# Poisson process and Markov jump processes

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# Outline

- Poisson processes
- Markov jump processes

# The Poisson distribution (Siméon-Denis Poisson, 1781-1840)

$\left\{e^{-\lambda}\frac{\lambda^n}{n!}\right\}_{n\in\mathbb{N}}$ As prevalent as Gaussian distribution

# The Poisson distribution (Siméon-Denis Poisson, 1781-1840)

$\lambda_n = \sum_i p_{n,i} = \mathbb{E} X_n.$

$d_{var}(X_n, \mathrm{Poi}(\lambda_n)) \leq 2 \min(1, \lambda_n^{-1}) \sum_i p_{n,i} (p_{n,i} + 0)$

$\leq 2 \min(1, \lambda_n^{-1}) (\sup_i p_{n,i}) \lambda_n \leq 2 \sup_i p_{n,i}$

$\left\{e^{-\lambda} \frac{\lambda^n}{n!}\right\}_{n \in \mathbb{N}}$ As prevalent as Gaussian distribution

Law of *rare events* (a.k.a. *law of small numbers*)

$p_{n,i} \geq 0$ such that $\lim_{n \to \infty} \sup_i p_{n,i} = 0$, $\lim_{n \to \infty} \sum_i p_{n,i} = \lambda > 0$

Then $X_n = \sum_i Z_{n,i}$ with $Z_{n,i}$: independent Bernoulli($p_{n,i}$) verifies

$$X_n \xrightarrow{\mathcal{D}} \mathrm{Poisson}(\lambda)$$

# Point process on $\mathbb{R}_+$

## Definition

**Point process** on $\mathbb{R}_+$:

Collection of random times $\{T_n\}_{n>0}$ with $0 < T_1 < T_2 \ldots$

Alternative description
Collection $\{N_t\}_{t\in\mathbb{R}_+}$ with $N_t := \sum_{n>0} \mathbb{I}_{T_n\in[0,t]}$

$N_t = N([0,t])$

Yet another description
Collection $\{N(C)\}$ for all measurable $C \subset \mathbb{R}_+$ where

$$N(C) := \sum_{n>0} \mathbb{I}_{T_n\in C}$$

# Poisson process on $\mathbb{R}_+$

$N_{s_i} - N_{s_{i-1}} = N((s_{i-1}, s_i])$

## Definition

Point process such that for all $s_0 = 0 < s_1 < s_2 < \ldots < s_n$,

1. Increments $\{N_{s_i} - N_{s_{i-1}}\}_{1 \le i \le n}$ independent
2. Law of $N_{t+s} - N_s$ only depends on $t$
3. for some $\lambda > 0$, $N_t \sim \text{Poisson}(\lambda t)$

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In fact, (3) follows from (1)–(2); see lecture notes.

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$\lambda$ is called the **intensity** of the process

# Structure of interarrival sequences

**Proposition**

For Poisson process $\{T_n\}_{n>0}$ of intensity $\lambda$, its interarrival times $\tau_i = T_{i+1} - T_i$, where $T_0 = 0$, verify $\{\tau_n\}_{n\geq 0}$ i.i.d. with common distribution $\text{Exp}(\lambda)$

Density of $\text{Exp}(\lambda)$: $\lambda e^{-\lambda x}\mathbb{I}_{x>0}$

Key property: Exponential random variable $\tau$ is **memoryless**, i.e. $\forall t > 0$, $\mathbb{P}(\tau - t \in \cdot|\tau > t) = \mathbb{P}(\tau \in \cdot)$

# Proof:

$\mathbb{E}\, e^{-\alpha\, \mathrm{Exp}(\lambda)} = \int_0^\infty e^{-\alpha x} \lambda e^{-\lambda x} dx = \frac{\lambda}{\lambda+\alpha}.$

Characterization of law of $\{\tau_i\}_{i\in[n]}$ by Laplace transform $\mathbb{E}\prod_{i\in[n]} e^{-\alpha_i\tau_i}$: show that for all $n \geq 1,\ \alpha_1^n \in \mathbb{R}_+^n$, $\mathbb{E}\prod_{i\in[n]} e^{-\alpha_i\tau_i} = \prod_{i\in[n]} \frac{\lambda}{\lambda+\alpha_i}$.

