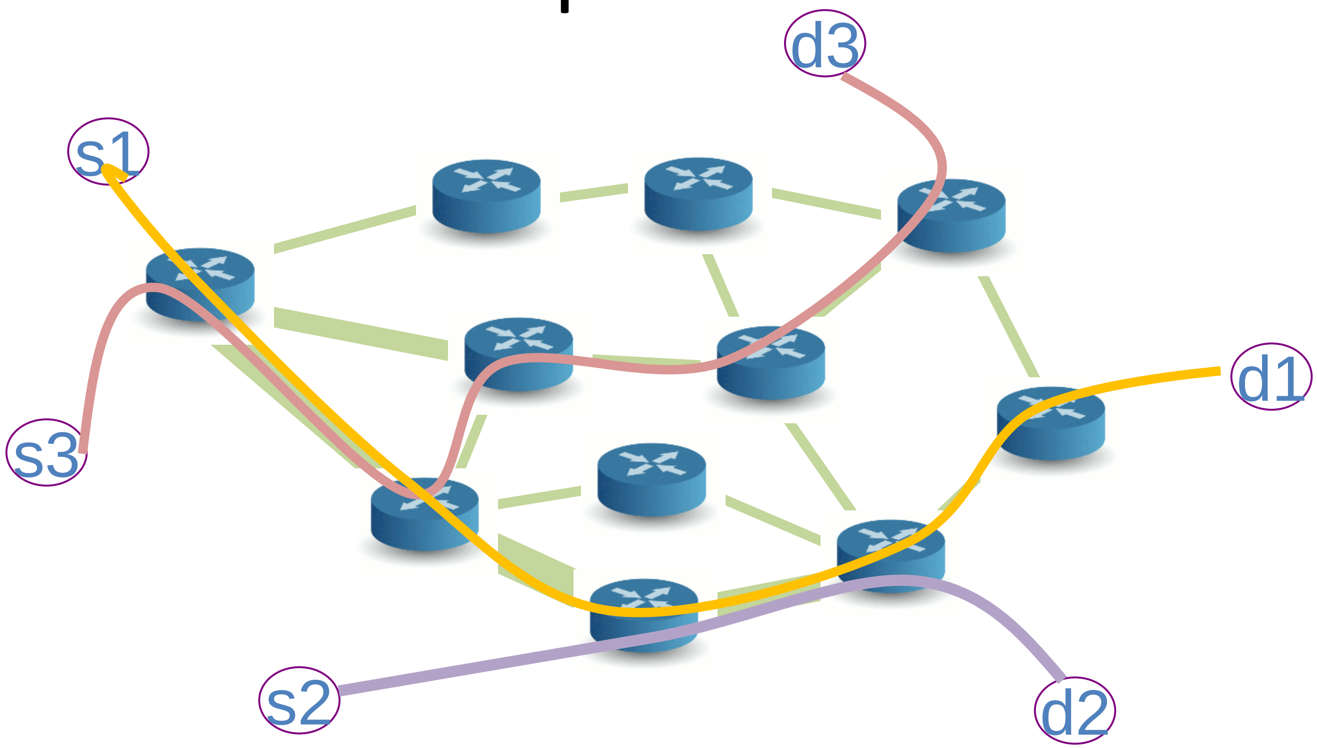
Network resource allocation: principles and algorithms

Mathematical framework:

Convex optimization,

dynamical systems (ordinary differential equations)

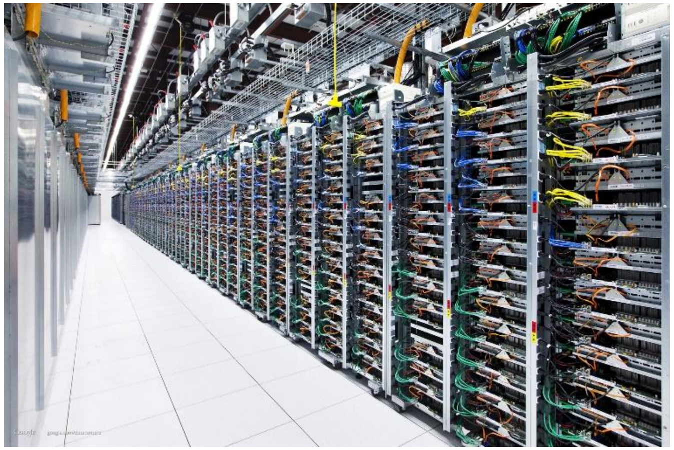
Motivating example: Distributed control of data transport in the Internet



