Description Logics and Reasoning on Data
2: Reasoning in $\mathcal{ALC}$

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Outline

Reminder: $\mathcal{ALC}$

The $\mathcal{ALC}$ DL is defined as follows:

- if A is an atomic concept, then A is an $\mathcal{ALC}$ concept
- if C, D are $\mathcal{ALC}$ concepts and R is an atomic role, then the following are $\mathcal{ALC}$ concepts:
  - $C \sqcap D$ (conjunction)
  - $C \sqcup D$ (disjunction)
  - $\neg C$ (negation)
  - $\exists R.C$ (existential restriction)
  - $\forall R.C$ (universal restriction)
- an $\mathcal{ALC}$ TBox contains only concept inclusions

Note that $A \sqcap \neg A$ can be abbreviated by $\bot$ and $A \sqcup \neg A$ by $\top$.

Reminder: Concept and KB Satisfiability

- Concept satisfiability w.r.t. an empty TBox: Given a concept C, is there an interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ such that $C^{\mathcal{I}} \neq \emptyset$?
  - $A \sqcap B$ is satisfiable, $A \sqcap \neg A$ is not satisfiable
- Concept satisfiability w.r.t. a TBox: Given a concept C and a TBox $\mathcal{T}$, is there a model $\mathcal{I}$ of $\mathcal{T}$ such that $C^{\mathcal{I}} \neq \emptyset$?
  - $A \sqcap B$ is not satisfiable w.r.t. $\mathcal{T} = \{A \sqsubseteq \neg B\}$
- KB satisfiability: Given a KB $\langle \mathcal{T}, \mathcal{A} \rangle$, does $\langle \mathcal{T}, \mathcal{A} \rangle$ have a model?
  - $\langle \{A \sqsubseteq \neg B\},\ \{A(a), B(a)\} \rangle$ is not satisfiable,
    $\langle \{A \sqsubseteq \neg B\},\ \{A(a), B(b)\} \rangle$ is satisfiable
- Important in practice to build and debug ontologies
  - we usually don't want to use an unsatisfiable concept when defining an ontology
  - we may want to check that the model is sufficiently constrained to prevent some situation captured by a concept that should be unsatisfiable w.r.t. the TBox
  - an unsatisfiable KB indicates a modelisation problem

Reminder: Reduction Between Reasoning Tasks in $\mathcal{ALC}$

- From subsumption to concept satisfiability:
  $\mathcal{T} \models C \sqsubseteq D$ iff $C \sqcap \neg D$ is not satisfiable w.r.t. $\mathcal{T}$
  - note that if C and D are $\mathcal{ALC}$ concepts, so is $C \sqcap \neg D$
- From concept satisfiability to KB satisfiability:
  C is satisfiable w.r.t. $\mathcal{T}$ iff $\langle \mathcal{T} \cup \{A \sqsubseteq C\}, \mathcal{A} \cup \{C(a)\} \rangle$ is satisfiable
- From instance checking to KB satisfiability:
  $\langle \mathcal{T}, \mathcal{A} \rangle \models C(a)$ iff $\langle \mathcal{T} \cup \{C \sqsubseteq \neg A\}, \mathcal{A} \cup \{A(a)\} \rangle$ is not satisfiable

In this course: Algorithms to decide concept satisfiability w.r.t. an empty TBox and KB satisfiability
$\rightarrow$ concept satisfiability w.r.t. a non-empty TBox, subsumption and instance checking can be solved via reduction to KB satisfiability

Tableau Algorithms

- Tableau-based methods are used to decide satisfiability of a formula or theory by using rules to construct a model
  - if it succeeds, the theory is satisfiable
  - if it fails, despite having considered all possibilities, the theory is unsatisfiable
- Classical approach used for different kinds of logics (propositional, FOL, modal...)
- Popular approach for reasoning in expressive DLs ($\mathcal{ALC}$ and its extensions), implemented in state-of-the-art DL reasoners (with variants and optimizations)

Negation Normal Form

- The algorithms we consider need $\mathcal{ALC}$ concepts to be in negation normal form (NNF):
  An $\mathcal{ALC}$ concept C is in NNF if the symbol $\neg$ appears only in front of atomic concepts:
  - in NNF: $A \sqcap \neg B$, $\exists R.\neg A$, $A \sqcup B$
  - not in NNF: $\neg(A \sqcap B)$, $\exists R.\neg(\forall S.B)$, $A \sqcap \neg(B \sqcup C)$
- Every $\mathcal{ALC}$ concept C is equivalent to an $\mathcal{ALC}$ concept $\mathsf{nnf}(C)$ in NNF
  - $C^{\mathcal{I}} = \mathsf{nnf}(C)^{\mathcal{I}}$ for every interpretation $\mathcal{I}$
- $\mathsf{nnf}(C)$ can be computed in linear time by "pushing the negation inside" using the following equivalences

$$\neg(C \sqcap D) \equiv \neg C \sqcup \neg D \qquad \neg(\exists R.C) \equiv \forall R.\neg C \qquad \neg(\neg C) \equiv C$$

$$\neg(C \sqcup D) \equiv \neg C \sqcap \neg D \qquad \neg(\forall R.C) \equiv \exists R.\neg C$$

Negation Normal Form

Given an $\mathcal{ALC}$ concept C, $\text{nnf}(C)$ is computed by the recursive algorithm:

- $\text{nnf}(A) = A$ for A atomic concept
- $\text{nnf}(\neg A) = \neg A$ for A atomic concept
- $\text{nnf}(C \sqcap D) = \text{nnf}(C) \sqcap \text{nnf}(D)$
- $\text{nnf}(C \sqcup D) = \text{nnf}(C) \sqcup \text{nnf}(D)$
- $\text{nnf}(\exists R.C) = \exists R.\text{nnf}(C)$
- $\text{nnf}(\forall R.C) = \forall R.\text{nnf}(C)$
- $\text{nnf}(\neg(\neg C)) = \text{nnf}(C)$
- $\text{nnf}(\neg(C \sqcap D)) = \text{nnf}(\neg C) \sqcup \text{nnf}(\neg D)$
- $\text{nnf}(\neg(C \sqcup D)) = \text{nnf}(\neg C) \sqcap \text{nnf}(\neg D)$
- $\text{nnf}(\neg(\exists R.C)) = \forall R.\text{nnf}(\neg C)$
- $\text{nnf}(\neg(\forall R.C)) = \exists R.\text{nnf}(\neg C)$

Tableau Algorithm for Concept Satisfiability

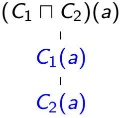
Overview

- Take as input an $\mathcal{ALC}$ concept C in NNF
- Decide the satisfiability of C by trying to construct an interpretation $\mathcal{I}$ such that $C^{\mathcal{I}} \neq \emptyset$
- Represent an interpretation $\mathcal{I}$ by an ABox $\mathcal{A}_{\mathcal{I}}$ such that $a \in A^{\mathcal{I}}$ (resp. $(a, b) \in R^{\mathcal{I}}$) iff $A(a) \in \mathcal{A}_{\mathcal{I}}$ (resp. $R(a, b) \in \mathcal{A}_{\mathcal{I}}$)
- Initialize a set S of ABoxes, containing a single ABox $\{C(a_0)\}$
- At each stage, apply a tableau rule to some $\mathcal{A} \in S$ (see rules next slide)
- A rule application replaces $\mathcal{A}$ by one or two ABoxes that extend $\mathcal{A}$ with new assertions
- Stop applying rules when either:
  1. every $\mathcal{A} \in S$ contains a clash, that is, a pair $\{A(a_i), \neg A(a_i)\}$
  2. some $\mathcal{A} \in S$ is clash-free and complete, meaning that no rule can be applied to $\mathcal{A}$
- Return "yes" if some $\mathcal{A} \in S$ is clash-free, "no" otherwise

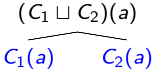
Tableau Algorithm for Concept Satisfiability

Tableau rules

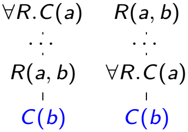
$\sqcap$-rule: if $(C_1 \sqcap C_2)(a) \in \mathcal{A}$
and $\{C_1(a), C_2(a)\} \not\subseteq \mathcal{A}$
replace $\mathcal{A}$ with $\mathcal{A} \cup \{C_1(a), C_2(a)\}$.



$\sqcup$-rule: if $(C_1 \sqcup C_2)(a) \in \mathcal{A}$
and $\{C_1(a), C_2(a)\} \cap \mathcal{A} = \emptyset$
replace $\mathcal{A}$ with $\mathcal{A} \cup \{C_1(a)\}$ and $\mathcal{A} \cup \{C_2(a)\}$.



$\forall$-rule: if $\{\forall R.C(a), R(a,b)\} \subseteq \mathcal{A}$
and $C(b) \notin \mathcal{A}$
replace $\mathcal{A}$ with $\mathcal{A} \cup \{C(b)\}$.