Fix $h > 0$. Let $Z_n = \mathbb{I}_{N_{nh}-N_{(n-1)h}\geq 1}$, $S_1 = \inf\{n > 0 : Z_n \geq 1\}$, $S_i = \inf\{n > S_{i-1} : Z_n = 1\}$, $i > 1$.
Approximation of interarrivals: $\tau_i^{(h)} := h(S_{i+1} - S_i)$.

$\tau_i = T_{i+1} - T_i$

$\{S_{i+1} - S_i\}$: i.i.d. $\mathrm{Geom}(1 - e^{-\lambda h})$, so that $\mathbb{E}\prod_{i\in[n]} e^{-\alpha_i\tau_i^{(h)}} = (1 + O(h))\prod_{i\in[n]} \frac{\lambda}{\lambda+\alpha_i}$.

On event $\mathcal{A}_h := \{\forall i \in [n], \tau_i > h\}$, $\forall i \in [n]$, $|\tau_i - \tau_i^{(h)}| \leq h$ and hence $\prod_{i\in[n]} e^{-\alpha_i\tau_i} = (1 + O(h))\prod_{i\in[n]} e^{-\alpha_i\tau_i^{(h)}}$.

Conclude by noting that $\mathcal{A}_h \uparrow \{\forall i \in [n], \tau_i > 0\}$ hence $\lim_{h\to 0}\mathbb{P}(\mathcal{A}_h) = 1$.

# Alternative construction

**Proposition**

Process with i.i.d., $\mathrm{Exp}(\lambda)$ interarrival times $\{\tau_i\}_{i\geq 0}$ can be constructed on $[0,t]$ by
1) Drawing $N_t \sim \mathrm{Poisson}(\lambda t)$
2) Putting $N_t$ points $U_1,\ldots,U_{N_t}$ on $[0,t]$ where $U_i$: i.i.d. uniform on $[0,t]$


chgt de variables:

$$t_1 = s_0,\ t_2 = s_0 + s_1, \ldots, t_n = s_0 + \cdots s_{n-1}$$

$$e^{-\lambda t}\lambda^n \int \phi(t_1,\ldots,t_n)\, \mathbb{1}_{0<t_1<\cdots<t_n<t}\, dt_1 \ldots dt_n$$

$$= e^{-\lambda t}\frac{(\lambda t)^n}{n!}\iint_{\mathbb{R}_+^n}\frac{n!}{t^n}\mathbb{1}_{0<t_1<\cdots<t_n<t}\,\phi(t_1,\dots,t_n)\,dt_1\cdots dt_n$$

$$\mathbb{P}(\mathrm{Poi}(\lambda t)=n)$$

$$\mathbb{E}[\phi(U^{(1)},U^{(2)},\dots,U^{(n)})] \qquad n!\,\mathbb{E}\left\{\phi(U_1,\dots,U_n)\,\mathbb{1}_{U_1<U_2<\cdots<U_n}\right\}$$

$U^{(i)}$ : échantillon trié des $U_j$ iid $\sim \mathcal{U}([0,t])$

où $U_i$ : iid $\sim \mathcal{U}([0,t])$.

$$\mathbb{E}\left[\phi(\tau_0, \dots, \tau_0 + \dots + \tau_{n-1}) \mathbb{1}_{N_t = n}\right]$$

$$\iint_{\mathbb{R}_+^n} \prod_{i=0}^{n-1} \lambda e^{-\lambda s_i} ds_i \; \phi(s_0, s_0 + s_1, \dots, s_0 + \dots + s_{n-1}) \; \mathbb{1}_{\sum_{i=0}^{n-1} s_i \leq t} \; e^{-\lambda(t-(s_0 + \dots + s_{n-1}))}$$

$$= e^{-\lambda t} \lambda^n \int_{\mathbb{R}_+^n} \phi(s_0, s_0 + s_1, \dots, s_0 + \dots + s_{n-1}) \; \mathbb{1}_{s_0 + \dots + s_{n-1} < t} \; ds_0 \cdots ds_{n-1}$$

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## Proof.

