B \in \text{sat}(\mathcal{T})$ then $\mathcal{T} \models A \sqsubseteq B$.

Show that if β is added to $\mathcal{T}$ by applying a saturation rule whose premises are entailed by $\mathcal{T}$, then $\mathcal{T} \models \beta$

- $\text{CR}_1^{\mathcal{T}}$ or $\text{CR}_2^{\mathcal{T}}$ case: β is of the form $A \sqsubseteq A$ or $A \sqsubseteq \top$ and holds in every interpretation, so $\mathcal{T} \models \beta$
- $\text{CR}_3^{\mathcal{T}}$ case: $\beta = A_1 \sqsubseteq A_2$, $\mathcal{T} \models A_1 \sqsubseteq B$ and $\mathcal{T} \models B \sqsubseteq A_2$
  - let $\mathcal{I}$ be a model of $\mathcal{T}$: $A_1^{\mathcal{I}} \subseteq B^{\mathcal{I}}$ and $B^{\mathcal{I}} \subseteq A_2^{\mathcal{I}}$ so $A_1^{\mathcal{I}} \subseteq A_2^{\mathcal{I}}$, yielding $\mathcal{I} \models A_1 \sqsubseteq A_2$
  - hence $\mathcal{T} \models A_1 \sqsubseteq A_2$
- $\text{CR}_4^{\mathcal{T}}$ and $\text{CR}_5^{\mathcal{T}}$ cases: left as practice

The property follows by induction on the number of rule applications before $A \sqsubseteq B$ has been added to $\text{sat}(\mathcal{T})$

Classification Algorithm

Completeness

If $\mathcal{T} \models A \sqsubseteq B$ then $A \sqsubseteq B \in \text{sat}(\mathcal{T})$.

Show the contrapositive: if $A \sqsubseteq B \notin \text{sat}(\mathcal{T})$, then $\mathcal{T} \not\models A \sqsubseteq B$

- Define an interpretation $\mathcal{I}_{\text{sat}(\mathcal{T})}$ from $\text{sat}(\mathcal{T})$
  - $\Delta^{\mathcal{I}_{\text{sat}(\mathcal{T})}} = \{e_A \mid A \text{ is an atomic concept in } \mathcal{T}\} \cup \{e_\top\}$
  - $A^{\mathcal{I}_{\text{sat}(\mathcal{T})}} = \{e_B \mid B \sqsubseteq A \in \text{sat}(\mathcal{T})\}$
  - $R^{\mathcal{I}_{\text{sat}(\mathcal{T})}} = \{(e_A, e_B) \mid A \sqsubseteq C \in \text{sat}(\mathcal{T}), C \sqsubseteq \exists R.B \in \text{sat}(\mathcal{T})\}$
- Claim: $\mathcal{I}_{\text{sat}(\mathcal{T})}$ is a model of $\mathcal{T}$ and $A \sqsubseteq B \notin \text{sat}(\mathcal{T})$ implies that $\mathcal{I}_{\text{sat}(\mathcal{T})} \not\models A \sqsubseteq B$
- If $A \sqsubseteq B \notin \text{sat}(\mathcal{T})$, then $\mathcal{I}_{\text{sat}(\mathcal{T})} \not\models A \sqsubseteq B$, so $\mathcal{T} \not\models A \sqsubseteq B$

Remark: $\mathcal{I}_{\text{sat}(\mathcal{T})}$ is actually the compact canonical model of $\langle \mathcal{T}, \emptyset \rangle$

Classification Algorithm

Completeness – Proof of the claim

$\mathcal{I}_{\mathsf{sat}(\mathcal{T})} \models \mathcal{T}$ and $A \sqsubseteq B \notin \mathsf{sat}(\mathcal{T})$ implies that $\mathcal{I}_{\mathsf{sat}(\mathcal{T})} \not\models A \sqsubseteq B$

