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Non-backtracking matrix

$B$: $2m \times 2m$ matrix where $m$: number of edges of $G$, defined as $B_{i \rightarrow j, k \rightarrow \ell} = \mathbb{I}_{j=k} \mathbb{I}_{i \neq \ell}$.

Allows counting of non-backtracking paths in $G$: $(B^t)_{i \rightarrow j, k \rightarrow \ell} = \cdots \cdots |\{\text{NB paths with } t + 1 \text{ edges, started at } i \rightarrow j, \text{ ending at } k \rightarrow \ell\}|$.

Spectrum of $B$: $\lambda_1(B) \geq |\lambda_2(B)| \geq \cdots \geq |\lambda_{2m}(B)|$.
Spectrum of NBM $B$ for sparse SBM $G \sim G(n, P, \alpha)$

Mean progeny matrix $M = \alpha P$, spectrum:
$$
\lambda_1(M) = \alpha \geq |\lambda_2(M)| = \alpha |\lambda_2(P)| \geq \cdots \geq |\lambda_q(M)| = \alpha |\lambda_q(P)|.
$$

Let $x_i \in \mathbb{R}^q$: eigenvector of $M$ associated with $\lambda_i(M)$.
For $e = u \rightarrow v \in \tilde{E}$, define $y_i(e) = x_i(\sigma_u)$.
For $\ell = c \ln(n)$, $c > 0$ fixed constant, let $z_i = B^\ell B^{\top \ell} y_i$.

**Theorem**

Let $r_0 = \sup \{ i \in [q] : \lambda_i(M)^2 > \lambda_1(M) \}$.
(Note: $r_0 \geq 2 \iff \alpha \lambda_2(P)^2 > 1$, i.e. above Kesten-Stigum threshold).
Then $\forall i \in [r_0]$, eigenpair $(\lambda_i(B), \xi_i)$ of $B$ verifies:

$$
\lambda_i(B) \xrightarrow{\text{proba.}}_{n \to \infty} \lambda_i(M).
$$

$$
\exists x_i \in \mathbb{R}^q: \text{eigenvector of } M \leftrightarrow \lambda_i(M) \text{ such that for associated } z_i \in \mathbb{R}^{2m},
\lim_{n \to \infty} \frac{\langle z_i, \xi_i \rangle}{\|z_i\|,\|\xi_i\|} = 1.
$$

For $i > r_0$, $|\lambda_i(B)| \leq \sqrt{\lambda_1(M)} + o(1)$. 
Spectrum of NBM for $q = 2$, above Kesten-Stigum threshold.
Intuition for form of eigenvectors:

For large $\ell$,

$$z_i := B^\top \ell y_i(u \to v) = \sum_{(u',v')} x_i(\sigma_{v'}) B^\top \ell (u \to v, u' \to v')$$

$$\approx \sum_{(u',v')} x_i(\sigma_{v'})(\exists\text{ length } (\ell + 1)-\text{NBP } (u'\to v') \to (u \to v))$$

$$\approx [\alpha \lambda_i(P)]^\ell Z_i(u),$$

where $Z_i(u)$: martingale limit as in analysis of Census reconstruction for tree model.

Then by construction $B^\top z_i \approx \alpha \lambda_i(P) z_i$, i.e. $z_i$: approximate eigenvector.
Relation to Ramanujan graphs

Definition (Lubotzky, Phillips and Sarnak, 1995)

Ramanujan graphs = \(d\)-regular graphs with adjacency matrix \(A\) such that

\[
\sup_{\lambda \in \text{Sp}(A), |\lambda| \neq d} |\lambda| \leq 2\sqrt{d - 1}.
\]

Recall Alon-Boppana's inequality, \(\lambda_2(A) \geq 2\sqrt{d - 1}(1 - O(\Delta^{-2}))\) where \(\Delta\): graph diameter. Hence Ramanujan graphs: regular graphs with optimal spectral gap.