Establish identity for all $n \in \mathbb{N}$, $\phi : \mathbb{R}_+^n \to \mathbb{R}$:

$$\mathbb{E}[\phi(\overset{T_1}{\tau_0}, \overset{T_2}{\tau_0 + \tau_1}, \overset{\ldots}{\ldots}, \overset{T_n}{\tau_0 + \ldots + \tau_{n-1}})\mathbb{I}_{N_t = n}] = \cdots$$

$$e^{-\lambda t}\frac{(\lambda t)^n}{n!} \times n! \int_{(0,t]^n} \phi(s_1, s_2, \ldots, s_n)\mathbb{I}_{s_1 < s_2 < \ldots < s_n} \prod_{i=1}^n \frac{1}{t} ds_i$$

$$= \mathbb{P}(\text{Poisson}(\lambda t) = n) \times \mathbb{E}[\phi(S_1, \ldots, S_n)]$$

where $S_1^n$: sorted version of i.i.d. variables uniform on $[0,t]$ □

# Laplace transform of Poisson processes

## Definition

For function $f : \mathbb{R}_+ \to \mathbb{R}_+$, let

$$N(f) := \sum_{n>0} f(T_n).$$

The Laplace transform of point process $N \leftrightarrow \{T_n\}_{n>0}$ is the functional whose evaluation at $f : \mathbb{R}_+ \to \mathbb{R}_+$ is

$$\mathcal{L}_N(f) := \mathbb{E}\exp(-N(f)) = \mathbb{E}(\exp(-\sum_{n>0} f(T_n))).$$

$0 < s_1 < \dots < s_n$, $\quad X_i = N_{s_i} - N_{s_{i-1}}$,

$f(x) = \alpha_i$ sur $(s_{i-1}, s_i]$, $0$ en dehors de $[0, s_n]$.

$N(f) = \sum_{i=1}^{n} \alpha_i X_i \to \mathbb{E} e^{-N(f)} = \mathbb{E} e^{-\sum_i \alpha_i X_i}$.

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Proposition: Knowledge of $\mathcal{L}_N(f)$ on sufficiently rich class of functions $f : \mathbb{R}_+ \to \mathbb{R}_+$ characterizes law of point process $N$.

Exemples of such classes of functions:

- non-negative piecewise constant with compact support
- non-negative piecewise continuous with compact support
- non-negative continuous with compact support

# Laplace transform of Poisson processes

**Proposition**

Poisson process with intensity $\lambda$ admits Laplace transform

$$\mathcal{L}_N(f) = \exp(-\int_{\mathbb{R}_+} \lambda(1 - e^{-f(x)})dx)$$

$f$ : à support $[0, t]$. $N_t \sim \text{Poi}(\lambda t)$ ; $U_1 \dots U_i$ : i.i.d $\sim \mathcal{U}([0,t])$

$$\mathbb{E}\, e^{-N(f)} = \mathbb{E}\left(\prod_{i=1}^{N_t} e^{-f(U_i)}\right) = \sum_{n \geq 0} e^{-\lambda t}\frac{(\lambda t)^n}{n!}\left(\frac{1}{t}\int_0^t e^{-f(u)}du\right)^n$$

$$= e^{-\lambda t}\, e^{\lambda \int_0^t e^{-f(u)}du} = e^{-\int_0^t \lambda(1 - e^{-f(u)})du}$$

$$\doteq e^{-\int_{\mathbb{R}_+} \lambda(1 - e^{-f(u)})du}.$$

# Laplace transform of Poisson processes

## Proposition

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Proof: Previous construction yields expression for $\mathcal{L}_N(f)$

Corollary: For $f = \sum_i \alpha_i \mathbb{I}_{C_i} \Rightarrow N(C_i) \sim \text{Poisson}(\lambda \int_{C_i} dx)$, with independence for disjoint $C_i$. Hence existence of Poisson process.

$N_{s_i} - N_{s_{i-1}}$ : ind$^{ts}$ $\sim$ Poi$(\lambda(s_i - s_{i-1}))$.

# Poisson process with general space and intensity

## Definition

Point process on $\mathbb{R}^d$: countable or finite collection $\{T_i\}_{i\geq 1}$ of distinct points of $\mathbb{R}^d$

## Definition

For $\lambda : \mathbb{R}^d \to \mathbb{R}_+$ locally integrable function, $N \leftrightarrow \{T_n\}_{n>0}$ point process on $\mathbb{R}^d$ is Poisson with intensity function $\lambda$ if and only if
for measurable, disjoint $C_i \subset \mathbb{R}^d, i = 1, \ldots, n$,
$N(C_i)$ independent, $\sim \text{Poisson}(\int_{C_i} \lambda(x)dx)$

① Supposer $\lambda(x)$ : à support compact, $C \subseteq \mathbb{R}^d$.

