Geometry of Nonholonomic Systems

A. Bellaïche F. Jean J.J. Risler

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Jean-Paul Laumond (Editor)

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Geometry of Nonholonomic Systems

A. Bellaïche[1], F. Jean[2] and J.-J. Risler[3]

[1] Paris 7 University
[2] Paris 6 University
[3] Ecole Normale Supérieure and Paris 6 University

Nonholonomic motion planning is best understood with some knowledge of the underlying geometry. In this chapter, we first introduce in Section 1 the basic notions of the geometry associated to control systems without drift. In the following sections, we present a detailed study of an example, the car with n trailers, then some general results on polynomial systems, which can be used to bound the complexity of the decision problem and of the motion planning for these systems.

1 Symmetric control systems: an introduction

1.1 Control systems and motion planning

Regardless of regularity hypotheses, control systems may be introduced in two ways. By ascribing some condition

$$\dot{x} \in V_x$$

where V_x is, for every x, some subset of the tangent space $T_x M$, or in a parametric way, as

$$\dot{x} = f(x, u)$$

where, for every x, the map $u \mapsto f(x, u)$ has V_x as its image.

In mechanics or robotics, conditions of the first kind occur as linear constraints on the velocities, such as rolling constraints, as well in free movement—the classical object of study in mechanics, as in the case of systems propulsed by motors.

Equations of the second kind may represent the action of "actuators" used to move the state of the system in the configuration space. One can show that if the action of two actuators are represented by $\dot{x} = f_1(x)$ and $\dot{x} = f_2(x)$, we may also consider the action of any convex combination of vector fields f_1 and f_2, and add it to the possible actions without changing in an essential way the accessible set $A(x)$ or $A(x, T)$. For this reason, one may suppose V_x to be convex, or equivalently, $u \mapsto f(x, u)$ to be affine, of the form $(u_1, \ldots, u_m) \mapsto$

$X_0(x) + u_1 X_1(x) + \cdots + u_m X_m(x)$, and defined on some convex subset K_x of $\mathbf{R}^m$, for some m. This is responsible for the form

$$\dot{x} = X_0(x) + u_1 X_1(x) + \cdots + u_m X_m(x)$$

under which control systems are often encountered in the literature. (It makes no harm to suppose m and K_x to be independent of x, and to suppose that the origin is an interior point of $K = K_x$.) The vector field X_0 is called the drift.

Now, we will use only systems without drift, that is with $X_0 = 0$, for the study of the problem of motion planning for robots. We may content with such systems as long as no dynamics is involved. That is, if the state of the system represents its position, and if we control directly its velocity. As opposed to a system whose state would represent position *and* velocities, and where the control is exerted on accelerations. Consider the simplest possible of such a system: a mobile point on a line, submitted to the control equation

$$\ddot{x} = u.$$

Introducing the velocity $y = \dot{x}$, we see that this system is equivalent to a system governed by the equation

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= u \end{aligned}$$

which can be written as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ 0 \end{pmatrix} + u \begin{pmatrix} 0 \\ 1 \end{pmatrix} = X_0 + u X_1,$$

that is, with a non-zero drift X_0.

For some applications, our study will be valid in the case of slow motion only, and resemble to the thermodynamics of equilibriums, where all transformation are supposed to be infinitely slow.

1.2 Definitions. Basic problems

To sum up, we shall be interested in control systems of the form

$$\dot{x} = \sum_{i=1}^{m} u_i X_i(x), \qquad x \in M, \tag{Σ}$$

where the configuration space M of the system is a C^∞ manifold, $X_1, \ldots, X_m$ are C^∞ vector fields on M, and the control function $u(t) = (u_1(t), \ldots, u_t(t))$ takes values in a fixed compact convex K of $\mathbf{R}^m$, with nonempty interior, and *symmetric* with respect to the origin. Such systems are called *symmetric* (or *driftless*). One also says that controls enter *linearly* in (Σ).

For any choice of u as a measurable function defined on some interval $[0, T]$, with value in K, equation (Σ) becomes a differential equation

$$\dot{x} = \sum_{i=1}^{m} u_i(t) X_i(x). \tag{1}$$

Given any point x_0 on M, we can integrate (1), taking

$$x(0) = x_0 \tag{2}$$

as an initial condition. For the sake of simplicity, we shall suppose that this equation has a well-defined solution on $[0, T]$ for all choices of u (this is guaranteed if M is compact or if $M = \mathbf{R}^n$, and vector fields X_i are bounded). Call this solution x_u. One says that x_u is the path with initial point x_0 and controlled by u. We shall mainly be interested in its final value $x_u(T)$. Classically, points in M are called the *states* of the system. One says for example that the system is *steered* from state x_0 to state $x_u(T)$ by means of the control function u.

One also says that $x_u(T)$ is *accessible*, or *reachable*, in time T from x_0. We shall denote by $A(x, T)$ the set of points of M accessible from x in time T (or in time $\leq T$, it is the same thing for symmetric systems), and by $A(x)$ the set of points accessible from x, that is

$$A(x) = \bigcup_{T>0} A(x, T).$$

Basic problems of Control Theory are:

- determine the accessible set $A(x)$;
- given a point y, accessible from x, find control functions steering the system from x to y;
- do the preceding in minimal time;
- more generally, find control function u ensuring any given property of $x_u(t)$, the path controlled by u.

Given x_0, the control function $u(t)$ is considered as the *input* of the system, and $x_u(t)$ as the *output*. In a more general setting, the output is only some function $h(x)$ of the state x, h being called the observation: the state is only partially known. Here we will take as observation $h = \mathrm{Id}$, and call indifferently x the state or the output.

We can now state another basic problem:

– can one find a map $k : M \to K$ such that the differential equation

$$\dot{x} = f(x, k(x)) \tag{3}$$

has a determined behaviour, for example, has a given point x_0 as an attractor?

Since in this problem, the output is reused as an input, such a map k is called a *feedback control law*, or a *closed-loop control*. If (3) has x_0 as an attractor, one says that k is a *stabilizing feedback* at x_0.

1.3 The control distance

Return to the control system (Σ). For x, $y \in M$, define $d(x, y)$ as the infimum of times T such that y is accessible from x in time T, so $d(x, y) = +\infty$ if y is not accessible from x. It is immediate to prove that $d(x, y)$ is a distance [distance function] on M. Of course, this is the case only because we supposed that K is symmetric with respect to the origin in $\mathbf{R}^m$.

Distance d will be called the *control distance.*

We can define d in a different way. First, observe that since K is convex, symmetric, with nonempty interior, we can associate to it a norm $\| \cdot \|_K$ on $\mathbf{R}^m$, such that K is the unit ball $\|u\|_K \leq 1$. Now, for a controlled path $c = x_u : [a, b] \to M$ obtained by means of a control function $u \in L^1([a, b], \mathbf{R}^m)$, we set

$$\text{length}(c) = \int_a^b \|u(t)\|_K \, dt. \tag{4}$$

If c can be obtained in such a way from several different u's, we take the infimum of the corresponding integrals. Then, $d(x, y)$ is the infimum of the lengths of controlled paths joining x to y (and, of course, this is intended in the definition of an infimum, $+\infty$ if no such path exists).

A slightly variant construction may be useful. Transfer the function $\| \cdot \|_K$ to T_xM, by setting

$$\|v\|_K = \inf\{ \|(u_1, \ldots, u_m)\|_K \mid v = u_1 X_1(x) + \cdots + u_m X_m(x) \}.$$

We get in this way a function on T_xM which is a norm on $\text{span}(X_1(x), \ldots, X_m(x))$ and takes the value $+\infty$ for vector not in this subspace. We can now define the length of any absolutely continuous path $c : [a, b] \to M$ as

$$\text{length}(c) = \int_a^b \|\dot{c}(t)\|_K \, dt$$

and the distance $d(x, y)$ as the infimum of length of paths joining x and y.

Note that distances corresponding to different K, say K_1 and K_2, are equivalent: there exists some positive constants A and B such that

$$Ad_1(x,y) \leq d_2(x,y) \leq Bd_1(x,y).$$

The most convenient version of the control distance is obtained by taking for K the unit ball of $\mathbf{R}^m$, which gives

$$\|u\| = (u_1^2 + \cdots + u_m^2)^{1/2}.$$

In this case, the distance d is called *the sub-Riemannian distance attached to the system of vector fields* $X_1, \ldots, X_m$. As a justification for this name, observe that, locally, any Riemannian distance can be recovered in such a way by taking $m = n$, and as $X_1(x), \ldots, X_n(x)$ an orthonormal basis, depending on x, of the tangent space T_xM. A more general, more abstract, definition of sub-Riemannian metrics can be given, but we shall not use it in this book.

Now, observe that $d(x,y) < \infty$ if and only if x and y are reachable from one another, that $A(x,T)$ is nothing else that the ball of center x and radius T (for d), and that controlled paths joining x to y in minimal time are simply minimizing geodesics.

Many problems of control theory, or path planning, get in this way a geometric interpretation. For another example, one could think to obtain a feedback law $k(x)$ stabilizing the system at $x = x_0$ by choosing k so as to ensure $f(x, k(x))$ to be the gradient of $d(x, x_0)$. Unfortunately, this does not work, even if we take the good version of the gradient, *i.e.*, the sub-Riemannian one:

$$\operatorname{grad} f = (X_1 f)X_1 + \cdots + (X_m f)X_m.$$

and take $k(x) = (X_1 f, \ldots, X_m f)$ for that purpose. But studying the reasons of this failure is very instructive. Such a geometric interpretation, using the sub-Riemaniann distance, really brings a new insight in theory, and it will in several occasions be very useful to us.

1.4 Accessibility. The theorems of Chow and Sussmann

We shall deduce the classical theorem of Chow (Chow [7], Rashevskii [28]) from a more precise result by Sussmann. Sussmann's theorem will be proved using L^1 controls. However, it can be shown that the results obtained are, to a great extend, independent of the class of control used (see Bellaïche [2]).

Consider a symmetric control system, as described above,

$$\dot{x} = \sum_{i=1}^{m} u_i X_i(x), \qquad x \in M, \quad u \in K. \tag{Σ}$$

Recall the configuration space M is a C^∞ manifold, $X_1, \dots, X_m$ are C^∞ vector fields on M, and K, the control set, or parameter set, is a fixed compact convex of $\mathbf{R}^m$, with nonempty interior, *symmetric* with respect to the origin.

In all this section, we fix a point $x_0 \in M$, the initial point, and a positive time T. Set

$$\mathcal{H}_T = L^1([0, T], \mathbf{R}^m).$$

We shall call this space the *space of controls*. It may be considered as a normed space by setting

$$\|u\| = \int_0^T \|u(t)\|_K \, dt.$$

Given $u \in \mathcal{H}_T$, we consider the differential equation

$$\begin{cases} \dot{x} = \sum_{i=1}^m u_i(t) X_i(x), \; 0 \le t \le T \\ x(0) = x_0 \end{cases} \tag{5}$$

Under suitable hypotheses, the differential equation (5) has a well defined solution $x_u(t)$. We will denote by

$$\mathrm{End}_{x_0,T} : \mathcal{H}_T \to M$$

the mapping wich maps u to $x_u(T)$. We will call $\mathrm{End}_{x_0,T}$, or End for short, the end-point map.

