Correction of Exercise Sheet
Description Logics

Correction of Exercise 1: Modelisation

1. PhD students are students and researchers.
   $\mathsf{PhDStudent} \sqsubseteq \mathsf{Student} \sqcap \mathsf{Researcher}$: in $\mathcal{ALC}$

2. Professors are not PhD students.
   $\mathsf{Professor} \sqsubseteq \neg\mathsf{PhDStudent}$: in $\mathcal{ALC}$

3. PhD students are employed by some university.
   $\mathsf{PhDStudent} \sqsubseteq \exists \mathsf{employedBy}.\mathsf{University}$: in $\mathcal{ALC}$

4. Those who are employed by some university are researchers, professors, administrative staff workers or technical staff workers.
   $\exists \mathsf{employedBy}.\mathsf{University} \sqsubseteq \mathsf{Researcher} \sqcup \mathsf{Professor} \sqcup \mathsf{AdminStaff} \sqcup \mathsf{TechnicalStaff}$: in $\mathcal{ALC}$

5. Teachers are exactly the persons that teach some course.
   $\mathsf{Teacher} \equiv \mathsf{Person} \sqcap \exists \mathsf{teach}.\mathsf{Course}$: in $\mathcal{ALC}$

6. Professors teach at least two courses.
   $\mathsf{Professor} \sqsubseteq \geq 2\mathsf{teach}.\mathsf{Course}$: not in $\mathcal{ALC}$ (number restriction)

7. PhD students are supervised by a researcher.
   $\mathsf{PhDStudent} \sqsubseteq \exists \mathsf{supervise}^{-}.\mathsf{Researcher}$: not in $\mathcal{ALC}$ (inverse role)

8. PhD students teach only tutorials or hands-on-sessions.
   $\mathsf{PhDStudent} \sqsubseteq \forall \mathsf{teach}.(\mathsf{Tutorial} \sqcup \mathsf{HandsOnSession})$: in $\mathcal{ALC}$

9. Administrative staff workers do not supervise PhD students.
   $\mathsf{AdminStaff} \sqsubseteq \forall \mathsf{supervise}.(\neg\mathsf{PhDStudent})$: in $\mathcal{ALC}$

10. Researchers are members of a department which is part of a university.
    $\mathsf{Researcher} \sqsubseteq \exists \mathsf{memberOf}.(\mathsf{Department} \sqcap \exists \mathsf{partOf}.\mathsf{University})$: in $\mathcal{ALC}$

11. Students that are not PhD students are not employed by a university.
    $\mathsf{Student} \sqcap \neg\mathsf{PhDStudent} \sqsubseteq \neg(\exists \mathsf{employedBy}.\mathsf{University})$: in $\mathcal{ALC}$

12. Things that are taught are courses.
    $\top \sqsubseteq \forall \mathsf{teach}.\mathsf{Course}$: in $\mathcal{ALC}$ (equivalent to $\exists \mathsf{teach}^{-}.\top \sqsubseteq \mathsf{Course}$ which is not in $\mathcal{ALC}$)

13. Courses are attended by at most 50 students.
    $\mathsf{Course} \sqsubseteq \leq 50\mathsf{attend}^{-}.\mathsf{Student}$: not in $\mathcal{ALC}$ (number restriction, inverse role)

14. Courses taught by Ana are not hands-on-sessions.
    $\mathsf{Course} \sqcap \exists \mathsf{teach}^{-}.\{ana\} \sqsubseteq \neg\mathsf{HandsOnSession}$: not in $\mathcal{ALC}$ (nomimals, inverse role)

15. Ana is a researcher.
    $\mathsf{Researcher}(ana)$

16. John is a PhD student who teaches logic and is supervised by Ana.
    $\mathsf{PhDStudent}(john)$, $\mathsf{teach}(john, logic)$, $\mathsf{supervise}(ana, john)$

Can you express that PhD students are employed by the same university that the one the department they are member of is part of ?

No. Best try: $\mathsf{PhDStudent} \sqsubseteq \exists \mathsf{employedBy}.(\mathsf{University} \sqcap (\exists \mathsf{partOf}^-.(\mathsf{Department} \sqcap \exists \mathsf{memberOf}^-.\mathsf{PhDStudent})))$ but no way to say that the PhD student is the same.