- $\mathcal{I}_{\mathsf{sat}(\mathcal{T})}$ is a model of $\mathsf{sat}(\mathcal{T})$: let $\beta \in \mathsf{sat}(\mathcal{T})$
  - Case $\beta = A \sqsubseteq B$: if $e_D \in A^{\mathcal{I}_{\mathsf{sat}(\mathcal{T})}}$, then $D \sqsubseteq A \in \mathsf{sat}(\mathcal{T})$
    By $\mathsf{CR}_3^{\mathcal{T}}$, $D \sqsubseteq B \in \mathsf{sat}(\mathcal{T})$, so $e_D \in B^{\mathcal{I}_{\mathsf{sat}(\mathcal{T})}}$
  - Case $\beta = A_1 \sqcap A_2 \sqsubseteq B$: if $e_D \in (A_1 \sqcap A_2)^{\mathcal{I}_{\mathsf{sat}(\mathcal{T})}}$, then
    $D \sqsubseteq A_1 \in \mathsf{sat}(\mathcal{T})$ and $D \sqsubseteq A_2 \in \mathsf{sat}(\mathcal{T})$
    By $\mathsf{CR}_4^{\mathcal{T}}$, $D \sqsubseteq B \in \mathsf{sat}(\mathcal{T})$, so $e_D \in B^{\mathcal{I}_{\mathsf{sat}(\mathcal{T})}}$
  - Case $\beta = A \sqsubseteq \exists R.B$: if $e_D \in A^{\mathcal{I}_{\mathsf{sat}(\mathcal{T})}}$, then $D \sqsubseteq A \in \mathsf{sat}(\mathcal{T})$
    By construction of $\mathcal{I}_{\mathsf{sat}(\mathcal{T})}$, it follows that $(e_D, e_B) \in R^{\mathcal{I}_{\mathsf{sat}(\mathcal{T})}}$
    By $\mathsf{CR}_1^{\mathcal{T}}$, $B \sqsubseteq B \in \mathsf{sat}(\mathcal{T})$ so $e_B \in B^{\mathcal{I}_{\mathsf{sat}(\mathcal{T})}}$: $e_D \in \exists R.B^{\mathcal{I}_{\mathsf{sat}(\mathcal{T})}}$
  - Case $\beta = \exists R.B \sqsubseteq A$: if $e_D \in \exists R.B^{\mathcal{I}_{\mathsf{sat}(\mathcal{T})}}$, then there exists
    $e_C \in B^{\mathcal{I}_{\mathsf{sat}(\mathcal{T})}}$ such that $(e_D, e_C) \in R^{\mathcal{I}_{\mathsf{sat}(\mathcal{T})}}$
    Hence $C \sqsubseteq B \in \mathsf{sat}(\mathcal{T})$ and $D \sqsubseteq \exists R.C \in \mathsf{sat}(\mathcal{T})$
    By $\mathsf{CR}_5^{\mathcal{T}}$, $D \sqsubseteq A \in \mathsf{sat}(\mathcal{T})$, so $e_D \in A^{\mathcal{I}_{\mathsf{sat}(\mathcal{T})}}$
- Since $\mathcal{T} \subseteq \mathsf{sat}(\mathcal{T})$, it follows that $\mathcal{I}_{\mathsf{sat}(\mathcal{T})} \models \mathcal{T}$
- If $A \sqsubseteq B \notin \mathsf{sat}(\mathcal{T})$, then $e_A \notin B^{\mathcal{I}_{\mathsf{sat}(\mathcal{T})}}$ while $e_A \in A^{\mathcal{I}_{\mathsf{sat}(\mathcal{T})}}$
  (since $A \sqsubseteq A \in \mathsf{sat}(\mathcal{T})$ by $\mathsf{CR}_1^{\mathcal{T}}$) so $\mathcal{I}_{\mathsf{sat}(\mathcal{T})} \not\models A \sqsubseteq B$

Instance Checking

Add rules to derive assertions to the saturation rules

$$\mathrm{CR}_1^{\mathcal{T}}\ \frac{}{A \sqsubseteq A} \qquad \mathrm{CR}_2^{\mathcal{T}}\ \frac{}{A \sqsubseteq \top} \qquad \mathrm{CR}_3^{\mathcal{T}}\ \frac{A_1 \sqsubseteq B \quad B \sqsubseteq A_2}{A_1 \sqsubseteq A_2}$$

$$\mathrm{CR}_4^{\mathcal{T}}\ \frac{A \sqsubseteq A_1 \quad A \sqsubseteq A_2 \quad A_1 \sqcap A_2 \sqsubseteq B}{A \sqsubseteq B} \qquad \mathrm{CR}_5^{\mathcal{T}}\ \frac{A \sqsubseteq \exists R.A_1 \quad A_1 \sqsubseteq B_1 \quad \exists R.B_1 \sqsubseteq B}{A \sqsubseteq B}$$