Now, the accessible set $A(x_0)$ (the set of points accessible from x_0 for the system Σ, regardless of time) is exactly the image of $\mathrm{End}_{x_0,T}$. Indeed, every controlled path $c : [0, T'] \to M$, defined by the control $u : [0, T'] \to M$ may be reparametrized by $[0, T]$. Conversely, if $u \in \mathcal{H}$, and $L = \mathrm{length}(x_u)$, the control function

$$v(t) = \frac{u(\phi(t))}{\|u(\phi(t))\|_K}, \qquad 0 \le t \le L,$$

where ϕ is defined as a right inverse to the mapping

$$s \mapsto \int_0^s \|u(\tau)\|_K \, d\tau$$

from $[0, T]$ to $[0, L]$ takes its values in K, and defines the same geometric path as u.

Theorem 1.1 (Sussmann [36], Stefan [35]). *The set $A(x_0)$ of points accessible from a given point x_0 in M is an immersed submanifold.*

We shall prove this theorem using arguments from differential calculus in Banach spaces, taking advantage from the fact that the end-point map is a differentiable mapping from $\mathcal{H}$ to M, a finite dimensional manifold.

Recall the rank of a differentiable mapping at a given point is by definition the rank of its differential at that point. The theorem of the constant rank asserts that the image of a differential map with constant rank is an immersed submanifold (for more details about this part of the proof, see Bellaïche [2]).

Definition. Let ρ the maximal rank of the end-point map $\mathrm{End}_{x_0,T} : \mathcal{H}_T \to M$. We say that a control function $u \in \mathcal{H}_T$ is *normal* if the rank of $\mathrm{End}_{x_0,T}$ at u is equal to ρ. We shall say that the path x_u defined by u is a normal path. Otherwise, u is said to be an *abnormal* control, and x_u an abnormal path. A point which can be joined to x_0 by a normal path is said to be *normally accessible from* x_0.

Lemma 1.2. *Every point accessible from* x_0 *is normally accessible from* x_0.

Proof. Let y be a point accessible from x_0, and let $u \in \mathcal{H}_T$ a control steering x_0 to y. Choose a normal control $z \in \mathcal{H}_T$, steering x_0 to some point z. Such a control exists by definition. We claim that the control function $w \in \mathcal{H}_{3T}$ defined by

$$w(t) = \begin{cases} v(t) & \text{if } 0 \le t \le T \\ v(2T - t) & \text{if } T \le t \le 2T \\ u(t - 2T) & \text{if } 2T \le t \le 3T \end{cases}$$

is normal and steers x_0 to y.

The second part of our assertion is evident: the path x_w steers x_0, first to z, then back to x_0, then to y. Now, the image of $D\,\mathrm{End}_{x_0,3T}$ consists of the infinitesimal variations $\delta x_w(3T)$ obtained from infinitesimal variations δw of w. We can consider special variations of w, namely variations of the first part of w only, leaving the two other parts unchanged. In other words, we consider the control functions

$$w(t) + \delta w(t) = \begin{cases} v(t) + \delta v(t) & \text{if } 0 \le t \le T \\ v(2T - t) & \text{if } T \le t \le 2T \\ u(t - 2T) & \text{if } 2T \le t \le 3T \end{cases}$$

Since v is a normal control, these variations yield infinitesimal variations of $\delta x_w(T) = x_{w+\delta w}(T) - x_w(T)$ which form a subspace of dimension ρ at that point. Now, the corresponding variations of $x_w(3T)$ are obtained from those of $x_w(T)$ by applying the flow of the time-dependent vector field $\sum_{1\le i\le m} w_i(t) X_i(x)$ between time T and time $3T$. Since this flow is a diffeomorphism of M, these variations of $x_w(3T)$ form a subspace of dimension ρ of the tangent space T_yM. The space formed by variations of the $x_w(3T)$ caused by unrestricted variations of the control w has thus dimension $\ge \rho$, an so has dimension ρ. This proves that w is normal.

Of course, the fact that w is in $\mathcal{H}_{3T}$ instead of being in $\mathcal{H}_T$ is harmless. ∎

Proof of Sussmann's theorem. The normal controls form an open subset $N_{x_0,T}$ of $\mathcal{H}_T$. From Theorem 1.2, the accessible set $A(x_0)$ is the image of $N_{x_0,T}$ by a constant rank map. By using the Theorem of constant rank, the proof is done. ∎

Theorem 1.3 (Chow [7], Rashevskii [28]). *If M is connected (for its original topology), and if the vector fields $X_1, \dots, X_m$ and their iterated brackets $[X_i, X_j]$, $[[X_i, X_j], X_k]$, etc. span the tangent space T_xM at every point of M, then any two points of M are accessible from one another.*

Proof. Since the relation $y \in A(x)$ is clearly an equivalence relation, we can speak of *accessibility components.* Since

$$y \in A(x) \Longleftrightarrow d(x,y) < \infty,$$

the set $A(x)$ is the union of open balls $B(x,R)$ (for d), so it is itself an open set. Whence it results that the accessibility components are also the connected component of M for the topology defined by d.

It is clear that the accessibility components of M are stable under the flow $\exp tX_i$ of vector field X_i ($i = 1, \dots, m$). Therefore, the vector fields $X_1, \dots, X_m$ are, at any point, tangent to the accessibility component through that point (see [2] for details). And so are their brackets $[X_i, X_j]$, their iterated brackets $[[X_i, X_j], X_k]$, etc.

If the condition on the brackets is fulfilled, then

$$T_xA(x) = T_xM$$

at every point x, as the preceding discussion shows. In that case, the accessibility components are open. Since M is connected, there can be only one accessibility component. ∎

Definition. The following condition

(C) The vector fields $X_1, \dots, X_m$ and their iterated brackets $[X_i, X_j]$, $[[X_i, X_j], X_k]$, etc. span the tangent space T_xM at every point of M,

is called Chow's Condition.

When the Chow's Condition holds, one says that system (Σ) is *controllable.* The reciprocal of Chow's theorem, that is, if (Σ) is controllable, the X_i's and their iterated brackets span the tangent space at every point of M, is true if M and the vector fields are analytic, and false in the C^∞ case (see Sussmann [36]).

Chow's Condition is also known under the name of Lie Algebra Rank Condition (LARC) since it states that the rank at every point x of the Lie algebra generated by the X_i's is full (self-evident definition). In the context of PDE, it is known under the name of Hörmander's Condition: if it is verified, the differential operator $X_1^2 + \cdots + X_m^2$ is hypoelliptic (Hörmander's Theorem).

1.5 The shape of the accessible set in time ε

The purpose of this section is to study the geometric structure of $A(x, \varepsilon)$ for small ε. Let us recall that $A(x, \varepsilon)$ denotes the set of points accessible from x in time ε (or in time $\leq \varepsilon$, it is the same thing) by means of control u_i such that $\sum u_i^2 \leq 1$. In other words, $A(x, \varepsilon)$ is equal to $B(x, \varepsilon)$, the sub-Riemaniann closed ball centered at x with radius ε.

We suppose in the sequel that Chow's condition is satisfied for the control system (Σ). Choosing some chart in a neighbourhood of x_0, we may write (1) as

$$\dot{x} = \sum_{i=1}^{m} u_i(t)\Big(X_i(0) + O(\|x\|)\Big) \tag{6}$$

The differential equation (1) thus appear as a perturbation of the trivial equation

$$\dot{x} = \sum_{i=1}^{m} u_i(t) X_i(0) \tag{7}$$

Classical arguments on perturbation of differential equations show that the solution of (6) is given by

$$x(T) = x(0) + \sum_{i=1}^{m} \Big(\int_0^T u_i(t)\, dt\Big)\, X_i(0) + O(\|u\|^2), \tag{8}$$

where, for u, we use the L^1 norm. Thus, with a linear change of coordinates, the set of points accessible from $x(0) = 0$ in time $T \leq \varepsilon$ satisfies, for small ε

$$A(x, \varepsilon) \subset C[-\varepsilon, \varepsilon]^{n_1} \times [-\varepsilon^2, \varepsilon^2]^{n-n_1},$$

where n_1 is the rank of the family $X_1(0), \ldots, X_m(0)$. As a first step, the set $A(x, \varepsilon)$ is then included in a flat pancake.

The expression (8) implies also that the differential of the end-point mapping at the origin in $\mathcal{H}$ is the linear map

$$u \mapsto \sum_{i=1}^{m} \Big(\int_0^T u_i(t)\, dt\Big)\, X_i(0).$$

Since, typically, we suppose $m < n$, this linear map has rank $n_1 < n$ and the end-point mapping is not a submersion at $0 \in \mathcal{H}$. Following our definition, this means that the constant path at x_0 is an abnormal path. This result has a lot of consequences.

Given a neighbourhood U of x_0, there may not exist a *smooth* mapping $x \mapsto u^x$ of U into $\mathcal{H}$ such that the control u^x steers x_0 to x, or, as well, x to x_0. A stronger result is Brockett's theorem asserting the non-existence of a continuous feedback law, stabilizing system (Σ) at a given point x_0, when $m < n$.

To go further in the description of the set $A(x, \varepsilon)$ we can use the so-called *iterated integrals*. For example, the system

$$\begin{cases} \dot{x}_1 = u_1 \\ \dot{x}_2 = u_2 \\ \dot{x}_3 = x_1 u_2 - x_2 u_1 \\ x_1(0) = x_2(0) = x_3(0) = 0 \end{cases} \tag{9}$$

is solved by

$$\begin{aligned} x_1(T) &= \int_0^T u_1(t)\,dt \\ x_2(T) &= \int_0^T u_2(t)\,dt \\ x_3(T) &= \int_0^T \left(\int_0^{t_1} u_1(t_2)\,dt_2 \right) u_2(t_1)\,dt_1 - \int_0^T \left(\int_0^{t_1} u_2(t_2)\,dt_2 \right) u_1(t_1)\,dt_1 \end{aligned} \tag{10}$$

This scheme works for chained or triangular systems, that is, $\dot{x}_j$ depends only on the controls and $x_1, \ldots, x_{j-1}$. But we shall see that it can be put to work for any system. To begin with, let us rewrite (9) as

$$\dot{x} = u_1 X_1(x) + u_2 X_2(x), \quad x(0) = 0.$$

Then (10) can be read as

$$\begin{aligned} x(T) = x(0) &+ \left(\int_0^T u_1(t)\,dt \right) X_1(0) + \left(\int_0^T u_2(t)\,dt \right) X_2(0) + \\ &\left(\int_0^T \left(\int_0^{t_1} u_1(t_2)\,dt_2 \right) u_2(t_1)\,dt_1 - \right. \\ &\left. \int_0^T \left(\int_0^{t_1} u_2(t_2)\,dt_2 \right) u_1(t_1)\,dt_1 \right) [X_1, X_2](0). \end{aligned}$$

Put this way, the formula for $x(T)$ can readily be generalized to any control system of the form

$$\dot{x} = \sum_{1\le i\le m} u_i X_i(x).$$

One gets (the proof is not hard, cf. Brockett [5])

$$x(T) = x(0) + \sum_{1\le i\le m} \left(\int_0^T u_1(t)\,dt + O(\|u\|^2) \right) X_i(0) + \sum_{1\le i<j\le m} \left(\int_0^T \left(\int_0^{t_1} u_i(t_2)\,dt_2 \right) u_j(t_1)\,dt_1 - \int_0^T \left(\int_0^{t_1} u_j(t_2)\,dt_2 \right) u_i(t_1)\,dt_1 \right) [X_i, X_j](0) + O(\|u\|^3),$$

or, written in a more civilized manner

$$x(T) = x(0) + \sum_{1\le i\le m} (A_i^T(u) + O(\|u\|^2))X_i(0) + \sum_{1\le i<j\le m} A_{ij}^T(u)[X_i, X_j](0) + O(\|u\|^3)$$

which can, for given T, be considered as a limited expansion of order 2 of the end-point mapping about 0 in $\mathcal{H}$. Observe that $A_i^T(u)$ is a linear function with respect to $u \in \mathcal{H}$, and $A_{ij}^T(u)$ is a quadratic function on $\mathcal{H}$. This expansion generalizes the expansion (8) and the set $A(x,\varepsilon)$ satisfies now

$$A(x,\varepsilon) \subset C[-\varepsilon,\varepsilon]^{n_1} \times [-\varepsilon^2,\varepsilon^2]^{n_2-n_1} \times [-\varepsilon^3,\varepsilon^3]^{n-n_2}. \tag{11}$$

Having shown that $A(x,\varepsilon)$ is contained in some box, one can ask whether it contains some other box of the same kind. Of course, before this question can be taken seriously, one has to replace inclusion (11) by

$$A(x,\varepsilon) \subset C[-\varepsilon,\varepsilon]^{n_1} \times [-\varepsilon^2,\varepsilon^2]^{n_2-n_1} \times [-\varepsilon^3,\varepsilon^3]^{n_3-n_2} \times \cdots$$

where the integers n_1, n_2, n_3, ... are the best possible.

