# Susceptible-Infective and SIS Epidemic propagation models

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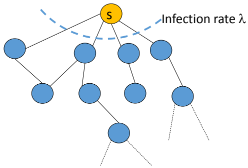
# outline

- SI process on complete graph
- Markovian transforms of Markov processes
- Application: control of SI process on general graph via isoperimetric constant
- SIS process on general graphs:
  Fast extinction and spectral radius
  Long survival and isoperimetric constant
- SIR process on general graph and spectral radius

# Susceptible-Infective epidemic propagation

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- Infected nodes keep attempting to infect graph neighbors
- Models "push"-based distributed information dissemination mechanism (example of a "gossip" algorithm); variants used in Peer-to-peer systems (e.g. Bittorrent): pull, push-pull...

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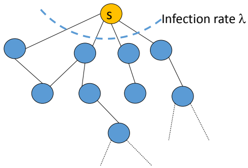


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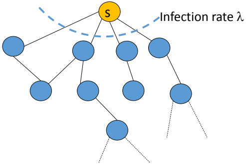


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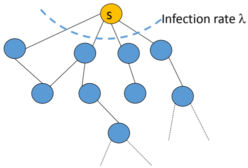


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- Variant: each node = origin of its own specific epidemics; each propagation: forwards all epidemics currently held
  $\Rightarrow$Time till everyone heard from everyone else ("all-to-all" broadcast)? Useful e.g. for estimating graph size

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  $\Rightarrow$ $X_t$ a Markov jump process with non-zero jump rate $q_{x,x+1} = \lambda x(n-x)/(n-1)$

# Time to total infection

Let $E_x$: i.i.d. Exponential(1) random variables, $T_n$: time to total infection (or broadcast)
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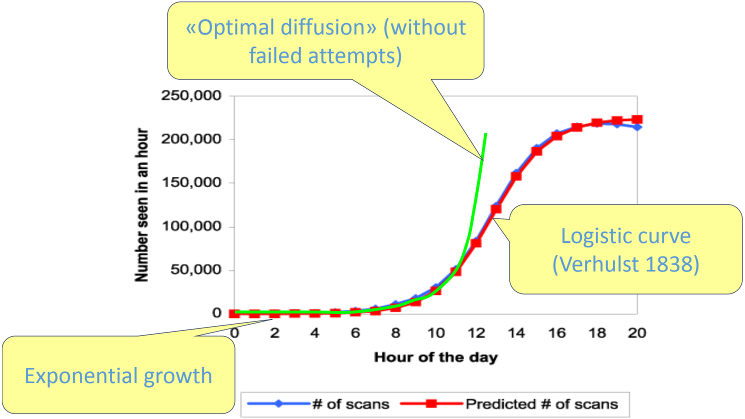
Heuristic inversion: starting from $X_0 = an$, $X_t \approx n\frac{ae^{\lambda t}}{1-a+ae^{\lambda t}}$
$\Rightarrow$ The celebrated **logistic function**, or S-curve




Time to total infection order-optimal
(logarithmic in number of targets) despite random target selection

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# Controlling fluctuations

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$\mathbb{E}(\exp(\theta S_n)) \leq \exp(4\pi^2\theta^2/3) =: C_\theta < +\infty$
hence (Chernoff bound argument):
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**Proof**: For $r_x = x(n-x)/(n-1) = q_x/\lambda$,
$\mathbb{E}e^{\theta S_n} = \prod_{x=1}^{n-1} \frac{r_x}{r_x - \theta} e^{-\theta/r_x}$
For $u \in (0, 1/2]$, $\frac{e^{-u}}{1-u} \leq 1 + 2u^2$, hence:
$\mathbb{E}e^{\theta S_n} \leq \prod_{x=1}^{n-1}[1 + 2(\theta/r_x)^2] \leq e^{\sum_{x=1}^{n-1} 2(\theta/r_x)^2} \leq e^{8\theta^2 \sum_{x\geq 1} x^{-2}}$