Theorem (Ihara-Bass formula; see lecture notes)

For graph with \(n\) vertices, \(m\) edges, adjacency matrix \(A \in \{0, 1\}^{n \times n}\), non-backtracking matrix \(B \in \{0, 1\}^{2m \times 2m}\), matrix \(Q = \text{Diag}([d_i - 1]) \in \mathbb{R}^{n \times n}\), where \(d_i\): degree of node \(i\), then

\[
\forall u \in \mathbb{C}, (1 - u^2)^{n-m} \text{Det}(I - uB) = \text{Det}(I - uA + u^2Q).
\]
Relation to Ramanujan graphs

**Corollary**

A \(d\)-regular graph, \(d \geq 2\) is Ramanujan iff its non-backtracking matrix \(B\) is such that all eigenvalues \(\lambda\) of \(B\) satisfy either \(|\lambda| = d - 1 = \rho(B)\), or \(|\lambda| \leq \sqrt{\rho(B)}\).

**Definition (extended)**

Ramanujan graphs: not necessarily regular graphs with NBM \(B\) such that for \(\lambda \in \text{Sp}(B)\), either \(|\lambda| = \rho(B)\), or \(|\lambda| \leq \sqrt{\rho(B)}\).

Theorem’s result implies: for \(G \sim G(n, \alpha/n)\) Erdős-Rényi graph, its NBM has w.h.p. spectrum: \(\rho(B) = \alpha + o(1)\), and all other eigenvalues \(|\lambda| \leq \sqrt{\alpha} + o(1)\). Hence up to \(o(1)\) error, Ramanujan according to extended definition.

Result is a (non-regular) counterpart of [Friedman 2008]: for \(d \geq 3\), random \(d\)-regular graphs such that w.h.p.,

\[
\sup_{\lambda \in \text{Sp}(A), |\lambda| \neq d} |\lambda| \leq 2\sqrt{d - 1} + o(1).
\]
Existence of hard phase [Banks et al. 2016]

For symmetric Potts model with $q$ blocks, parametrized by $\alpha P_{ii} = c_{in}$, $\alpha P_{ij} = c_{out}$, $j \neq i \in [q]$, average degree $\alpha = \frac{c_{in} + (q-1)c_{out}}{q} = \lambda_1(M)$, $\lambda_2(M) = \frac{c_{in} - c_{out}}{q}$.

**Definition**

Partition of $[n]$ is **good** if it splits $[n]$ into $q$ equal-size groups, such that the number $m_{in}$ (resp. $m_{out}$) of edges intra-groups (resp. inter-groups) verifies $|m_{in} - \bar{m}_{in}| \leq n^{2/3}$, $|m_{out} - \bar{m}_{out}| \leq n^{2/3}$, where $\bar{m}_{in} = \frac{n c_{in}}{2q}$, $\bar{m}_{out} = \frac{n(q-1)c_{out}}{2q}$.

Then: for $q \geq 4$, can find parameters $c_{in}$, $c_{out}$ such that: $\lambda_2^2 < \lambda_1$, i.e. below Kesten-Stigum threshold, and for some $\epsilon > 0$, $\lim_{n \to \infty} \mathbb{E} |\{ \text{good partitions with overlap} \leq \epsilon \}| = 0$. Since there exists a good partition w.h.p. (partition $[q]$ according to true spin values $\sigma$), this implies that partial reconstruction is feasible below K-S threshold.
Remark

Partial reconstruction of communities in model $G(n, \alpha, P)$ is polynomial-time feasible above KS threshold, and believed to be infeasible in poly-time below KS threshold.

Hence existence of two or three phases: reconstruction

- Infeasible
- feasible (information-theoretically) but computationally hard
- poly-time feasible (above KS threshold)
Example: for graph $G$, decide whether

$H_0 : G \sim \mathbb{P}_n : G(n, \alpha/n)$ (Erdős-Rényi graph), or

$H_1 : G \sim \mathbb{Q}_n : G(n, \alpha, P)$ (sparse SBM).