$\exists$-rule: if $\exists R.C(a) \in \mathcal{A}$
and there is no b with $\{R(a,b), C(b)\} \subseteq \mathcal{A}$
create a new individual name c and
replace $\mathcal{A}$ with $\mathcal{A} \cup \{R(a,c), C(c)\}$.

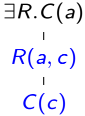


Tableau Algorithm for Concept Satisfiability

Example

$$(A \sqcup B) \sqcap ((\neg B \sqcup D) \sqcap \neg A)(a_0)$$

Tableau Algorithm for Concept Satisfiability

Example

$$(A \sqcup B) \sqcap ((\neg B \sqcup D) \sqcap \neg A)(a_0)$$

$$|$$

$$A \sqcup B(a_0)$$

$$((\neg B \sqcup D) \sqcap \neg A)(a_0)$$

Tableau Algorithm for Concept Satisfiability

Example

$$(A \sqcup B) \sqcap ((\neg B \sqcup D) \sqcap \neg A)(a_0)$$

$$A \sqcup B(a_0)$$

$$((\neg B \sqcup D) \sqcap \neg A)(a_0)$$

$$A(a_0) \qquad B(a_0)$$

Tableau Algorithm for Concept Satisfiability

Example

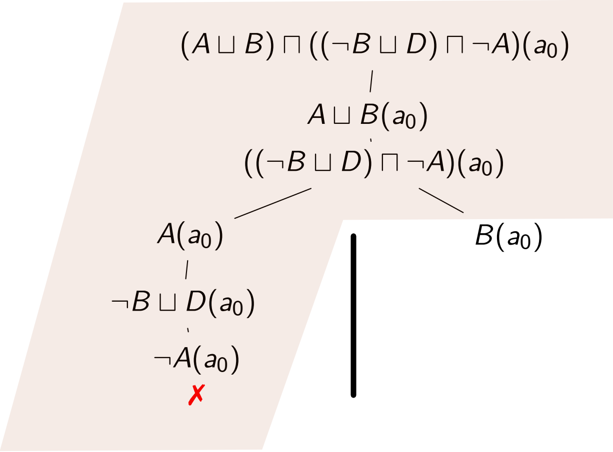


Tableau Algorithm for Concept Satisfiability

Example

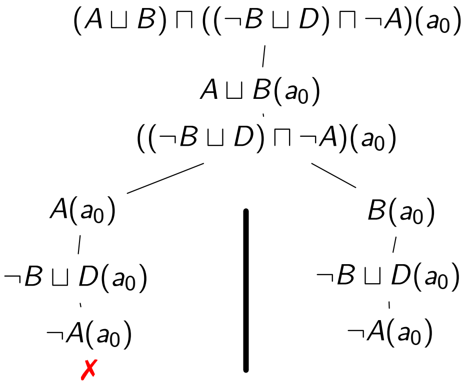


Tableau Algorithm for Concept Satisfiability

Example

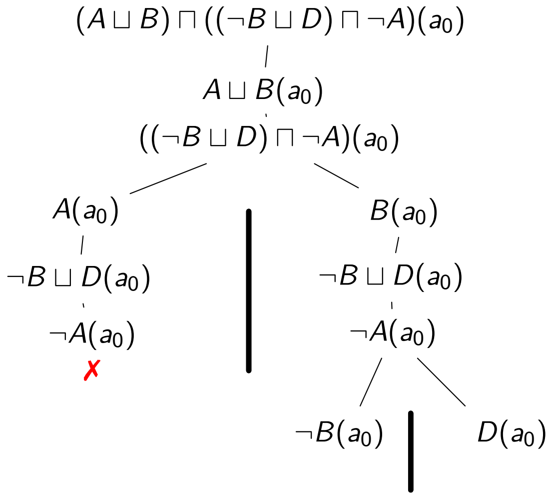


Tableau Algorithm for Concept Satisfiability

Example

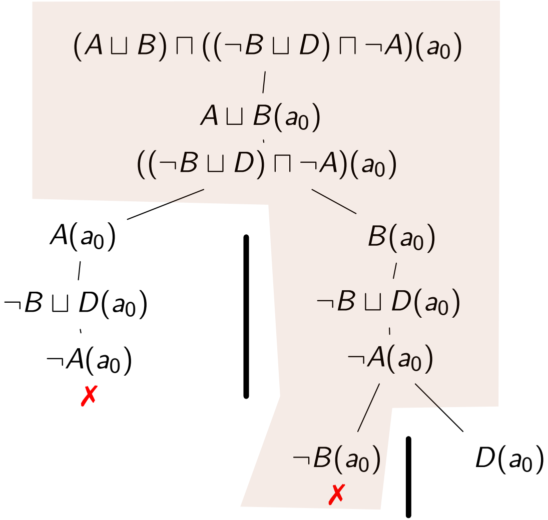


Tableau Algorithm for Concept Satisfiability

Example

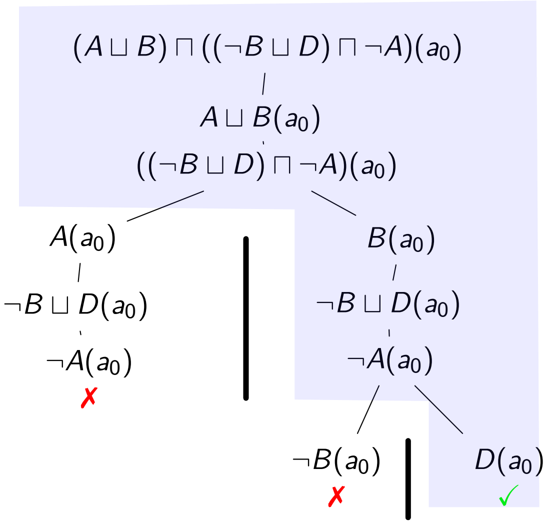


Tableau Algorithm for Concept Satisfiability

Another example

$$(\exists R.A \sqcap \forall R.\neg A)(a_0)$$

Tableau Algorithm for Concept Satisfiability

Another example

$$(\exists R.A \sqcap \forall R.\neg A)(a_0)$$

$$\exists R.A(a_0) \qquad \sqcap\text{-rule}$$

$$\forall R.\neg A(a_0)$$

Tableau Algorithm for Concept Satisfiability

Another example

$$(\exists R.A \sqcap \forall R.\neg A)(a_0)$$

$$\exists R.A(a_0)$$

$$\forall R.\neg A(a_0)$$

$$R(a_0, a_1) \qquad \exists\text{-rule}$$

$$A(a_1)$$

Tableau Algorithm for Concept Satisfiability

Another example

$(\exists R.A \sqcap \forall R.\neg A)(a_0)$

$\exists R.A(a_0)$

$\forall R.\neg A(a_0)$

$R(a_0, a_1)$

$A(a_1)$

$\neg A(a_1)$ ∀-rule

✗

Exercise

Use the tableau algorithm to decide which of the following concepts is satisfiable:

- $\exists R.(A \sqcap B) \sqcap \forall R.(\neg A \sqcup C) \sqcap \forall R.(\neg B \sqcup \neg C)$
- $\exists R.A \sqcap \forall R.(\exists R.A \sqcup \neg A)$

Tableau Algorithm for Concept Satisfiability

Let us call our tableau algorithm `CSat` (for concept satisfiability)

Theorem

CSat terminates and it answers yes if and only if the input concept is satisfiable.

To prove this theorem, we must show:

- termination: `CSat` always terminates
- soundness: if `Csat` outputs "yes" on input C_0, then the concept C_0 is satisfiable
- completeness: if C_0 is satisfiable, then `CSat` outputs "yes" on input C_0

Preliminary Definitions

Subconcepts of a concept

$$
\begin{aligned}
\mathrm{sub}(A) &= \{A\} \\
\mathrm{sub}(\neg C) &= \{\neg C\} \cup \mathrm{sub}(C) \\
\mathrm{sub}(\exists R.C) &= \{\exists R.C\} \cup \mathrm{sub}(C) \\
\mathrm{sub}(\forall R.C) &= \{\forall R.C\} \cup \mathrm{sub}(C) \\
\mathrm{sub}(C_1 \sqcup C_2) &= \{C_1 \sqcup C_2\} \cup \mathrm{sub}(C_1) \cup \mathrm{sub}(C_2) \\
\mathrm{sub}(C_1 \sqcap C_2) &= \{C_1 \sqcap C_2\} \cup \mathrm{sub}(C_1) \cup \mathrm{sub}(C_2)
\end{aligned}
$$

Preliminary Definitions

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\text{sub}(\forall R.C) &= \{\forall R.C\} \cup \text{sub}(C) \\
\text{sub}(C_1 \sqcup C_2) &= \{C_1 \sqcup C_2\} \cup \text{sub}(C_1) \cup \text{sub}(C_2) \\
\text{sub}(C_1 \sqcap C_2) &= \{C_1 \sqcap C_2\} \cup \text{sub}(C_1) \cup \text{sub}(C_2)
\end{aligned}
$$