# Application: All-to-all scenario (one epidemic per user)

**Lemma**

*Let random variables* $S^1, \dots S^n$ *be such that for some* $a, b > 0$ :
$\forall t > 0, \forall i \in [n], \mathbb{P}(S^i \geq t) \leq ae^{-bt}$
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**Corollary**

All-to-all propagation time $T$ satisfies for all $\theta \in (0, 1/2]$
$\mathbb{E}\,T \leq \frac{1}{\lambda}\left[2(\ln(n)+\gamma) + o(1) + \frac{\ln(C_\theta n)+1}{\theta}\right] = O(\ln(n))$,
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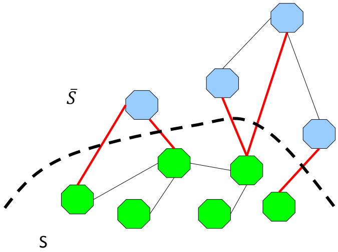
Indeed: $T =$ supremum of $n$ propagation times corresponding each to single epidemic propagation

# Isoperimetric constant of a graph

## Definition

For a graph $G = (V, E)$ and any $m < n$, the isoperimetric constant $\eta_m$ is defined as $\eta_m = \min_{S \subset V, |S| \leq m} \frac{|E(S, \overline{S})|}{|S|}$,
where $E(S, \overline{S})$ denotes the set of edges between $S$ and its complement $\overline{S} = V \setminus S$.

**Remark:** When $m$ not specified, $\eta = \eta_{n/2}$



# Markovian transforms of Markov chains

Let $\{X_n\}_{n\in\mathbb{N}}$ be a Markov chain on countable set $E$ with transition matrix $(p_{ij})_{i,j\in E}$.
For countable set $F$ and $f : E \to F$, let $Y_n := f(X_n),\ n \in \mathbb{N}$.

**Theorem**

*If for some transition matrix $\hat{P} = (\hat{p}_{uv})_{u,v\in F}$, one has*

$$\forall x \in E,\ v \in F, \quad \sum_{y\in E: f(y)=v} p_{xy} = \hat{p}_{f(x),v},$$

*then $\{Y_n\}_{n\in\mathbb{N}}$ is a Markov chain on $F$ with transition matrix $\hat{P}$.*

**Proof:** by evaluating $\mathbb{P}(Y_0^k = y_0^k)$ for arbitrary $y_0^k \in F^{k+1}$...

**Remark:** In general, image of Markov chain fails to be Markovian.
Example: $X_0^\infty = \{0,1,2,0,1,2,\ldots\}$,
$f(x) = \mathbb{I}_{x=2} \Rightarrow Y_0^\infty = \{0,0,1,0,0,1,\ldots,\}$

# Markovian transforms of Markovian jump processes

Let $\{X(t)\}_{t\in\mathbb{R}_+}$ be a non-explosive Markov jump process on countable set $E$ with infinitesimal generator $(q_{ij})_{i,j\in E}$.
For countable set $F$ and $f : E \to F$, let $Y(t) := f(X(t)),\ t \in \mathbb{R}_+$.

**Theorem**

*If for some generator* $\hat{Q} = (\hat{q}_{uv})_{u,v\in F}$ *such that* $\forall u \in F, \hat{q}_{u,u} = -\sum_{v\neq u} \hat{q}_{uv} =: -\hat{q}(u)$, *one has*

$$\forall x \in E,\ v \in F : f(x) \neq v, \quad \sum_{y\in E: f(y)=v} q_{xy} = \hat{q}_{f(x),v},$$

*then* $\{Y(t)\}_{t\in\mathbb{R}_+}$ *is a Markov jump process on* $F$ *with infinitesimal generator* $\hat{Q}$.

# Application: SI epidemics on general graphs and isoperimetric constant

Consider SI process on $G = (V, E)$ with infection rate $\lambda$ along each edge.

Then propagation is at least as fast as SI process on complete graph with per-node infection rate $\lambda\eta_{n/2}$.

