Fast and Robust Stability Region Estimation for Nonlinear Dynamical Systems
European Control Conference 2021

Eloïse Berthier, Justin Carpentier, Francis Bach

July 1, 2021
Motivation: Feedback Motion Planning

Three steps in *LQR-trees* [TMTR10]:

1. create a tree of trajectories [LK01];
2. **find a stability region around each trajectory**;
3. deduce a global controller [BRK99, TMT11].
1 Stability of the LQR Feedback

2 First-Order Perturbation

3 Second-Order Perturbation

4 Numerical Experiments
Let a nonlinear controlled dynamical system:

\[ \dot{x} = f(x, u) \]

with equilibrium point: \( f(0, 0) = 0 \). Then:

\[ f(x, u) = \frac{\partial f}{\partial x}(0, 0)x + \frac{\partial f}{\partial u}(0, 0)u + o(x) + o(u). \]

Consider the LQR problem for the linearized system \( \dot{x} = Ax + Bu \):

\[ \min_{u(\cdot)} \int_0^{+\infty} (x^T(t)Qx(t) + u^T(t)Ru(t))dt, \text{ with } x(0) = x. \]
The cost-minimizing controller is:

\[ u(x) = -R^{-1}B^\top Sx =: -Kx, \]

where \( S \) is the symmetric positive definite solution of the algebraic Riccati equation, which exists if \( (A, B) \) is controllable:

\[ A^\top S + SA - SBR^{-1}B^\top S = -Q. \]

Under the **closed-loop controller** \( u = -Kx \), the system is **autonomous** with

\[ \dot{x} = f(x, -Kx) =: g(x). \]
Stability Region Estimation (fixed controller)

Find a maximal region $\mathcal{R}$ containing states $x_0$ such that

$$x(0) = x_0, \quad \dot{x} = f(x, -Kx) \implies \lim_{t \to +\infty} x(t) = 0.$$ 

One approach is to find a Lyapunov function $V$ and a region $\mathcal{R}$ such that:

- $V(0) = 0$,
- $\forall x \in \mathcal{R} \setminus \{0\}, \quad V(x) > 0$,
- $\forall x \in \mathcal{R} \setminus \{0\}, \quad \dot{V}(x) < 0$. 

Lyapunov Functions
A natural candidate Lyapunov function is the **LQR cost-to-go**:

\[ V(x) = x^\top Sx \geq 0. \]

The candidate \( \mathcal{R} \) are the **sublevel sets of** \( V \):

\[ \mathcal{B}_\rho := \{ x \mid x^\top Sx \leq \rho \}. \]

**Stability Region Estimation (fixed controller, fixed Lyapunov)**

Find the maximal \( \rho \) such that

\[ V(x) \leq \rho, \ x \neq 0 \implies \dot{V}(x) < 0. \]
If the dynamics is **polynomial**, a sum-of-squares relaxation of the condition is that there exists a SOS polynomial $\sigma(x)$ such that:

$$\dot{V}(x) + \sigma(x)(\rho - V(x)) < 0.$$ 

In practice, this is solved with a hierarchy of SDPs, with a matrix of size $C_{n+d}^d \times C_{n+d}^d$, for $n \geq 1$.

$\rightarrow$ **intractable** in (not so) large dimensions $d \approx 10$. 


Robustness for Nonlinear Systems

Back to the linear system:

\[ g(x) = f(x, -Kx) = (A - BK)x. \]

Can we say something if the closed-loop system is *almost* linear, locally around the equilibrium?

\[ g(x) = (A - BK)x + \delta(x). \]

This could account for uncertainties or model misspecifications, and hence the method is robust. We study two cases:

- First-order perturbation: \( \delta(x) = \bar{A}x, \bar{A} \in \Omega \),
- Second-order perturbation: \( \delta(x) = x^\top \bar{H}x, \bar{H} \in \Xi \).
1 Stability of the LQR Feedback

2 First-Order Perturbation

3 Second-Order Perturbation

4 Numerical Experiments
Linear Differential Inclusion

Consider the uncertain linear system [AC84]:

\[ g(x) = Ax, \ A \in \Omega, \]

where we know bounds on each entry of \( A \):

\[ \Omega = \{ A_0 + \Gamma \mid \forall i, j, |\Gamma_{ij}| \leq U_{ij} \}. \]

This can be recast as \( \Omega = \{ A_0 + C\Delta E \mid \|\Delta\| \leq 1, \ \Delta \text{ diagonal} \} \), for suitable \( C \) and \( E \). \( \Delta \) has size \( d^2 \times d^2 \).
Suppose we have an uncertain linear system:

\[ g(x) = Ax, \ A \in \Omega, \]

with \( \Omega = \{ A_0 + C \Delta E \mid \|\Delta\| \leq 1, \ \Delta \text{ diagonal} \} \).