Correction of Exercise 2: Interpretations

1. $(A \sqcap \exists S.C)^\mathcal{I} = \{b\}$
2. $(B \sqcup (C \sqcap \exists S^-.\top))^\mathcal{I} = \{b, c\}$
3. $(\forall R.C)^\mathcal{I} = \{b, c, d\}$
4. $(\forall S.C)^\mathcal{I} = \{b, c, d\}$
5. $(A \sqcap \neg\exists R.\top)^\mathcal{I} = \{b\}$
6. $(\exists R.\exists S.\top)^\mathcal{I} = \{a\}$

1. No: $\mathcal{I} \not\models A \sqsubseteq B \sqcup C$ because $\{a, b\} \not\subseteq \{b, c, d\}$
2. Yes: $\mathcal{I} \models A \sqsubseteq \exists S.\top$ because $\{a, b\} \subseteq \{a, b\}$
3. Yes: $\mathcal{I} \models \exists S^-.B \sqsubseteq C$ because $\{c\} \subseteq \{c, d\}$
4. Yes: $\mathcal{I} \models A \sqsubseteq \neg C$ because $\{a, b\} \subseteq \{a, b\}$

Correction of Exercise 3: Basic reasoning

1. No. Consider the following interpretation $\mathcal{I}$ on domain $\Delta^\mathcal{I} = \{a\}$: $A^\mathcal{I} = \{a\}$, $B^\mathcal{I} = \emptyset$, $C^\mathcal{I} = \emptyset$, $R^\mathcal{I} = \emptyset$. $\mathcal{I}$ is a model of $\mathcal{T}$ and $A^\mathcal{I} \not\subseteq C^\mathcal{I}$ so $\mathcal{T} \not\models A \sqsubseteq C$.

2. Yes. Let $\mathcal{I}$ be a model of $\mathcal{T}$ and e be an element of $\Delta^\mathcal{I}$ such that $e \in (A \sqcap \exists R.\top)^\mathcal{I} = A^\mathcal{I} \cap (\exists R.\top)^\mathcal{I}$. Since $e \in (\exists R.\top)^\mathcal{I}$, there exists $d \in \Delta^\mathcal{I}$ such that $(e, d) \in R^\mathcal{I}$. Since $\mathcal{I}$ is a model of $\mathcal{T}$, $\mathcal{I} \models A \sqsubseteq \forall R.B$, so $e \in A^\mathcal{I}$ and $(e, d) \in R^\mathcal{I}$ implies that $d \in B^\mathcal{I}$. Hence $e \in (\exists R.B)^\mathcal{I}$. Since $\mathcal{I} \models \exists R.B \sqsubseteq C$, it follows that $e \in C^\mathcal{I}$. Finally, since $\mathcal{I} \models B \sqsubseteq \neg C$, $B^\mathcal{I} \subseteq \Delta^\mathcal{I} \setminus C^\mathcal{I}$ so $e \notin B^\mathcal{I}$. We have shown that for every model $\mathcal{I}$ of $\mathcal{T}$, $(A \sqcap \exists R.\top)^\mathcal{I} \subseteq \Delta^\mathcal{I} \setminus B^\mathcal{I}$, i.e. $\mathcal{I} \models A \sqcap \exists R.\top \sqsubseteq \neg B$. This is exactly the definition of $\mathcal{T} \models A \sqcap \exists R.\top \sqsubseteq \neg B$.

3. No. Assume for a contradiction that there exists a model $\mathcal{I}$ of $\mathcal{T}$ such that $(B \sqcap \exists R.B)^\mathcal{I}$ is non-empty and let $e \in (B \sqcap \exists R.B)^\mathcal{I}$. Since $\mathcal{I} \models \exists R.B \sqsubseteq C$ and $e \in (\exists R.B)^\mathcal{I}$, then $e \in C^\mathcal{I}$. It follows that e belongs to $B^\mathcal{I}$ and to $C^\mathcal{I}$, so $B^\mathcal{I} \not\subseteq \Delta^\mathcal{I} \setminus C^\mathcal{I}$, which contradicts $\mathcal{T} \models B \sqsubseteq \neg C$.