Example

$$
\begin{aligned}
\text{sub}(\exists R.(A \sqcap \forall S.(B \sqcup \neg C))) = \{ \\
& \exists R.(A \sqcap \forall S.(B \sqcup \neg C)), \quad A \sqcap \forall S.(B \sqcup \neg C), \quad A, \\
& \forall S.(B \sqcup \neg C), \quad B \sqcup \neg C, \quad B, \quad \neg C, \quad C \\
& \}
\end{aligned}
$$

Preliminary Definitions

Role depth of a concept

$$
\begin{aligned}
\text{depth}(A) &= 0\\
\text{depth}(\neg C) &= \text{depth}(C)\\
\text{depth}(\exists R.C) &= \text{depth}(\forall R.C) = \text{depth}(C) + 1\\
\text{depth}(C_1 \sqcup C_2) &= \text{depth}(C_1 \sqcap C_2) = \max(\text{depth}(C_1), \text{depth}(C_2))
\end{aligned}
$$

Preliminary Definitions

Role depth of a concept

$$
\begin{aligned}
\text{depth}(A) &= 0 \\
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\end{aligned}
$$

Example

$$
\text{depth}(\exists R.(A \sqcap \forall S.(B \sqcup C))) = 2
$$

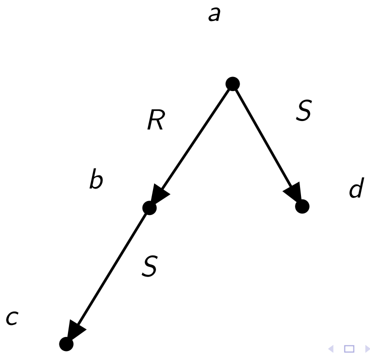
Preliminary Definitions

Tree-shaped ABox

Graph representation of an ABox $\mathcal{A}$: graph whose vertices are individual names of $\mathcal{A}$ and such that there is a (directed) edge from a to b labelled by R iff $R(a, b) \in \mathcal{A}$.
If this graph is a tree, $\mathcal{A}$ is tree-shaped.

Example

$\{R(a, b), S(b, c), S(a, d)\}$ is tree-shaped



Termination of CSat (Informal Proof)

Suppose we run CSat starting from $S = \{\{C(a_0)\}\}$. Let us make the following observations for every ABox $\mathcal{A}$ generated by CSat:

1. $\mathcal{A}$ is tree-shaped
2. The depth of the tree is bounded by the role depth of C: each individual in $\mathcal{A}$ is at distance $k \leq \text{depth}(C)$ from a_0
   - if $D(b) \in \mathcal{A}$ and the unique path from a_0 to b has length k, then $\text{depth}(D) \leq \text{depth}(C) - k$
3. The degree of the tree is bounded by the number of existentials in C
4. The number of concept assertions per individual is bounded by the number of subconcepts $|\text{sub}(C)|$
   - if $D(b) \in \mathcal{A}$, then $D \in \text{sub}(C)$

Hence there is a bound on the size of generated ABoxes. Since CSat only adds assertions to ABoxes, every generated ABox will eventually be complete or contain a clash. Hence CSat terminates.

Soundness of CSat

Assume that CSat returns “yes” on input C.

- Then S must contain a complete and clash-free ABox $\mathcal{A}$.
- Define an interpretation $\mathcal{I}$ as follows:
  - $\Delta^{\mathcal{I}} = \{a \mid a \text{ is an individual in } \mathcal{A}\}$
  - $A^{\mathcal{I}} = \{a \mid A(a) \in \mathcal{A}\}$
  - $R^{\mathcal{I}} = \{(a, b) \mid R(a, b) \in \mathcal{A}\}$
- Claim: $\mathcal{I}$ is such that $C^{\mathcal{I}} \neq \emptyset$

  To show the claim, we prove by induction on the size of concepts that:

  $$D(b) \in \mathcal{A} \Rightarrow b \in D^{\mathcal{I}}$$

  Since the completion algorithm never deletes assertions, $C(a_0) \in \mathcal{A}$ for every $\mathcal{A} \in S$ and the claim follows.

It follows from the claim that C is satisfiable.

Soundness of CSat

Proof of the claim: $D(b) \in \mathcal{A} \Rightarrow b \in D^{\mathcal{I}}$

Base Case: $D = A$ or $D = \neg A$

- If $D = A$, then $b \in A^{\mathcal{I}}$ by definition of $\mathcal{I}$
- If $D = \neg A$, then $A(b) \notin \mathcal{A}$ because $\mathcal{A}$ is clash-free, hence $b \notin A^{\mathcal{I}}$, i.e., $b \in \neg A^{\mathcal{I}}$

Soundness of CSat

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Induction Hypothesis: statement holds whenever $|D| \leq k$

Induction Step: show statement holds for D with $|D| = k + 1$

Soundness of CSat

Proof of the claim: $D(b) \in \mathcal{A} \Rightarrow b \in D^{\mathcal{I}}$

Base Case: $D = A$ or $D = \neg A$

- If $D = A$, then $b \in A^{\mathcal{I}}$ by definition of $\mathcal{I}$
- If $D = \neg A$, then $A(b) \notin \mathcal{A}$ because $\mathcal{A}$ is clash-free, hence $b \notin A^{\mathcal{I}}$, i.e., $b \in \neg A^{\mathcal{I}}$

Induction Hypothesis: statement holds whenever $|D| \leq k$

Induction Step: show statement holds for D with $|D| = k + 1$

- $D = E \sqcap F$: since $\mathcal{A}$ is complete, $\mathcal{A}$ contains $E(b)$ and $F(b)$. By the induction hypothesis, $b \in E^{\mathcal{I}}$ and $b \in F^{\mathcal{I}}$, so $b \in (E \sqcap F)^{\mathcal{I}}$

Soundness of CSat

Proof of the claim: $D(b) \in \mathcal{A} \Rightarrow b \in D^{\mathcal{I}}$

Base Case: $D = A$ or $D = \neg A$

- If $D = A$, then $b \in A^{\mathcal{I}}$ by definition of $\mathcal{I}$
- If $D = \neg A$, then $A(b) \notin \mathcal{A}$ because $\mathcal{A}$ is clash-free, hence $b \notin A^{\mathcal{I}}$, i.e., $b \in \neg A^{\mathcal{I}}$

Induction Hypothesis: statement holds whenever $|D| \leq k$

Induction Step: show statement holds for D with $|D| = k + 1$

- $D = E \sqcap F$: since $\mathcal{A}$ is complete, $\mathcal{A}$ contains $E(b)$ and $F(b)$. By the induction hypothesis, $b \in E^{\mathcal{I}}$ and $b \in F^{\mathcal{I}}$, so $b \in (E \sqcap F)^{\mathcal{I}}$
- $D = \exists R.E$: since $\mathcal{A}$ is complete, there is some c such that $R(b, c) \in \mathcal{A}$ and $E(c) \in \mathcal{A}$. Then $(b, c) \in R^{\mathcal{I}}$, and by the induction hypothesis, we get that $c \in E^{\mathcal{I}}$, so $b \in (\exists R.E)^{\mathcal{I}}$

Soundness of CSat

Proof of the claim: $D(b) \in \mathcal{A} \Rightarrow b \in D^{\mathcal{I}}$

Base Case: $D = A$ or $D = \neg A$

- If $D = A$, then $b \in A^{\mathcal{I}}$ by definition of $\mathcal{I}$
- If $D = \neg A$, then $A(b) \notin \mathcal{A}$ because $\mathcal{A}$ is clash-free, hence $b \notin A^{\mathcal{I}}$, i.e., $b \in \neg A^{\mathcal{I}}$

Induction Hypothesis: statement holds whenever $|D| \leq k$

Induction Step: show statement holds for D with $|D| = k + 1$

- $D = E \sqcap F$: since $\mathcal{A}$ is complete, $\mathcal{A}$ contains $E(b)$ and $F(b)$. By the induction hypothesis, $b \in E^{\mathcal{I}}$ and $b \in F^{\mathcal{I}}$, so $b \in (E \sqcap F)^{\mathcal{I}}$
- $D = \exists R.E$: since $\mathcal{A}$ is complete, there is some c such that $R(b, c) \in \mathcal{A}$ and $E(c) \in \mathcal{A}$. Then $(b, c) \in R^{\mathcal{I}}$, and by the induction hypothesis, we get that $c \in E^{\mathcal{I}}$, so $b \in (\exists R.E)^{\mathcal{I}}$
- $D = E \sqcup F$: left as practice
- $D = \forall R.E$: left as practice

Completeness of `CSat`

Suppose that C is satisfiable.