**Corollary**: Time to total infection in $O\left(\frac{\ln(n)}{\lambda\eta_{n/2}}\right)$

# Coupling proof:

Define process $(X, Z)$ on $E := \{(x, y) \in \{0,1\}^V \times [n] : \sum_{i\in V} x_i \geq z\}$ (hence for all $t \in \mathbb{R}_+, \sum_{i\in V} X_i(t) \geq Z(t)$) so that:
$X$: SI process on $G$ with per edge infection rate $\lambda$, and
$Z$: number of infected nodes in SI on complete graph, with per node infection rate $\lambda\eta_{n/2}$.

Process $(X, Z)$ specified by non-zero transition rates: for each $(x, z) \in E,\ i \in V$,

$$
\begin{array}{ll}
\sum_{j\in V} x_j > z \Rightarrow & q_{(x,z),(x+e_i,z)} = \lambda \sum_{j\sim i}(1-x_i)x_j, \\
& q_{(x,z),(x,z+1)} = \lambda\eta_{n/2}\frac{z(n-z)}{n-1}, \\
\sum_{j\in V} x_j = z \Rightarrow & q_{(x,z),(x+e_i,z+1)} = C\lambda \sum_{j\sim i}(1-x_i)x_j, \\
& q_{(x,z),(x+e_i,z)} = [1-C]\lambda \sum_{j\sim i}(1-x_i)x_j,
\end{array}
$$

where $C := \frac{z(n-z)\eta_{n/2}}{(n-1)\sum_{i\in V}\sum_{j\sim i} x_i(1-x_j)}$

# Coupling proof:

Well-defined: non-negative rates, as $C \leq 1$ because $\sum_{i \in V} \sum_{j \sim i} x_i(1 - x_j) \geq \eta_{n/2} \mathrm{Min}(z, n - z)$

Component processes have desired distributions: use criterion for transform of jump process to be itself jump process, with $f((X, Z)) = X$ and $f((X, Z)) = Z$.

Examples of verification: for process $X$,
$\tilde{q}_{x,x+e_i} = [C + 1 - C]\lambda \sum_{j \sim i}(1 - x_i)x_j = \lambda \sum_{j \sim i}(1 - x_i)x_j$;

For process $Z$,
$\tilde{q}_{z,z+1} = \sum_{i \in V} C\lambda \sum_{j \sim i} x_i(1 - x_j) = \lambda \eta_{n/2} \frac{z(n-z)}{(n-1)}$.

# SIS model

- Basic model: graph $G = (V, E)$
- Each infected node infects each of its neighbors at rate $\beta$ , and becomes healthy at rate $\delta$

May model propagation of mutating virus, or replication of data in volatile memories

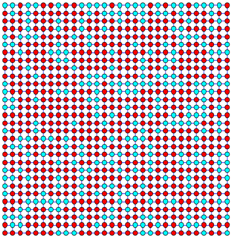
$\Rightarrow$ Markov jump process on $\{0,1\}^V$ with non-zero transition rates

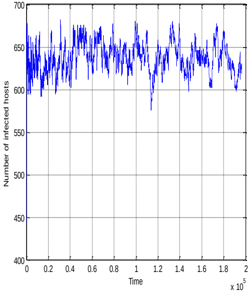
$$\begin{aligned} q(x, x+e_i) &= \beta \textstyle\sum_{j \sim i} x_j,\ i \in V,\ x \in \{0,1\}^V,\ x_i = 0; \\ q(x, x-e_i) &= \delta,\ i \in V,\ x \in \{0,1\}^V,\ x_i = 1. \end{aligned}$$

Stationary regime: complete extinction (absorbing state)

Goal: understand impact of $\beta$, $\delta$ and topology of $G$ on time to extinction

# Example of a grid network






Behaviour characterized by [Durrett-Liu,Durrett-Schonmann,'88]: there is a critical threshold $c > 0$ such that:

$\beta/\delta > c \Rightarrow$ long survival (expected time to extinction: exponential in $n = |V|$),

$\beta/\delta < c \Rightarrow$ fast extinction (expected time to extinction logarithmic in $n = |V|$),

# Fast extinction and spectral radius

**Definition**

The spectral radius $\rho(A)$ of matrix $A$ is the largest modulus of its eigenvalues.