Now, except for the case $n_2 = n$ which can be dealt with directly, the proof of an estimate like

$$C'[-\varepsilon,\varepsilon]^{n_1} \times [-\varepsilon^2,\varepsilon^2]^{n_2-n_1} \times [-\varepsilon^3,\varepsilon^3]^{n_3-n_2} \times \cdots \subset A(x,\varepsilon)$$

requires new techniques and special sets of coordinates. Instead of computing limited expansion up to order r, we will compute an expansion to order 1 only, but by assigning weights to the coordinates. This will be done in §§1.6–1.8.

1.6 Regular and singular points

In the sequel we will fix a manifold M, of dimension n, a system of vector fields $X_1, \ldots, X_m$ on M. We will suppose that $X_1, \ldots, X_m$ verify the condition of Chow. We will denote by d the distance defined on M by means of vector fields $X_1, \ldots, X_m$.

Let $\mathcal{L}^1 = \mathcal{L}^1(X_1, \ldots, X_m)$ be the set of linear combinations, with real coefficients, of the vector fields $X_1, \ldots, X_m$. We define recursively $\mathcal{L}^s = \mathcal{L}^s(X_1, \ldots, X_m)$ by setting

$$\mathcal{L}^s = \mathcal{L}^{s-1} + \sum_{i+j=s} [\mathcal{L}^i, \mathcal{L}^j]$$

for $s = 2, 3, \ldots,$ as well as $\mathcal{L}^0 = 0$. The union $\mathcal{L} = \mathcal{L}(X_1, \ldots, X_m)$ of all $\mathcal{L}^s$ is a Lie subalgebra of the Lie algebra of vector fields on M which is called the *control Lie algebra* associated to (Σ).

Now, for p in M, let $L^s(p)$ be the subspace of T_pM which consists of the values $X(p)$ taken, at the point p, by the vector fields X belonging to $\mathcal{L}^s$. Chow's condition states that for each point $p \in M$, there is a smallest integer $r = r(p)$ such that $L^{r(p)}(p) = T_pM$. This integer is called the *degree of nonholonomy* at p. It is worth noticing that $r(q) \leq r(p)$ for q near p. For each point $p \in M$, there is in fact an increasing sequence of vector subspaces, or flag:

$$\{0\} = L^0(p) \subset L^1(p) \subset \cdots \subset L^s(p) \subset \cdots \subset L^{r(p)}(p) = T_pM.$$

We shall denote this flag by $\mathcal{F}(p)$.

Points of the control system split into two categories: regular states, around which the behaviour of the system does not change in a qualitative way, and singular states, where some qualitative changes occur.

Definition. We say that p is a *regular point* if the integers $\dim L^s(q)$ ($s = 1, 2, \ldots$) remain constant for q in some neighbourhood of p. Otherwise we say that p is a *singular point.*

Let us give an example. Take $M = \mathbf{R}^2$, and

$$X_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, X_2 = \begin{pmatrix} 0 \\ x^k \end{pmatrix}$$

(k is some integer). Then for $c = (x, y)$ we have $\dim L^1(c) = 1$ if $x = 0$, $\dim L^1(c) = 2$ if $x \neq 0$, so all points on the line $x = 0$ are singular and the others are regular. For other examples, arising in the context of mobile robot with trailers, see Section 2.

It is worth to notice that, when M and vector fields $X_1, \ldots, X_m$ are analytic, regular points form an open dense set in M. Moreover, the sequence $\dim L^s(p)$, $s = 0, 1, 2, \ldots$, is the same for all regular points in a same connected component of M and is streactly increasing for $0 \leq s \leq r(p)$. Thus the degree of nonholonomy at a regular point is bounded by $n - m + 1$ (if we suppose that no one of the X_i's is at each point a linear combination of the other vector fields). It may be easily computed when the definition of the X_i's allows symbolic computation, as for an analytic function, being non-zero at the formal level is equivalent to being non-zero at almost every point.

Computing, or even bounding the degree of nonholonomy at singular points is much harder, and motivated, for some part, sections 2 and 3 (see also [9,11,19,24]).

1.7 Distance estimates and privileged coordinates

Now, fix a point p in M, regular or singular. We set $n_s = \dim L^s(p)$ $(s = 0, 1, \ldots, r)$.

Consider a system of coordinates centered at p, such that the differentials $dy_1, \ldots, dy_n$ form a basis of T_p^*M adapted to $\mathcal{F}(p)$ (we will see below how to build such coordinates). If $r = 1$ or 2, then it is easy to prove the following local estimate for the sub-Riemannian distance. For y closed enough to 0, we have

$$d(0, (y_1, \ldots, y_n)) \asymp |y_1| + \cdots + |y_{n_1}| + |y_{n_1+1}|^{1/2} + \cdots + |y_n|^{1/2} \tag{12}$$

where $n_1 = \dim L^1(p)$ (the notation $f(y) \asymp g(y)$ means that there exists constants $c,\ C > 0$ such that $cg(y) \leq f(y) \leq Cg(y)$). Coordinates $y_1, \ldots, y_{n_1}$ are said to be of weight 1, and coordinates $y_{n_1+1}, \ldots, y_n$ are said to be of weight 2.

In the general case, we define the weight w_j as the smallest integer s such that dy_j is non identically zero on $L^s(p)$. (So that $w_j = s$ if $n_{s-1} < j \leq n_s$.) Then the proper generalization of (12) would be

$$d(0, (y_1, \ldots, y_n)) \asymp |y_1|^{1/w_1} + \cdots + |y_n|^{1/w_n}. \tag{13}$$

It turns out that this estimate is generically *false* as soon as $r \geq 3$. A simple counter-example is given by the system

$$X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 \\ 1 \\ x^2 + y \end{pmatrix} \tag{14}$$

on $\mathbf{R}^3$. We have

$$L^1(0) = L^2(0) = \mathbf{R}^2 \times \{0\}, \quad L^3(0) = \mathbf{R}^3,$$

so that $y_1 = x$, $y_2 = y$, $y_3 = z$ are adapted coordinates and have weight 1, 1 and 3. In this case, the estimates (13) cannot be true. Indeed, this would imply

$$|z| \leq \text{const.}\big(d(0,(x,y,z))\big)^3,$$

whence

$$\Big|z\big(\exp(tX_2)p\big)\Big| \leq \text{const.}\, t^3,$$

but this is impossible since

$$\frac{d^2}{dt^2}\, z\Big(\exp(tX_2)(p)\Big)\Big|_{t=0} = (X_2^2 z)(p) = 1.$$

However a slight nonlinear change of coordinates allows for (13) to hold. It is sufficient to replace y_1, y_2, y_3 by $z_1 = x$, $z_2 = y$, $z_3 = z - y^2/2$.

In the above example, the point under consideration is singular, but one can give similar examples with regular p in dimension ≥ 4. To formulate conditions on coordinate systems under which estimates like (13) may hold, we introduce some definitions.

Call $X_1 f, \ldots, X_m f$ the *nonholonomic partial derivatives of order* 1 *of* f relative to the considered system (compare to $\partial_{x_1} f, \ldots, \partial_{x_n} f$). Call further $X_i X_j f$, $X_i X_j X_k f$, ... the nonholonomic derivatives of order 2, 3, ... of f.

Proposition 1.4. *For a smooth function f defined near p, the following conditions are equivalent:*

(i) *One has $f(q) = O\big(d(p,q)^s\big)$ for q near p.*
(ii) *The nonholonomic derivatives of order $\leq s-1$ of f vanish at p*

This is proven by the same kind of computations as in the study of example (14).

Definition. If Condition (i), or (ii), holds, we say that f is of order $\geq s$ at p.

Definition. We call local coordinates $z_1, \ldots, z_n$ centered at p a system of *privileged coordinates* if the order of z_j at p is equal to w_j $(j = 1, \ldots, n)$.

If $z_1, \ldots, z_n$ are privileged coordinates, then $dz_1, \ldots, dz_n$ form a basis of $T_p^* M$ adapted to $\mathcal{F}(p)$. The converse is not true. Indeed, if $dz_1, \ldots, dz_n$ form an adapted basis, one can show that the order of z_j is $\leq w_j$, but it may be $< w_j$: for the system (14), the order of coordinate $y_3 = z$ at 0 is 2, while $w_3 = 3$.

To prove the existence, in an effective way, of privileged coordinates, we first choose vector fields $Y_1, \ldots, Y_n$ whose values at p form a basis of T_pM in the following way.

First, choose among $X_1, \ldots, X_m$ a number n_1 of vector fields such that their values form a basis of $L^1(p)$. Call them $Y_1, \ldots, Y_{n_1}$. Then for each s ($s = 2, \ldots, r$) choose vector fields of the form

$$Y_{i_1 i_2 \ldots i_{s-1} i_s} = [X_{i_1}, [X_{i_2}, \ldots [X_{i_{s-1}}, X_{i_s}] \ldots]] \tag{15}$$

which form a basis of $L^s(p)$ mod $L^{s-1}(p)$, and call them $Y_{n_{s-1}+1}, \ldots, Y_{n_s}$.

Choose now any system of coordinates $y_1, \ldots, y_n$ centered at p such that the differentials $dy_1, \ldots, dy_n$ form a basis dual to $Y_1(p), \ldots, Y_n(p)$. (Starting from any system of coordinates $x_1, \ldots, x_n$ centered at p, one can obtain such a system $y_1, \ldots, y_n$ by a linear change of coordinates.)

Theorem 1.5. *The functions $z_1, \ldots, z_n$ recursively defined by*

$$z_q = y_q - \sum_{\{\alpha \,|\, w(\alpha) < w_q\}} \frac{1}{\alpha_1! \ldots \alpha_{q-1}!} (Y_1^{\alpha_1} \ldots Y_{q-1}^{\alpha_{q-1}} y_q)(p)\; z_1^{\alpha_1} \ldots z_{q-1}^{\alpha_{q-1}} \tag{16}$$

form a system of privileged coordinates near p. (We have set $w(\alpha) = w_1\alpha_1 + \cdots + w_n\alpha_n$.)

The proof is based on the following lemma.

Lemma 1.6. *For a function f to be of order $> s$ at p, it is necessary and sufficient that*

$$(Y_1^{\alpha_1} \ldots Y_n^{\alpha_n} f)\,(p) = 0$$

for all $\alpha = (\alpha_1, \ldots, \alpha_n)$ such that $w_1\alpha_1 + \cdots + w_n\alpha_n \le s$.

