Power-law random graphs and small-world phenomenon

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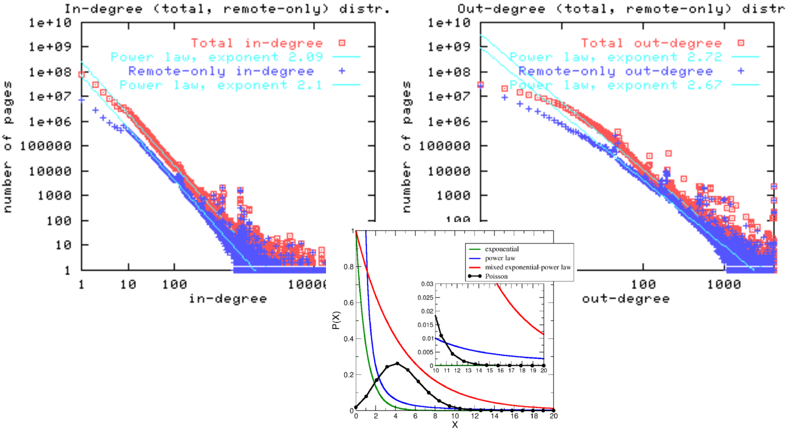
Power-law random graphs

Rough definition: graphs such that number X_i of degree i-nodes verifies $X_i \approx C \times i^{-\beta}$ for some exponent $\beta > 0$ over some *wide* range of values i

Examples: Web graph, FaceBook graph, Hollywood graph, protein interaction graph, Internet router-level graph,...



Also known as **scale-free graphs**: no natural scale for node degrees.

Contrast with E-R graphs $\mathcal{G}(n, d/n)$: for $d >> \ln(n)$, with high probability all node degrees close to d (Exercise!)

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Outline

- The Barabási-Albert (BA) preferential attachment model
- Power-law property of the BA random graph
- Azuma-Hoeffding concentration inequality
- The small-world phenomenon and Kleinberg's navigable graphs

BA preferential attachment model

Iterative construction of graphs $G_t = (\mathcal{V}_t, \mathcal{E}_t),\ t \geq 0$ from initial graph $G_0 = (\mathcal{V}_0, \mathcal{E}_0)$

Step t:

- Add new node t to $\mathcal{V}_{t-1}$, hence $\mathcal{V}_t = \mathcal{V}_0 \cup \{1, \ldots, t\}$, and $n_t := |\mathcal{V}_t| = n_0 + t$
- Connect node t by single edge to *anchor node* $V_t \in \mathcal{V}_{t-1}$, hence $\mathcal{E}_t = \mathcal{E}_{t-1} \cup \{(V_t, t)\}$, and $e_t := |\mathcal{E}_t| = |\mathcal{E}_0| + t$
- Selection procedure of anchor node V_t:
  $\forall v \in \mathcal{V}_{t-1}, \mathbb{P}(V_t = v | \mathcal{F}_{t-1}) = \alpha \frac{1}{n_{t-1}} + (1-\alpha) \frac{D_{t-1}(v)}{2e_{t-1}}$,
  where $\mathcal{F}_{t-1} = \sigma(V_1^{t-1})$ and $D_{t-1}(v)$: degree of node v in G_t.

Main result

Theorem

Let $X_i(t) =$ *number of degree* i*-nodes in* G_t*. Let*

$$c_1 = \frac{2}{3+\alpha}, \ \forall i > 1, \ \frac{c_i}{c_{i-1}} = 1 - \frac{3-\alpha}{2+2\alpha+(1-\alpha)i}.$$

Then for any fixed $i \geq 1$*, almost surely one has* $\lim_{t\to\infty} \frac{1}{t} X_i(t) = c_i$*.*

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Corollary

The Barabási-Albert random graph model for $\alpha \in [0,1[$ *is approximately power-law with exponent* $\beta = \frac{3-\alpha}{1-\alpha}$
in that for some constant $C > 0$*,* $c_i \sim C \times i^{-\beta}$ *as* $i \to \infty$

Comments

- Model extensions: new node creates fixed number (not necessarily one) of edges; edges can be oriented $\Rightarrow$ possibility to induce distinct exponents β_{in}, β_{out} for node in-degree and out-degree distributions
- Precursors of BA model for explaining power-laws by preferential attachment dynamics: Yule model of evolution (Yule, 1925) of number of species in each genera (family of species)
- Alternative explanations of power-laws: Mandelbrot's argument that power laws optimize some criterion (e.g., power-law distribution of word frequencies in a language optimizes information content per symbol)

The Yule model

Species grouped in *genera*. Mutation within a species induces creation of a new species, assumed to belong to same genera with probability $1 - \alpha$ (mild mutation), or to initiate a new genera with probability α (radical mutation)

Discrete time model: at each step choose one species uniformly and add corresponding mutant species. Preferential attachment: bigger genera increase more than smaller ones.

