Exercise Sheet
Description Logics

Exercise 1: Modelisation

Build a description logic knowledge base that models the following statements, using the following concept, role, and individual names. For each axiom, say whether it belongs to the language $\mathcal{ALC}$ (recall that an $\mathcal{ALC}$ TBox is a set of concept inclusions that use only the constructors $\sqcap, \sqcup, \neg, \exists, \forall$).

$$\begin{aligned}
\text{Concept names} \quad & \{\mathsf{Person}, \mathsf{Teacher}, \mathsf{Professor}, \mathsf{Researcher}, \mathsf{AdminStaff}, \\
& \mathsf{TechnicalStaff}, \mathsf{PhDStudent}, \mathsf{Student}, \mathsf{Course}, \\
& \mathsf{Tutorial}, \mathsf{HandsOnSession}, \mathsf{University}, \mathsf{Department}\} \\
\text{Role names} \quad & \{\mathsf{teach}, \mathsf{supervise}, \mathsf{employedBy}, \mathsf{memberOf}, \mathsf{attend}, \mathsf{partOf}\} \\
\text{Individual names} \quad & \{\mathit{john}, \mathit{ana}, \mathit{logic}\}
\end{aligned}$$

1. PhD students are students and researchers.
2. Professors are not PhD students.
3. PhD students are employed by some university.
4. Those who are employed by some university are researchers, professors, administrative staff workers or technical staff workers.
5. Teachers are exactly the persons that teach some course.
6. Professors teach at least two courses.
7. PhD students are supervised by a researcher.
8. PhD students teach only tutorials or hands-on-sessions.
9. Administrative staff workers do not supervise PhD students.
10. Researchers are members of a department which is part of a university.
11. Students that are not PhD students are not employed by a university.
12. Things that are taught are courses.
13. Courses are attended by students.
14. Courses taught by Ana are not hands-on-sessions.
15. Ana is a researcher.
16. John is a PhD student who teaches logic and is supervised by Ana.

Can you express that PhD students are employed by the same university that the one the department they are member of is part of ?

Exercise 2: Interpretations

Consider the following interpretation $\mathcal{I}$ with $\Delta^{\mathcal{I}} = \{a, b, c, d\}$.

$$A^{\mathcal{I}} = \{a, b\} \qquad B^{\mathcal{I}} = \{b\} \qquad C^{\mathcal{I}} = \{c, d\} \qquad R^{\mathcal{I}} = \{(a, b), (a, c)\} \qquad S^{\mathcal{I}} = \{(a, b), (b, c)\}$$

For each of the following concepts D, list all elements x of $\Delta^{\mathcal{I}}$ such that $x \in D^{\mathcal{I}}$:

1. $A \sqcap \exists S.C$
2. $B \sqcup (C \sqcap \exists S^{-}.\top)$
3. $\forall R.C$
4. $\forall S.C$
5. $A \sqcap \neg\exists R.\top$
6. $\exists R.\exists S.\top$

For each of the following TBox statements, say whether it is satisfied in $\mathcal{I}$.

1. $A \sqsubseteq B \sqcup C$
2. $A \sqsubseteq \exists S.\top$
3. $\exists S^{-}.B \sqsubseteq C$
4. $A \sqsubseteq \neg C$

Exercise 3: Basic reasoning

Consider the following TBox and ABoxes.

$$\mathcal{T} = \{A \sqsubseteq \forall R.B,\ \exists R.B \sqsubseteq C,\ B \sqsubseteq \neg C\}$$

$$\mathcal{A}_1 = \{A(a)\} \qquad \mathcal{A}_2 = \{A(a), R(a, b)\} \qquad \mathcal{A}_3 = \{A(a), R(a, b), C(b)\}$$

1. Does $\mathcal{T} \models A \sqsubseteq C$?
2. Does $\mathcal{T} \models A \sqcap \exists R.\top \sqsubseteq \neg B$?
3. Is $B \sqcap \exists R.B$ satisfiable w.r.t. $\mathcal{T}$? If so give a model of $\mathcal{T}$ where $B \sqcap \exists R.B$ is non-empty.
4. Is $A \sqcap \forall R.C$ satisfiable w.r.t. $\mathcal{T}$? If so give a model of $\mathcal{T}$ where $A \sqcap \forall R.C$ is non-empty.
5. Is the knowledge base $\langle \mathcal{T}, \mathcal{A}_1 \rangle$ satisfiable ? If so, give a model of $\langle \mathcal{T}, \mathcal{A}_1 \rangle$.
6. Is the knowledge base $\langle \mathcal{T}, \mathcal{A}_2 \rangle$ satisfiable ? If so, give a model of $\langle \mathcal{T}, \mathcal{A}_2 \rangle$.
7. Is the knowledge base $\langle \mathcal{T}, \mathcal{A}_3 \rangle$ satisfiable ? If so, give a model of $\langle \mathcal{T}, \mathcal{A}_3 \rangle$.
8. Does $\langle \mathcal{T}, \mathcal{A}_1 \rangle \models C(a)$?
9. Does $\langle \mathcal{T}, \mathcal{A}_2 \rangle \models C(a)$?
10. Does $\langle \mathcal{T}, \mathcal{A}_3 \rangle \models C(a)$?