This is is an immediate consequence of the following, proved by J.-J. Risler [4]: any product $X_{i_1} X_{i_2} \ldots X_{i_s}$, where $i_1, \ldots, i_s$ are integers, can be rearranged as a sum of ordered monomials

$$\sum c_{\alpha_1 \ldots \alpha_n}(x) Y_1^{\alpha_1} \ldots Y_n^{\alpha_n}$$

with $w_1\alpha_1 + \cdots + w_n\alpha_n \le s$, and where the $c_{\alpha_1 \ldots \alpha_n}$'s are smooth functions. This result reminds of the Poincaré-Birkhoff-Witt theorem.

Observe that the coordinates $z_1, \ldots, z_n$ supplied by the construction of Theorem 1.5 are given from original coordinates by expressions of the form

$$\begin{aligned} z_1 &= y_1 \\ z_2 &= y_2 + \mathrm{pol}(y_1) \\ &\ldots \\ z_n &= y_n + \mathrm{pol}(y_1, \ldots, y_{n-1}) \end{aligned}$$

where pol denotes a polynomial, without constant or linear term, and that the reciprocal change of coordinates has exactly the same form.

Other ways of getting privileged coordinates are to use the mappings

$$(z_1, \dots, z_n) \mapsto \exp(z_1 Y_1 + \dots + z_n Y_n)\, p \quad \text{(see [14])},$$
$$(z_1, \dots, z_n) \mapsto \exp(z_n Y_n) \cdots \exp(z_1 Y_1)\, p \quad \text{(see [18])}.$$

Following the usage in Lie group theory, such coordinates are called canonical coordinates of the first (resp. second) kind.

1.8 Ball-Box Theorem

Using privileged coordinates, the control system (Σ) may be rewritten near p as

$$\dot{z}_j = \sum_{i=1}^{m} u_i \big[f_{ij}(z_1, \dots, z_{j-1}) + O\big(\|z\|^{w_j}\big) \big] \qquad (j = 1, \dots, n),$$

where the functions f_{ij} are weighted homogeneous polynomials of degree $w_j - 1$. By dropping the $O\big(\|z\|^{w_j}\big)$, we get a control system ($\widehat{\Sigma}$)

$$\dot{z}_j = \sum_{i=1}^{m} u_i \big[f_{ij}(z_1, \dots, z_{j-1}) \big] \qquad (j = 1, \dots, n),$$

or, in short,

$$\dot{z} = \sum_{i=1}^{m} u_i \widehat{X}_i(z),$$

by setting $\widehat{X}_i = \sum_{j=1}^{n} f_{ij}(z_1, \dots, z_n) \partial_{z_j}$. This system is nilpotent and the vector fields $\widehat{X}_i$ are homogeneous of degree -1 under the non-isotropic dilations $(z_1, \dots, z_n) \mapsto (\lambda^{w_1} z_1, \dots, \lambda^{w_n} z_n)$. The system ($\widehat{\Sigma}$) is called the *nilpotent homogeneous approximation* of the system (Σ). For the sub-Riemaniann distance $\hat{d}$ associated to the nilpotent approximation, the estimate (17) below can be shown by homogeneity arguments. The following theorem is then proved by comparing the distances d and $\hat{d}$ (for a detailed proof, see Bellaïche [2]).

Theorem 1.7. *The estimate*

$$d\,(0, (z_1, \dots, z_n)) \asymp |z_1|^{1/w_1} + \dots + |z_n|^{1/w_n} \tag{17}$$

holds near p if and only if $z_1, \dots, z_n$ form a system of privileged coordinates at p.

The estimate (17) of the sub-Riemannian distance allows to describe the shape of the accessible set in time ε. $A(x,\varepsilon)$ can indeed be viewed as the sub-Riemannian ball of radius ε and Theorem 1.7 implies

$$A(x,\varepsilon) \asymp [-\varepsilon^{w_1}, \varepsilon^{w_1}] \times \cdots \times [-\varepsilon^{w_n}, \varepsilon^{w_n}].$$

Then $A(x,\varepsilon)$ looks like a box, the sides of the box being of length proportionnal to $\varepsilon^{w_1}, \ldots, \varepsilon^{w_n}$. By the fact, Theorem 1.7 is called the Ball-Box Theorem (see Gromov [16]).

1.9 Application to complexity of nonholonomic motion planning

The Ball-Box Theorem can be used to address some issues in complexity of motion planning. The problem of nonholonomic motion planning with obstacle avoidance has been presented in Chapter [Laumond-Sekhavat]. It can be formulated as follows. Let us consider a nonholonomic system of control in the form (Σ). We assume that Chow's Condition is satisfied. The constraints due to the obstacles can be seen as closed subsets F of the configuration space M. The open set $M - F$ is called the *free space*. Let $a,\ b \in M - F$. The motion planning problem is to find a trajectory of the system linking a and b contained in the free space.

From Chow's Theorem (§1.4), deciding the existence of a trajectory linking a and b is the same thing as deciding if a and b are in the same connected component of $M - F$. Since $M - F$ is an open set, the connexity is equivalent to the arc connexity. Then the problem is to decide the existence of a path in $M - F$ linking a and b. In particular this implies that the decision part of the motion planning problem is the same for nonholonomic controllable systems as for holonomic ones.

For the complete problem, some algorithms are presented in Chapter [Laumond–Sekhavat]. In particular we see that there is a general method (called "Approximation of a collision-free holonomic path"). It consists in dividing the problem in two parts:

- find a path in the free space linking the configurations a and b (this path is called also the collision-free holonomic path);
- approximate this path by a trajectory of the system close enough to be contained in the free space.

The existence of a trajectory approximating a given path can be shown as follows. Choose an open neighbourhood U of the holonomic path small enough to be contained in $M - F$. We can assume that U is connected and then, from Chow's Theorem, there is a trajectory lying in U and linking a and b.

What is the complexity of this method?

The complexity of the first part (*i.e.*, the motion planning problem for holonomic systems) is very well modeled and understood. It depends on the geometric complexity of the environment, that is on the complexity of the geometric primitives modeling the obstacles and the robot in the real world (see [6,30]).

The complexity of the second part requires more developments. It can be seen actually as the "complexity" of the output trajectory. We have then to define this complexity for a trajectory approximating a given path.

Let γ be a collision-free path (provided by solving the first part of the problem). For a given ρ, we denote by $\mathrm{Tube}(\gamma, \rho)$ the reunion of the balls of radius ρ centered at q, for any point q of γ. Let ε be the biggest radius ρ such that $\mathrm{Tube}(\gamma, \rho)$ is contained in the free space. We call ε the *size of the free space around the path* γ. The output trajectories will be the trajectories following γ to within ε, that is the trajectories contained in $\mathrm{Tube}(\gamma, \varepsilon)$.

Let us assume that we have already defined a complexity $\sigma(c)$ of a trajectory c. We denote by $\sigma(\gamma, \varepsilon)$ the infimum of $\sigma(c)$ for c trajectory of the system linking a and b and contained in $\mathrm{Tube}(\gamma, \varepsilon)$. $\sigma(\gamma, \varepsilon)$ gives a complexity of an output trajectory. Thus we can choose it as a definition of the complexity of the second part of our method.

It remains to define the complexity of a trajectory. We will present here some possibilities.

Let us consider first bang-bang trajectories, that is trajectories obtained with controls in the form $(u_1, \ldots, u_m) = (0, \ldots, \pm 1, \ldots, 0)$. For such a trajectory the complexity $\sigma(c)$ can be defined as the number of switches in the controls associated to c.

We will now extend this definition to any kind of trajectory. Following [3], a complexity can be derived from the topological complexity of a real-valued function (*i.e.*, the number of changes in the sign of variation of the function). The complexity $\sigma(c)$ appears then as the total number of sign changes for all the controls associated to the trajectory c. Notice that, for a bang-bang trajectory, this definition coincides with the previous one. We will call *topological complexity* the complexity $\sigma_t(\gamma, \varepsilon)$ obtained with this definition.

Let us recall that the complexity of an algorithm is the number of elementary steps needed to get the result. For the topological complexity, we have chosen as elementary step the construction of a piece of trajectory without change of sign in the controls (that is without manoeuvring, if we think to a car-like robot).

Another way to define the complexity is to use the length introduced in §1.3 (see Formula (4)). For a trajectory c contained in $\mathrm{Tube}(\gamma, \varepsilon)$, we set

$$\sigma_\varepsilon(c) = \frac{\mathrm{length}(c)}{\varepsilon}$$

and we call *metric complexity* the complexity $\sigma_m(\gamma, \varepsilon)$ obtained with $\sigma_\varepsilon(c)$. Let us justify this definition on an example. Consider a path γ such that, for any $q \in \gamma$ and any $i \in \{1, \ldots, m\}$, the angle between $T_q\gamma$ and $X_i(q)$ is greater than a given $\theta \neq 0$. Then, for a bang-bang trajectory without switches contained in $\mathrm{Tube}(\gamma, \varepsilon)$, the length cannot exceed $\varepsilon / \sin\theta$. Thus, the number of switches in a bang-bang trajectory ($\subset \mathrm{Tube}(\gamma, \varepsilon)$) is not greater than the length of the trajectory divided by ε (up to a constant). This links $\sigma_\varepsilon(c)$ and $\sigma_m(\gamma, \varepsilon)$ to the topological complexity.

Let us give an estimation of these complexities for the system of the car-like robot (see Chapter [Laumond–Sekhavat]). The configurations are parametrized by $q = (x, y, \theta)^T \in \mathbf{R}^2 \times \mathcal{S}^1$ and the system is given by:

$$\dot{q} = u_1 X_1 + u_2 X_2, \quad \text{with} \quad X_1 = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}, \; X_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

It is well-known that, for all $q \in \mathbf{R}^2 \times \mathcal{S}^1$, the space $L^2(q)$ has rank 3 (see Section 2).

Let us consider a non-feasible path $\gamma \subset \mathbf{R}^2 \times \mathcal{S}^1$. When γ is C^1 and its tangent vector is never in $L^1(q)$, one can link the complexity $\sigma_m(\gamma, \varepsilon)$ to the number of ε-balls needed to cover γ. By the Ball-Box Theorem (§1.8), this number is greater than $K\varepsilon^{-2}$, where the constant depends on γ.

More precise results have been proven by F. Jean (see also [22] for weaker estimates). Let $T(q)$ ($\|T\| = 1$) be the tangent vector to γ. Assume that $T(q)$ belongs to $L^2(q) - L^1(q)$ almost everywhere and that γ is parametrized by its arclength s. Then we have, for small $\varepsilon \neq 0$:

$$\sigma_t(\gamma, \varepsilon) \text{ and } \sigma_m(\gamma, \varepsilon) \asymp \varepsilon^{-2} \int_0^L \det(X_1, X_2, T)(\gamma(s))\, ds$$

(let us recall that the notation $\sigma(\gamma, \varepsilon) \asymp f(\gamma, \varepsilon)$ means that there exist c, $C > 0$ independant on γ and ε such that $cf(\gamma, \varepsilon) \leq \sigma(\gamma, \varepsilon) \leq Cf(\gamma, \varepsilon)$).

2 The car with n trailers

2.1 Introduction

This section is devoted to the study of an example of nonholonomic control system: the car with n trailers. This system is nonholonomic since it is subject to non integrable constraints, the rolling without skiding of the wheels. The states of the system are given by two planar coordinates and $n+1$ angles: the configuration space is then $\mathbf{R}^2 \times (\mathcal{S}^1)^{n+1}$, a $(n+3)$-dimensional manifold. There are only two inputs, namely one tangential velocity and one angular velocity which represent the action on the steering wheel and on the accelerator of the car.

Historically the problem of the car is important, since it is the first nonholonomic system studied in robotics. It has been intensively treated in many papers throughout the litterature, in particular from the point of view of finding stabilizing control laws: see e.g. Murray and Sastry ([25]), Fliess *et al.* ([8]), Laumond and Risler ([23]).