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Theorem

Let $Y_i(t)$: number of genera with i species
Let $d_1 = \frac{\alpha}{2-\alpha}$, $\forall i > 1$, $\frac{d_i}{d_{i-1}} = 1 - \frac{2-\alpha}{1+i(1-\alpha)}$, $i > 1$
Then almost surely $\forall i \geq 1$, $\lim_{t\to\infty} \frac{Y_i(t)}{t} = d_i$

Hence, power-law distribution with exponent $\beta = (2 - \alpha)/(1 - \alpha)$

Proof elements: controlling the mean

Proposition

For fixed $i \geq 1$ let $x_i(t) := \mathbb{E}X_i(t)$ and $\delta_i(t) := x_i(t) - c_i t$.
Then for all $\epsilon > 0$, $\delta_i(t) = o(t^{\epsilon})$ as $t \to \infty$.

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Evolution equations

$$\begin{aligned}
\mathbb{P}(X_1(t+1) = X_1(t)|\mathcal{F}_t) &= \mathbb{P}(D_t(V_{t+1}) = 1|\mathcal{F}_t)\\
&= \alpha\frac{X_1(t)}{n_t} + (1-\alpha)\frac{1\times X_1(t)}{2e_t},\\
\mathbb{P}(X_1(t+1) = X_1(t)+1|\mathcal{F}_t) &= 1 - \alpha\frac{X_1(t)}{n_t} - (1-\alpha)\frac{X_1(t)}{2e_t}
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Hence $x_1(t+1) - x_1(t) = 1 - \left[\frac{\alpha}{n_t} + \frac{1-\alpha}{2e_t}\right] x_1(t)$.

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$$\begin{aligned} \mathbb{P}(X_1(t+1) = X_1(t) | \mathcal{F}_t) &= \mathbb{P}(D_t(V_{t+1}) = 1 | \mathcal{F}_t) \\ &= \alpha \frac{X_1(t)}{n_t} + (1-\alpha) \frac{1 \times X_1(t)}{2e_t}, \\ \mathbb{P}(X_1(t+1) = X_1(t) + 1 | \mathcal{F}_t) &= 1 - \alpha \frac{X_1(t)}{n_t} - (1-\alpha) \frac{X_1(t)}{2e_t} \end{aligned}$$

Hence $x_1(t+1) - x_1(t) = 1 - \left[\frac{\alpha}{n_t} + \frac{1-\alpha}{2e_t}\right] x_1(t)$. This yields

$$\begin{aligned} \delta_1(t+1) - \delta_1(t) &= -c_1 + 1 - \left[\frac{\alpha}{n_t} + \frac{1-\alpha}{2e_t}\right] (c_1 t + \delta_1(t)) \\ &= -c_1 + 1 - (\alpha + \tfrac{1-\alpha}{2}) c_1 + O(t^{-1}) \\ &\quad - \left[\frac{\alpha}{n_t} + \frac{1-\alpha}{2e_t}\right] \delta_1(t), \end{aligned}$$

Proof elements: controlling the mean 2

Hence $\delta_1(t+1) = O(t^{-1}) + [1 - \frac{\alpha}{n_t} - \frac{1-\alpha}{2e_t}]\delta_1(t) = \sum_{s=1}^{t} O(s^{-1}) = o(t^{\epsilon})$

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\mathbb{P}(X_i(t+1) = X_i(t)+1|\mathcal{F}_t) &= \mathbb{P}(D_t(V_{t+1}) = i-1|\mathcal{F}_t) \\
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\mathbb{P}(X_i(t+1) = X_i(t)-1|\mathcal{F}_t) &= \mathbb{P}(D_t(V_{t+1}) = i|\mathcal{F}_t) \\
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hence difference $x_i(t+1) - x_i(t)$ equals

$$\left[\frac{\alpha}{n_t} + \frac{(1-\alpha)(i-1)}{2e_t}\right] x_{i-1}(t) - \left[\frac{\alpha}{n_t} + \frac{i(1-\alpha)}{2e_t}\right] x_i(t)$$

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Writing $x_j(t) = c_j t + \delta_j(t)$, and using induction hypothesis $\delta_{i-1}(t) = o(t^{\epsilon})$ yields

$$|\delta_i(t+1)| \leq |\delta_i(t)| + O(t^{\epsilon-1}).$$

Azuma-Hoeffding inequality

Definition

A sequence $\{M_s\}_{0\leq s\leq t}$ is a martingale with respect to an increasing sequence $\{\mathcal{F}_s\}_{0\leq s\leq t}$ of σ-fields if for all s, M_s is $\mathcal{F}_s$-measurable, and $\mathbb{E}(M_s|\mathcal{F}_{s-1}) = M_{s-1}$.