$$\mathrm{CR}_1^{\mathcal{A}}\ \frac{}{\top(a)} \qquad \mathrm{CR}_2^{\mathcal{A}}\ \frac{A \sqsubseteq B \quad A(a)}{B(a)}$$

$$\mathrm{CR}_3^{\mathcal{A}}\ \frac{A_1 \sqcap A_2 \sqsubseteq B \quad A_1(a) \quad A_2(a)}{B(a)} \qquad \mathrm{CR}_4^{\mathcal{A}}\ \frac{\exists R.A \sqsubseteq B \quad R(a,b) \quad A(b)}{B(a)}$$

- Take as input an $\mathcal{EL}$ KB $\langle \mathcal{T}, \mathcal{A} \rangle$ with $\mathcal{T}$ in normal form and an atomic concept A
- Exhaustively apply instantiated saturation rules to $\langle \mathcal{T}, \mathcal{A} \rangle$
  - the resulting KB $\mathrm{sat}(\mathcal{T}, \mathcal{A}) = \langle \mathcal{T}^\star, \mathcal{A}^\star \rangle$ is the saturated KB
- For every individual a, return $\langle \mathcal{T}, \mathcal{A} \rangle \models A(a)$ iff $A(a) \in \mathcal{A}^\star$

Instance Checking

- The instance checking algorithm adds a number of concept inclusions and concept assertions which is at most quadratic in the size of the KB, hence runs in polynomial time
- Soundness: left as practice
- Completeness: Show the contrapositive: if $A(a) \notin \mathcal{A}^\star$, then $\langle \mathcal{T}, \mathcal{A} \rangle \not\models A(a)$
  - Define an interpretation $\mathcal{I}^\star$ from $\mathsf{sat}(\mathcal{T}, \mathcal{A}) = \langle \mathcal{T}^\star, \mathcal{A}^\star \rangle$
    - $\Delta^{\mathcal{I}^\star} = \{c \mid c \text{ individual from } \mathcal{A}\} \cup \{e_A \mid A \text{ is an atomic concept in } \mathcal{T}\} \cup \{e_\top\}$
    - $c^{\mathcal{I}^\star} = c$ for every individual c from $\mathcal{A}$
    - $A^{\mathcal{I}^\star} = \{c \mid A(c) \in \mathcal{A}^\star\} \cup \{e_B \mid B \sqsubseteq A \in \mathcal{T}^\star\}$
    - $R^{\mathcal{I}^\star} = \{(c, d) \mid R(c, d) \in \mathcal{A}^\star\} \cup \{(a, e_B) \mid A \sqsubseteq \exists R.B \in \mathcal{T}^\star, A(a) \in \mathcal{A}^\star\} \cup \{(e_A, e_B) \mid A \sqsubseteq C \in \mathcal{T}^\star, C \sqsubseteq \exists R.B \in \mathcal{T}^\star\}$
  - Claim: $\mathcal{I}^\star$ is a model of $\langle \mathcal{T}, \mathcal{A} \rangle$ and $A(a) \notin \mathcal{A}^\star$ implies that $\mathcal{I}^\star \not\models A(a)$: left as practice

Exercise

Normalize $\mathcal{T}$ and apply the saturation algorithm to classify $\mathcal{T}$ and find the assertions entailed by $\langle \mathcal{T}, \mathcal{A} \rangle$

$$\mathcal{T} = \{\exists S.B \sqsubseteq D,\ \exists R.D \sqsubseteq E,\ \exists R.A \sqsubseteq \exists R.\exists S.(B \sqcap C)\}$$

$$\mathcal{A} = \{R(a, b),\ A(b)\}$$

A Saturation Algorithm for $\mathcal{ELI}$

- $\mathcal{ELI} = \mathcal{EL}$ + inverse roles

$$C := \top \mid A \mid C \sqcap C \mid \exists R.C \mid \exists R^{-}.C$$

- Axiom entailment is ExpTime-complete
- However, $\mathcal{ELI}$ retains some nice properties
  - canonical model (no case-based reasoning)
  - can extend the saturation algorithm to handle $\mathcal{ELI}$
    - may produce an exponential number of concept inclusions
    - deduce $A \sqcap D \sqsubseteq \exists R.(B \sqcap E)$ from $A \sqsubseteq \exists R.B$ and $\exists R^{-}.D \sqsubseteq E$
- The same holds for $\mathcal{ELHI}_{\bot} = \mathcal{ELI}$ + role inclusions + $\bot$