4. Yes. Consider the model $\mathcal{I}$ of $\mathcal{T}$ given in the correction of question 1. $(A \sqcap \forall R.C)^\mathcal{I} = \{a\}$ is non-empty.

5. Yes. We just need to extend the interpretation given in the correction of question 1 by setting $a^\mathcal{I} = a$ to obtain a model of $\langle \mathcal{T}, \mathcal{A}_1 \rangle$.

6. Yes. Consider the following interpretation $\mathcal{I}$ on domain $\Delta^\mathcal{I} = \{a, b\}$: $a^\mathcal{I} = a$, $b^\mathcal{I} = b$, $A^\mathcal{I} = \{a\}$, $B^\mathcal{I} = \{b\}$, $C^\mathcal{I} = \{a\}$, $R^\mathcal{I} = \{(a, b)\}$. $\mathcal{I}$ is a model of $\langle \mathcal{T}, \mathcal{A}_2 \rangle$.

7. No. Assume for a contradiction that $\langle \mathcal{T}, \mathcal{A}_3 \rangle$ has a model $\mathcal{I}$. We must have $a^\mathcal{I} \in A^\mathcal{I}$ and $(a^\mathcal{I}, b^\mathcal{I}) \in R^\mathcal{I}$ so since $\mathcal{I} \models A \sqsubseteq \forall R.B$, it follows that $b \in B^\mathcal{I}$. However, we also must have $b \in C^\mathcal{I}$, which contradicts $\mathcal{I} \models B \sqsubseteq \neg C$.

8. No. The model of $\langle \mathcal{T}, \mathcal{A}_1 \rangle$ given in question 5 does not satisfy $C(a)$.

9. Yes. Let $\mathcal{I}$ be a model of $\langle \mathcal{T}, \mathcal{A}_2 \rangle$. Since $\mathcal{I} \models A(a)$ and $\mathcal{I} \models R(a, b)$, then $a^\mathcal{I} \in A^\mathcal{I}$ and $(a^\mathcal{I}, b^\mathcal{I}) \in R^\mathcal{I}$. Since $\mathcal{I} \models A \sqsubseteq \forall R.B$, it follows that $b \in B^\mathcal{I}$. Hence $a^\mathcal{I} \in (\exists R.B)^\mathcal{I}$, so since $\mathcal{I} \models \exists R.B \sqsubseteq C$, $a^\mathcal{I} \in C^\mathcal{I}$. We have shown that for every model $\mathcal{I}$ of $\langle \mathcal{T}, \mathcal{A}_2 \rangle$, $a^\mathcal{I} \in C^\mathcal{I}$. This is exactly the definition of $\langle \mathcal{T}, \mathcal{A}_2 \rangle \models C(a)$.

10. Yes. Since $\langle \mathcal{T}, \mathcal{A}_3 \rangle$ has no model, it is true that $a^\mathcal{I} \in C^\mathcal{I}$ in every model of $\langle \mathcal{T}, \mathcal{A}_3 \rangle$. An unsatisfiable knowledge base entails every logical axiom.

Correction of Exercise 4: DL fragments

Minimal fragments of $\mathcal{ALC}$: $\{\sqcap, \neg, \exists\}$, $\{\sqcap, \neg, \forall\}$, $\{\sqcup, \neg, \exists\}$, $\{\sqcup, \neg, \forall\}$.

Proof for the $\{\sqcap, \neg, \exists\}$ fragment:

Let C be an $\mathcal{ALC}$ concept. We first show by induction on the structure of C that there exists C' in the $\{\sqcap, \neg, \exists\}$ fragment that is equivalent to C.

Base case: If C is an atomic concept, then C is in the $\{\sqcap, \neg, \exists\}$ fragment.