**Stability of an LDI**

The asymptotic stability of this system with a fixed Lyapunov function is equivalent to the feasibility of the following linear matrix inequality (LMI) [BEGFB94]:

Find \( \Lambda \succeq 0 \in \mathbb{R}^{d^2 \times d^2} \) diagonal such that:

\[
\begin{bmatrix}
A_0^\top S + SA_0 + E^\top \Lambda E & SC \\
C^\top S & -\Lambda
\end{bmatrix} \prec 0.
\]
1 Stability of the LQR Feedback
2 First-Order Perturbation
3 Second-Order Perturbation
4 Numerical Experiments
Suppose that we have:
\[ g_k(x) = (A - BK)_k \cdot x + \frac{1}{2} x^\top \bar{H}^k(x) x. \]

Then:
\[
\dot{V}(x) = 2x^\top S \left( (A - BK)x + \frac{1}{2} (x^\top H^k(x) x)_{k \in \{1, \ldots, d\}} \right) \\
= x^\top (-Q - SBR^{-1} B^\top S + \sum_{k=1}^{d} (S_k \cdot x) \bar{H}^k(x)) x.
\]

A sufficient condition for \( \dot{V}(x) < 0 \) is:
\[
- Q - SBR^{-1} B^\top S + \sum_{k=1}^{d} (S_k \cdot x) \bar{H}^k(x) \prec 0, \quad \forall x \neq 0.
\]
Let $M := Q + SBR^{-1}B^\top S$, and $T^k(x) := M^{-1/2}H^k(x)M^{-1/2}$.

If we know entrywise bounds on the rescaled Hessian $T$:

$$
\Xi := \prod_{k=1}^{d} \Xi^k, \quad \Xi^k = \{ T \in \mathbb{R}^{d \times d} \mid \forall i, j, |T_{ij}| \leq U^k_{ij} \}.
$$

Then we can extract the maximal sublevel set $\rho$:

**Stability Region Estimation (bounded Hessian)**

$$
\rho = \frac{1}{\lambda^2}, \quad \text{where } \lambda := \sup_{\|y\|_2 \leq 1} \sup_{T \in \Xi} \lambda_{\text{max}} \left( \sum_{k=1}^{d} (S_k^{1/2} y) T^k \right).
$$
Theorem (A Second-Order Stability Certificate)

$B_{\rho_a}$ is a stability region for $\rho_a := 1/\lambda_a^2$ and

$$\lambda_a := \lambda_{\max} \left( \sum_{k=1}^{d} \sqrt{S_k . S^{-1} S_k^\top U^k} \right).$$

Theorem (Another Second-Order Stability Certificate)

$B_{\rho_b}$ is a stability region for

$$\rho_b := \frac{1}{d \|DS^{1/2}\|_2^2}, \quad \text{with} \ D := \text{Diag}(\|U^k\|_2).$$
Iterative Algorithm

Two ingredients:

- $\mathcal{O}$ is an oracle bounding the derivatives of $g$ on some region;
- $\mathcal{C}$ returns a stability certificate, given a candidate Lyapunov function $S$ and bounds on the derivatives $U$.

**Input:** $S$, $\mathcal{C}()$, $\mathcal{O}()$, $\rho_0 > 0$, $\eta \in (0, 1)$

**Output:** A stability certificate on $\{x \mid x^\top Sx \leq \rho\}$

1: $\rho_{up} \leftarrow \rho_0$
2: repeat
3: $U \leftarrow \mathcal{O}(B_{\rho_{up}})$
4: $\rho \leftarrow \mathcal{C}(S, \rho_{up}, U)$
5: $\rho_{up} \leftarrow \eta \rho_{up}$
6: until $\rho \geq \rho_{up}$
7: return $\rho$
1. Stability of the LQR Feedback
2. First-Order Perturbation
3. Second-Order Perturbation
4. Numerical Experiments
Examples of Dynamical Systems

We consider controlled dynamical systems of various dimensions around an equilibrium:

- **Pendulum**
  
  \[ d = 4 \]

- **UR-5 Robotic Arm**
  
  6 joints \( \Rightarrow \) dimension \( d = 12 \)
Table 1. Radius and volume of the certified ROA for the different methods, relative to the values obtained by sampling for reference.