- This implies that the ABox $\{C(a_0)\}$ is satisfiable.
- Claim: Tableau rules are satisfiability-preserving:
  - if an ABox $\mathcal{A}$ is satisfiable and $\mathcal{A}'$ is the result of applying a rule to $\mathcal{A}$, then $\mathcal{A}'$ is also satisfiable
  - if an ABox $\mathcal{A}$ is satisfiable and $\mathcal{A}_1$ and $\mathcal{A}_2$ are obtained when applying a rule to $\mathcal{A}$, then either $\mathcal{A}_1$ or $\mathcal{A}_2$ is satisfiable
- Since ABoxes containing a clash are not satisfiable and we start with the satisfiable ABox $\{C(a_0)\}$, `CSat` will eventually generate a complete satisfiable (thus clash-free) ABox.

Hence `CSat` returns “yes” on input C.

Completeness of CSat

Proof of the claim: Tableau rules are satisfiability-preserving

Let $\mathcal{A}$ be a satisfiable ABox and $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be a model of $\mathcal{A}$

- If $\mathcal{A}'$ is the result of applying the $\sqcap$-rule to $\mathcal{A}$, there is $(C_1 \sqcap C_2)(b) \in \mathcal{A}$ and $\mathcal{A}' = \mathcal{A} \cup \{C_1(b), C_2(b)\}$
  - since $b^{\mathcal{I}} \in (C_1 \sqcap C_2)^{\mathcal{I}}$, then $b^{\mathcal{I}} \in C_1^{\mathcal{I}}$ and $b^{\mathcal{I}} \in C_2^{\mathcal{I}}$
  - it follows that $\mathcal{I}$ is a model of $\mathcal{A}'$, thus $\mathcal{A}'$ is satisfiable
- If $\mathcal{A}_1$ and $\mathcal{A}_2$ are the result of applying the $\sqcup$-rule to $\mathcal{A}$, there is $(C_1 \sqcup C_2)(b) \in \mathcal{A}$, $\mathcal{A}_1 = \mathcal{A} \cup \{C_1(b)\}$, and $\mathcal{A}_2 = \mathcal{A} \cup \{C_2(b)\}$
  - since $b^{\mathcal{I}} \in (C_1 \sqcup C_2)^{\mathcal{I}}$, then $b^{\mathcal{I}} \in C_1^{\mathcal{I}}$ or $b^{\mathcal{I}} \in C_2^{\mathcal{I}}$
  - it follows that $\mathcal{I}$ is a model of $\mathcal{A}_1$ or of $\mathcal{A}_2$, thus $\mathcal{A}_1$ or $\mathcal{A}_2$ is satisfiable
- $\forall$-rule: left as practice
- $\exists$-rule: left as practice

Tree Model Property

`CSat` produces tree-shaped ABoxes, so we get that for every $\mathcal{ALC}$ concept C, if C has a model, then it has a tree-shaped model

This is an important property

- We only need to look at tree-shaped structures when reasoning about $\mathcal{ALC}$ concepts
- Trees are computationally "friendly"
- This property exposes a limitation in the expressive power of $\mathcal{ALC}$ (for example they cannot describe structures with cycles)

Extension to KB Satisfiability

We want to modify `CSat` to check the satisfiability of a knowledge base $\langle \mathcal{T}, \mathcal{A} \rangle$

Adding the ABox is easy:

- start from $S = \{\mathcal{A}\}$ instead of $S = \{\{C(a)\}\}$

Extension to KB Satisfiability

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For the TBox, note that $C \sqsubseteq D \equiv \top \sqsubseteq \neg C \sqcup D$ and add the following rule to the tableau rules:

TBox-rule:
if $C \sqsubseteq D \in \mathcal{T}$,
a is an individual of $\mathcal{A}$
and $(\text{nnf}(\neg C \sqcup D))(a) \notin \mathcal{A}$
replace $\mathcal{A}$ with $\mathcal{A} \cup \{(\text{nnf}(\neg C \sqcup D))(a)\}$.

$$X(a)$$
$$|$$
$$(\text{nnf}(\neg C \sqcup D))(a)$$

Exercise

Use the tableau algorithm to check whether the following KBs are satisfiable:

- $\langle \mathcal{T}, \{A(a)\} \rangle$
- $\langle \mathcal{T}, \{R(c,a), B(a)\} \rangle$

where

$$\mathcal{T} = \{A \sqsubseteq \exists R.B,\ B \sqsubseteq D,\ \exists R.D \sqsubseteq \neg A\}$$

Exercise

Now try on the following KB: $\langle \{A \sqsubseteq \exists R.A\}, \{A(a_0)\} \rangle$

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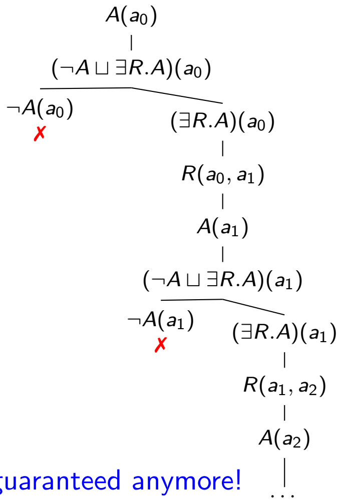

Termination is not guaranteed anymore!

Making the Algorithm Terminate

Basic idea: if two individuals "look the same", explore only one

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Blocking

An individual a blocks an individual b in an ABox $\mathcal{A}$ if:

- $\{C \mid C(b) \in \mathcal{A}\} \subseteq \{C \mid C(a) \in \mathcal{A}\}$
- a was in $\mathcal{A}$ when b has been introduced

An individual b is blocked if some a blocks b

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An individual b is blocked if some a blocks b

The blocked individual b can use the role successors of a instead of generating new ones

Modify the tableau rules to apply them only to individuals that are not blocked

Tableau Algorithm for KB Satisfiability

Tableau rules

$\sqcap$-rule: if $(C_1 \sqcap C_2)(a) \in \mathcal{A}$, a is not blocked, and $\{C_1(a), C_2(a)\} \not\subseteq \mathcal{A}$, replace $\mathcal{A}$ with $\mathcal{A} \cup \{C_1(a), C_2(a)\}$.

$\sqcup$-rule: if $(C_1 \sqcup C_2)(a) \in \mathcal{A}$, a is not blocked, and $\{C_1(a), C_2(a)\} \cap \mathcal{A} = \emptyset$ replace $\mathcal{A}$ with $\mathcal{A} \cup \{C_1(a)\}$ and $\mathcal{A} \cup \{C_2(a)\}$.

$\forall$-rule: if $\{\forall R.C(a), R(a,b)\} \subseteq \mathcal{A}$, a is not blocked, and $C(b) \notin \mathcal{A}$, replace $\mathcal{A}$ with $\mathcal{A} \cup \{C(b)\}$.

$\exists$-rule: if $\exists R.C(a) \in \mathcal{A}$, a is not blocked, and there is no b with $\{R(a,b), C(b)\} \subseteq \mathcal{A}$, create a new individual name c and replace $\mathcal{A}$ with $\mathcal{A} \cup \{R(a,c), C(c)\}$.

TBox-rule: if $C \sqsubseteq D \in \mathcal{T}$, a is not blocked, and $(\text{nnf}(\neg C \sqcup D))(a) \notin \mathcal{A}$, replace $\mathcal{A}$ by $\mathcal{A} \cup \{(\text{nnf}(\neg C \sqcup D)(a))\}$.

Tableau Algorithm for KB Satisfiability

Example

Apply blocking to the previous KB: $\langle\{A \sqsubseteq \exists R.A\}, \{A(a_0)\}\rangle$

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$$A(a_0)$$

$$|$$

$$(\neg A \sqcup \exists R.A)(a_0)$$

Tableau Algorithm for KB Satisfiability

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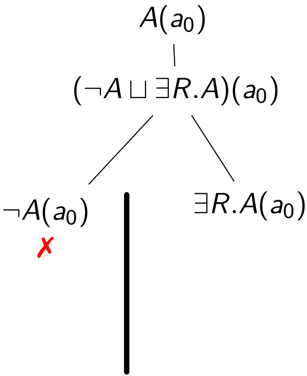

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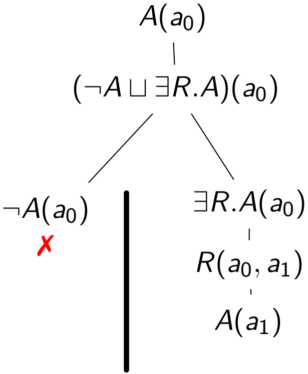

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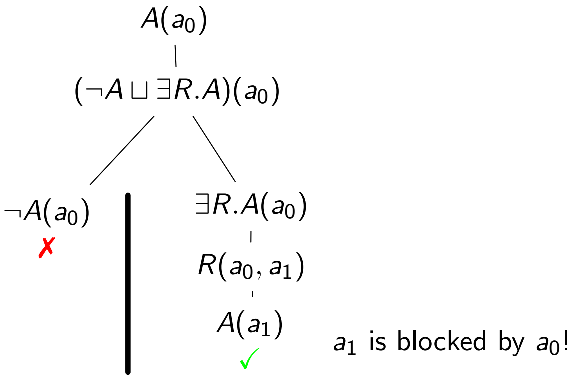

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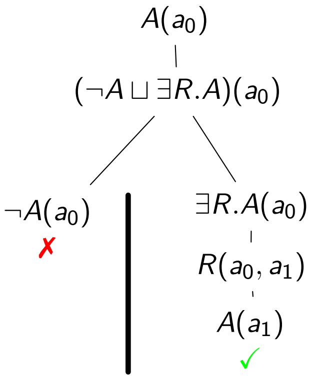

a_1 is blocked by a_0!