- If $C = C_1 \sqcap C_2$, and C_1, C_2 are $\mathcal{ALC}$ concepts equivalent to C_1' and C_2' in the $\{\sqcap, \neg, \exists\}$ fragment, then C is equivalent to $C' = C_1' \sqcap C_2'$ which belongs to the fragment.
- If $C = C_1 \sqcup C_2$, and C_1, C_2 are $\mathcal{ALC}$ concepts equivalent to C_1' and C_2' in the $\{\sqcap, \neg, \exists\}$ fragment, then C is equivalent to $C' = \neg(\neg C_1' \sqcap \neg C_2')$ which belongs to the fragment.
- If $C = \neg C_1$ and C_1 is an $\mathcal{ALC}$ concept equivalent to C_1' in the $\{\sqcap, \neg, \exists\}$ fragment, then C is equivalent to $C' = \neg C_1'$ which belongs to the fragment.
- If $C = \exists R.C_1$ and C_1 is an $\mathcal{ALC}$ concept equivalent to C_1' in the $\{\sqcap, \neg, \exists\}$ fragment, then C is equivalent to $C' = \exists R.C_1'$ which belongs to the fragment.
- If $C = \forall R.C_1$ and C_1 is an $\mathcal{ALC}$ concept equivalent to C_1' in the $\{\sqcap, \neg, \exists\}$ fragment, then C is equivalent to $C' = \neg(\exists R.\neg C_1')$ which belongs to the fragment.

We now show that every sub-fragment of the $\{\sqcap, \neg, \exists\}$ fragment does not capture $\mathcal{ALC}$. Let A and B be atomic concepts.

- $A \sqcap B$ cannot be expressed on $\{\neg, \exists\}$
- $\neg A$ cannot be expressed on $\{\sqcap, \exists\}$
- $\exists R.A$ cannot be expressed on $\{\sqcap, \neg\}$

Correction of Exercise 5: Translation to FOL

1. $\forall x\ (\exists y\ (R(x, y) \wedge \exists z\ S(y, z)) \Rightarrow B(x) \vee C(x))$
2. $\forall x\ (A(x) \wedge \neg B(x) \Rightarrow \forall y\ (R(x, y) \Rightarrow C(y)))$
3. $\forall x\ (\exists y\ (R(y, x) \wedge A(y)) \Rightarrow \neg C(x))$
4. $\forall x\ (A(x) \vee \exists y\ (R(x, y) \wedge B(y)) \Rightarrow \exists z\ S(x, z))$

Correction of Exercise 6: Negation normal form

$$
\begin{aligned}
1. \qquad \mathsf{nnf}(\neg(\neg A \sqcup \forall R.(\neg(B \sqcap \neg C)))) =& \mathsf{nnf}(\neg(\neg A) \sqcap \mathsf{nnf}(\neg(\forall R.(\neg(B \sqcap \neg C))))\\
=& \mathsf{nnf}(A) \sqcap \exists R.\mathsf{nnf}(\neg(\neg(B \sqcap \neg C))))\\
=& A \sqcap \exists R.\mathsf{nnf}(B \sqcap \neg C)\\
=& A \sqcap \exists R.(\mathsf{nnf}(B) \sqcap \mathsf{nnf}(\neg C))\\
=& A \sqcap \exists R.(B \sqcap \neg C)\\
2. \qquad \mathsf{nnf}(\neg(\exists R.(\neg\exists S.A)) \sqcap \neg(\forall R.B)) =& \mathsf{nnf}(\neg(\exists R.(\neg\exists S.A))) \sqcap \mathsf{nnf}(\neg(\forall R.B))\\
=& \forall R.\mathsf{nnf}(\neg(\neg\exists S.A)) \sqcap \exists R.\mathsf{nnf}(\neg B)\\
=& \forall R.\mathsf{nnf}(\exists S.A) \sqcap \exists R.(\neg B)\\
=& \forall R.\exists S.\mathsf{nnf}(A) \sqcap \exists R.(\neg B)\\
=& \forall R.\exists S.A \sqcap \exists R.(\neg B)
\end{aligned}
$$