<table>
<thead>
<tr>
<th>Dynamics</th>
<th>$C_1$</th>
<th>$C_a$</th>
<th>$C_b$</th>
<th>SOS</th>
<th>sampling</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\rho/\rho_s$</td>
<td>$v/v_s$</td>
<td>$\rho/\rho_s$</td>
<td>$v/v_s$</td>
<td>$\rho/\rho_s$</td>
</tr>
<tr>
<td>Vanderpol</td>
<td>0.20</td>
<td>0.20</td>
<td>0.14</td>
<td>0.14</td>
<td>0.10</td>
</tr>
<tr>
<td>Satellite</td>
<td>$2.9 \times 10^{-2}$</td>
<td>$2.6 \times 10^{-5}$</td>
<td>$9.3 \times 10^{-2}$</td>
<td>$9.4 \times 10^{-4}$</td>
<td>$7.9 \times 10^{-2}$</td>
</tr>
<tr>
<td>Pend. (bot.)</td>
<td>$3.2 \times 10^{-2}$</td>
<td>$1.1 \times 10^{-3}$</td>
<td>$3.5 \times 10^{-2}$</td>
<td>$1.2 \times 10^{-3}$</td>
<td><strong>4.2 \times 10^{-2}</strong></td>
</tr>
<tr>
<td>Pend. (top)</td>
<td>$5.1 \times 10^{-3}$</td>
<td>$2.6 \times 10^{-5}$</td>
<td>$4.5 \times 10^{-2}$</td>
<td>$2.0 \times 10^{-3}$</td>
<td><strong>4.7 \times 10^{-2}</strong></td>
</tr>
<tr>
<td>Robot</td>
<td>$2.4 \times 10^{-3}$</td>
<td>$1.8 \times 10^{-16}$</td>
<td>$7.1 \times 10^{-3}$</td>
<td>$1.5 \times 10^{-13}$</td>
<td><strong>1.5 \times 10^{-2}</strong></td>
</tr>
</tbody>
</table>
Table 2. CPU time (s) per iteration, except for SOS (total time).

<table>
<thead>
<tr>
<th>Dynamics</th>
<th>$\Theta + C_1$</th>
<th>$\Theta + C^a_2$</th>
<th>$\Theta + C^b_2$</th>
<th>SOS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vanderpol</td>
<td>$1.8 \times 10^{-3}$</td>
<td>$1.1 \times 10^{-4}$</td>
<td>$1.6 \times 10^{-4}$</td>
<td>0.05</td>
</tr>
<tr>
<td>Satellite</td>
<td>1.2</td>
<td>0.17</td>
<td>0.17</td>
<td>32</td>
</tr>
<tr>
<td>Pend. (bot.)</td>
<td>2.3</td>
<td>15</td>
<td>15</td>
<td>132</td>
</tr>
<tr>
<td>Robot</td>
<td>2.3</td>
<td>32</td>
<td>33</td>
<td>N.A.</td>
</tr>
</tbody>
</table>
A funnel $\mathcal{B}(t)$ around a trajectory, obtained with $C_1$. \( \mathcal{R} \) is a region of attraction around 0, and $\mathcal{B}_f$ is the target region. The reference trajectory is displayed with arrows.
\( \rho(t) \) with different certificates, around a trajectory of Vanderpol. The total CPU time is 7s for two iterations of SOS, 1s for \( C_1, C_2^a \).
A general method to compute stability certificates, that is:

- simple to implement;
- fast to compute;
- tractable in large dimensions;
- applicable to non-polynomial systems;
- BUT less precise than SOS based certificates.

When integrated into the \textbf{LQR-trees framework}, is it better to compute:

- a lot of low-quality certificates with this method,
- or a few tight certificates with SOS?