Exercise 4: DL fragments

Recall the definition of $\mathcal{ALC}$ concepts:

- if A is an atomic concept, then A is an $\mathcal{ALC}$ concept
- if C, D are $\mathcal{ALC}$ concepts and R is an atomic role, then the following are $\mathcal{ALC}$ concepts:
  - $C \sqcap D$ (conjunction)
  - $C \sqcup D$ (disjunction)
  - $\neg C$ (negation)
  - $\exists R.C$ (existential restriction)
  - $\forall R.C$ (universal restriction)

Each subset of the set of constructors $\{\sqcap, \sqcup, \neg, \exists, \forall\}$ defines a *fragment* of $\mathcal{ALC}$. Identify all minimal fragments that are equivalent to $\mathcal{ALC}$ in the sense that, for every $\mathcal{ALC}$ concept, there is an equivalent concept in the fragment.

Exercise 5: Translation to FOL

Translate the following TBox statements into first-order logic.

1. $\exists R.\exists S.\top \sqsubseteq B \sqcup C$
2. $A \sqcap \neg B \sqsubseteq \forall R.C$
3. $\exists R^{-}.A \sqsubseteq \neg C$
4. $A \sqcup \exists R.B \sqsubseteq \exists S.\top$

Exercise 6: Negation normal form

Put the following concepts in negation normal form:

1. $\neg(\neg A \sqcup \forall R.(\neg(B \sqcap \neg C)))$
2. $\neg(\exists R.(\neg\exists S.A)) \sqcap \neg(\forall R.B)$

Exercise 7: Tableau algorithm for concept satisfiability

Use the tableau algorithm to decide which of the following concepts are satisfiable:

1. $\exists R.\exists S.A \sqcap \forall R.\forall S.\neg A$
2. $\exists R.B \sqcap \forall R.\forall R.A \sqcap \forall R.\neg A$

Exercise 8: Tableau algorithm for KB satisfiability

Consider the following TBox:

$$\mathcal{T} = \{A \sqsubseteq \exists R.(A \sqcup \neg B),\ \exists R.\neg B \sqsubseteq C,\ \exists R.C \sqsubseteq C\}.$$

Using the tableau algorithm, decide whether $\mathcal{T} \models A \sqsubseteq C$. If $\mathcal{T} \not\models A \sqsubseteq C$, provide a model of $\mathcal{T}$ that shows it.

Exercise 9: Tableau algorithm for KB satisfiability – Optimization

Consider the following TBox:

$$\mathcal{T} = \{A \sqsubseteq \forall R.B,\ B \sqsubseteq \neg F,\ E \sqsubseteq G,\ A \sqsubseteq D \sqcup E,\ D \sqsubseteq \exists R.F,\ \exists R.\neg B \sqsubseteq G\}.$$

Using the tableau algorithm with the optimization that replaces the TBox-rule for concept inclusions with atomic left- or right-hand side, decide whether:

1. $\mathcal{T} \models A \sqsubseteq E$
2. $\mathcal{T} \models E \sqsubseteq F$
3. $\mathcal{T} \models A \sqsubseteq G$
4. $\mathcal{T} \models D \sqsubseteq G$
5. $\mathcal{T} \models G \sqsubseteq F$

Exercise 10: Negation normal form algorithm

Prove that the algorithm for computing the negation normal form of $\mathcal{ALC}$ concept given in the lecture is correct, that is, prove that for every $\mathcal{ALC}$ concept C, $\mathsf{nnf}(C)$ is in NNF and for every interpretation $\mathcal{I}$, $C^{\mathcal{I}} = \mathsf{nnf}(C)^{\mathcal{I}}$, where $\mathsf{nnf}(C)$ is defined as follows.