Correction of Exercise 7: Tableau algorithm for concept satisfiability

1. $\exists R.\exists S.A \sqcap \forall R.\forall S.\neg A$ is not satisfiable. Indeed, every ABox generated by the tableau algorithm contains a clash:

$$\begin{array}{c}
(\exists R.\exists S.A \sqcap \forall R.\forall S.\neg A)(a_0) \\
| \\
(\exists R.\exists S.A)(a_0) \\
| \\
(\forall R.\forall S.\neg A)(a_0) \\
| \\
R(a_0, a_1) \\
| \\
(\exists S.A)(a_1) \\
| \\
(\forall S.\neg A)(a_1) \\
| \\
S(a_1, a_2) \\
| \\
A(a_2) \\
\neg A(a_2) \\
\text{✗}
\end{array}$$

2. $\exists R.B \sqcap \forall R.\forall R.A \sqcap \forall R.\neg A$ is satisfiable. The interpretation $\mathcal{I}$ defined by $B^{\mathcal{I}} = \{a_1\}$, $A^{\mathcal{I}} = \emptyset$ and $R^{\mathcal{I}} = \{(a_0, a_1)\}$ is such that $(\exists R.B \sqcap \forall R.\forall R.A \sqcap \forall R.\neg A)^{\mathcal{I}}$ is non-empty.

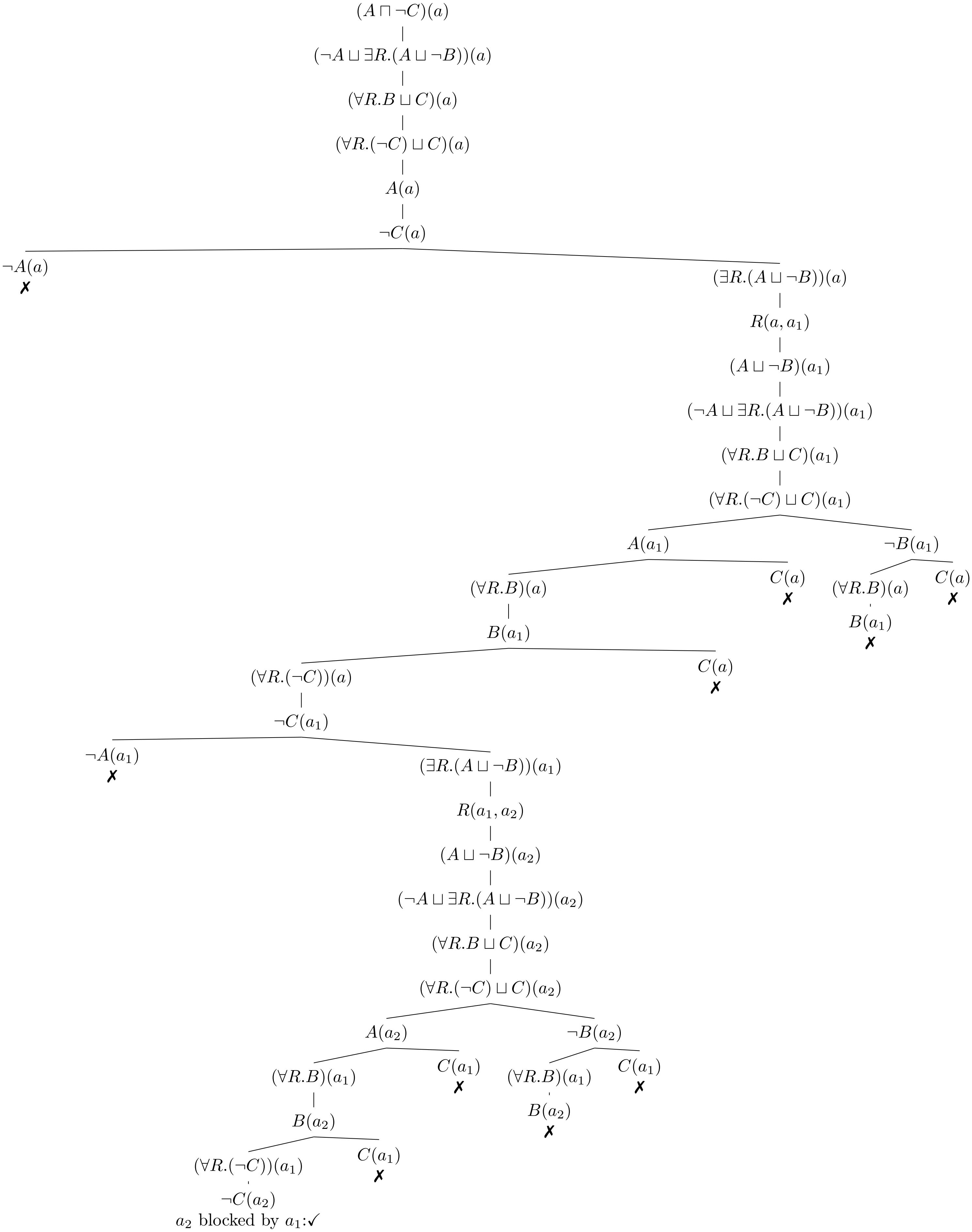
$$\begin{array}{c}
(\exists R.B \sqcap \forall R.\forall R.A \sqcap \forall R.\neg A)(a_0) \\
| \\
(\exists R.B)(a_0) \\
| \\
(\forall R.\forall R.A \sqcap \forall R.\neg A)(a_0) \\
| \\
(\forall R.\forall R.A)(a_0) \\
| \\
(\forall R.\neg A)(a_0) \\
| \\
R(a_0, a_1) \\
| \\
B(a_1) \\
| \\
\forall R.A(a_1) \\
\neg A(a_1) \\
\checkmark
\end{array}$$