We obtain a complete, clash-free ABox

$\rightarrow \langle \{A \sqsubseteq \exists R.A\}, \{A(a_0)\} \rangle$ is satisfiable

Tableau Algorithm for KB Satisfiability

Another example

Consider

$$\mathcal{T} = \{A \sqsubseteq \exists R.A,\ A \sqsubseteq B,\ \exists R.B \sqsubseteq D\}$$

We want to test whether $\mathcal{T} \models A \sqsubseteq D$ using the tableau algorithm

$\rightarrow$ check whether the following KB is satisfiable

$$\langle \mathcal{T}, \{(A \sqcap \neg D)(a_0)\}\rangle$$

Tableau Algorithm for KB Satisfiability

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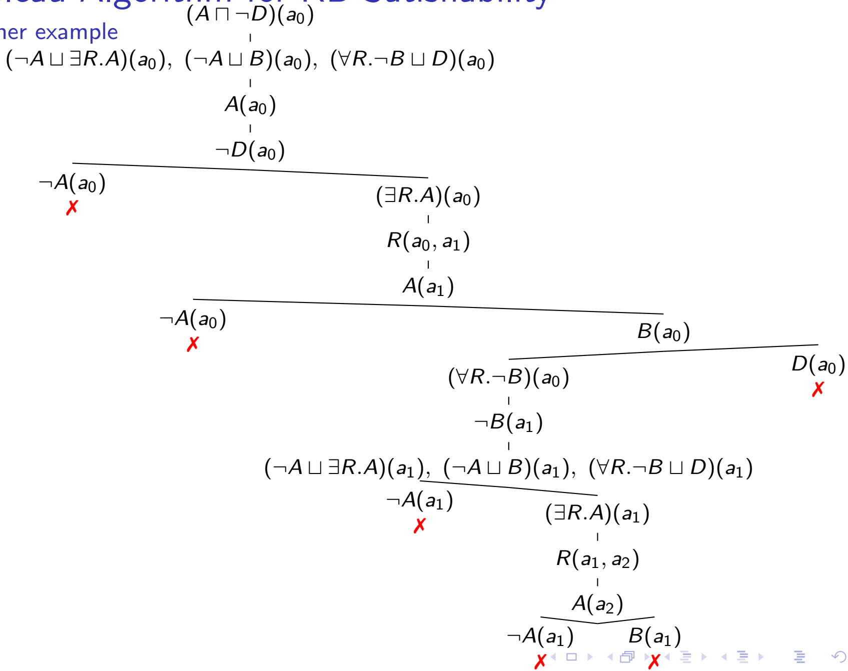

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$\rightarrow$ check whether the following KB is satisfiable

$$\langle \mathcal{T}, \{(A \sqcap \neg D)(a_0)\} \rangle$$

$\langle \mathcal{T}, \{(A \sqcap \neg D)(a_0)\} \rangle$ is unsatisfiable so $\mathcal{T} \models A \sqsubseteq D$

Remark: an individual can be blocked then later become unblocked

Tableau Algorithm for KB Satisfiability

Let us call our tableau algorithm `KBSat` (for KB satisfiability)

Theorem

`KBSat` terminates and it answers yes if and only if the input KB is satisfiable.

Termination of KBSat (Informal Proof)

KBSat terminates on every input $\langle \mathcal{T}, \mathcal{A} \rangle$.

Similar to the proof of termination for CSat: Show that there is a bound on the size of the generated ABoxes

For every ABox $\mathcal{A}'$ generated by KBSat:

1. The number of concept assertions per individual is bounded by the total number of subconcepts of concepts that occur in $\mathcal{A}$ or in $\{\mathsf{nnf}(\neg C \sqcup D) \mid C \sqsubseteq D \in \mathcal{T}\}$
2. The individuals generated by the $\exists$-rule form trees whose roots are individuals from $\mathcal{A}$
3. Blocking ensures that the depth of each tree is finite (bounded by the number of sets of subconcepts of concepts that occur in $\mathcal{A}$ or in $\{\mathsf{nnf}(\neg C \sqcup D) \mid C \sqsubseteq D \in \mathcal{T}\}$)
4. The degree of each tree is bounded by the number of existentials in $\mathcal{T}$

Soundness of `KBSat`

If `KBSat` returns "yes" on input $\langle\mathcal{T},\mathcal{A}\rangle$, then $\langle\mathcal{T},\mathcal{A}\rangle$ is satisfiable.

- Build a model $\mathcal{I}$ from a complete and clash-free ABox $\mathcal{A}'$
- Difference with `CSat`: deal with the blocked individuals
  - $\Delta^{\mathcal{I}} = \{a \mid a$ is an individual in $\mathcal{A}'$ which is not blocked$\}$
  - $A^{\mathcal{I}} = \{a \mid A(a) \in \mathcal{A}', a$ not blocked$\}$
  - $R^{\mathcal{I}} = \{(a,b) \mid R(a,b) \in \mathcal{A}', a, b$ not blocked$\} \cup \{(a,b) \mid R(a,c) \in \mathcal{A}', a$ not blocked$, c$ blocked by b, b not blocked$\}$
- Claim: $\mathcal{I}$ is a model of $\langle\mathcal{T},\mathcal{A}\rangle$

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- Claim: $\mathcal{I}$ is a model of $\langle\mathcal{T},\mathcal{A}\rangle$
  - Since individuals from $\mathcal{A}$ are never blocked, $\mathcal{I} \models \mathcal{A}$
  - Let $C \sqsubseteq D \in \mathcal{T}$ and $b \in C^{\mathcal{I}}$
    - since b is not blocked in $\mathcal{A}'$ and $\mathcal{A}'$ is complete, $\mathrm{nnf}(\neg C \sqcup D)(b) \in \mathcal{A}'$ (TBox-rule) so $\mathrm{nnf}(\neg C)(b)$ or $\mathrm{nnf}(D)(b)$ is in $\mathcal{A}'$ ($\sqcup$-rule)
    - we prove that $E(b) \in \mathcal{A}'$ and b not blocked $\Rightarrow b \in E^{\mathcal{I}}$ for every concept E by induction on the size of E
    - since $b \in C^{\mathcal{I}}$ (so that $b \notin \mathrm{nnf}(\neg C)^{\mathcal{I}}$), it follows that $\mathrm{nnf}(\neg C)(b) \notin \mathcal{A}'$
    - thus $\mathrm{nnf}(D)(b)$ is in $\mathcal{A}'$ and $b \in \mathrm{nnf}(D)^{\mathcal{I}} = D^{\mathcal{I}}$

    It follows that $\mathcal{I} \models C \sqsubseteq D$
  - Hence $\mathcal{I} \models \mathcal{T}$

Soundness of KBSat

Proof of the claim: $E(b) \in \mathcal{A}'$ and b not blocked $\Rightarrow b \in E^{\mathcal{I}}$

Base Case: $E = A$ or $E = \neg A$

- If $E = A$, then $b \in A^{\mathcal{I}}$, by definition of $\mathcal{I}$
- If $E = \neg A$, then $A(b) \notin \mathcal{A}'$ because $\mathcal{A}'$ is clash-free, hence $b \in \neg A^{\mathcal{I}}$

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Induction Hypothesis: statement holds whenever $|E| \leq k$

Induction Step: show statement holds for $|E| = k + 1$

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- $E = \exists R.F$: since $\mathcal{A}'$ is complete and b is not blocked, there is some c such that $R(b, c) \in \mathcal{A}'$ and $F(c) \in \mathcal{A}'$
  - if c is not blocked, $(b, c) \in R^{\mathcal{I}}$, and by the induction hypothesis, $c \in F^{\mathcal{I}}$, so $b \in (\exists R.F)^{\mathcal{I}}$
  - if c is blocked, it must be blocked by some d which is not blocked, so $(b, d) \in R^{\mathcal{I}}$, and $F(d) \in \mathcal{A}'$ so by the induction hypothesis, $d \in F^{\mathcal{I}}$, so $b \in (\exists R.F)^{\mathcal{I}}$

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- $E = \forall R.F$: left as practice
- $E = F \sqcap G$: left as practice
- $E = F \sqcup G$: left as practice

Completeness of KBSat

If $\langle \mathcal{T}, \mathcal{A} \rangle$ is satisfiable, then KBSat returns "yes" on input $\langle \mathcal{T}, \mathcal{A} \rangle$.