$N(C) \sim \mathrm{Poi}\left(\int_C \lambda(x)\,dx\right)$.

$U_1 \dots U_i \dots$ : iid, de densité $\dfrac{\lambda(x)}{\int_C \lambda(y)dy}$ sur $\mathbb{R}^d$.

$\rightsquigarrow N \longleftrightarrow \{U_1, \dots, U_{N_t}\}$.

$f : \mathbb{R}^d \to \mathbb{R}_+$. $\mathbb{E}\, e^{-N(f)} = \mathbb{E} \prod_{i=1}^{N(C)} e^{-f(U_i)}$

$$= \sum_{n \geq 0} e^{-\int_C \lambda} \frac{\cancel{(\int_C \lambda)^n}}{n!} \left[\frac{1}{\cancel{\int_C \lambda}} \int_C \lambda(x) e^{-f(x)} dx\right]^n$$

$$= e^{-\int_C \lambda}\, e^{\int_C \lambda(x) e^{-f(x)} dx} = e^{-\int_C \lambda(x)(1-e^{-f(x)})dx}$$

② Cas général : $= e^{-\int_{\mathbb{R}^d} \lambda(x)(1-e^{-f(x)})dx}$

Construire $N_k$ : processus de Poisson $\lambda^{(k)}(x) = \lambda(x) \mathbb{1}_{\|x\| \in [k, k+1)}$

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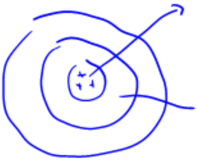
$N(C_i)$ independent, $\sim$ Poisson$(\int_{C_i} \lambda(x)dx)$

## Proposition

Such a process exists and admits Laplace transform

$$\mathcal{L}_N(f) = \exp\left(-\int_{\mathbb{R}^d} \lambda(x)(1 - e^{-f(x)})dx\right)$$



$N^{(0)} \rightsquigarrow \lambda^{(0)}(x) = \lambda(x)\,\mathbb{1}_{\|x\|<1}$

$N^{(1)}$ ...

$C_k = \{x : \|x\| \in [k, k+1)\}$

$$\to N = \sum_{k\geq 0} N^{(k)}.$$

$$\mathbb{E}\, e^{-N^{(k)}(f)} = e^{-\int_{C_k} \lambda(x)[1-e^{-f(x)}]dx}$$

$$\mathbb{E}\, e^{-N(f)} = \prod_{k\geq 0} e^{-N^{(k)}(f)}$$

$$= e^{-\int_{\mathbb{R}^d} \lambda(x)(1-e^{-f(x)})dx}.$$

# Markov jump processes

Process $\{X_t\}_{t\in\mathbb{R}_+}$ with values in $E$, countable or finite, is

**Markov** if
$\mathbb{P}(X_{t_n} = x_n | X_{t_{n-1}} = x_{n-1}, \dots X_{t_1} = x_1) = \mathbb{P}(X_{t_n} = x_n | X_{t_{n-1}} = x_{n-1})$,
$t_1^n \in \mathbb{R}_+^n,\ t_1 < \cdots < t_n,\ x_1^n \in E^n$

**Homogeneous** if $\mathbb{P}(X_{t+s} = y | X_s = x) =: p_{xy}(t)$ independent of $s$,
$s, t \in \mathbb{R}_+,\ x, y \in E$

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$\mathbb{P}(X_{t+s} = y | X_0 = x) = \sum_z \mathbb{P}(X_{t+s} = y, X_t = z | X_0 = x)$

$\Rightarrow$ Semi-group property $p_{xy}(t+s) = \sum_{z \in E} p_{xz}(t) p_{zy}(s)$,
or $P(t+s) = P(t)P(s)$ with $P(t) = \{p_{xy}(t)\}_{x,y \in E}$

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## Definition

$\{X_t\}_{t\in\mathbb{R}_+}$ is a **pure jump** Markov process if in addition
(i) It spends with probability 1 a strictly positive time in each state
(ii) Trajectories $t \to X_t$ are right-continuous