Correction of Exercise 8: Tableau algorithm for KB satisfiability

To decide whether $\mathcal{T} \models A \sqsubseteq C$ with the tableau algorithm, we need to check whether $\{A \sqcap \neg C\}$ is satisfiable w.r.t. $\mathcal{T}$, i.e., whether $\langle \mathcal{T}, \{(A \sqcap \neg C)(a)\}\rangle$ is satisfiable.

$\langle \mathcal{T}, \{(A \sqcap \neg C)(a)\}\rangle$ is satisfiable so $\mathcal{T} \not\models A \sqsubseteq C$. A model of $\mathcal{T}$ that shows it is:

$$\begin{aligned}
\Delta^{\mathcal{I}} =& \{a, a_1\} \\
A^{\mathcal{I}} =& \{a, a_1\} \\
B^{\mathcal{I}} =& \{a_1\} \\
C^{\mathcal{I}} =& \emptyset \\
R^{\mathcal{I}} =& \{(a, a_1), (a_1, a_1)\}
\end{aligned}$$


$(A \sqcap \neg C)(a)$
$(\neg A \sqcup \exists R.(A \sqcup \neg B))(a)$
$(\forall R.B \sqcup C)(a)$
$(\forall R.(\neg C) \sqcup C)(a)$
$A(a)$
$\neg C(a)$
$\neg A(a)$
✗
$(\exists R.(A \sqcup \neg B))(a)$
$R(a, a_1)$
$(A \sqcup \neg B)(a_1)$
$(\neg A \sqcup \exists R.(A \sqcup \neg B))(a_1)$
$(\forall R.B \sqcup C)(a_1)$
$(\forall R.(\neg C) \sqcup C)(a_1)$
$A(a_1)$
$\neg B(a_1)$
$(\forall R.B)(a)$
$C(a)$
✗
$(\forall R.B)(a)$
$C(a)$
✗
$B(a_1)$
$B(a_1)$
✗
$(\forall R.(\neg C))(a)$
$C(a)$
✗
$\neg C(a_1)$
$\neg A(a_1)$
✗
$(\exists R.(A \sqcup \neg B))(a_1)$
$R(a_1, a_2)$
$(A \sqcup \neg B)(a_2)$
$(\neg A \sqcup \exists R.(A \sqcup \neg B))(a_2)$
$(\forall R.B \sqcup C)(a_2)$
$(\forall R.(\neg C) \sqcup C)(a_2)$
$A(a_2)$
$\neg B(a_2)$
$(\forall R.B)(a_1)$
$C(a_1)$
✗
$(\forall R.B)(a_1)$
$C(a_1)$
✗
$B(a_2)$
$B(a_2)$
✗
$(\forall R.(\neg C))(a_1)$
$C(a_1)$
✗
$\neg C(a_2)$
a_2 blocked by a_1:✓