Similar to the proof of completeness of CSat: Show that tableau rules are satisfiability-preserving

Let $\langle \mathcal{T}, \mathcal{A} \rangle$ be a satisfiable KB and $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be a model of $\langle \mathcal{T}, \mathcal{A} \rangle$

- For the new TBox-rule: If $\mathcal{A}'$ is the result of applying the TBox-rule to $\mathcal{A}$, there is $C \sqsubseteq D \in \mathcal{T}$ and $\mathcal{A}' = \mathcal{A} \cup \{(\mathsf{nnf}(\neg C \sqcup D)(a))\}$
  - if $a^{\mathcal{I}} \notin (\neg C)^{\mathcal{I}}$, i.e., $a^{\mathcal{I}} \in C^{\mathcal{I}}$, since $\mathcal{I} \models \mathcal{T}$, then $a^{\mathcal{I}} \in D^{\mathcal{I}}$
  - hence $a^{\mathcal{I}} \in (\neg C)^{\mathcal{I}} \cup D^{\mathcal{I}}$, i.e., $a^{\mathcal{I}} \in (\neg C \sqcup D)^{\mathcal{I}} = \mathsf{nnf}(\neg C \sqcup D)^{\mathcal{I}}$
  - it follows that $\mathcal{I} \models \langle \mathcal{T}, \mathcal{A}' \rangle$, thus $\langle \mathcal{T}, \mathcal{A}' \rangle$ is satisfiable
- Adding the condition that a is not blocked only restricts the rules applicability

Forest Model Property

- An interpretation $\mathcal{I}$ is forest-shaped if the graph whose vertices are the domain elements and edges are

$$\{(d, d') \mid (d, d') \in R^{\mathcal{I}} \text{ for some } R \text{ and } d, d' \notin \{a^{\mathcal{I}} \mid a \text{ individual name}\}\}$$

  is a set of (disconnected) trees
- The model built in the proof of tableau algorithm soundness need not be forest-shaped because of the way it handles blocked individuals
- It can be shown that every satisfiable $\mathcal{ALC}$ KB has a forest-shaped model
- Unlike the case of $\mathcal{ALC}$ concepts, trees may be infinite

Tableau Algorithm for Expressive DLs

Tableau algorithm can be modified to handle extensions of $\mathcal{ALC}$ (with number restrictions, role inclusions, transitive roles...)

- additional tableau rule for each constructor
- new types of clashes
- different blocking conditions

Complexity Issues

- CSat decides whether an $\mathcal{ALC}$ concept is satisfiable
- KBSat decides whether an $\mathcal{ALC}$ KB is satisfiable
  - also concept satisfiability w.r.t. a TBox, subsumption and instance checking via polynomial reduction

Two questions for each case:

- What is the complexity of the algorithm?
  - what amount of ressources (time, memory) is required to run the algorithm, expressed as a function of the input size, in the worst possible case?
- What is the complexity of the decision problem solved?
  - what is the complexity of the best algorithms that solve the problem?

Complexity of CSat

CSat needs exponential time and space:

- Due to the $\sqcup$-rule, exponentially many complete ABoxes may be generated
  - consider $C = \bigsqcap_{i=1}^{n}(A_i \sqcup B_i)$

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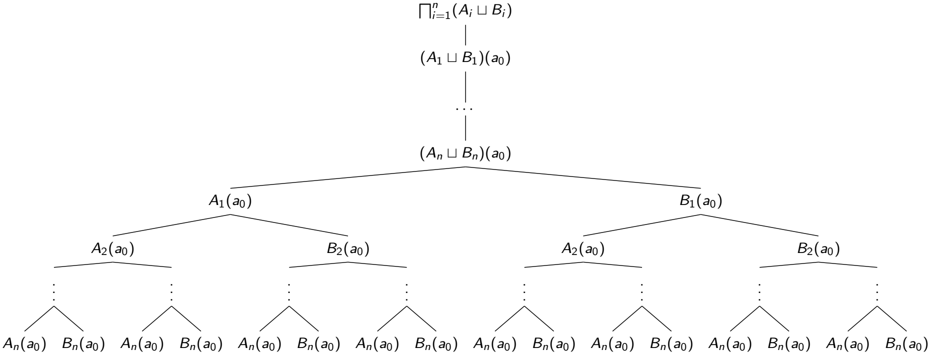
  - $|C|$ is linear w.r.t. n and $\texttt{CSat}(C)$ generates 2^n complete ABoxes



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  - consider $C = \bigsqcap_{i=0}^{n} \underbrace{\forall R. \dots \forall R}_{i \text{ times}}(\exists R.B \sqcap \exists R.\neg B)$

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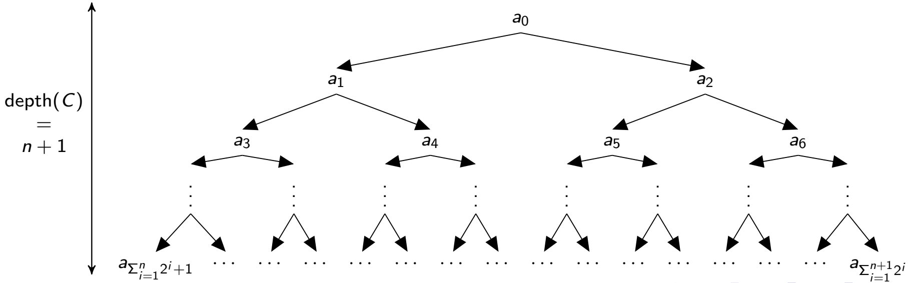
  - consider $C = \bigsqcap_{i=0}^{n} \underbrace{\forall R. \ldots \forall R}_{i \text{ times}}(\exists R.B \sqcap \exists R.\neg B)$
  - $|C|$ is polynomial w.r.t. n and CSat(C) generates a complete ABox with $2^{n+2} - 1$ individuals



Complexity of CSat

CSat can be modified so that it runs in polynomial space

- Keep only one ABox in memory at a time:
  - when applying the $\sqcup$-rule, first examine $\mathcal{A}_1$, then afterwards examine $\mathcal{A}_2$
  - keep in memory that the second disjunct needs to be checked

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  - keep in memory that the second disjunct needs to be checked
- Keep at most $\mathsf{depth}(C) + 1$ individuals in memory:
  - explore the children of an individual one at a time, in a depth-first manner
  - possible because no interaction between individuals in different branches
  - store which $\exists R.C$ have been explored and which are left to do

Complexity of $\mathcal{ALC}$ Concept Satisfiability (No TBox)

- `CSat` runs in polynomial space so the problem of deciding whether an $\mathcal{ALC}$ concept is satisfiable is in PSPACE
- Any hope for better algorithms?

  PTIME $\subseteq$ NP $\subseteq$ PSPACE $\subseteq$ EXPTIME $\subseteq$ NEXPTIME $\subseteq$ EXPSPACE
  - inclusions are believed to be strict
- It can be shown that deciding whether an $\mathcal{ALC}$ concept is satisfiable is PSPACE-hard
  - reduction from a PSPACE-complete problem (for instance deciding whether a quantified Boolean formula is valid)

Theorem

Checking the satisfiability of an $\mathcal{ALC}$ concept in the absence of a TBox is PSPACE-complete.

Complexity of KBSat

We cannot make KBSat run in polynomial space as we did for CSat because we may need to generate exponentially many individuals on a single "branch"

- consider $\mathcal{A} = \{F_1(a_0), \ldots, F_n(a_0)\}$ and

$$\begin{aligned}
\mathcal{T} = & \{\bigsqcup_{i=1}^{n} F_i \sqsubseteq \exists R.\top\} \cup \{F_i \sqsubseteq \neg T_i \mid 1 \leq i \leq n\} \\
& \cup \{T_1 \sqcap \cdots \sqcap T_{k-1} \sqcap F_k \sqsubseteq \\
& \quad \forall R.(F_1 \sqcap \cdots \sqcap F_{k-1} \sqcap T_k) \sqcap \\
& \bigsqcap_{k+1 \leq \ell \leq n} ((T_\ell \sqcap \forall R.T_\ell) \sqcup (F_\ell \sqcap \forall R.F_\ell)) \mid 1 \leq k \leq n\}
\end{aligned}$$

Complexity of $\mathcal{ALC}$ KB Satisfiability

- What is the complexity of KB satisfiability?

Theorem

Checking the satisfiability of an $\mathcal{ALC}$ KB is ExpTime-complete.