Let Likelihood ratio $Y_n := \frac{d\mathbb{Q}_n}{d\mathbb{P}_n}$. By Neyman-Pearson’s lemma, for all $\epsilon > 0$, $\exists$ test $T \in \{0, 1\}$ maximizing $\mathbb{Q}_n(T = 1)$ among tests such that $\mathbb{P}_n(T = 1) \leq \epsilon$ given by

- $T = 1$ if $Y_n > t$,
- $T = 1(0)$ with prob. $p(1-p)$ if $Y_n = t$,
- $T = 0$ if $Y_n < t$.

Definition

Detection between $H_0, H_1$ (i.e. $\mathbb{P}_n, \mathbb{Q}_n$) is feasible if $\exists$ tests $\{T_n\}_{n>0}$ such that $\mathbb{P}_n(T_n = 1) \xrightarrow{n \to \infty} 0$ and $\mathbb{Q}_n(T_n = 0) \xrightarrow{n \to \infty} 0$. 
Definition (Contiguity)

Sequence \( \{P_n\}_{n>0} \) contiguous with respect to sequence \( \{Q_n\}_{n>0} \) iff for all sequence of events \( \{E_n\}_{n>0} \),
\[
\lim_{n \to \infty} Q_n(E_n) = 0 \Rightarrow \lim_{n \to \infty} P_n(E_n) = 0
\]

Lemma

If for some \( c > 0 \), \( \sup_{n>0} E_{Q_n} Y_n^2 \leq c \), then \( \{P_n\}_{n>0} \) contiguous with respect to \( \{Q_n\}_{n>0} \).

Proof: 
\[
P_n(E_n) = E_{Q_n} [Y_n \mathbb{1}_{E_n}] \leq \sqrt{Q_n(E_n) E_{Q_n} Y_n^2} \leq \sqrt{c} \sqrt{Q_n(E_n)}.
\]
**Property:** If contiguity (of $\{P_n\}_{n>0}$ w.r.t. $\{Q_n\}_{n>0}$, or of $\{Q_n\}_{n>0}$ w.r.t. $\{P_n\}_{n>0}$) then detection is infeasible.

Indeed, let $E_n = \{T_n = 0\}$, where $\{T_n\}_{n>0}$: tests supposed to achieve detection. Thus $Q_n(E_n) \to 0$, hence by contiguity, $P_n(E_n) \to 0$. Thus impossible to have $P_n(E_n) \to 1$ as required by detectability.

**Lemma**

For $Y = \frac{dP}{dQ}$, $d_{\text{var}}(P, Q) \leq 2\sqrt{E_Q Y^2 - 1}$, hence

$$\lim_{n \to \infty} E_{Q_n} Y_n^2 = 1 \Rightarrow \lim_{n \to \infty} d_{\text{var}}(P_n, Q_n) = 0.$$ 

**Proof:**

$$d_{\text{var}}(P, Q) = 2 \sup_A |P(A) - Q(A)| = 2 \sup_A |E_Q(I_A(Y - 1))|$$

$$\leq 2 \sup_A \sqrt{Q(A)E_Q(Y - 1)^2} \leq 2\sqrt{E_Q Y^2 - 1}.$$
Detection between $\mathcal{G}(n, \alpha/n)$ and binary symmetric SBM

$\sigma_i$ i.i.d. uniform on $\{-, +\}$; $\mathbb{P}((u, v) \in E|\sigma[n]) = \begin{cases} \frac{a}{n} & \text{if } \sigma_u \sigma_v = +, \\ \frac{b}{n} & \text{if } \sigma_u \sigma_v = -. \end{cases}$

Spectrum of mean progeny matrix $M = \frac{1}{2} \begin{pmatrix} a & b \\ b & a \end{pmatrix}$: $\alpha = \frac{a+b}{2}, \ \beta = \frac{a-b}{2}$.

Kesten-Stigum condition: $\tau := \frac{\beta^2}{\alpha} = \frac{(a-b)^2}{2(a+b)} > 1$.

Theorem (Mossel-Neyman-Sly 2015)

Distinction between $\mathbb{P}_n : \mathcal{G}(n, \alpha/n)$ and $\mathbb{Q}_n$: symmetric binary SBM is feasible if $\tau > 1$, infeasible if $\tau < 1$.