A Saturation Algorithm for $\mathcal{ELI}$

$$\mathrm{CR}_1^T \; \frac{}{A \sqsubseteq A} \qquad \mathrm{CR}_2^T \; \frac{}{A \sqsubseteq \top}$$

$$\mathrm{CR}_3^T \; \frac{\{A \sqsubseteq B_i\}_{i=1}^n \quad B_1 \sqcap \cdots \sqcap B_n \sqsubseteq B}{A \sqsubseteq B} \qquad \mathrm{CR}_4^T \; \frac{M \sqsubseteq \exists S.(N \sqcap N') \quad N \sqsubseteq A}{M \sqsubseteq \exists S.(N \sqcap N' \sqcap A)}$$

$$\mathrm{CR}_5^T \; \frac{M \sqsubseteq \exists S.(N \sqcap A) \quad \exists S.A \sqsubseteq B}{M \sqsubseteq B} \qquad \mathrm{CR}_6^T \; \frac{M \sqsubseteq \exists S.N \quad \exists \mathrm{inv}(S).A \sqsubseteq B}{M \sqcap A \sqsubseteq \exists S.(N \sqcap B)}$$

$$\mathrm{CR}_1^A \; \frac{}{\top(a)} \qquad \mathrm{CR}_2^A \; \frac{A_1 \sqcap \cdots \sqcap A_n \sqsubseteq B \quad \{A_i(a)\}_{i=1}^n}{B(a)}$$

$$\mathrm{CR}_3^A \; \frac{\exists R.A \sqsubseteq B \quad R(a,b) \quad A(b)}{B(a)} \qquad \mathrm{CR}_4^A \; \frac{\exists R^-.A \sqsubseteq B \quad R(b,a) \quad A(b)}{B(a)}$$

- R is an atomic role, $S := R \mid R^-$, $\mathrm{inv}(R) = R^-$ and $\mathrm{inv}(R^-) = R$
- A, B, A_i, B_i are atomic concepts or $\top$
- M, N, N' are conjunctions of atomic concepts or $\top$, treated as sets (no repetition, the order does not matter)

A Saturation Algorithm for $\mathcal{ELI}$

Example

$$\text{CR}_1^{\mathcal{T}} \ \frac{}{A \sqsubseteq A} \qquad \text{CR}_2^{\mathcal{T}} \ \frac{}{A \sqsubseteq \top}$$

$$\text{CR}_3^{\mathcal{T}} \ \frac{\{A \sqsubseteq B_i\}_{i=1}^{n} \quad B_1 \sqcap \cdots \sqcap B_n \sqsubseteq B}{A \sqsubseteq B} \qquad \text{CR}_4^{\mathcal{T}} \ \frac{M \sqsubseteq \exists S.(N \sqcap N') \quad N \sqsubseteq A}{M \sqsubseteq \exists S.(N \sqcap N' \sqcap A)}$$

$$\text{CR}_5^{\mathcal{T}} \ \frac{M \sqsubseteq \exists S.(N \sqcap A) \quad \exists S.A \sqsubseteq B}{M \sqsubseteq B} \qquad \text{CR}_6^{\mathcal{T}} \ \frac{M \sqsubseteq \exists S.N \quad \exists \text{inv}(S).A \sqsubseteq B}{M \sqcap A \sqsubseteq \exists S.(N \sqcap B)}$$

$$\mathcal{T} = \{A \sqsubseteq R.B,\ \exists R^{-}.C \sqsubseteq D,\ D \sqsubseteq E,\ \exists R.E \sqsubseteq F,\ G \sqsubseteq A,\ G \sqsubseteq C\}$$