**Theorem**

*Let $\rho$ be the spectral radius of the adjacency matrix $A$ of graph $G = (V, E)$. The time to extinction $T$ verifies for all $t > 0$:*

$$\mathbb{P}(T \geq t) \leq n e^{(\beta\rho-\delta)t},$$

*where $n = |V|$.*

**Corollary**

*If $\beta\rho < \delta$, then $\mathbb{E}(T) \leq \frac{\ln(n)+1}{\delta-\beta\rho}$*

# Long survival and isoperimetric constants

## Theorem

*Assume that for some* $r \in ]0,1[$ *and some* $m < n$, $\beta\eta_m \geq \frac{\delta}{r}$.
*Then there is a function* $f : \mathbb{N} \to \mathbb{R}$ *such that* $\lim_{k\to\infty} f(k) = 0$ *and for any* $k \in \mathbb{N}$,

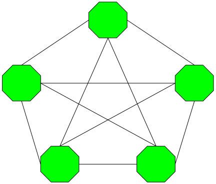
$$\mathbb{P}(T \geq \frac{k}{2\delta m}) \geq \frac{1-r}{1-r^m}\left(\frac{1-r^{m-1}}{1-r^m}\right)^k (1-f(k))$$

## Corollary

*If for fixed* $r \in ]0,1[$ *and a sequence of graphs* $G_n$ *each on* $n$ *nodes one has for some* $m = m(n)$ *with* $\lim_{n\to\infty} m(n) = +\infty$ : $\beta\eta_m(G_n) \geq \frac{\delta}{r}$, *then the time* $T_n$ *to extinction of the* $(\beta,\delta)$*- epidemic process on* $G_n$ *verifies:*

$$\mathbb{E}[\delta T_n] \geq \frac{(1-r)^2}{3m}\lfloor r^{-m+2}\rfloor = e^{\Omega(m)}.$$

# Example: complete graph



For complete graph on $n$ nodes, $\rho = n - 1$ and $\eta_m = n - m$.

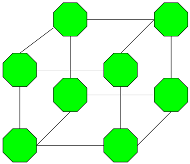
$\Rightarrow$ for fixed $\epsilon \in ]0, 1[$,

if $\beta(n-1) \leq \delta(1-\epsilon)$, $\mathbb{E}[\delta T_n] \leq \frac{\ln(n)+1}{\epsilon} = O(\ln(n))$;
if $\beta(n-1) \geq \delta(1+\epsilon)$, for $m = n\epsilon/2$ one has $\beta\eta_m \geq \delta/r$ with $r^{-1} = (1+\epsilon)(1-\epsilon/2) > 1$, so that

$\mathbb{E}[\delta T_n] \geq e^{\Omega(n)}$

A sharp transition with respect to $(\beta n/\delta)$ at 1.

# Example: hypercube



Hypercube $G = \{0,1\}^d$ on $n = 2^d$ nodes:
$\rho = d$, $\eta_m \geq d - k$ for $m = 2^k, k < d$ (ref: [Harper'64])

Fix $\epsilon \in ]0,1[$.
If $\beta d \leq \delta(1-\epsilon)$, then $\mathbb{E}[\delta T_n] \leq \frac{\ln(n)+1}{\epsilon} = O(\ln(n))$;

If $\beta d \geq \delta(1+\epsilon)$, for $m = 2^{\epsilon d/2}$, $\eta_m \geq (1-\epsilon/2)d$.
Hence $\beta\eta_m \geq \delta/r$ with $r < 1$, so that
$\mathbb{E}[\delta T_n] \geq e^{\Omega(m)} = e^{\Omega(n^{\epsilon/2})}$.

A sharp transition with respect to $(\beta d/\delta)$ at 1.