Case $\tau > 1$: By previous theorem (spectrum of NBM $B$ for SBM), eigenvalue of second largest modulus of $\lambda_2(B)$ verifies w.h.p.

under $\mathbb{Q}_n$: $|\lambda_2(B)| = |\beta| + o(1)$;

under $\mathbb{P}_n$, $|\lambda_2(B)| \leq \sqrt{\alpha} + o(1)$.

Hence test $T_n = \mathbb{I}_{|\lambda_2(B)|^2 \geq |\beta|\sqrt{\alpha}}$ successful at detection.
Lemma

For $Y_n = \frac{dQ_n}{dP_n}$, and $\tau < 1$, one has $\mathbb{E}_{P_n} Y_n^2 = (1 + o(1)) \frac{e^{-\tau/2} - \tau^2/4}{\sqrt{1-\tau}}$.

Thus $\sup_{n>0} \mathbb{E}_{P_n} Y_n^2 < +\infty$, hence infeasibility of detection.

$Y_n(g) = \sum_{s \in \{\pm\}^n} \frac{Q_n(G = g; \sigma[n] = s)}{P_n(G = g)} = 2^{-n} \sum_{s \in \{\pm\}^n} \prod_{(uv)} W_{uv}(s)$, where

$W_{uv}(s) = \begin{cases} 
\frac{2a}{a+b} & \text{if } s_us_v = +, (uv) \in E(g), \\
\frac{2b}{a+b} & \text{if } s_us_v = -, (uv) \in E(g), \\
\frac{1-a/n}{1-(a+b)/2n} & \text{if } s_us_v = +, (uv) \notin E(g), \\
\frac{1-b/n}{1-(a+b)/2n} & \text{if } s_us_v = -, (uv) \notin E(g).
\end{cases}$

Fix $s, t \in \{\pm\}^n$. Note $W_{uv} = W_{uv}(s)$, $V_{uv} = W_{uv}(t)$, and $\epsilon = s_us_v t_ut_v \in \{-, +\}$.

Then: $\mathbb{E}_{P_n} W_{uv} V_{uv} = 1 + \epsilon \frac{(a-b)^2}{2n(a+b)} + \epsilon \frac{(a-b)^2}{4n^2} + O(n^{-3})$. 
Let $\gamma := \frac{\tau}{n} + \frac{(a-b)^2}{4n^2}$, $S_{\pm} = |\{(uv) : s_usvt_utv = \pm\}|$. Then:

\[
E_P \mathcal{Y}_n^2 = 2^{-2n} \sum_{s,t} \prod_{(uv)} E_P W_{uv} V_{uv} \\
= 2^{-2n} \sum_{s,t} (1 + \gamma + O(n^{-3}))S_+ (1 - \gamma + O(n^{-3}))S_- .
\]

Let $\rho = \rho(s, t) := \frac{1}{n} \sum_{i \in [n]} s_it_i$.

Then: $\rho^2 = \frac{1}{n} + \frac{2}{n^2} \sum_{u \neq v} s_us_v t_ut_v = \frac{1}{n} + \frac{2}{n^2} (S_+ - S_-)$

Also: \[ \frac{2}{n^2} (S_+ + S_-) = \frac{2}{n^2} \binom{n}{2} = 1 - \frac{1}{n} \]

Hence:

\[ S_+ = (1 + \rho^2) \frac{n^2}{4} - \frac{n}{2}, \quad S_- = (1 - \rho^2) \frac{n^2}{4} . \]

For fixed $x \in \mathbb{R}$, one has: \((1 + x/n)^{n^2} = (1 + o(1)) e^{nx - x^2/2} . \)

Thus:

\[
(1 + \gamma + O(n^{-3}))S_+ \sim e^{-\tau/2} (1 + \gamma) n^2 (1 + \rho^2)/4 \\
\sim e^{-\tau/2} [e^{n\tau + (a-b)^2/4 - n^2 \gamma^2/2} (1 + \rho^2)/4
\]