A Saturation Algorithm for $\mathcal{ELI}$

Example

$$\text{CR}_1^{\mathcal{T}} \ \frac{}{A \sqsubseteq A} \qquad \text{CR}_2^{\mathcal{T}} \ \frac{}{A \sqsubseteq \top}$$

$$\text{CR}_3^{\mathcal{T}} \ \frac{\{A \sqsubseteq B_i\}_{i=1}^n \quad B_1 \sqcap \cdots \sqcap B_n \sqsubseteq B}{A \sqsubseteq B} \qquad \text{CR}_4^{\mathcal{T}} \ \frac{M \sqsubseteq \exists S.(N \sqcap N') \quad N \sqsubseteq A}{M \sqsubseteq \exists S.(N \sqcap N' \sqcap A)}$$

$$\text{CR}_5^{\mathcal{T}} \ \frac{M \sqsubseteq \exists S.(N \sqcap A) \quad \exists S.A \sqsubseteq B}{M \sqsubseteq B} \qquad \text{CR}_6^{\mathcal{T}} \ \frac{M \sqsubseteq \exists S.N \quad \exists \text{inv}(S).A \sqsubseteq B}{M \sqcap A \sqsubseteq \exists S.(N \sqcap B)}$$

$$\mathcal{T} = \{A \sqsubseteq R.B,\ \exists R^-.C \sqsubseteq D,\ D \sqsubseteq E,\ \exists R.E \sqsubseteq F,\ G \sqsubseteq A,\ G \sqsubseteq C\}$$

$$\frac{A \sqsubseteq \exists R.B \quad \exists R^-.C \sqsubseteq D}{A \sqcap C \sqsubseteq \exists R.(B \sqcap D)} \quad (\text{CR}_6^{\mathcal{T}})$$

A Saturation Algorithm for $\mathcal{ELI}$

Example

$$\text{CR}_1^{\mathcal{T}} \ \frac{}{A \sqsubseteq A} \qquad \text{CR}_2^{\mathcal{T}} \ \frac{}{A \sqsubseteq \top}$$

$$\text{CR}_3^{\mathcal{T}} \ \frac{\{A \sqsubseteq B_i\}_{i=1}^{n} \quad B_1 \sqcap \cdots \sqcap B_n \sqsubseteq B}{A \sqsubseteq B} \qquad \text{CR}_4^{\mathcal{T}} \ \frac{M \sqsubseteq \exists S.(N \sqcap N') \quad N \sqsubseteq A}{M \sqsubseteq \exists S.(N \sqcap N' \sqcap A)}$$

$$\text{CR}_5^{\mathcal{T}} \ \frac{M \sqsubseteq \exists S.(N \sqcap A) \quad \exists S.A \sqsubseteq B}{M \sqsubseteq B} \qquad \text{CR}_6^{\mathcal{T}} \ \frac{M \sqsubseteq \exists S.N \quad \exists \text{inv}(S).A \sqsubseteq B}{M \sqcap A \sqsubseteq \exists S.(N \sqcap B)}$$

$$\mathcal{T} = \{A \sqsubseteq R.B,\ \exists R^{-}.C \sqsubseteq D,\ D \sqsubseteq E,\ \exists R.E \sqsubseteq F,\ G \sqsubseteq A,\ G \sqsubseteq C\}$$

$$\frac{A \sqsubseteq \exists R.B \quad \exists R^{-}.C \sqsubseteq D}{A \sqcap C \sqsubseteq \exists R.(B \sqcap D)} \quad (\text{CR}_6^{\mathcal{T}})$$

$$\frac{A \sqcap C \sqsubseteq \exists R.(B \sqcap D) \quad D \sqsubseteq E}{A \sqcap C \sqsubseteq \exists R.(B \sqcap D \sqcap E)} \quad (\text{CR}_4^{\mathcal{T}})$$

A Saturation Algorithm for $\mathcal{ELI}$

Example

$$\text{CR}_1^{\mathcal{T}} \ \frac{}{A \sqsubseteq A} \qquad \text{CR}_2^{\mathcal{T}} \ \frac{}{A \sqsubseteq \top}$$

$$\text{CR}_3^{\mathcal{T}} \ \frac{\{A \sqsubseteq B_i\}_{i=1}^{n} \quad B_1 \sqcap \cdots \sqcap B_n \sqsubseteq B}{A \sqsubseteq B} \qquad \text{CR}_4^{\mathcal{T}} \ \frac{M \sqsubseteq \exists S.(N \sqcap N') \quad N \sqsubseteq A}{M \sqsubseteq \exists S.(N \sqcap N' \sqcap A)}$$