# Example: Erdős-Rényi graph with super-logarithmic average degree

## proposition

Let $G = (n, d/n)$ with $d \gg \ln(n)$, and some fixed $\alpha \in ]0,1[$. One then has the convergences in probability

$$\lim_{n\to\infty} \frac{\rho(A)}{d} = 1, \quad \lim_{n\to\infty} \frac{\eta_{\alpha n}}{(1-\alpha)d} = 1.$$

## Corollary

*Let $\epsilon > 0$ be fixed. One has the following with high probability with respect to $G$:*
*If $\beta d \leq (1-\epsilon)\delta$, then $\mathbb{E}\frac{T_n}{\delta} \leq \frac{2\ln(n)}{\epsilon} = O(\ln(n))$.*
*If $\beta d \geq (1+\epsilon)\delta$, then $\mathbb{E}\frac{T_n}{\delta} \geq e^{\Omega(\epsilon n)} = e^{\Omega(n)}$.*

# Fast extinction and spectral radius: proof elements

Define branching random walk on graph $G = (V, E)$ as process $X'$ on $\mathbb{N}^V$ with non-zero transition rates $q'_{x,x+e_i} = \beta \sum_{j \sim i} x_j$ and $q'_{x,x-e_i} = \delta x_i$.

Couple two processes $X, X'$, where $X$: SIS on $G = (V, E)$ with initial conditions $x(0) \in \{0, 1\}^V$ so that $\forall t \in \mathbb{R}_+,\ X(t) \le X'(t)$
Bound probability of SIS survival:

$$\mathbb{P}(T > t) \le \mathbb{E}(\sum_i X_i(t)) \le \mathbb{E} \sum_i X'_i(t).$$

Linearity of rates $q'$ in $x'$:

$$\begin{aligned} \frac{d}{dt}\mathbb{E}(X'(t)) &= \beta A \mathbb{E}(X'(t)) - \delta \mathbb{E}(X'(t)) \\ \Rightarrow \mathbb{E}(X'(t)) &= e^{t(\beta A - \delta I)} x(0). \end{aligned}$$

# SIR epidemics and spectral radius

Consider Reed-Frost process with neighbor infection parameter $\beta$ on graph $G = (V, E)$, $X_i(t) = \mathbb{I}_{i \text{ infectious at } t}$, $Y_i(t) = \mathbb{I}_{i \text{ removed at } t}$. Then:

**Theorem**

*Suppose $\beta\rho < 1$. Then the total number of nodes eventually removed verifies*

$$\mathbb{E} \sum_{i \in V} Y_i(\infty) \leq \frac{1}{1 - \beta\rho} \sqrt{n \sum_{i \in V} X_i(0)}.$$

*If moreover $G$ is $d$- regular, then*

$$\mathbb{E} \sum_{i \in V} Y_i(\infty) \leq \frac{1}{1 - \beta\rho} \sum_{i \in V} X_i(0).$$

# SIR epidemics: proof

By union bound,

$$\begin{aligned}\mathbb{P}(Y_u(\infty)=1) &\leq \sum_{t\geq 0}\sum_{u_0,\ldots,u_t}\beta^t X_{u_0}(0)\\ &= \sum_{t\geq 0}\sum_{v\in V}(\beta A)^t_{uv}X_v(0)\end{aligned}$$

where $u_0,\ldots,u_t$: graph path with $u_t=u$.
Hence

$$\begin{aligned}\mathbb{E}\sum_u Y_u(\infty) &\leq \sum_{t\geq 0} e^T(\beta A)^t X(0)\\ &= e^T(I-\beta A)^{-1}X(0)\\ &= \sum_i \langle x_i, e\rangle \frac{1}{1-\beta\lambda_i}\langle x_i, X(0)\rangle\end{aligned}$$

# Takeaway messages

- SI Epidemics spreads in logarithmic time on well-connected graphs (as measured by isoperimetric constant) for single propagation and for all-to-all propagation, same order as if infection targets were chosen optimally
- Epidemic (or gossip) algorithms good candidates for managing information dissemination in P2P systems
- Behaviour of SIS epidemics undergoes phase transitions as ratio $\beta/\delta$ crosses thresholds
- Graph topology determines thresholds; in several scenarios (complete graph, hypercube, E-R graphs), spectral radius and isoperimetric constants are close, hence a single threshold
- Coupling constructions allow control of complex process by simpler ones