We are interested here in the properties of the control system (see below §2.2). The first question is indeed the controllability. We will prove in §2.4 that the system is controllable at each point of the configuration space. The second point is the study of the degree of nonholonomy. We will give in §2.6 an upper bound which is exponential in terms of the number of trailers. This bound is the sharpest one since it is a maximum. We give also the value of the degree of nonholonomy at the regular points (§2.5). The last problem is the singular locus. We have to find the set of all the singular points (it is done in §2.5) and also to determinate its structure. We will see in §2.7 that one has a natural stratification of the singular locus related to the degree of nonholonomy.

2.2 Equations and notations

Different representations have been used for the car with n trailers. The problem is to choose the variables in such a way that simple induction relation may appear. The kinematic model introduced by Fliess [8] and Sørdalen [33] satisfies this condition. A car in this context will be represented by two driving wheels connected by an axle. The kinematic model of a car with two degrees of freedom pulling n trailers can be given by:

$$
\begin{aligned}
\dot{x} &= \cos\theta_0 v_0, \\
\dot{y} &= \sin\theta_0 v_0, \\
\dot{\theta}_0 &= \sin(\theta_1 - \theta_0)\tfrac{v_1}{r_1}, \\
&\ \vdots \\
\dot{\theta}_i &= \sin(\theta_{i+1} - \theta_i)\tfrac{v_{i+1}}{r_{i+1}}, \\
&\ \vdots \\
\dot{\theta}_{n-1} &= \sin(\theta_n - \theta_{n-1})\tfrac{v_n}{r_n}, \\
\dot{\theta}_n &= \omega,
\end{aligned}
\tag{18}
$$

where the two inputs of the system are the angular velocity ω of the car and its tangential velocity $v = v_n$. The state of the system is parametrized by $q = (x, y, \theta_0, \ldots, \theta_n)^T$ where:

- (x, y) are the coordinates of the center of the axle between the two wheels of the *last* trailer,
- θ_n is the orientation angle of the pulling car with respect to the x-axis,
- θ_i, for $0 \le i \le n-1$, is the orientation angle of the trailer $(n-i)$ with respect to the x-axis.

Finally r_i is the distance from the wheels of trailer $n-i+1$ to the wheels of trailer $n-i$, for $1 \le i \le n-1$, and r_n is the distance from the wheels of trailer 1 to the wheels of the car.

The point of this representation is that the system is viewed from the last trailer to the car: the numbering of the angles is made in this sense and the position coordinates are those of the last trailer. The converse notations would be more natural but unfortunately it would lead to complicated computations.

The tangential velocity v_i of the trailer $n-i$ is given by:

$$v_i = \prod_{j=i+1}^{n} \cos(\theta_j - \theta_{j-1})\ v,$$

or $v_i = f_i v$ where

$$f_i = \prod_{j=i+1}^{n} \cos(\theta_j - \theta_{j-1}).$$

The motion of the system is then characterized by the equation:

$$\dot{q} = \omega X_1(q) + v X_2(q)$$

with the control system $\{X_1, X_2\}$ given by:

$$X_1 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \quad X_2 = \begin{pmatrix} \cos\theta_0\ f_0 \\ \sin\theta_0\ f_0 \\ \vdots \\ \frac{1}{r_n}\sin(\theta_n - \theta_{n-1}) \\ 0 \end{pmatrix}$$

2.3 Examples: the car with 1 and 2 trailers

Let us study first the example of the car with one trailer. The state is $q = (x, y, \theta_0, \theta_1)^T$ and the vector fields are:

$$X_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad X_2 = \begin{pmatrix} \cos\theta_0 \cos(\theta_1 - \theta_0) \\ \sin\theta_0 \cos(\theta_1 - \theta_0) \\ \frac{1}{r_1}\sin(\theta_1 - \theta_0) \\ 0 \end{pmatrix}$$

We want to solve the three problems above (controllability, degree of nonholonomy, singular set). For that, we have to study the Lie Algebra generated by the control system (see §1.6). Let us compute the first brackets of X_1 and X_2:

$$[X_1, X_2] = \begin{pmatrix} -\cos\theta_0 \sin(\theta_1 - \theta_0) \\ -\sin\theta_0 \sin(\theta_1 - \theta_0) \\ \frac{1}{r_1}\cos(\theta_1 - \theta_0) \\ 0 \end{pmatrix}, \; [X_2, [X_1, X_2]] = \frac{1}{r_1} \begin{pmatrix} \sin\theta_0 \\ -\cos\theta_0 \\ -\frac{1}{r_1} \\ 0 \end{pmatrix}.$$

It is straightforward that, for any q, the vectors $X_1(q)$, $X_2(q)$, $[X_1, X_2](q)$ and $[X_2, [X_1, X_2]](q)$ are independant. This implies that, for each q:

$$\begin{aligned} \dim L_1(X_1, X_2)(q) &= 2, \\ \dim L_2(X_1, X_2)(q) &= 3, \\ \dim L_3(X_1, X_2)(q) &= 4, \end{aligned}$$

where $L_k(X_1, X_2)(q)$ is the linear subspace generated by the values at q taken by the brackets of X_1 and X_2 of length $\leq k$.

These dimensions allow us to resolve our three problems. First, the conditions of the Chow theorem are satisfied at each point (since the configuration space is 4-dimensional), so the car with one trailer is controllable. On the other hand, the dimensions of the $L_k(X_1, X_2)(q)$ doesn't depend on q, so all the points are regular and the degree of nonholonomy is always equal to 3.

Let us consider now the car with 2 trailers. If we compute the first brackets, we obtain the following results:

– if $\theta_2 - \theta_1 \neq \pm\frac{\pi}{2}$, then the first independant brackets are $X_1(q)$, $X_2(q)$, $[X_1, X_2](q)$, $[X_2, [X_1, X_2]](q)$ and $[X_2, [X_2, [X_1, X_2]]](q)$;
– if $\theta_2 - \theta_1 = \pm\frac{\pi}{2}$, then the first independant brackets are $X_1(q)$, $X_2(q)$, $[X_1, X_2](q)$, $[X_2, [X_1, X_2]](q)$ and $[X_1 [X_2, [X_2, [X_1, X_2]]]](q)$.

Thus the car with 2 trailers is also controllable since, in both cases, the subspace $L_5(X_1, X_2)(q)$ is 5-dimensional. However we have now a singular set, the points q such that $\theta_2 - \theta_1 = \pm\frac{\pi}{2}$. At these points, the degree of nonholonomy equals 5 and at the regular points it equals 4.

2.4 Controllability

The controllability of the car with n trailers has first been proved by Laumond ([21]) in 1991. He used the kinematic model (18) but a slightly different parametrization where the equation were given in terms of $\varphi_i = \theta_i - \theta_{i-1}$ and (x', y') ((x', y') is the position of the pulling car). The proof of the controllability given here is an adaptation of the proof of Laumond for our parametrization. This adaptation has been presented by Sørdalen ([33]).

Theorem 2.1. *The kinematic model of a car with n trailers is controllable.*

Proof. Let us recall some notations introduced in §1.6.

Let $\mathcal{L}_1(X_1, X_2)$ be the set of linear combinations with real coefficients of X_1 and X_2. We define recursively the distribution $\mathcal{L}_k = \mathcal{L}_k(X_1, X_2)$ by:

$$\mathcal{L}_k = \mathcal{L}_{k-1} + \sum_{i+j=k} [\mathcal{L}_i, \mathcal{L}_j] \tag{19}$$

where $[\mathcal{L}_i, \mathcal{L}_j]$ denotes the set of all brackets $[V, W]$ for $V \in \mathcal{L}_i$ and $W \in \mathcal{L}_j$. The union $\mathcal{L}(X_1, X_2)$ of all $\mathcal{L}_k(X_1, X_2)$ is the Control Lie Algebra of the system $\{X_1, X_2\}$.

Let us now denote $\mathcal{L}'_1(X_1, X_2)$ the set of linear combinations of X_1 and X_2 which coefficients are *smooth functions*. By the induction (19) we construct from $\mathcal{L}'_1(X_1, X_2)$ the sets $\mathcal{L}'_k(X_1, X_2)$ and $\mathcal{L}'(X_1, X_2)$.

For a given state q, we denote by $L_k(X_1, X_2)(q)$, resp. $L'_k(X_1, X_2)(q)$, the subspace of $T_q(\mathbf{R}^2 \times (\mathcal{S}^1)^{n+1})$ wich consists of the values at q taken by the vector fields belonging to $\mathcal{L}_k(X_1, X_2)$, resp. $\mathcal{L}'_k(X_1, X_2)$.

Obviously, the sets $\mathcal{L}_k(X_1, X_2)$ and $\mathcal{L}'_k(X_1, X_2)$ are different. However, for each $k \geq 1$ and each q, the linear subspaces $L_k(X_1, X_2)(q)$ and $L'_k(X_1, X_2)(q)$ are equal. We are going to prove this equality for $k = 2$ (the proof for any k can be easily deduced from this case).

By definition $L_2(X_1, X_2)(q_0)$ is the linear subspace generated by $X_1(q_0)$, $X_2(q_0)$ and $[X_1, X_2](q_0)$. $L'_2(X_1, X_2)(q_0)$ is generated by $X_1(q_0)$, $X_2(q_0)$ and all the $[f(q)X_1, g(q)X_2](q_0)$ with f and g smooth functions. Then $L_2(X_1, X_2)(q_0) \subset L'_2(X_1, X_2)(q_0)$.

From the other hand a bracket $[fX_1, gX_2](q_0)$ is equal to:

$$fg[X_1, X_2](q_0) - g(X_2.f)X_1(q_0) + f(X_1.g)X_2(q_0).$$

Thus $[fX_1, gX_2](q_0)$ is a linear combination with real coefficients of $X_1(q_0)$, $X_2(q_0)$ and $[X_1, X_2](q_0)$. Then $L'_2(X_1, X_2)(q_0) = L_2(X_1, X_2)(q_0)$, which prove our statement for $k = 2$.

To establish the controllability, we want to apply Chow's theorem (see §1.4): we have then to show that the dimension of $L(X_1, X_2)(q)$ is $n + 3$. For that, we are going to prove that $L'(X_1, X_2)(q)$ is $n+3$-dimensionnal and use the relation $L'(X_1, X_2)(q) = L(X_1, X_2)(q)$.

Let us introduce the following vector fields, for $i \in \{0, \ldots, n-1\}$, which belong to $\mathcal{L}'(X_1, X_2)$:

$$\begin{array}{ll} W_0 = X_1 & W_{i+1} = r_{i+1}(\sin\varphi_i V_i + \cos\varphi_i Z_i) \\ V_0 = X_2 & V_{i+1} = \cos\varphi_i V_i - \sin\varphi_i Z_i) \\ Z_0 = [X_1, X_2] & Z_{i+1} = [W_{i+1}, V_{i+1}]. \end{array}$$

The form of these vector fields can be computed by induction. We give only the expression of the interesting ones:

$$\begin{cases} W_i = (\underbrace{0, \ldots, 0}_{n-i+2}, 1, \underbrace{0, \ldots, 0}_{i})^T, \; i = 0, \ldots, n \\ V_n = (\cos\varphi_0, \frac{1}{r_1}\sin\varphi_0, 0, \ldots, 0)^T, \\ Z_n = (-\sin\varphi_0, \frac{1}{r_1}\cos\varphi_0, 0, \ldots, 0)^T. \end{cases} \tag{20}$$

We have $n+3$ vector fields which values at each point of the configuration space are independant since their determinant equals $1/r_1$. Therefore $L'(X_1, X_2)(q)$, and then $L(X_1, X_2)(q)$, are equal to $T_q(\mathbf{R}^2 \times (\mathcal{S}^1)^{n+1})$. We can then apply Chow's theorem and get the result. ∎

Remark. A stronger concept than controllability is given by the following definition: the system $\{X_1, X_2\}$ is called well-controllable if there exists a basis of $n+3$ vector fields in $L(X_1, X_2)(q)$ such that the determinant of the basis is constant for each point q of the configuration space.
The $n+3$ vector fields that we have constructed in the proof satisfy this condition. So the car with n trailers is well-controllable.