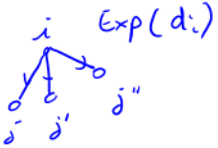
# Markov jump processes: examples

$$\mathbb{P}(N_{t_n} = x_n \mid N_{t_i} = x_i,\ i = 1 \cdots n-1)$$
$$= \mathbb{P}(N_{t_n} - N_{t_{n-1}} = x_n - x_{n-1})$$

- Poisson process $\{N_t\}_{t \in \mathbb{R}_+}$: then Markov jump process with $p_{xy}(t) = \mathbb{P}(\text{Poisson}(\lambda t) = y - x)$

# Markov jump processes: examples

$X_t \in V$




- Poisson process $\{N_t\}_{t\in\mathbb{R}_+}$: then Markov jump process with $p_{xy}(t) = \mathbb{P}(\text{Poisson}(\lambda t) = y - x)$
- Continuous-time random walk on finite, undirected graph $G = (V, E)$: Sojourn time at node $i \in V$: exponentially distributed with parameter $d_i$, degree of node $i$,
  after which: jump to neighbor of $i$ selected uniformly at random

# Markov jump processes: more examples

arrivées : $N$

- Single-server queue, FIFO ("First-in-first-out") discipline, arrival times: $N$ Poisson $(\lambda)$, service times: i.i.d. $\text{Exp}(\mu)$ independent of $N$
  $X_t$= number of customers present at time $t$: Markov jump process by Memoryless property of Exponential distribution + Markov property of Poisson process
  (the $M/M/1/\infty$ queue)

nb de serveurs

tps de service $\text{Exp}(\mu)$

arrivées Poissoniennes

arrivées entre $t$ et $t+s$ :

processus translaté $N^{(t)}_s = N_{t+s} - N_t$ :

processus de Poisson, ind$^t$ des $N_{t_i}$, $t_i \le t$.

$(\lambda)$

taille de file d'attente.

# Markov jump processes: more examples

- Single-server queue, FIFO ("First-in-first-out") discipline, arrival times: $N$ Poisson $(\lambda)$, service times: i.i.d. Exp$(\mu)$ independent of $N$
  $X_t=$ number of customers present at time $t$: Markov jump process by Memoryless property of Exponential distribution + Markov property of Poisson process
  (the $M/M/1/\infty$ queue)
- Infinite server queue with Poisson arrivals and Exponential service times: customer arrived at $T_n$ stays in system till $T_n + \sigma_n$, where $\sigma_n$: service time
  $X_t=$ number of customers present at time $t$:
  Markov jump process (the $M/M/\infty/\infty$ queue)

# Structure of Markov jump processes

## Infinitesimal Generator

$\forall x, y,\ y \neq x \in E$, limits $q_x := \lim_{t\to 0} \frac{1-p_{xx}(t)}{t}$, $q_{xy} = \lim_{t\to 0} \frac{p_{xy}(t)}{t}$ exist in $\mathbb{R}_+$ and satisfy $\sum_{y\neq x} q_{xy} = q_x$

$q_{xy}$: **Jump rate** from $x$ to $y$

$Q := \{q_{xy}\}_{x,y\in E}$ where $q_{xx} = -q_x$: **Infinitesimal Generator** of process $\{X_t\}_{t\in\mathbb{R}_+}$

Formally: $Q = \lim_{h\to 0} \frac{1}{h}[P(h) - I]$ where $I$: identity matrix

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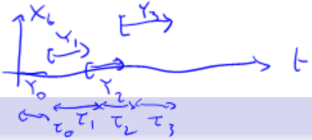


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## Structure of Markov jump processes

Sequence $\{Y_n\}_{n\in\mathbb{N}}$ of visited states: Markov chain with transition matrix $p_{xy} = \mathbb{I}_{x\neq y} \frac{q_{xy}}{q_x}$

Conditionally on $\{Y_n\}_{n\in\mathbb{N}}$, sojourn times $\{\tau_n\}_{n\in\mathbb{N}}$ in successive states $Y_n$: independent, with distributions $\text{Exp}(q_{Y_n})$

# Proof elements

$0 < T_1 < \cdots < T_n < \cdots$: jump times; $\tau_i = T_{i+1} - T_i$ sojourn time

Discrete time chain for $h > 0$: $Z_n := X_{nh}$

$\tau_0^{(h)} := h \inf\{n > 0 : Z_n \neq Z_{n-1}\}$.