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Theorem

Let $\{M_s\}_{0\leq s\leq t}$ *be a martingale with bounded increments: there exist constants* c_s *such that almost surely,* $\forall s > 0, |M_s - M_{s-1}| \leq c_s$*.*
Then for all $x > 0$, $\mathbb{P}(M_t - M_0 \geq x) \leq \exp\left(-\frac{x^2}{2\sum_{s=1}^t c_s^2}\right)$.

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Corollary

Under the same assumptions, $\mathbb{P}(|M_t - M_0| \geq x) \leq 2\exp\left(-\frac{x^2}{2\sum_{s=1}^t c_s^2}\right)$.

Azuma-Hoeffding inequality – Proof

Write $M_t - M_{t-1} = Z \cdot c_t + (1 - Z)(-c_t)$ for some random Z.
Necessarily $Z \in [0, 1]$ and $\mathbb{E}(Z|\mathcal{F}_{t-1}) = 1/2$.

For $\theta > 0$ write

$$\begin{aligned}\mathbb{E}[\exp(\theta(M_t - M_{t-1}))|\mathcal{F}_{t-1}] &\leq \mathbb{E}[Ze^{\theta c_t} + (1 - Z)e^{-\theta c_t}|\mathcal{F}_{t-1}] \\ &= \frac{e^{\theta c_t} + e^{-\theta c_t}}{2} \\ &\leq \exp\left(\frac{(\theta c_t)^2}{2}\right).\end{aligned}$$

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Result follows by Chernoff's argument:

$$\mathbb{P}(M_t - M_0 \geq x) \leq \exp\left(-\sup_{\theta>0}[\theta x - \ln \mathbb{E} e^{\theta(M_t - M_0)}]\right)$$

Azuma-Hoeffding inequality – Remarks

- A Gaussian-like bound on tail probabilities $\mathbb{P}(M_t - M_0 \geq x)$

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Corollary

Let $f : \Omega_1 \times \cdots \times \Omega_t \to \mathbb{R}$: measurable function. Assume there exist constants $c_1, \ldots, c_t$ such that for all $x_1^T \in \Omega_1 \times \cdots \times \Omega_t$, all $s \in [t], y_s \in \Omega_s$,

$$|f(x_1^t) - f(x_1^{s-1}, y_s, x_{s+1}^t)| \leq c_t.$$

Then given independent random variables $X_1, \ldots, X_t \in \Omega_1 \times \cdots \times \Omega_t$, random variable $Y := f(X_1^t)$ satisfies for all $x > 0$:

$$\mathbb{P}(Y - \mathbb{E}(Y) \geq x) \leq \exp\left(-\frac{x^2}{2\sum_{s=1}^t c_s^2}\right)$$

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Proof: Apply Azuma-Hoeffding to $M_s := \mathbb{E}[Y|X_1^s]$

Key Lemma (see lecture notes for proof)

Lemma

For fixed $i, t \in \mathbb{N}$ *construct from variable* $X_i(t)$ *in the BA graph model the martingale* $M_s := \mathbb{E}[X_i(t)|\mathcal{F}_s]$ *where* $\mathcal{F}_s = \sigma(V_1^s)$.
Then this martingale has increments bounded by 2.

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For all $i, t \in \mathbb{N}, x \in \mathbb{R}_+$, $\mathbb{P}(|X_i(t) - x_i(t)| \geq x) \leq 2 \exp\left(-\frac{x^2}{8t}\right)$.

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$$\Rightarrow \mathbb{P}\left(|X_i(t) - x_i(t)| \geq 4\sqrt{t\ln(t)}\right) \leq \frac{1}{t^2}.$$

By Borel-Cantelli lemma almost surely only finitely many events $\mathcal{A}_t := \{|X_i(t) - x_i(t)| \geq 4\sqrt{t\ln(t)}\}$ occur. Thus for all $\epsilon > 0$, large enough t:

$$|X_i(t) - c_i t| \leq |\delta_i(t)| + 4\sqrt{t\ln(t)} = O(t^\epsilon) + 4\sqrt{t\ln(t)}.$$

Hence $\lim_{t\to\infty} \frac{X_i(t)}{t} = c_i$ almost surely.