Similarly,

\[
(1 - \gamma + O(n^{-3}))S_- \sim \left[ e^{n\tau - (a-b)^2/4 - n^2 \gamma^2/2} \right] (1 - \rho^2)/4
\]
\[ \mathbb{E}_{\mathbb{P}_n} Y_n^2 \sim 2^{-2n} e^{-\tau/2-\tau^2/4} \sum_{s,t} e^{(\rho^2/2)(n\tau+(a-b)^2/4)} \]
\[ \sim e^{-\tau/2-\tau^2/4} \mathbb{E} e(Z_n^2/2)(\tau+(a-b)^2/(4n)), \]

where \( Z_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \sigma_i \), with \( \sigma_i \) i.i.d. uniform random signs.

By CLT, \( Z_n \xrightarrow{\mathcal{L}} N(0,1) \) so that \( e^{Z_n^2/2(\tau+(a-b)^2/(4n))} \xrightarrow{\mathcal{L}} e^{Z^2\tau/2} \), with \( Z \sim N(0,1) \).

Uniform integrability of r.v. \( e^{\tau Z_n^2/2} \):
\[ \mathbb{P}(e^{\tau Z_n^2/2} \geq M) = \mathbb{P}(\vert Z_n \vert \geq \sqrt{2 \ln(M)/\tau}) \leq 2e^{-2\ln(M)/(2\tau)} = 2M^{-1/\tau}, \]
by Hoeffding’s inequality.

Thus
\[ \mathbb{E} e^{\tau Z_n^2/2} I_{e^{\tau Z_n^2/2} \geq M} = M \mathbb{P}(e^{\tau Z_n^2/2} \geq M) + \int_{M}^{\infty} \mathbb{P}(e^{\tau Z_n^2/2} \geq x)dx \]
\[ \leq 2M^{1-1/\tau} + 2 \int_{M}^{\infty} x^{-1/\tau} dx \]
\[ \rightarrow 0 \text{ as } M \rightarrow \infty \text{ for } \tau < 1. \]

By uniform integrability, \( \mathbb{E}_{\mathbb{P}_n} Y_n^2 \rightarrow \mathbb{E} e^{Z^2\tau/2} = \frac{1}{\sqrt{1-\tau}}. \)
The planted clique detection problem

Under $H_0 (\mathbb{P}_n)$, Erdős-Rényi graph $G \sim G(n, 1/2)$;
Under $H_1 (\mathbb{Q}_n)$, $G : G(n, 1/2) \cup$ clique of size $K$, on subset $\mathcal{K}$ uniformly chosen in $\binom{[n]}{K}$.

**Theorem (Informational threshold)**

For any $\epsilon > 0$, if $K \leq (1 - \epsilon)2 \log_2(n)$ then $d_{\text{var}}(\mathbb{P}_n, \mathbb{Q}_n) \to 0$, hence detection infeasible.

If $K \geq K_0 := (1 + \epsilon)2 \log_2(n)$, detection feasible, based on test $T = \mathbb{I}_G$ contains clique of size $K_0$.

**Proof** (infeasibility):

$Y_n(g) = \frac{\mathbb{Q}_n(G=g)}{\mathbb{P}_n(G=g)} = \frac{1}{\binom{n}{K}} \sum_{C \in \binom{[n]}{K}} \frac{\mathbb{Q}_n(G=g | \mathcal{K}=C)}{\mathbb{P}_n(G=g)}$

$= \frac{1}{\binom{n}{K}} \sum_{C \in \binom{[n]}{K}} 2^{\binom{K}{2}} \mathbb{I}_C$ clique of $g$
\[ \mathbb{E}_{p_n} Y_n^2 = \frac{1}{(n^K)^2} \sum_{C, C' \in [n]^K} 2^2 \binom{K}{2} \mathbb{P}_n(C, C' \text{ cliques of } G) \]
\[ = \frac{1}{(n^K)} \sum_{C \in [n]^K} 2^2 \binom{K}{2} \mathbb{P}_n(C, [K] \text{ cliques of } G) \]
\[ = \frac{1}{(n^K)} \sum_{K} 2^2 \binom{K}{2} \binom{n-K}{K-k} \left(\frac{1}{2}\right) 2^2 \binom{K}{2} - \binom{k}{2} \]
\[ \leq \frac{1}{(n^K)} \sum_{K} K \binom{K}{k} \binom{n-K}{K-k} 2^{kK/2} \]