- How to assign bandwidth in networks
  - Understanding TCP, the protocol regulating most Internet traffic
  - Still an active research topic, in the context of datacenter networks (see « DC-TCP »)

Other application scenarios of current interest

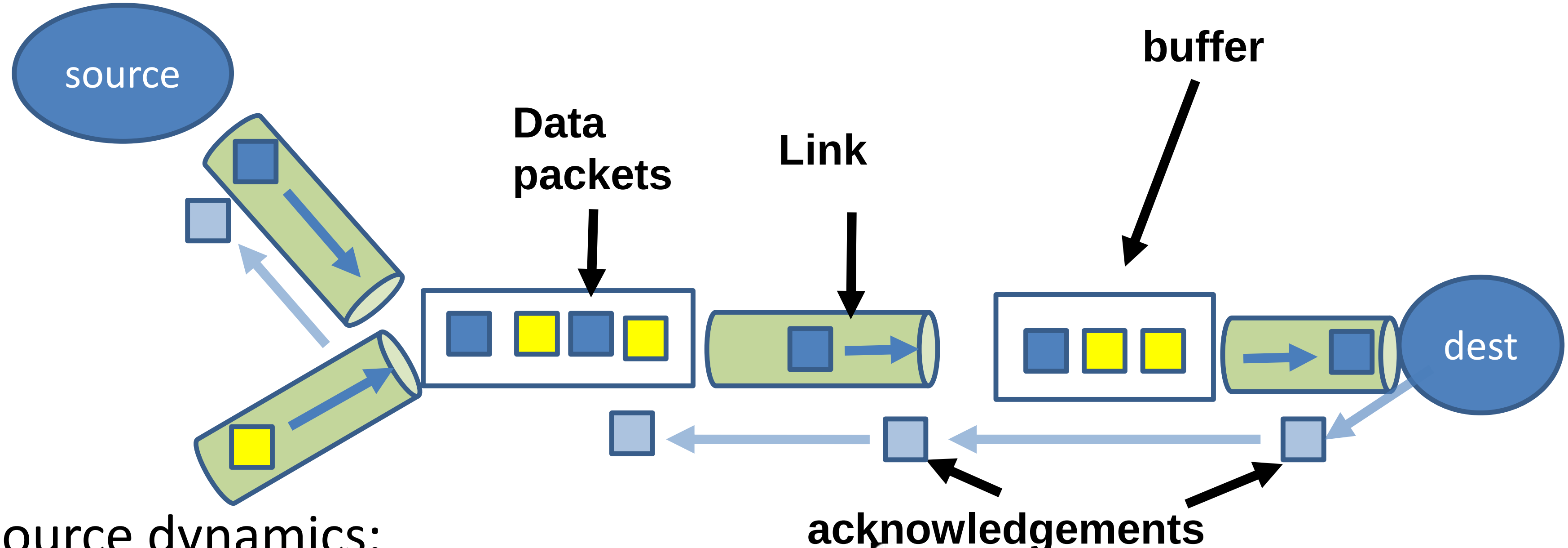
- Allocation of {storage,bandwidth,CPU} resources in **cloud computing**



A Google datacenter

- Allocation of energy to consumers in the **smart grid**, under demand-response scenarios

TCP in one slide



Source dynamics:

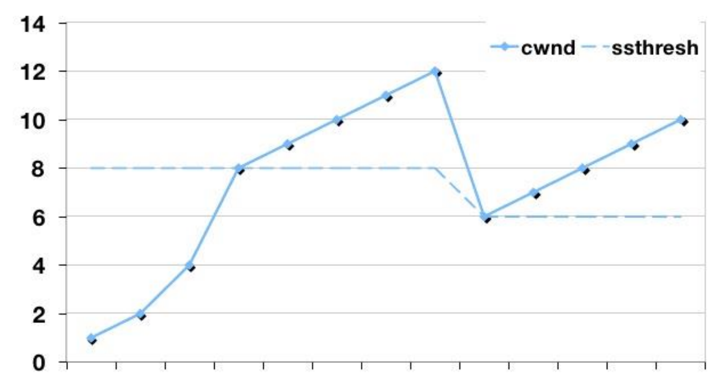
- Maintain Nb of (sent&not acked pkts)=:cwnd (congestion window)
- Update cwnd

← cwnd+1/cwnd upon receipt of pkt ack

← cwnd/2 upon detection of pkt loss

“Congestion avoidance” alg introduced in 1993

After Internet congestion collapse



Outline

- ❑ Resource allocation principles
  - ❑ Fairness criteria
  - ❑ Utility optimization models inspired by micro-economics
- ❑ A "primal" algorithm
- ❑ Reverse-engineering TCP
- ❑ Lagrangian, duality and Lagrange multipliers
- ❑ A "dual" algorithm

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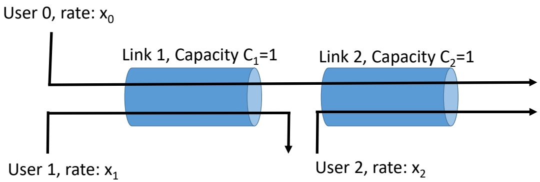
Network model



- Resources, or links, $\ell \in \mathcal{L}$, each with capacity $C_\ell > 0$
- Users, or transmissions, or flows, $s \in \mathcal{S}$
- User s uses same rate at all $\ell \in s$ ($s \leftrightarrow$ subset of $\mathcal{L}$)

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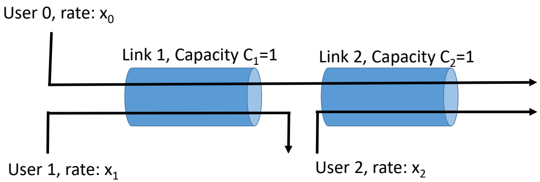

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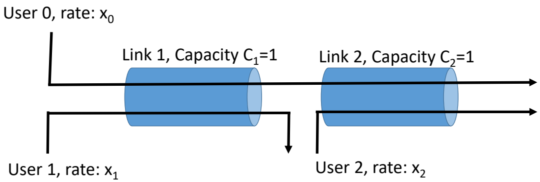

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POTENTIAL APPLICATIONS

- Links on single path from source to destination
- Links on tree of transmission from source to set of receivers

Allocation principles 1

- **max-min fairness**: feasible x^{mm} such that
  $\forall s \in \mathcal{S}, \exists \ell \in s$ with $\sum_{t \ni \ell} x_t^{mm} = C_\ell$ and $x_s^{mm} = \max_{t \ni \ell} x_t^{mm}$
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Notion introduced by F. Kelly (Cambridge University) in 1997

Allocation principles 1

Alternative characterization: Nash's bargaining solution (1950)

i.e. unique vector $\phi(\mathcal{C})$ in feasible convex set $\mathcal{C} \subset \mathbb{R}_+^{\mathcal{S}}$ s.t.

- Pareto efficiency: $\phi(\mathcal{C}) \leq x \in \mathcal{C} \Rightarrow x = \phi(\mathcal{C})$
- independence of irrelevant alternatives: $\phi(\mathcal{C}) \in \mathcal{C}' \subset \mathcal{C} \Rightarrow \phi(\mathcal{C}) = \phi(\mathcal{C}')$
- symmetry: $\mathcal{C}$ symmetric $\Rightarrow \phi(\mathcal{C})_i \equiv \phi(\mathcal{C})_1$
- scale invariance: for diagonal D with $D_{ii} > 0$, $\phi(D\mathcal{C}) = D\phi(\mathcal{C})$