Power laws as optimal design

Mandelbrot's argument for power-law distribution of word occurrences

Words $w_1, \ldots, w_i, \ldots$ over alphabet of a letters,
p_i: frequency of i-th word (in lexicographic order)

Per-word information content: Shannon's entropy
$H(\{p_i\}) := \sum_i p_i \ln(1/p_i)$

Let $|w_i|$: length of word w_i.
Average information per character: $\frac{H(\{p_i\})}{L(\{p_i\})}$ where $L(\{p_i\}) := \sum_i p_i |w_i|$

Power laws as optimal design 2

Result:
Average information per character $\frac{H(\{p_i\})}{L(\{p_i\})}$ maximized by frequencies $p_i \propto e^{-C|w_i|}$.

Power laws as optimal design 2

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Using $a\frac{a^{|w_i|}-1}{a-1} \geq i > a\frac{a^{|w_i|-1}-1}{a-1}$, this yields
$\ln(p_i) \sim -C|w_i| \sim -C\ln(i)/\ln(a)$,

i.e. a power-law shape for the optimal distribution $\{p_i\}$

Small-world graphs–Milgram's experiment (67)

S. Milgram asked people to transmit letter (based on destination name, city and profession) to some friend, recursively, until letter reaches destination

Most letters reached destination in ≤ 6 hops

Now known as the "small-world", or "six degrees of separation" phenomenon: everyone connected to everyone in at most 6 hops

Consequence: the "social graph" has *diameter* at most 6

Small-world graphs–models

First modeling attempt:
view social links as Erdős-Rényi $\mathcal{G}(n, d/n)$ graph.
$\rightarrow$ For $d \gg \ln(n)$, diameter of $\mathcal{G}(n, d/n)$: $O(\ln(n)/\ln(d))$, hence small

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Augment structured graph (eg "path" graph: nodes=$[n]$; edge (i, j) if and only if $|i - j| = 1$)
with random edges, e.g. each $i \in [n]$ creates with probability $\epsilon > 0$ edge towards short-cut destination Z_i.

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Then for any $\epsilon > 0$, and Z_i i.i.d. on $[n]$, diameter is w.h.p. $O(\log(n))$, hence small

Kleinberg's navigable graphs

Strogatz-Watts model: explains small diameter as result of structured (large diameter) graph augmented with random edges

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Kleinberg's model: similar to Strogatz-Watts, with $\mathbb{P}(Z_i = j) \propto |i - j|^{-\alpha}$, for exponent $\alpha \geq 0$

For neighbor selection (or routing) algorithm *Alg*, let $T_{Alg}(s, d)$: number of steps for letter started at $s \in [n]$ to reach destination $d \in [n]$

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Does not explain how people **found** short paths in Milgram's experiment

Kleinberg's model: similar to Strogatz-Watts, with $\mathbb{P}(Z_i = j) \propto |i - j|^{-\alpha}$, for exponent $\alpha \geq 0$

For neighbor selection (or routing) algorithm *Alg*, let $T_{Alg}(s, d)$: number of steps for letter started at $s \in [n]$ to reach destination $d \in [n]$

Simplest distributed routing algorithm: *greedy*, i.e. letter for $d \in [n]$ forwarded by $i \in [n]$ to $\text{Argmin}_{j \sim i} |d - j|$

Kleinberg's navigable graphs

For $\alpha = 1$ (so-called *Harmonic* distribution of shortcuts Z_i),
$\forall s, d \in [n], \;\; \mathbb{E} T_{greedy}(s, d) = O(\ln(n)^2)$

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For $\alpha \neq 1$, and **any** distributed algorithm *Alg* (relying only on information $d, \{j : j \sim i\}$ to choose next hop when at node i), one has
$\mathbb{E}\, T_{alg}(s, d) = \Omega(n^\beta)$
for fraction $\Omega(1)$ of pairs $s, d \in [n]$, and positive exponent β

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For D-dimensional initial structured graph ($V = [n]^D$,
$i \sim j \Leftrightarrow \|i - j\|_1 = 1$), same result holds where now harmonic distribution defined as $\mathbb{P}(Z_i = j) \propto \|j - i\|^{-D}$

Takeaway messages

- Preferential attachment dynamics induce scale-free, power-law distributions
- Examples: Barabási-Albert random graph model, Yule model of number of species per genera
- Azuma-Hoeffding inequality: Chernoff-like bound for martingales with bounded increments, an example of a *concentration inequality*
- Navigable graphs: obtained from harmonic distribution of shortcuts, provide fast routing based on greedy algorithm
- A potential model of information location in social networks (as in Milgram's experiment), but also a potential design for engineered (computer) networks