Since \( K \leq (1 - \epsilon)2 \log_2(n) \), \( 2^{kK/2} \leq n^{(1-\epsilon)k} \), so that
\[ \mathbb{E}_{p_n} Y_n^2 \leq (1 + o(1)) \frac{K!}{n^K} \sum_{k=0}^{K} \binom{K}{k} \frac{n^{K-k}}{(K-k)!} n^{(1-\epsilon)k} \]
\[ (1 + o(1)) \sum_{k=0}^{K} \binom{K}{k} \frac{K!}{(K-k)!} n^{-\epsilon k} \]

Let \( f(k) = \binom{K}{k} \frac{K!}{(K-k)!} n^{-\epsilon k} \). Then \( \frac{f(k+1)}{f(k)} = \frac{(K-k)^2}{k+1} n^{-\epsilon} < 1 \) for large enough \( n \).

Hence:
\[ \mathbb{E}_{p_n} Y_n^2 \leq (1 + o(1))(1 + Kf(1)) \]
\[ = (1 + o(1))(1 + K^3 n^{-\epsilon}) \]
\[ = 1 + o(1). \]
**Proof (feasibility):**
For $K \geq k_0 := (1 + \epsilon)2 \log_2(n)$, and $T_n = \mathbb{I}_G$ contains a $k_0$-clique, obviously $Q_n(T_n = 1) = 1$.

Write
\[
\mathbb{P}_n(T_n = 1) \leq \binom{n}{k_0} 2^{-\binom{k_0}{2}} \\
\leq n^{k_0} 2^{-k_0(k_0-1)/2} \\
\leq n^{k_0} n^{-(1+\epsilon)(k_0-1)} \\
\leq n^{-\epsilon k_0 + 1 + \epsilon} \\
\rightarrow 0 \text{ as } n \rightarrow \infty.
\]

**Remark**

*No known polynomial-time implementation of this test: finding a maximum size-clique in a graph is NP-hard (it is even NP-hard to find an approximation of the size of the largest clique up to an approximation factor $n^{1-\epsilon}$ for any $\epsilon \in (0, 1)$, [Hastad, 1999]).*
Planted clique detection, computational threshold

Theorem (Alon-Krivelevitch-Sudakov, 1998)

For constant $c > 0$, detection of planted cliques of size $K = c\sqrt{n}$ can be done in polynomial time, using a spectral method.

To graph $G$ with adjacency matrix $A$, associate matrix $W_{uv} = 2A_{uv} - 1$. Let $G^0 \sim G(n, 1/2)$, with adjacency matrix $A^0$ and for $u \neq v \in [n]$ let $W^0_{uv} = 2A^0_{uv} - 1$.

For set $K \subset [n], |K| = K = c\sqrt{n}$, let $\Delta_{uv} = \begin{cases} 0 & \text{if } u \text{ or } v \notin K, \\ 1 - W^0_{uv} & \text{if both } u, v \in K. \end{cases}$

Then for $G^1 = G^0 \cup$ clique on $K$, $W^1 = W^0 + \Delta$.

Note also: $\overline{\Delta}_{uv} = \mathbb{I}_{u,v \in K}$. 
Theorem (Anderson-Guionnet-Zeitouni, proof of Theorem 2.1.22)

For \((Y_i)_{i \in \mathbb{N}}\) i.i.d., \((Z_{ij})_{i<j \in \mathbb{N}}\) i.i.d., \(\mathbb{E} Y_1 = \mathbb{E} Z_{12} = 0, \mathbb{E} Z^2_{12} = 1\),

let Wigner matrix \(W^{(n)}: W^{(n)}_{ii} = n^{-1/2} Y_i, W^{(n)}_{ij} = W^{(n)}_{ji} = n^{-1/2} Z_{ij}, i < j \in \mathbb{N}\).