Allocation principles 2

Network Utility Maximization x^*: solution of

$$\begin{array}{rcc} \text{Max} & \sum_s U_s(x_s) & \\ \text{Over} & x_s \geq 0 & (P) \\ \text{Such that} & \forall \ell, \sum_{s \ni \ell} x_s \leq C_\ell & \end{array}$$

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[Exercise: $\lim_{\alpha \to 1} x(1, \alpha) = x^{pf}$ and $\lim_{\alpha \to +\infty} x(1, \alpha) = x^{mm}$]

Relaxed constraints and a "primal" algorithm

$$\begin{aligned} \text{Relaxed problem:} \quad \text{Max} \quad & \sum_s U_s(x_s) - \sum_\ell C_\ell(y_\ell) \\ \text{Over} \quad & x_s \geq 0 \\ \text{with} \quad & y_\ell = \sum_{s \ni \ell} x_s \end{aligned} \tag{RP}$$

for concave increasing utility functions U_s and convex increasing cost functions C_ℓ

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primal algorithm: for U_s and C_ℓ differentiable, and positive gain function $\kappa_s : \mathbb{R}_+ \to \mathbb{R}_+$, let

$$\frac{d}{dt} x_s = \kappa_s(x_s) \left(U_s'(x_s) - \sum_{\ell \in s} C_\ell'(y_\ell) \right) \quad \text{"gradient ascent"}$$

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$\rightarrow$ Implementable in a distributed fashion

Stability via Lyapunov functions

Criterion for convergence of ODE $\dot{x} = F(x)$ with trajectories in $O \subset \mathbb{R}^n$

Theorem

Assume F continuous on O, and $\exists V : O \to \mathbb{R}$ such that:

(i) V continuously differentiable

(ii) $\forall a \leq A$, $\{x \in O : V(x) \leq A\}$ and $\{x \in O : V(x) \in [a, A]\}$ either compact or empty

(iii) $\forall x \in O \setminus B$, $\langle \nabla V(x), F(x) \rangle < 0$, where $B = argmin_{x \in O}\{V(x)\}$

Then $\lim_{t\to\infty} V(x(t)) = \inf_{x\in O} V(x)$, $\lim_{t\to\infty} d(x(t), B) = 0$.

If $B = \{x^*\}$ then $\lim_{t\to\infty} x(t) = x^*$.

Application to gradient ascent / descent dynamics

$$\frac{d}{dt}x_s = \kappa_s(x_s)\left(U_s'(x_s) - \sum_{\ell \in s} C_\ell'(y_\ell)\right)$$

Let $W(x) = \sum_s U_s(x_s) - \sum_\ell C_\ell(y_\ell)$ (system *welfare*)
and $V(x) = -W(x)$

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Theorem

For U_s strictly concave, differentiable with $U_s'(0^+) = +\infty$,
C_ℓ convex, continuously differentiable,
[$\Rightarrow$ strict concavity and continuous differentiability of W]
$\kappa_s > 0$, continuous [$\Rightarrow$continuity of F]
$\exists x_s > 0$ s.t. $U_s'(x_s) < \sum_{\ell \in s} C_\ell'(x_s)$
[$\Rightarrow$ Max of W achieved at single point $x^* \in O := (0, \infty)^{\mathcal{S}}$]
Then "primal" dynamics converge to unique maximizer x^* of W

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Approx. $x_s \approx cwnd_s / T_s$ where T_s: packet round-trip time

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Can tweak congestion avoidance alg. if want e.g. proportional fairness ($\alpha = 1$) instead

Convex optimization: Lagrangian, duality, multipliers

Generic convex optimization program
For convex set $\mathcal{C}^0$, convex functions J, $f_\ell : \mathcal{C}^0 \to \mathbb{R}$,

$$\begin{array}{rcc} \text{Min} & J(x) & \\ \text{Over} & x \in \mathcal{C}^0 & (P) \\ \text{Such that} & \forall \ell \in \mathcal{L}, f_\ell(x) \leq 0 & \end{array}$$

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Dual problem (D): Max $D(\lambda)$ Over $\lambda \geq 0$
where $D(\lambda) := \inf_{x \in \mathcal{C}^0} L(x,\lambda)$

Kuhn-Tucker theorem and strong duality

Def: $\lambda^* \geq 0$ a Kuhn-Tucker vector iff $\forall x \in \mathcal{C}^0, L(x, \lambda^*) \geq J^*$
where J^*: optimal value of (P).