- we will next show membership
- hardness can be shown by reduction from an ExpTime-complete problem (for instance the problem of deciding the existence of a winning strategy for infinite Boolean games)

Complexity of $\mathcal{ALC}$ KB Satisfiability

ExpTime membership

- Show that concept satisfiability w.r.t. a TBox is in ExpTime
  - to decide whether a KB $\langle \mathcal{T}, \mathcal{A} \rangle$ is satisfiable, let $\mathcal{T}' = \mathcal{T} \cup \{C_a \sqsubseteq A \mid A(a) \in \mathcal{A}\} \cup \{C_a \sqsubseteq \exists R.C_b \mid R(a,b) \in \mathcal{A}\}$ and decide whether $\sqcap_{a \text{ individual of } \mathcal{A}} \exists S.C_a$ is satisfiable w.r.t. $\mathcal{T}'$ where S and all C_a are fresh role and concept names

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- We consider an atomic concept A_0
  - C is satisfiable w.r.t. $\mathcal{T}$ iff A_0 is satisfiable w.r.t. $\mathcal{T} \cup \{A_0 \sqsubseteq C\}$
- We assume that $\mathcal{T}$ contains a single axiom of the form $\top \sqsubseteq C_\mathcal{T}$ with $C_\mathcal{T}$ an $\mathcal{ALC}$ concept in NNF
  - A_0 is satisfiable w.r.t. $\mathcal{T}$ iff A_0 is satisfiable w.r.t. $\{\top \sqsubseteq \bigsqcap_{C \sqsubseteq D \in \mathcal{T}} \mathsf{nnf}(\neg C \sqcup D)\}$
- We assume that $A_0 \in \mathsf{sub}(C_\mathcal{T})$
  - otherwise A_0 is satisfiable w.r.t. $\{\top \sqsubseteq C_\mathcal{T}\}$ iff $C_\mathcal{T}$ is satisfiable

Complexity of $\mathcal{ALC}$ KB Satisfiability

Type elimination algorithm

We use a type elimination algorithm to decide whether A_0 is satisfiable w.r.t. $\{\top \sqsubseteq C_\mathcal{T}\}$

- A $\mathcal{T}$-type is a set of concepts $\tau \subseteq \text{sub}(C_\mathcal{T})$ such that
  - $C \in \tau$ implies $\text{nnf}(\neg C) \notin \tau$ for all $C \in \text{sub}(C_\mathcal{T})$
  - $C \sqcap D \in \tau$ implies $C \in \tau$ and $D \in \tau$
  - $C \sqcup D \in \tau$ implies $C \in \tau$ or $D \in \tau$
  - $C_\mathcal{T} \in \tau$
- There are at most $2^{|\text{sub}(C_\mathcal{T})|}$ types

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We use a type elimination algorithm to decide whether A_0 is satisfiable w.r.t. $\{\top \sqsubseteq C_\mathcal{T}\}$

- A $\mathcal{T}$-type is a set of concepts $\tau \subseteq \text{sub}(C_\mathcal{T})$ such that
  - $C \in \tau$ implies $\text{nnf}(\neg C) \notin \tau$ for all $C \in \text{sub}(C_\mathcal{T})$
  - $C \sqcap D \in \tau$ implies $C \in \tau$ and $D \in \tau$
  - $C \sqcup D \in \tau$ implies $C \in \tau$ or $D \in \tau$
  - $C_\mathcal{T} \in \tau$
- There are at most $2^{|\text{sub}(C_\mathcal{T})|}$ types
- The algorithm starts with the set of all types and iteratively removes the bad types that contain some existential restriction that cannot be satisfied in models of $\mathcal{T}$
  - Given a set of types T, τ is bad in T if there exists $\exists R.C \in \tau$ such that the set $\{C\} \cup \{D \mid \forall R.D \in \tau\}$ is not a subset of any type in T
- If at the end of the algorithm there remains some type that contains A_0, return "satisfiable", otherwise return "not satisfiable"

Complexity of $\mathcal{ALC}$ KB Satisfiability

Type elimination algorithm: Complexity

The type elimination algorithm runs in exponential time w.r.t. the size of $C_{\mathcal{T}}$

- At most $2^{|\mathsf{sub}(C_{\mathcal{T}})|}$ iterations and $|\mathsf{sub}(C_{\mathcal{T}})|$ is linear in the size of $C_{\mathcal{T}}$
- Each step takes polynomial time in the number of remaining types, thus is in $O(2^{|\mathsf{sub}(C_{\mathcal{T}})|})$
- Hence the algorithm runs in $O(2^{2*|\mathsf{sub}(C_{\mathcal{T}})|})$

Complexity of $\mathcal{ALC}$ KB Satisfiability

Type elimination algorithm: Soundness

The type elimination algorithm is sound

- Assume that the algorithm returns "satisfiable"
- At the end of the algorithm, T is a set of types such that every $\tau \in T$ is good in T and there exists $\tau_0 \in T$ such that $A_0 \in \tau_0$
- Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ with
  - $\Delta^{\mathcal{I}} = T$
  - $A^{\mathcal{I}} = \{\tau \mid A \in \tau\}$
  - $R^{\mathcal{I}} = \{(\tau_1, \tau_2) \mid \exists R.C \in \tau_1, \{C\} \cup \{D \mid \forall R.D \in \tau_1\} \subseteq \tau_2\}$
- Since $A_0 \in \tau_0$, $\tau_0 \in A_0^{\mathcal{I}}$ and $A_0^{\mathcal{I}} \neq \emptyset$
- Claim: $\mathcal{I} \models \top \sqsubseteq C_{\mathcal{T}}$
- Hence A_0 is satisfiable w.r.t. $\{\top \sqsubseteq C_{\mathcal{T}}\}$

Complexity of $\mathcal{ALC}$ KB Satisfiability

Type elimination algorithm: Soundness – Proof of the claim

$\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ with $\Delta^{\mathcal{I}} = T$, $A^{\mathcal{I}} = \{\tau \mid A \in \tau\}$ and
$R^{\mathcal{I}} = \{(\tau_1, \tau_2) \mid \exists R.C \in \tau_1, \{C\} \cup \{D \mid \forall R.D \in \tau_1\} \subseteq \tau_2\}$

- Show by induction that for every concept E, for every $\tau \in T$ such that $E \in \tau$, $\tau \in E^{\mathcal{I}}$
  - Base case: $E = A$ or $E = \neg A$.
    - if $E = A$, $A \in \tau$ implies $\tau \in A^{\mathcal{I}}$ by definition of $\mathcal{I}$
    - if $E = \neg A$, $\neg A \in \tau$ implies that $A \notin \tau$ because τ is a type, so $\tau \notin A^{\mathcal{I}}$

Complexity of $\mathcal{ALC}$ KB Satisfiability

Type elimination algorithm: Soundness – Proof of the claim

$\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ with $\Delta^{\mathcal{I}} = T$, $A^{\mathcal{I}} = \{\tau \mid A \in \tau\}$ and
$R^{\mathcal{I}} = \{(\tau_1, \tau_2) \mid \exists R.C \in \tau_1, \{C\} \cup \{D \mid \forall R.D \in \tau_1\} \subseteq \tau_2\}$

- Show by induction that for every concept E, for every $\tau \in T$ such that $E \in \tau$, $\tau \in E^{\mathcal{I}}$
  - Base case: $E = A$ or $E = \neg A$.
    - if $E = A$, $A \in \tau$ implies $\tau \in A^{\mathcal{I}}$ by definition of $\mathcal{I}$
    - if $E = \neg A$, $\neg A \in \tau$ implies that $A \notin \tau$ because τ is a type, so $\tau \notin A^{\mathcal{I}}$
  - Induction step:
    - if $E = C \sqcap D$, since τ is a type, then C and D are in τ, and by induction hypothesis, $\tau \in C^{\mathcal{I}}$ and $\tau \in D^{\mathcal{I}}$ so $\tau \in (C \sqcap D)^{\mathcal{I}}$
    - if $E = \exists R.C$, since τ is good in T, there exists τ' such that $(\tau, \tau') \in R^{\mathcal{I}}$ and $C \in \tau'$, so by induction hypothesis $\tau' \in C^{\mathcal{I}}$ so $\tau \in \exists R.C^{\mathcal{I}}$
    - $E = C \sqcup D$: left as practice
    - $E = \forall R.C$: left as practice

Complexity of $\mathcal{ALC}$ KB Satisfiability

Type elimination algorithm: Soundness – Proof of the claim

$\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ with $\Delta^\mathcal{I} = T$, $A^\mathcal{I} = \{\tau \mid A \in \tau\}$ and
$R^\mathcal{I} = \{(\tau_1, \tau_2) \mid \exists R.C \in \tau_1, \{C\} \cup \{D \mid \forall R.D \in \tau_1\} \subseteq \tau_2\}$