$$\text{CR}_5^{\mathcal{T}} \ \frac{M \sqsubseteq \exists S.(N \sqcap A) \quad \exists S.A \sqsubseteq B}{M \sqsubseteq B} \qquad \text{CR}_6^{\mathcal{T}} \ \frac{M \sqsubseteq \exists S.N \quad \exists \text{inv}(S).A \sqsubseteq B}{M \sqcap A \sqsubseteq \exists S.(N \sqcap B)}$$

$$\mathcal{T} = \{A \sqsubseteq R.B,\ \exists R^{-}.C \sqsubseteq D,\ D \sqsubseteq E,\ \exists R.E \sqsubseteq F,\ G \sqsubseteq A,\ G \sqsubseteq C\}$$

$$\frac{A \sqsubseteq \exists R.B \quad \exists R^{-}.C \sqsubseteq D}{A \sqcap C \sqsubseteq \exists R.(B \sqcap D)} \quad (\text{CR}_6^{\mathcal{T}})$$

$$\frac{A \sqcap C \sqsubseteq \exists R.(B \sqcap D) \quad D \sqsubseteq E}{A \sqcap C \sqsubseteq \exists R.(B \sqcap D \sqcap E)} \quad (\text{CR}_4^{\mathcal{T}})$$

$$\frac{A \sqcap C \sqsubseteq \exists R.(B \sqcap D \sqcap E) \quad \exists R.E \sqsubseteq F}{A \sqcap C \sqsubseteq F} \quad (\text{CR}_5^{\mathcal{T}})$$

A Saturation Algorithm for $\mathcal{ELI}$

Example

$$\mathrm{CR}_1^{\mathcal{T}} \ \frac{}{A \sqsubseteq A} \qquad \mathrm{CR}_2^{\mathcal{T}} \ \frac{}{A \sqsubseteq \top}$$

$$\mathrm{CR}_3^{\mathcal{T}} \ \frac{\{A \sqsubseteq B_i\}_{i=1}^{n} \quad B_1 \sqcap \cdots \sqcap B_n \sqsubseteq B}{A \sqsubseteq B} \qquad \mathrm{CR}_4^{\mathcal{T}} \ \frac{M \sqsubseteq \exists S.(N \sqcap N') \quad N \sqsubseteq A}{M \sqsubseteq \exists S.(N \sqcap N' \sqcap A)}$$

$$\mathrm{CR}_5^{\mathcal{T}} \ \frac{M \sqsubseteq \exists S.(N \sqcap A) \quad \exists S.A \sqsubseteq B}{M \sqsubseteq B} \qquad \mathrm{CR}_6^{\mathcal{T}} \ \frac{M \sqsubseteq \exists S.N \quad \exists \mathrm{inv}(S).A \sqsubseteq B}{M \sqcap A \sqsubseteq \exists S.(N \sqcap B)}$$

$$\mathcal{T} = \{A \sqsubseteq R.B,\ \exists R^{-}.C \sqsubseteq D,\ D \sqsubseteq E,\ \exists R.E \sqsubseteq F,\ G \sqsubseteq A,\ G \sqsubseteq C\}$$

$$\frac{A \sqsubseteq \exists R.B \quad \exists R^{-}.C \sqsubseteq D}{A \sqcap C \sqsubseteq \exists R.(B \sqcap D)} \quad (\mathrm{CR}_6^{\mathcal{T}})$$

$$\frac{A \sqcap C \sqsubseteq \exists R.(B \sqcap D) \quad D \sqsubseteq E}{A \sqcap C \sqsubseteq \exists R.(B \sqcap D \sqcap E)} \quad (\mathrm{CR}_4^{\mathcal{T}})$$

$$\frac{A \sqcap C \sqsubseteq \exists R.(B \sqcap D \sqcap E) \quad \exists R.E \sqsubseteq F}{A \sqcap C \sqsubseteq F} \quad (\mathrm{CR}_5^{\mathcal{T}})$$

$$\frac{G \sqsubseteq A \quad G \sqsubseteq C \quad A \sqcap C \sqsubseteq F}{G \sqsubseteq F} \quad (\mathrm{CR}_3^{\mathcal{T}})$$

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