Assume \(r_k := \max(\mathbb{E}|Z_{12}|^k, \mathbb{E}|Y_1|^k) \leq k^{ak}\) for some constant \(a > 0\). Then for all \(\delta > 0:\)

\[\mathbb{P}(\rho(W^{(n)}) \leq 2 - \delta) \xrightarrow{n \to \infty} 0,\]

and for any constant \(b > 0,\)

\[\mathbb{P}(\rho(W^{(n)}) \geq 2 + \delta) = o(n^{-b}).\]

Thus w.h.p., \(\rho(W^0) = (1 + o(1))2\sqrt{n}\) \((W^0: \text{Wigner matrix of size } n)\), and \(\rho(\Delta - \bar{\Delta}) = (1 + o(1))2\sqrt{K}\) \((\text{conditionally on } K, \Delta - \bar{\Delta}: \text{Wigner block of size } K)\)

\(\bar{\Delta}\): rank 1 matrix with spectral radius \(K\), and associated eigenvector \(x = (\mathbb{I}_{u \in \mathcal{K}})_{u \in \mathbb{N}}\).
Write $W^1 = W^0 + (\Delta - \bar{\Delta}) + \bar{\Delta} = \tilde{W} + \bar{\Delta}$, where $\tilde{W} = W^0 + (\Delta - \bar{\Delta})$.

Thus w.h.p., $\rho(\tilde{W}) \leq (1 + o(1))2[\sqrt{n} + \sqrt{K}] = (1 + o(1))2\sqrt{n}$

Assume $K = c\sqrt{n}$, $c > 4$. Then:

Under $\mathbb{P}_n$, $\rho(W^0) = (1 + o(1))2\sqrt{n}$, and by Weyl's inequalities, under $\mathbb{Q}_n$, $\rho(W^1) \geq K - (1 + o(1))2\sqrt{n} \geq (c - 2)(1 + o(1))\sqrt{n}$.

Hence test $T_n = \mathbb{I}_{\rho(W) > (1 + \epsilon)2\sqrt{n}}$ detects w.h.p. between $H_0$ and $H_1$.

For $c > 0$ not necessarily $> 4$:

For each set of size $\ell$ of nodes $\{i_1, \ldots, i_\ell\} \in [n]$, consider subgraph $G^{i_\ell}_i$ of $G$ among nodes $i$ neighbours of all $i_1, \ldots, i_\ell$.

Then under $H_0$, for each choice $i_1, \ldots, i_\ell$, with probability $1 - o(n^{-\ell})$, $\rho(W^{i_\ell}_i) \leq (1 + o(1))2\sqrt{2^{-\ell}n}$

Thus (union bound) w.h.p., for all $i_\ell^1 \in [n], \rho(W^{i_\ell^1}_i) \leq (1 + o(1))2\sqrt{2^{-\ell}n}$.

Under $H_1$, for choice $G^{i_\ell^1}_i$: $\sim 2^{-\ell}n$ vertices, planted clique of size $c\sqrt{n}$.

Hence $\rho(W^{i_\ell^1}_i) \geq [c - 2(1 + o(1))2^{-\ell/2}]\sqrt{n}$.

Thus for $\ell : c > 4\sqrt{2^{-\ell}}$, test $T_n = \mathbb{I}_{\exists i_\ell^1 \in [n] : \rho(W^{i_\ell^1}_i) \geq (1 + \epsilon)2\sqrt{2^{-\ell}n}}$ detects between $H_0$ and $H_1$. 
Remark

No polynomial-time algorithm known for detection of planted clique of size $K = o(\sqrt{n})$. Conjecture: no polynomial-time algorithm exists in that case. Hence three phases: detection is

- infeasible for $K \leq (1 - \epsilon)2 \log_2(n)$,
- feasible (in an information-theoretic sense), but computationally hard for $(1 + \epsilon)2 \log_2(n) \leq K \leq o(\sqrt{n})$,
- poly-time feasible for $K = \Omega(\sqrt{n})$. 
Clique reconstruction under $H_1$ and $K = c\sqrt{n}$, $c > 4$

$\rho(\bar{\Delta}) = c\sqrt{n}$, and $\rho(\bar{\Delta})$ separated from other eigenvalue 0 by $c\sqrt{n}$. $\rho(\tilde{W}) \leq 2(1 + o(1))\sqrt{n}$.