- $\mathsf{nnf}(A) = A$ for A atomic concept
- $\mathsf{nnf}(\neg A) = \neg A$ for A atomic concept
- $\mathsf{nnf}(C \sqcap D) = \mathsf{nnf}(C) \sqcap \mathsf{nnf}(D)$
- $\mathsf{nnf}(C \sqcup D) = \mathsf{nnf}(C) \sqcup \mathsf{nnf}(D)$
- $\mathsf{nnf}(\exists R.C) = \exists R.\mathsf{nnf}(C)$
- $\mathsf{nnf}(\forall R.C) = \forall R.\mathsf{nnf}(C)$
- $\mathsf{nnf}(\neg(\neg C)) = \mathsf{nnf}(C)$
- $\mathsf{nnf}(\neg(C \sqcap D)) = \mathsf{nnf}(\neg C) \sqcup \mathsf{nnf}(\neg D)$
- $\mathsf{nnf}(\neg(C \sqcup D)) = \mathsf{nnf}(\neg C) \sqcap \mathsf{nnf}(\neg D)$
- $\mathsf{nnf}(\neg(\exists R.C)) = \forall R.\mathsf{nnf}(\neg C)$
- $\mathsf{nnf}(\neg(\forall R.C)) = \exists R.\mathsf{nnf}(\neg C)$

Exercise 11: Adapting tableau algorithm for another DL

Modify the tableau algorithm to decide satisfiability of KBs $\langle \mathcal{T}, \mathcal{A} \rangle$ where $\mathcal{T}$ is a TBox that contains only role inclusions of the form $R \sqsubseteq S$ or $R \sqsubseteq \neg S$. Show that your algorithm terminates and is sound and complete.

Exercise 12: Normal form of $\mathcal{EL}$ TBoxes

Normalize the following $\mathcal{EL}$ TBox.

$$\mathcal{T} = \{A \sqsubseteq \exists R.\exists S.C, \quad A \sqcap \exists R.\exists S.C \sqsubseteq B \sqcap C, \quad \exists R.\top \sqcap B \sqsubseteq \exists S.\exists R.D\}$$

Exercise 13: Compact canonical model

Construct the compact canonical model of the following KB and use it to classify $\mathcal{T}$ and find all assertions entailed by $\langle \mathcal{T}, \mathcal{A} \rangle$.

$$\begin{aligned}
&\mathcal{T} = \{\ A \sqsubseteq \exists R.B, \quad B \sqsubseteq \exists R.D, \quad C \sqsubseteq \exists S.C, \quad A \sqcap C \sqsubseteq D, \quad B \sqcap C \sqsubseteq D, \quad \exists R.\top \sqsubseteq C\ \} \\
&\mathcal{A} = \{\ A(a), \quad R(b, a)\ \}
\end{aligned}$$

Exercise 14: Saturation algorithm

Consider the following $\mathcal{EL}$ TBox and ABox

$$\begin{aligned}
\mathcal{T} = \{\ & A \sqsubseteq B, \quad \exists R.\top \sqsubseteq D, \quad H \sqsubseteq \exists P.A, \quad D \sqsubseteq M, \\
& B \sqsubseteq \exists R.E, \quad D \sqcap M \sqsubseteq H, \quad A \sqsubseteq \exists S.B, \quad \exists S.M \sqsubseteq G\ \} \\
\mathcal{A} = \{\ & D(a), \quad S(a, b), \quad R(b, a)\ \}
\end{aligned}$$

1. Classify $\mathcal{T}$ and find all assertions entailed by $\langle \mathcal{T}, \mathcal{A} \rangle$ using the saturation algorithm.

2. Construct the compact canonical model from the saturated KB.

Exercise 15: Properties of conservative extensions

Prove the following properties stated in the course:

1. If $\mathcal{T}_2$ is a conservative extension of $\mathcal{T}_1$ and $\mathcal{T}_3$ is a conservative extension of $\mathcal{T}_2$, then $\mathcal{T}_3$ is a conservative extension of $\mathcal{T}_1$.

2. If $\mathcal{T}_2$ is a conservative extension of $\mathcal{T}_1$ and C and D are concepts containing only concept and role names from $\mathcal{T}_1$, then it holds that $\mathcal{T}_1 \models C \sqsubseteq D$ if and only if $\mathcal{T}_2 \models C \sqsubseteq D$.

3. If $\mathcal{T}_2$ is a conservative extension of $\mathcal{T}_1$, then for every ABox $\mathcal{A}$ and assertion α that use only atomic concepts and roles from $\mathcal{T}_1$, $\langle \mathcal{T}_1, \mathcal{A} \rangle \models \alpha$ iff $\langle \mathcal{T}_2, \mathcal{A} \rangle \models \alpha$.

Exercise 16: Conservative extensions

Let $\mathcal{T}_1$ be an $\mathcal{EL}$ TBox and C, D be two $\mathcal{EL}$ concepts. Let $\mathcal{T}_2 = \mathcal{T}_1 \cup \{A \sqsubseteq C,\ D \sqsubseteq B\}$ where A and B are atomic concepts that do not occur in $\mathcal{T}_1$.

1. Show that $\mathcal{T}_2$ is a conservative extension of $\mathcal{T}_1$.

2. Is $\mathcal{T}_2 \cup \{A \sqsubseteq B\}$ a conservative extension of $\mathcal{T}_1$?

3. Is $\mathcal{T}_2 \cup \{B \sqsubseteq A\}$ a conservative extension of $\mathcal{T}_1$?