$$\underbrace{\mathbb{P}_x(\tau_0 > t)} = e^{\underbrace{\lim_{h\to 0} \frac{t}{h}(p_{xx}(h)-1)}}$$

On $\{\tau_1 > h\}$, $\tau_0^{(h)} - h \leq \tau_0 \leq \tau_0^{(h)}$

By assumption, $\lim_{h\to 0} \mathbb{P}(\tau_1 > h) = 1$, hence

$$\begin{aligned} \mathbb{P}_x(\tau_0 > t) &= \lim_{h\to 0} \mathbb{P}_x(\tau_0^{(h)} > t; \tau_1 > h) \\ &= \lim_{h\to 0} p_{xx}(h)^{\lceil \frac{t}{h} \rceil} \\ &= e^{\lim_{h\to 0} \lceil \frac{t}{h} \rceil \ln p_{xx}(h)}. \end{aligned}$$

$\mathbb{P}_x(\tau_0 > t) = 0 \ \forall t.$

$\tau_0 = 0$ p.s.

$\mathbb{P}_x(\tau_0 > t) = 1 \ \forall t.$

Since $p_{xx}(h) \geq \mathbb{P}_x(\tau_0 > h) \stackrel{h\to 0}{\to} 1$, $\lceil \frac{t}{h} \rceil \ln p_{xx}(h) \sim \frac{t}{h}(p_{xx}(h) - 1)$.
Hence $\exists q_x := \lim_{h\to 0} \frac{1-p_{xx}(h)}{h}$, and $\mathbb{P}_x(\tau_0 > t) = e^{-q_x t}$. Case $q_x = +\infty$ ruled out ($\tau_0 > 0$ a.s.); case $q_x = 0$: $\tau_0 = +\infty$ a.s., i.e. process absorbed in $x$.

$\alpha > 0$ , $y \neq x$ , $y \in E$.

Assuming $q_x > 0$:

$\tau_0^{(h)} = n.h$

$$
\begin{aligned}
\mathbb{E}_x(e^{-\alpha\tau_0}\mathbb{I}_{Y_1=y}) & \sim \mathbb{E}_x[e^{-\alpha\tau_0^{(h)}}\mathbb{I}_{X_{\tau_0^{(h)}}=y}] \\
& = \sum_{n\geq 1} e^{-\alpha nh}\mathbb{P}_x(X_h = \cdots = X_{h(n-1)} = x, X_{hn} = y) \\
& = \sum_{n\geq 1} e^{-\alpha nh}[p_{xx}(h)]^{n-1}p_{xy}(h) \\
& = \sum_{n\geq 1} e^{-\alpha nh}\sum_{m\geq n}\mathbb{P}_x(\tau_0^{(h)} = hm)p_{xy}(h) \\
& = \sum_{m\geq 1}\mathbb{P}_x(\tau_0^{(h)} = m)e^{-\alpha h}\frac{1-e^{-\alpha hm}}{1-e^{-\alpha h}}p_{xy}(h) \\
& = e^{-\alpha h}[1 - \mathbb{E}_x e^{-\alpha\tau_0^{(h)}}]\frac{p_{xy}(h)}{1-e^{-\alpha h}} \\
& \sim [1 - \frac{q_x}{q_x+\alpha}]\frac{p_{xy}(h)}{\alpha h} \\
& = \frac{1}{q_x+\alpha}\frac{p_{xy}(h)}{h}.
\end{aligned}
$$

$$= \left(\frac{q_x}{q_x+\alpha}\right)\left(\frac{q_{xy}}{q_x}\right)$$

f° de $\alpha$

f° de $y$.

Implies existence of limit $\lim_{h\to 0}\frac{p_{xy}(h)}{h} = q_{xy}$, independence of $\tau_0$ and $Y_1$ and $\mathbb{P}_x(Y_1 = y) = \frac{q_{xy}}{q_x}$.

Similar arguments for joint law of $Y_1^n, \tau_0^{n-1}$.