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Theorem

Assume there exists λ^* a Kuhn-Tucker vector. Then
(i) λ^* solves (D), and $J^* = D^*$ (a.k.a. ***strong duality****)*
(ii) $x^* \in \mathcal{C}^0$ if optimal for (P) then achieves $\min_{x \in \mathcal{C}^0} L(x, \lambda^*)$
(iii) For $x^* \in int(\mathcal{C}^0)$ an optimum of (P) at which $\exists \nabla J, \nabla f_\ell$, then

$$\begin{array}{ll} \forall \ell, \lambda_\ell^* f_\ell(x^*) = 0 & \textbf{(complementarity)} \\ \nabla J(x^*) + \sum_\ell \lambda_\ell^* \nabla f_\ell(x^*) = 0 & \textbf{(stationarity)} \end{array}$$

Reciprocally assume stationarity + complementarity
for some $\lambda^* \geq 0$ and some x^* feasible for (P),
Then λ^*: Kuhn-Tucker and x^* optimal for (P)

Sufficient conditions for applying Kuhn-Tucker

Lemma

Assume $J^* > -\infty$ *and* $\exists \hat{x} \in \mathcal{C}^0$ *such that* $\forall \ell,\ f_\ell(\hat{x}) < 0$.
Then a Kuhn-Tucker vector λ^* *exists.*

In practice: verify Lemma's conditions + existence of optimum $x^* \in \mathrm{int}(\mathcal{C}^0)$ at which $\exists \nabla J$, ∇f_ℓ.
Then characterize x^* that verifies complementarity + stationarity (now guaranteed to exist)

Solving original problem: dual algorithm

Lagrangian: $L(x,\lambda) = \sum_s U_s(x_s) + \sum_\ell \lambda_\ell [C_\ell - \sum_{s \ni \ell} x_s]$

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$\Rightarrow \frac{\partial}{\partial \lambda_\ell} D(\lambda) = C_\ell - \sum_{s \ni \ell} g_s(\lambda^s)$

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Dual: $D(\lambda) = \sum_s U_s(g_s(\lambda^s)) + \sum_\ell \lambda_\ell [C_\ell - \sum_{s \ni \ell} g_s(\lambda^s)]$

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$\Rightarrow \frac{\partial}{\partial \lambda_\ell} D(\lambda) = C_\ell - \sum_{s \ni \ell} g_s(\lambda^s)$

Dual algorithm: $$\begin{aligned} x_s &\equiv g_s(\lambda^s), \\ \dot{\lambda}_\ell &= \kappa_\ell \left[\sum_{s \ni \ell} x_s - C_\ell\right]^+_{\lambda_\ell} \end{aligned}$$

where $[a]^+_b = a$ if $b > 0$, $\max(a, 0)$ if $b \le 0$

Solving original problem: dual algorithm

Theorem

Under suitable conditions
(U_s strictly concave, twice differentiable, $U_s'(0^+) = +\infty$, $U_s'(+\infty) = 0$)
Trajectories x_s of dual algorithm converge to unique maximizer x^* of primal problem.

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Potential implementation: multiplier dynamics $\equiv$ queue dynamics
$\Rightarrow$ Let $\lambda^\ell =$ queueing delay of packets and instantaneously let x_s to $g_s(\lambda^s)$
$\Rightarrow$ Principle underlying TCP-Vegas, an alternative to default TCP (TCP Reno)

Takeaway messages

- For unconstrained convex minimization, gradient descent converges to optimizer [Lyapunov stability]
- Admits distributed implementation in network optimization setting
- TCP implicitly achieves (w, α)-fair allocation by running gradient descent
- Kuhn-Tucker Theorem: Complementarity + Stationarity characterization of (P)'s optima
- Queue dynamics implicitly perform gradient descent for multipliers of constrained program

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Pending question: How to discriminate between allocation objectives?