2.5 Regular points

Let us denote $\beta^n(q)$ the degree of nonholonomy of the car with n trailers. It can be defined as:

$$\beta^n(q) = \min\{k \mid \dim L_k(X_1, X_2)(q) = n+3\}.$$

We have already computed (§2.3) the values of this degree for $n = 1$ and 2:

$$\beta^1(q) = 3, \; \beta^2(q) = 4 \text{ or } 5.$$

It appears, for $n = 2$, that the configurations where the car and the first trailer are perpendicular have particular properties. This fact can be generalized as follows ([19]):

Theorem 2.2. *The singular locus of the system is the set of the points for which there exists $k \in [2, n]$ such that $\theta_k - \theta_{k-1} = \pm\frac{\pi}{2}$.*

The regular points are then the configuration where no two consecutives trailers (except maybe the last two) are perpendicular. It results from §1.6 that the degree of nonholonomy at regular points is $\leq n+2$. In fact this degree is exactly $n+2$. It can be shown for instance by converting the system into the so-called chained form as in Sørdalen ([33]). This gives us a first result on the degree of nonholonomy:

Theorem 2.3. *At a regular point,* i.e., *a point such that $\theta_k - \theta_{k-1} \neq \pm\frac{\pi}{2}$ $\forall k = 2, \dots, n$, the degree of nonholonomy equals $n + 2$.*

2.6 Bound for the degree of nonholonomy

A first bound for this degree has been given by Laumond ([21]) as a direct consequence of the proof of controllability: we just have to remark that the vector fields (20) belong to $\mathcal{L}'_{2^{n+1}}(X_1, X_2)$. Thus the degree of nonholonomy is bounded by 2^{n+1}. However this bound is too large, as it can be seen in the examples with 1 or 2 trailers.

It has been proved in 1993 ([24,34]) that a better bound is the (n+3)-th Fibonacci number, which is defined by $F_0 = 0$, $F_1 = 1$, $F_{n+3} = F_{n+2} + F_{n+1}$. Luca and Risler have also proved that this bound is a maximum which is reached if and only if each trailer (except the last one) is perpendicular to the previous one.

Theorem 2.4. *The degree of nonholonomy $\beta^n(q)$ for the car with n trailers satisfies:*

$$\beta^n(q) \leq F_{n+3}.$$

Moreover, the equality happens if and only if $\theta_i - \theta_{i-1} = \pm\frac{\pi}{2}$, $i = 2, \dots, n$.

Let us remark that this bound is exponential in n since the value of the n-th Fibonacci number is given by:

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$

2.7 Form of the singular locus

The last problem is to determinate the form of the singular locus, which is given in Theorem 2.2. We already know the values of the degree of nonholonomy in two extremal cases:

– if no two consecutive trailers are perpendicular, $\beta^n(q) = n + 2$;
– if each trailer is perpendicular to the previous one, $\beta^n(q) = F_{n+3}$.

We have now to characterize the states intermediate between these both cases.

For a given state q, we have the following sequence of dimensions:

$$2 = \dim L_1(X_1, X_2)(q) \le \cdots \le \dim L_k(X_1, X_2)(q) \le \cdots \le n + 3. \tag{21}$$

Let us recall that, if this sequence stays the same in an open neighbourhood of q, the state q is a regular point of the control system; otherwise, q is a singular point (see §1.6). Thus to give the sequence (21) at any state q allows to characterize the singular locus.

To determinate the sequence (21), we only need the dimensions of the spaces $L_k(X_1, X_2)(q)$ such that $L_k(X_1, X_2)(q) \neq L_{k-1}(X_1, X_2)(q)$. For that we define, for $i \in \{1, n + 3\}$:

$$\beta_i^n(q) = \min\{k \mid \dim L_k(X_1, X_2)(q) \ge i\}$$

In other words, the fact that $k = \beta_i^n(q)$ is equivalent to:

$$\begin{cases} \dim L_k(X_1, X_2)(q) & \ge i \\ \dim L_{k-1}(X_1, X_2)(q) & < i \end{cases} \tag{22}$$

The sequence (21) can be entirely deduced from the sequence $\beta_i^n(q)$, $i = 1, \ldots, n+3$. Hence the singular locus is completly characterized by the $\beta_i^n(q)$'s which we are going to study. Let us remark that $\beta_{n+3}^n(q)$ is the degree of non-holonomy $\beta^n(q)$.

According to its definition, $\beta_i^n(q)$ increases with respect to i, for i lesser than $\dim L(X_1, X_2)(q)$ (when i is strictly greater than this dimension, $\beta_i^n(q)$ is equal to $-\infty$). In fact we will establish (in Theorem 2.5) that this sequence is strictly increasing with respect to i for $2 \le i \le n + 3$. In other words, we will prove that, for $2 \le i \le n + 3$, $\beta_i^n(q) > -\infty$ and that $k = \beta_i^n(q)$ is equivalent to (compare with (22)):

$$\begin{cases} \dim L_k(X_1, X_2)(q) & = i \\ \dim L_{k-1}(X_1, X_2)(q) & = i - 1 \end{cases}$$

We can also calculate easily the first values of these sequences. It is clear that the family $X_1,\ X_2,\ [X_1, X_2]$ is three dimensional for all q (see the examples $n = 1$ and 2). Then the dimensions of $L_1(X_1, X_2)(q)$ and $L_2(X_1, X_2)(q)$ are respectively 2 and 3 and we have, for all state q:

$$\beta_1^n(q) = 1 \ \beta_2^n(q) = 1 \ \beta_3^n(q) = 2. \tag{23}$$

Finally, for $q \in \mathbf{R}^2 \times (\mathcal{S}^1)^{n+1}$ and $1 \le p < n$, we will denote the projection on the first $(n+3-p)$ coordinates of q by q^p, that is $q^p = (x, y, \theta_0, \ldots, \theta_{n-p})^T$. q^p belongs to $\mathbf{R}^2 \times (\mathcal{S}^1)^{n-p+1}$ and it can be seen as the state of a car with $n-p$ trailers. Hence we can associate to this state the sequence $\beta_j^{n-p}(q^p)$, $j = 1, \ldots, n-p+3$.

We can now give the complete characterization of the singular locus, *i.e.*, the computation of the $\beta_i^n(q)$ and the determination of a basis of $T_q(\mathbf{R}^2 \times (\mathcal{S}^1)^{n+1})$. The following theorem has been proved by F. Jean in ([19]). We restrict us to the case where the distances r_i equal 1.

Theorem 2.5. *Let a_p defined by $a_1 = \pi/2$ and $a_p = \arctan \sin a_{p-1}$. $\forall q \in \mathbf{R}^2 \times (\mathcal{S}^1)^{n+1}$, for $2 \le i \le n+3$, $\beta_i^n(q)$ is streactly increasing with respect to i and can be computed, for $i \in \{3, n+3\}$, by the following induction formulae:*

1. *if $\theta_n - \theta_{n-1} = \pm\frac{\pi}{2}$, then*
$$\beta_i^n(q) = \beta_{i-1}^{n-1}(q^1) + \beta_{i-2}^{n-2}(q^2)$$
2. *if $\exists p \in [1, n-2]$ and $\epsilon = \pm 1$ such that $\theta_k - \theta_{k-1} = \epsilon a_{k-p}$ for every $k \in \{p+1, n\}$, then*
$$\beta_i^n(q) = 2\beta_{i-1}^{n-1}(q^1) - \beta_{i-2}^{n-2}(q^2)$$
3. *otherwise,*
$$\beta_i^n(q) = \beta_{i-1}^{n-1}(q^1) + 1.$$

Moreover, at a point q, we can construct a basis $\mathcal{B} = \{B_i, i = 1 \ldots n+3\}$ of $T_q(\mathbf{R}^2 \times (\mathcal{S}^1)^{n+1})$ by:

$$\begin{cases} B_1 = X_1 \\ B_2 = X_2 \\ B_i = [X_1, \underbrace{X_2, \ldots, X_2}_{\beta_{i-1}^{n-1}(q^1)}, \underbrace{X_1, \ldots, X_1}_{\beta_i^n(q) - \beta_{i-1}^{n-1}(q^1) - 1}] \text{ for } i > 2, \end{cases}$$

where $[X_{i_1}, \ldots, X_{i_s}]$ denotes $[[\ldots[X_{i_1}, X_{i_2}], \ldots, X_{i_{s-1}}], X_{i_s}]$.

Let us consider the sequence $(\beta_i^n(q))_{i=2,\ldots,n+3}$ (we remove $\beta_1^n(q)$ because it is always equal to $\beta_2^n(q)$). For example, for $n = 2$, the sequence $(\beta_i^2(q))$ is equal to $(1, 2, 3, 5)$ on the hyperplanes $\theta_2 - \theta_1 = \pm\frac{\pi}{2}$. The complementary of these two hyperplanes are the regular points of the system and corresponds to the values $(1, 2, 3, 4)$ of the sequence $(\beta_i^2(q))$.

As we have seen in Theorem 2.2, the singular locus is the union of the the hyperplanes $\theta_k - \theta_{k-1} = \pm\frac{\pi}{2}$, $2 \le k \le n$. On each hyperplane we have a generic sequence $(\beta_i^n(q))$ and the non generic points are:

- either in the intersection with another hyperplane $\theta_j - \theta_{j-1} = \pm\frac{\pi}{2}$ which corresponds to the case *1* of Theorem 2.5;
- either in the intersection with an hyperplane $\theta_{k+1} - \theta_k = \pm a_2$, ($a_2 = \frac{\pi}{4}$) which corresponds to the case *2* of Theorem 2.5.

For these "more singular" sets, we have again some generic and some singular points that we can find with Theorem 2.5. We have then a stratification of the singular locus by the sequence $(\beta_i^n(q))$. Let us consider for instance the hyperplane $\theta_2 - \theta_1 = \frac{\pi}{2}$. The generic sequence $(\beta_i^n(q))$ is equal to $(1, 2, \ldots, n+1, n+3)$ (it is a direct application of the recursion formulae of Theorem 2.5). The non generic points are at the intersection with the hyperplanes $\theta_j - \theta_{j-1} = \pm\frac{\pi}{2}$, $j = 3, \ldots, n$ and with $\theta_3 - \theta_2 = \pm\frac{\pi}{4}$. On $\theta_2 - \theta_1 = \frac{\pi}{2}$, $\theta_3 - \theta_2 = \frac{\pi}{4}$, the generic sequence is $(1, 2, \ldots, n+1, n+4)$ and we can continue the decomposition.

Let us remark at last that Theorem 2.5 contains all the previous results. For instance, it proves that $\beta_{n+3}^n(q)$ is always definite (*i.e.*, $> -\infty$): the rank of $L(X_1, X_2)(q)$ at any point is then $n+3$ and the system is controllable. We can also compute directly the values of $\beta_{n+3}^n(q)$ and then its maximum, and so on.