# Examples

$$X_t = N_t = \sum_{n>0} \mathbb{1}_{T_n \in [0,t]}.$$

- for Poisson process $(\lambda)$: only non-zero jump rate

$q_{x,x+1} = \lambda = q_x, \; x \in \mathbb{N}$

$$p_{xy}(t) = o(t), \; y-x \geq 2$$

$$p_{x,x+1}(t) = \lambda t + o(t).$$

$$p_{xy}(t) = \mathbb{P}(\text{Poi}(\lambda t) = y-x) = e^{-\lambda t}\frac{(\lambda t)^{y-x}}{(y-x)!}\mathbb{1}_{y \geq x}.$$

# Examples

$L = \mathrm{diag}(d_i) - A$ ; matrice Laplacienne du graphe


$\mathbb{P}(X_h \notin \{i, \text{voisins de } i\})$

- for Poisson process $(\lambda)$: only non-zero jump rate $q_{x,x+1} = \lambda = q_x,\ x \in \mathbb{N}$

$\leq \mathbb{P}(\mathrm{Exp}(d_i) \leq h \ \&\ \mathrm{Min}\, E_j \leq h)$ où $E_j : \mathrm{Exp}(d_j)$

- For continuous-time random walk on graph $G = (V, E)$, non-zero rates: $q_{i,j} = \mathbb{I}_{i \sim j}$, hence $q_i = d_i$. Generator $Q$ is opposite of so-called Laplacian matrix $L(G)$ of graph $G$

$j \sim i$ $\quad Q = -\mathrm{diag}(d_i) + A,$ $\quad j$ voisin de $i$.

$A$ : adjacence de $G$.

$\mathbb{P}(X_h = j \mid X_0 = i)$

$= \mathbb{P}(\mathrm{Exp}(d_i) \leq h) \times \mathbb{P}(j \text{ choisi})$

$= (1 - e^{-d_i h}) \cdot \frac{1}{d_i} + o(h) \sim h.$

$\leq \sum_{j \sim i} \mathbb{P}(\mathrm{Exp}(d_i) \leq h \ \&\ \mathrm{Exp}(d_j) \leq h)$

$\leq \sum_{j \sim i} (\underbrace{1 - e^{-d_i h}}_{\sim d_i h})(\underbrace{1 - e^{-d_j h}}_{\sim d_j h})$

$\leq O(h^2) = o(h).$

## Examples

$\mathbb{P}_x(X_t = x-1) = \mathbb{P}_x(\mathrm{Exp}(\mu) \le t) + o(t)$
$= \mu t + o(t).$

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$\mathbb{P}(X_t - X_0 \ge 2) \le \mathbb{P}(N_t \ge 2) = o(t).$

- For FIFO $M/M/1/\infty$ queue, non-zero rates: $q_{x,x+1} = \lambda$, $q_{x,x-1} = \mu \mathbb{I}_{x>0}, \; x \in \mathbb{N}$ hence $q_x = \lambda + \mu \mathbb{I}_{x>0}$

$\Rightarrow q_{x,y} = 0$ si $y - x \ge 2$.

$\boxed{\mathbb{P}_x(X_t - X_0 = 1) = \mathbb{P}(N_t = 1) + o(t)}$
$= \lambda t + o(t)$

$\mathbb{P}(X_t - X_0 \le -2) \le \mathbb{P}(\mathrm{Exp}(\mu) \le t, \mathrm{Exp}'(\mu) \le t) \le (1 - e^{-\mu t})^2 = o(t).$

# Examples

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# Structure of Markov jump processes (continued)

Let $T_n := \sum_{k=0}^{n-1} \tau_k$: time of $n$-th jump.

If $T_\infty = +\infty$ almost surely: trajectory determined on $\mathbb{R}_+$, hence generator $Q$ determines law of process $\{X_t\}_{t\in\mathbb{R}_+}$

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Sufficient conditions for non-explosiveness:

- $\sup_{x\in E} q_x < +\infty$
- Recurrence of induced chain $\{Y_n\}_{n\in\mathbb{N}}$
- For **Birth and Death** processes (i.e. $E = \mathbb{N}$, only non-zero rates: $\beta_n = q_{n,n+1}$, birth rate; $\delta_n = q_{n,n-1}$, death rate), non-explosiveness holds if

$$\sum_{n>0} \frac{1}{\beta_n + \delta_n} = +\infty$$

# Takeaway messages

- Poisson process a fundamental continuous-time process, adequate model for aggregate of infrequent independent events

- Markov jump processes: generator $Q$ characterizes distribution if not explosive