By results on perturbation of eigenvectors, $\exists x \leftrightarrow \rho(W^1)$ such that

$$\langle x, \bar{x} \rangle \geq \sqrt{1 - \frac{\rho(\tilde{W})^2}{(c\sqrt{n} - \rho(\tilde{W}))^2}} = \Omega(1),$$

where $\bar{x}_u = \frac{1}{\sqrt{K}} \mathbb{I}_{u \in \mathcal{K}}$

Hence $\sum_u x_u^2 = 1$, $\frac{1}{\sqrt{K}} \sum_{u \in \mathcal{K}} x_u \geq \beta = \sqrt{1 - \left[\frac{2}{(c - 2)}\right]^2} + o(1) = \Omega(1)$.

Let for constant $a \in (0, \beta)$: $C = \{u \in \mathcal{K} : x_u \geq \frac{a}{\sqrt{K}}\}$. Thus:

$$\sqrt{K} \beta \leq \sum_{u \in \mathcal{K}} x_u \leq (K - |C|) \frac{a}{\sqrt{K}} + \sum_{u \in C} x_u \leq K \frac{a}{\sqrt{K}} + \sqrt{|C|}$$

Hence $\sqrt{K}(\beta - a) \leq \sqrt{|C|}$, i.e. $|C| \geq K(\beta - a)^2$.

Let $D = \{u \in \bar{\mathcal{K}} : x_u \geq \frac{a}{\sqrt{K}}\}$. Necessarily, $|C| + |D| \leq \frac{K}{a^2}$.

Thus among nodes in $E = \{u : x_u \geq a/\sqrt{K}\} = C \cup D$, fraction at least $a^2(\beta - a)^2 = \Omega(1)$ belongs to $\mathcal{K}$. 
Set aside set \( \tilde{V} \) of \( \epsilon n \) vertices chosen at random, \( \epsilon > 0 \). Remaining graph \( G' \): \( n(1 - \epsilon) \) vertices, and planted clique of size \( c\sqrt{n}(1 - \epsilon - o(1)) \) whp.

Previous analysis applies to \( G' \) for \( \epsilon \) such that \( c\sqrt{1 - \epsilon} > 4 \): set \( E' \) of size \( m = \Theta(K) \) contains fraction \( \alpha = \Omega(1) \) of clique vertices in \( G' \).

For \( u \in \tilde{V} \cap K \), \( X_u := \sum_{v \in E'} I_{u \sim v} \geq \alpha m + \text{Bin}((1 - \alpha)m, 1/2) \);
For \( u \in \tilde{V} \setminus K \), \( X_u := \sum_{v \in E'} I_{u \sim v} = \text{Bin}(m, 1/2) \);
Let \( \hat{K} := \{ u \in \tilde{V} : X_u \geq (1 - \alpha)\frac{m}{2} + \frac{2}{3}\alpha m \} \).

Then Chernoff bounds for binomial random variables imply:
whp, \( \hat{K} = \tilde{V} \cap K \).
\( \rightarrow \) whp, exact reconstruction of \( K \) on \( \tilde{V} \).

Reconstruction of \( K \) on \( V' = [n] \setminus \tilde{V} \): whp, nodes in \( V' \) neighbours of all nodes in \( \hat{K} \) are exactly nodes in \( V' \cap K \).
Remark

Reconstruction in polynomial time also feasible for $K = c\sqrt{n}$, $c < 4$: identify subgraph $G_{1}^{\ell}$ with $2^{-\ell} n$ vertices and planted clique of size $c\sqrt{n}$, where $c > 4.2^{-\ell/2}$. Use previous reconstruction algorithm of $\hat{K}$ on this graph, giving whp subset of size $\Theta(\sqrt{n})$ of $K$. Add to it all nodes that are neighbours of everyone in $\hat{K}$. 