- Show by induction that for every concept E, for every $\tau \in T$ such that $E \in \tau$, $\tau \in E^\mathcal{I}$
  - Base case: $E = A$ or $E = \neg A$.
    - if $E = A$, $A \in \tau$ implies $\tau \in A^\mathcal{I}$ by definition of $\mathcal{I}$
    - if $E = \neg A$, $\neg A \in \tau$ implies that $A \notin \tau$ because τ is a type, so $\tau \notin A^\mathcal{I}$
  - Induction step:
    - if $E = C \sqcap D$, since τ is a type, then C and D are in τ, and by induction hypothesis, $\tau \in C^\mathcal{I}$ and $\tau \in D^\mathcal{I}$ so $\tau \in (C \sqcap D)^\mathcal{I}$
    - if $E = \exists R.C$, since τ is good in T, there exists τ' such that $(\tau, \tau') \in R^\mathcal{I}$ and $C \in \tau'$, so by induction hypothesis $\tau' \in C^\mathcal{I}$ so $\tau \in \exists R.C^\mathcal{I}$
    - $E = C \sqcup D$: left as practice
    - $E = \forall R.C$: left as practice
- For every $\tau \in T$, since τ is a $\mathcal{T}$-type, then $C_\mathcal{T} \in \tau$ so $\tau \in C_\mathcal{T}^\mathcal{I}$

Hence $\mathcal{I} \models \top \sqsubseteq C_\mathcal{T}$

Complexity of $\mathcal{ALC}$ KB Satisfiability

Type elimination algorithm: Completeness

The type elimination algorithm is complete

- Assume that A_0 is satisfiable w.r.t. $\{\top \sqsubseteq C_{\mathcal{T}}\}$
- There is a model $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ of $\top \sqsubseteq C_{\mathcal{T}}$ such that $A_0^{\mathcal{I}} \neq \emptyset$
- Claim: $T = \{\tau \mid e \in \Delta^{\mathcal{I}}, \tau = \{C \mid C \in \mathsf{sub}(C_{\mathcal{T}}), e \in C^{\mathcal{I}}\}\}$ is a set of $\mathcal{T}$-types such that there is $\tau \in T$ with $A_0 \in \tau$ and the type elimination algorithm does not remove any of the types in T

Complexity of $\mathcal{ALC}$ KB Satisfiability

Type elimination algorithm: Completeness – Proof of the claim

$$T = \{\tau \mid e \in \Delta^{\mathcal{I}}, \tau = \{C \mid C \in \mathsf{sub}(C_{\mathcal{T}}), e \in C^{\mathcal{I}}\}\}$$

- Since $A_0^{\mathcal{I}} \neq \emptyset$, there is $\tau \in T$ such that $A_0 \in \tau$
- T is a set of $\mathcal{T}$-types: for every $\tau \in T$
  - $e \in C^{\mathcal{I}}$ implies $e \notin \mathsf{nnf}(\neg C)^{\mathcal{I}}$, so $C \in \tau$ implies $\mathsf{nnf}(\neg C) \notin \tau$
  - $e \in (C \sqcap D)^{\mathcal{I}}$ implies $e \in C^{\mathcal{I}}$ and $e \in D^{\mathcal{I}}$, so $C \sqcap D \in \tau$ implies $C \in \tau$ and $D \in \tau$
  - similarly for $C \sqcup D$
  - $\mathcal{I} \models \top \sqsubseteq C_{\mathcal{T}}$, so $C_{\mathcal{T}} \in \tau$

Complexity of $\mathcal{ALC}$ KB Satisfiability

Type elimination algorithm: Completeness – Proof of the claim

$$T = \{\tau \mid e \in \Delta^{\mathcal{I}}, \tau = \{C \mid C \in \mathrm{sub}(C_{\mathcal{T}}), e \in C^{\mathcal{I}}\}\}$$

- Since $A_0^{\mathcal{I}} \neq \emptyset$, there is $\tau \in T$ such that $A_0 \in \tau$
- T is a set of $\mathcal{T}$-types: for every $\tau \in T$
  - $e \in C^{\mathcal{I}}$ implies $e \notin \mathrm{nnf}(\neg C)^{\mathcal{I}}$, so $C \in \tau$ implies $\mathrm{nnf}(\neg C) \notin \tau$
  - $e \in (C \sqcap D)^{\mathcal{I}}$ implies $e \in C^{\mathcal{I}}$ and $e \in D^{\mathcal{I}}$, so $C \sqcap D \in \tau$ implies $C \in \tau$ and $D \in \tau$
  - similarly for $C \sqcup D$
  - $\mathcal{I} \models \top \sqsubseteq C_{\mathcal{T}}$, so $C_{\mathcal{T}} \in \tau$
- Every $\tau \in T$ is good in T
  - let $\tau \in T$ and $\exists R.C \in \tau$
  - there is $e \in \Delta^{\mathcal{I}}$ such that $\tau = \{C \mid C \in \mathrm{sub}(C_{\mathcal{T}}), e \in C^{\mathcal{I}}\}$
  - $e \in \exists R.C^{\mathcal{I}}$ so there is $d \in \Delta^{\mathcal{I}}$ s.t. $(e, d) \in R^{\mathcal{I}}$ and $d \in C^{\mathcal{I}}$
  - for every D such that $\forall R.D \in \tau$, $e \in (\forall R.D)^{\mathcal{I}}$ so $d \in D^{\mathcal{I}}$
  - the type $\tau_d = \{E \mid E \in \mathrm{sub}(C_{\mathcal{T}}), d \in E^{\mathcal{I}}\}$ is such that $\{C\} \cup \{D \mid \forall R.D \in \tau\} \subseteq \tau_d$ and belongs to T
- The type elimination algorithm never removes any type $\tau \in T$: by induction on the number of iterations

In Practice: Optimizations

- Tableau algorithms are implemented and work well in practice
  - type elimination algorithm has optimal worst-case complexity but its best-case complexity is exponential!
- However, good performances crucially depends on optimizations
  - explore only one branch of one ABox at a time
  - strategies/heuristics for choosing next rule to apply
  - caching of results to reduce redundant computation
  - examine source of conflicts to prune search space
  - reduce numbers of $\sqcup$'s created by TBox inclusions
  - reduce number of satisfiability checks during classification

In Practice: Optimizations

Absorption: reduce number of disjunctions

If $\mathcal{T} = \{C_i \sqsubseteq D_i \mid 1 \leq i \leq n\}$, for each individual a, the TBox-rule builds n disjunctions $\text{nnf}(\neg C_i \sqcup D_i)(a)$
$\rightarrow$ Try to reduce this number

- When C_i or D_i is an atomic concept, trigger the TBox-rule only when we have information about this concept
  - for inclusions $A \sqsubseteq D$ with atomic left-hand side, replace the TBox-rule by
    TBox-atomic-left-rule: if $A(a) \in \mathcal{A}$, a is not blocked, $A \sqsubseteq D \in \mathcal{T}$ (A atomic), and $D(a) \notin \mathcal{A}$, replace $\mathcal{A}$ with $\mathcal{A} \cup \{D(a)\}$.
  - for inclusions $D \sqsubseteq A$ with atomic right-hand side, replace the TBox-rule by
    TBox-atomic-right-rule: if $\neg A(a) \in \mathcal{A}$, a is not blocked, $D \sqsubseteq A \in \mathcal{T}$ (A atomic), and $\neg D(a) \notin \mathcal{A}$, replace $\mathcal{A}$ with $\mathcal{A} \cup \{\neg D(a)\}$.

In Practice: Optimizations

Absorption: reduce number of disjunctions

- Preprocess the TBox
  - to decrease the number of concept inclusions with non-atomic left- and right-hand sides
    - $(A \sqcap C \sqsubseteq D) \equiv (A \sqsubseteq \neg C \sqcup D)$
    - $(D \sqsubseteq A \sqcup C) \equiv (D \sqcap \neg C \sqsubseteq A)$
    - ...
  - to obtain a single concept inclusion per atomic concept with this concept as right- or left-hand side ("absorption")
    - $A \sqsubseteq C_1, A \sqsubseteq C_2 \Rightarrow A \sqsubseteq C_1 \sqcap C_2$
    - $C_1 \sqsubseteq A, C_2 \sqsubseteq A \Rightarrow C_1 \sqcup C_2 \sqsubseteq A$

In Practice: Optimizations

Classification: reduce number of satisfiability checks

Classification consists in finding all pairs of atomic concepts A, B such that $\mathcal{T} \models A \sqsubseteq B$

- Naïve approach: test satisfiability of $A \sqcap \neg B$ w.r.t. $\mathcal{T}$ for every pair A, B
- Reduce the number of satisfiability checks
  - some subsumptions are obvious
    - $A \sqsubseteq A$
    - $A \sqsubseteq B \in \mathcal{T}$
  - use simple reasoning to obtain new (non-)subsumptions
    - if we found that $\mathcal{T} \models A \sqsubseteq B$ and $\mathcal{T} \models B \sqsubseteq C$, then $\mathcal{T} \models A \sqsubseteq C$
    - if we found that $\mathcal{T} \models A \sqsubseteq B$ and $\mathcal{T} \not\models A \sqsubseteq C$, then $\mathcal{T} \not\models B \sqsubseteq C$

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