3 Polynomial systems

3.1 Introduction

We will deal in this section with *polynomial systems*, *i.e.*, control systems in $\mathbf{R}^n$ made with vector fields $V_i = \sum_{j=1}^n P_i^j \partial X_i$, where the P_i^j's are polynomials in $X_1, \ldots, X_n$. Polynomial systems are important for "practical" (or "effective") purpose, because polynomials are the simplest class of functions for which symbolic computation can be used. Also, we can hope of global finiteness properties (on $\mathbf{R}^n$) for such systems, and more precisely of effective bounds in term of n and of a bound d on the degrees of the P_i^j.

In this section, we will study the degree of nonholonomy of an affine system without drift Σ made with polynomial vector fields $V_1, \ldots, V_s$ on $\mathbf{R}^n$, and prove that it is bounded by a function $\phi(n, d)$ depending only on the dimension n of the configuration space $\mathbf{R}^n$, and on a bound d on the degrees of the P_i^j. As a consequence, we have that the problem of controllability for a polynomial system $(V_1, \ldots, V_s)$ of degree $\leq d$ (with rational coefficients) is effectively decidable: take $x \in \mathbf{R}^n$, compute the value at x of the iterated brackets of $(V_1, \ldots, V_s)$ up to length $\phi(n, d)$. Then the system is controllable at x if and only if the vector space spaned by the values at x of these brackets is $\mathbf{R}^n$ (see above §1.4). For the controllability on $\mathbf{R}^n$, take a basis of $\mathcal{L}^{\phi(n,d)}$, *i.e.*, of the elements of degree $\leq \phi(n, d)$ of the Lie algebra $\mathcal{L}(V_1, \ldots, V_s)$. Then the system Σ is controllable on $\mathbf{R}^n$ if and only if this finite family of vector fields is of rank n at any point $x \in \mathbf{R}^n$. But this is known to be effectively decidable: one has to decide if a matrix M with polynomial entries is of rank n at any point of $\mathbf{R}^n$. The matrix M is the matrix $(V_1, \ldots, V_s, V_{s+1}, \ldots, V_k)$, where the V_i's are the vector fields of Σ for $1 \le i \le s$, and for $s+1 \le i \le k$ a set of brackets of the form $[[\ldots[V_{i_1}, V_{i_2}], V_{i_3},]\ldots]V_{i_p}]$, with $1 \le i_j \le s$, spaning $\mathcal{L}^{\phi(n,d)}$. Let I be the ideal of $\mathbf{R}[X_1, \ldots, X_n]$ spaned by all the $n \times n$ minors of the matrix M. Then Σ is controllable on all $\mathbf{R}^n$ if and only if the zero set of I is empty, and that is effectively decidable (see for instance [15] or [17]).

The bound described here for the degree of nonholonomy is doubly exponential in n. A better bound (and in fact an optimal one) would be a bound simply exponential in n, *i.e.*, of the form $O(d^n)$ or $d^{O(n)}$, or again $d^{n^{O(1)}}$. For an optimal bound in a particular case, see Section 2 of this chapter, for the case of the car with n trailers. Note that this system is not polynomial.

3.2 Contact between an integral curve and an algebraic variety in dimension 2

In this section, we will work over the field $\mathbf{C}$, but all the results will be the same over the field $\mathbf{R}$. By the contact (or intersection multiplicity) between a smooth analytic curve γ going through the origin O in $\mathbf{C}^n$ and an analytic germ of hypersurface at O, $\{Q = 0\}$, we mean the order of $Q_{|\gamma}$ at O. More precisely, let $X_1(t), \ldots, X_n(t)$, $X_i(0) = 0$ be a parametrization of the curve γ near the origin ($X_i(t)$ are convergent power series in t). Then the contact of γ and $\{Q = 0\}$ at O is the order at 0 of the power series $Q(X_1(t), \ldots, X_n(t))$ (*i.e.*, the degree of the non zero monomial of lowest degree of this series). Let us give an example, for the convenience of the reader. Set $n = 2$, $Q(X, Y) = Y^2 - X^3$, $\gamma(t)$ defined by $X(t) = t^2 + 2t^5$, $Y(t) = t^3 + t^4$. We have $Q_{|\gamma} = (t^3+t^4)^2 - (t^2+2t^5)^3 = 2t^7 +$ higher order terms; then the contact exponent between γ and the curve $\{Q = 0\}$ is 7.

Let us first recall some classical facts about intersection multiplicity. If $Q_1, \ldots, Q_p$ are analytic functions defined in a neighborhood of O, we will set $Z(Q_1, \ldots, Q_p)$ for the analytic germ at O defined by $Q_1 = \cdots = Q_p = 0$, and $\mathbf{C}\{X_1, \ldots, X_n\}$ for the ring of convergent power series.

Then, if in $\mathbf{C}^n$ we have $\{O\} = Z(Q_1, \ldots, Q_n)$, the intersection multiplicity at O of the analytic germ defined by $\{Q_i = 0\}$ $(1 \le i \le n)$ is by definition

$$\mu(Q_1, \ldots, Q_n) = \dim_{\mathbf{C}} \frac{\mathbf{C}\{X_1, \ldots, X_n\}}{(Q_1, \ldots, Q_n)} \tag{24}$$

Recall that the condition $\{O\} = Z(Q_1, \ldots, Q_n)$ (locally at O) is equivalent to the fact that the $\mathbf{C}$-vector space $\frac{\mathbf{C}\{X_1,\ldots,X_n\}}{(Q_1,\ldots,Q_n)}$ is of finite dimension. Recall at last that when $Q_1, \ldots, Q_n$ are polynomials of degrees $q_1, \ldots, q_n$, we have $\mu(Q_1, \ldots, Q_n) \leq q_1 \cdots q_n$ by Bézout's theorem, if $\dim_{\mathbf{C}} \frac{\mathbf{C}\{X_1,\ldots,X_n\}}{(Q_1,\ldots,Q_n)} < +\infty$.

Let $V = P_1 \partial/\partial X_1 + \cdots + P_n \partial/\partial X_n$ be a polynomial vector field such that $V(O) \neq 0$, $\deg(P_i) \leq d$. Let $Q(X_1, \ldots, X_n)$ be a polynomial of degree q. Set

$$\begin{aligned}
Q_1 &= P_1 \partial Q/\partial X_1 + \cdots + P_n \partial Q/\partial X_n \\
Q_2 &= P_1 \partial Q_1/\partial X_1 + \cdots + P_n \partial Q_1/\partial X_n \\
&\vdots \\
Q_{n-1} &= P_1 \partial Q_{n-2}/\partial X_1 + \cdots + P_n \partial Q_{n-2}/\partial X_n
\end{aligned}$$

(*i.e.*, $Q_0 = Q$ and $Q_i = < P, \mathrm{grad} Q_{i-1} > = \sum_{j=1}^n P_j \, \partial Q_{i-1}/\partial X_j$, for $1 \leq i \leq n-1$); Q_1 is the Lie derivative of Q along the vector field V, and more generally, Q_i is the Lie derivative of Q_{i-1} along the vector field V.
We have the following:

Theorem 3.1. *Let V be a vector field in $\mathbf{C}^n$ whose coordinates are polynomials of degree $\leq d$, and such that $V(O) \neq 0$. Let γ be the integral curve of V going through O, and Q a polynomial of degree q. Assume $Q_{|\gamma} \not\equiv 0$, and that O is isolated in the algebraic set $Z(Q, Q_1, \ldots, Q_{n-1})$ (which means that $\dim_{\mathbf{C}} \frac{\mathbf{C}\{X_1,\ldots,X_n\}}{(Q,\ldots,Q_{n-1})} < +\infty$). Then the contact exponent ν between Q and γ satisfies*

$$\nu \leq q q_1 \cdots q_{n-1} + n - 1, \tag{25}$$

where q_i is a bound for the degree of Q_i, namely $q_i = q + i(d-1)$.

Proof. We may assume $\nu \geq n$. Let $\gamma(t) \ : \ t \mapsto (X_1(t), \ldots, X_n(t))$ be a smooth analytic parametrization of γ. By definition, ν is the order of the power series $Q \circ \gamma(t) = Q(X_1(t), \ldots, X_n(t))$. Now, $Q_1 \circ \gamma(t) = Q_1(X_1(t), \ldots, X_n(t))$ is the derivative of $Q \circ \gamma(t)$, and therefore is of order $\nu - 1$ at O. Similarly $Q_i \circ \gamma(t)$ is of order $\nu - i$ for $1 \leq i \leq n-1$. We have that the series $Q(X_1(t), \ldots, X_n(t))$ is of the form $t^\nu v(t)$, *i.e.*, belongs to the ideal (t^ν) in $\mathbf{C}\{X_1, \ldots, X_n\}$. Similarly, $Q_i(X_1(t), \ldots, Q_n(t))$ belongs to the ideal $(t^{\nu - i})$.
Set γ^* for the ring homomorphism : $\mathbf{C}\{X_1, \ldots, X_n\} \longrightarrow \mathbf{C}\{t\}$ induced by the parametrization of γ. The image of γ^* contains by assumption a power series of order one, *i.e.*, of the form $v(t) = t u(t)$, with $u(O) \neq 0$. Then the inverse function theorem implies that t itself is in the image of γ^*, *i.e.*, that γ^* is surjective. Hence we have a commutative diagramm of ring homomorphisms:

$$\begin{array}{ccc}
\mathbf{C}\{X_1, \ldots, X_n\} & \xrightarrow{\gamma^*} & \mathbf{C}\{t\} \\
\downarrow & & \downarrow \\
\frac{\mathbf{C}\{X_1,\ldots,X_n\}}{(Q,Q_1,\ldots Q_{n-1})} & \xrightarrow{\bar{\gamma}^*} & \frac{\mathbf{C}\{t\}}{(t^{\nu-n+1})}
\end{array}$$

where the vertical arrows represent the canonical maps. Since γ^* is surjective, we have also that $\bar{\gamma}^*$ is surjective.

This implies that

$$\nu - n + 1 = \dim_{\mathbf{C}} \frac{\mathbf{C}\{t\}}{(t^{\nu-n+1})} \leq \dim_{\mathbf{C}} \frac{\mathbf{C}\{X_1, \ldots, X_n\}}{(Q, \ldots, Q_{n-1})} \leq q q_1 \cdots q_{n-1}$$

or $\nu \leq q q_1 \cdots q_{n-1} + n - 1$ as asserted, the last inequality coming from Bézout's theorem. ■

Remark. One may conjecture that such a kind of result is valid (may be with a slightly different bound) without the hypothesis $\dim_{\mathbf{C}} \frac{\mathbf{C}\{X_1,\ldots,X_n\}}{(Q,\ldots,Q_{n-1})}$ finite. This would imply a simply exponential bound (*i.e.*, of the form $C(n)d^n$, or $d^{n^{0(1)}}$) for the degree of nonholonomy.

Notice that for Theorem 3.1, we may always assume that the polynomial $Q(X_1, \ldots, X_n)$ is *reduced* (or even *irreducible*), because if $Q = R_1 \ldots R_s$, the bound (25) for the R_i's implies the same bound for Q. In fact, it is enough to prove that if $Q = RS$, $r = \deg R$, $s = \deg S$, $q = r + s$, then

$$r(r+d-1)\cdots(r+(n-1)(d-1)) + s(s+d-1)\cdots(s+(n-1)(d-1)) + 2(n-1) \leq q(q+d-1)\cdots(q+(n-1)(d-1)) + n - 1$$

which is immediate by induction on n.

If A is a $\mathbf{C}$-algebra, let us denote by $\dim A$ its dimension as a ring (it is its "Krull dimension"), and $\dim_{\mathbf{C}} A$ its dimension as a $\mathbf{C}$-vector space. If A is an analytic algebra, *i.e.*, $A = \frac{\mathbf{C}\{X_1,\ldots,X_n\}}{I}$ where I is an ideal, $I = (S_1, \ldots, S_q)$, then its dimension as a ring is the dimension (over $\mathbf{C}$) of the germ at O of the analytic space defined by $Z(S_1, \ldots, S_q)$. We have that $\dim_{\mathbf{C}} A < +\infty$ if and only if $\dim A = 0$.

Notice that Q_1 cannot be divisible by Q (since $Q \circ \gamma(t)$ is of order ν, and $Q_1 \circ \gamma(t)$ of order $\nu - 1$). Therefore, if Q is irreducible, we have

$$\dim \frac{\mathbf{C}\{X_1, \ldots, X_n\}}{(Q, Q_1)} = n - 2.$$

This implies that in Theorem 3.1, we may always assume that we have $\dim \frac{\mathbf{C}\{X_1,\ldots,X_n\}}{(Q,\ldots,Q_{n-1})} \leq n - 2$.

In particular, (25) is true in dimension 2 without additional hypothesis:

Corollary 3.2. *Let $V = P_1 \partial/\partial X + P_2 \partial/\partial Y$ be a polynomial vector field in the plane of degree $\leq d$, such that $V(O) \neq 0$, γ the integral curve of V by O, and $Q(X, Y)$ a polynomial of degree q such that $Q_{|\gamma} \not\equiv 0$. Then the contact exponent ν of Q and γ satisfies*

$$\nu \leq q(q + d - 1) + 1.$$

This corollary has first been proved by A. Gabrielov, J.-M. Lion and R. Moussu, [10].

3.3 The case of dimension n

We have the following result, due to Gabrielov ([12])

Theorem 3.3. *Let* $V = \sum P_i \partial/\partial X_i$ *be a polynomial vector field, with* $P_i \in \mathbf{C}[X_1, \ldots, X_n]$ *of degree* $\leq d$*, such that* $V(O) \neq 0$*,* $Q(X_1, \ldots, X_n)$ *a polynomial of degree* $\leq q$ *such that* $Q_{|\gamma} \not\equiv 0$*. Then the contact exponent* ν *between* Q *and* γ *satisfies*

$$\nu \leq 2^{2n-1} \sum_{k=1}^{n} [q + (k-1)(d-1)]^{2n} \tag{26}$$

Remark. This bound is polynomial in d and q and simply exponential in n. It is optimal (up to constants) since it comes from Example 2) below that there exists a lower bound also polynomial in d and q and simply exponential in n.

Remark. In 1988 Nesterenko ([27]) found a bound of the form

$$\nu \leq c(n) d^{n^2} q^n,$$

namely simply exponential in n when d is fixed, but doubly exponential in the general case.

Remark. In dimension 3, the following bound has been found by A. Gabrielov, F. Jean and J.-J. Risler, [9]:

$$\nu \leq q + 2q(q + d - 1)^2.$$

3.4 Bound for the degree of nonholonomy in the plane

In the two-dimensional case, we have the following bound for the degree of non-holonomy (see [29]):

Theorem 3.4. *Let* $\Sigma = (V_1, \ldots, V_s)$ *be a control system made with polynomial vector fields on* $\mathbf{R}^2$ *of degree* $\leq d$*; let* $r(x)$ *be the degree of nonholonomy of* Σ *at* $x \in \mathbf{R}^2$*. Then,*

$$r(x) \leq 6d^2 - 2d + 2 \tag{27}$$

Proof. Take $x = O$. Let as above (see §1.6) $L^i(O)$ be the vector space spaned by the values at O of the brackets of elements of Σ of length $\leq i$. We may assume $\dim L^1(O) = 1$, because otherwise the problem of computing $r(O)$ is trivial (if $\dim L^1(O) = 0$, then $L^s(O) = \{0\}$ $\forall s \geq 1$, and if $\dim L^1(O) = 2$, we have $L^1(O) = \mathbf{R}^2$ and $r(O) = 1$ by definition). We therefore assume that $V_1(O) \neq 0$, and set $V = V_1$. ■

Lemma 3.5. *Assume $r(O) > 1$, which in our case implies that the system Σ is controllable at O. Then there exists $Y \in \Sigma$ such that $\det(V, Y)_{|\gamma} \not\equiv 0$.*

Proof. Assume that $\det(V, Y)_{|\gamma} \equiv 0$ $\forall Y \in \Sigma$. Then, in some neighborhood of O, any vector field $Y \in \Sigma$ is tangent to the integral curve γ of V from O. This implies that the system cannot be controllable at O, since in some neighborhood of O the accessible set from O would be contained in γ. ■

Let us now state a Lemma in dimension n.

Lemma 3.6. *Let $V, Y_1, \ldots, Y_n$ be vector fields on $\mathbf{R}^n$. Then*

$$V.\det(Y_1, \ldots, Y_n) = \sum_{i=1}^{n} \det(Y_1, \ldots, [V, Y_i], \ldots, Y_n) + \mathrm{Div}(V).\det(Y_1, \ldots, Y_n).$$

Let us recall that $\mathrm{Div}(V) = \partial P_1/\partial X_1 + \partial P_2/\partial X_2 + \cdots + \partial P_n/\partial X_n$*, where* $V = P_1\partial/\partial X_1 + \cdots + P_n\partial/\partial X_n$.

Proof. This formula is classical. See for instance [13, Exercice page 93], or [26, Lemma 2.6]. ■

of Theorem 3.4, continued. Let γ be the integral curve of V by O. By Lemma 3.5, there exists $Y \in \Sigma$ such that $\det(V, Y)_{|\gamma} \not\equiv 0$. Set $Q = \det(V, Y)_{|\gamma}$.

By Lemma 3.6, we have $V.\det(V, Y) = \det(V, [V, Y]) + \mathrm{Div}V \det(V, Y)$. Let ν be the order of contact of Q and γ. We have that $Q_{|\gamma} = a_\nu t^\nu + \cdots$, with $a_\nu \neq 0$, and that $(V.Q)_{|\gamma} = \nu a_\nu t^{\nu-1} + \cdots$ because $V_{|\gamma}$ can be identified with $\partial/\partial t$. Then $\det(V, [V, Y])_{|\gamma}$ is of order $\nu - 1$ in t, and we see that when differentiating ν times in relation to t, we find that

$$\det(V, [V[V, \ldots [V, Y] \ldots]])(O) \neq 0,$$

the bracket inside the parenthesis being of length $\nu+1$. This means by definition of $r(O)$ that $r(O) \leq \nu + 1$.

The polynomial Q is of degree $\leq 2d$, and V is a polynomial vector field of degree $\leq d$. Then Corollary 3.2 gives $\nu \leq 2d(2d + d - 1) + 1 = 6d^2 - 2d + 1$. ■

Example. 1) Set

$$\Sigma \quad \begin{cases} V_1 = \partial/\partial X + X^d \partial/\partial Y \\ V_2 = Y^d \partial/\partial X \end{cases}$$

Then it should be easily seen that for this system, $r(O) = d^2 + 2d + 1$. The inequality $r(O) \geq d^2 + 2d + 1$ has been checked by F. Jean. This proves that the estimation (27) is asymptotically optimal in term of d, up to the constant 6.

2) Let in $\mathbf{R}^n$

$$\Sigma \quad \begin{cases} V_1 = \partial/\partial X_1 \\ V_2 = X_1^d \partial/\partial X_2 \\ \quad \vdots \\ V_n = X_{n-1}^d \partial/\partial X_n. \end{cases}$$

We see easily that for this system, $r(O) = d^{n-1}$, which means that in general $\phi(n,d)$ is at least exponential in n.

3.5 The general case

We have the following result, where for simplicity, and because it is the most important case, we assume the system controllable.

Theorem 3.7. *Let $n \geq 3$. With the above notation, let $r(x)$ be the degree of non-holonomy at $x \in \mathbf{R}^n$ for the control system Σ made with polynomial vector fields of degree $\leq d$. Let us assume that the system Σ is controllable. Then we have the following upper bound:*

$$r(x) \leq \phi(n,d), \quad \text{with } \phi(n,d) \leq 2^{n-2}(1 + 2^{2n(n-2)-2} d^{2n} \sum_{k=4}^{n+3} k^{2n}). \tag{28}$$

Proof. We first state without proof a result of Gabrielov [11].

Lemma 3.8. *Let $(V_1, \ldots, V_s)$ be a system of analytic vector fields controllable at O such that $V_1(O) \neq 0$. Let f be a germ of an analytic function such that $f(O) = 0$ and $f|_{\gamma(V_1)} \not\equiv 0$ ($\gamma(V_1)$ denotes the trajectory of V_1 going through O). Then there exists n vector fields $\chi_1, \ldots, \chi_n$ satisfying*

- *$\chi_1 = V_1$, χ_2 is one of the V_i, and, for $2 < k \leq n$, χ_k is either one of the V_i or belongs to the linear subspace generated by $[\chi_l, f\chi_m]$, for $l, m < k$;*
- *there exists a vector field $\chi_\epsilon = \chi_1 + \epsilon_2 \chi_2 + \cdots + \epsilon_{n-1}\chi_{n-1}$ such that*

$$\det(\chi_1, \ldots, \chi_n)|_{\gamma(\chi_\epsilon)} \not\equiv 0.$$

Let us assume $x = O$. For a generic linear function f, the conditions $f(O) = 0$ and $f|_{\gamma(V_1)} \not\equiv 0$ are ensured. We can then apply the lemma and obtain n vector fields $\chi_1, \ldots, \chi_n$. From the first point of Lemma 3.8, χ_k is a polynomial vector field of degree not exceeding $2^{k-2}d$. Thus the vector field χ_ϵ is polynomial of degree not exceeding $2^{n-3}d$ and the determinant $Q = \det(\chi_1, \ldots, \chi_n)$ is also polynomial. Its degree does not exceed $d + d + \cdots + 2^{n-2}d = 2^{n-1}d$.

The second point of Lemma 3.8 ensures that Q and χ_ϵ fulfill the conditions of Theorem 3.3. Then, applying (26), the contact exponent ν between Q and $\gamma(\chi_\epsilon)$ satisfies

$$\nu \leq 2^{2n^3-4n-1} \sum_{k=1}^{n} (4d + k - 1)^{2n}.$$

Each derivation of Q along χ_ϵ decreases this multiplicity by 1. Hence the result of ν consecutives derivations of Q along χ_ϵ does not vanish at O. By using Lemma 3.6, that means that there exists n brackets $\xi_k = [\chi_\epsilon, \ldots, [\chi_\epsilon, \chi_k] \ldots]$, with at most ν occurences of χ_ϵ, such that:

$$\det(\xi_1(O), \ldots, \xi_n(O)) \neq 0.$$

¿From the first point of Lemma 3.8, each χ_k is a linear combination with polynomial coefficient of brackets of the vector fields V_i of length not exceeding 2^{k-2}. This implies $\chi_k(O) \in L_{2^{k-2}}(\Sigma)(O)$ (this is the same reasoning as in the proof of Theorem 2.1). We have then $\chi_\epsilon(O) \in L_{2^{n-3}}(\Sigma)(O)$ and, $\forall k$, $\xi_k(O) \in L_{2^{n-2}+\nu 2^{n-3}}(\Sigma)(O)$.

Since $\det(\xi_1, \ldots, \xi_n) \neq 0$, the subspace $L_{2^{n-2}+\nu 2^{n-3}}(\Sigma)(O)$ is of dimension n and then

$$r(O) \leq 2^{n-2}(1 + 2^{2n(n-2)-2} d^{2n} \sum_{k=4}^{n+3} k^{2n}).$$

■

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