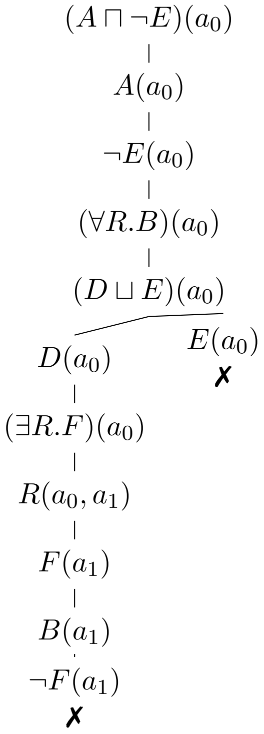
Correction of Exercise 9: Tableau algorithm for KB satisfiability – Optimization

$$\mathcal{T} = \{A \sqsubseteq \forall R.B,\ B \sqsubseteq \neg F,\ E \sqsubseteq G,\ A \sqsubseteq D \sqcup E,\ D \sqsubseteq \exists R.F,\ \exists R.\neg B \sqsubseteq G\}.$$

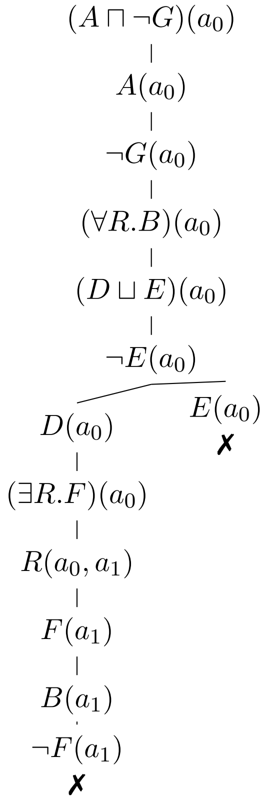
All axioms in $\mathcal{T}$ are inclusions with atomic left- or right-hand side.

- For inclusions $A \sqsubseteq D$ with atomic left-hand side, replace the TBox-rule by TBox-atomic-left-rule: if $A(a) \in \mathcal{A}$, a is not blocked, $A \sqsubseteq D \in \mathcal{T}$ (A atomic), and $D(a) \notin \mathcal{A}$, replace $\mathcal{A}$ with $\mathcal{A} \cup \{D(a)\}$.
- For inclusions $D \sqsubseteq A$ with atomic right-hand side, replace the TBox-rule by TBox-atomic-right-rule: if $\neg A(a) \in \mathcal{A}$, a is not blocked, $D \sqsubseteq A \in \mathcal{T}$ (A atomic), and $\neg D(a) \notin \mathcal{A}$, replace $\mathcal{A}$ with $\mathcal{A} \cup \{\neg D(a)\}$.

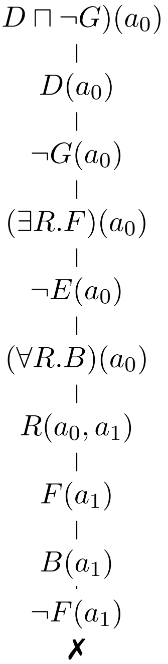
1. $\mathcal{T} \models A \sqsubseteq E$



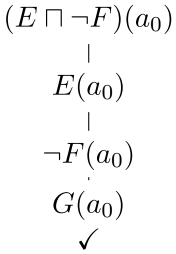
3. $\mathcal{T} \models A \sqsubseteq G$



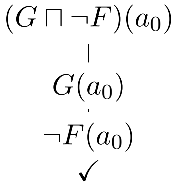
4. $\mathcal{T} \models D \sqsubseteq G$



2. $\mathcal{T} \not\models E \sqsubseteq F$



5. $\mathcal{T} \not\models G \sqsubseteq F$



Correction of Exercise 10: Negation normal form algorithm

Let C be an $\mathcal{ALC}$ concept. We show by structural induction that

1. $\mathsf{nnf}(C)$ is in NNF;
2. for every interpretation $\mathcal{I}$, $C^{\mathcal{I}} = \mathsf{nnf}(C)^{\mathcal{I}}$;
3. $\mathsf{nnf}(\neg C)$ is in NNF;
4. for every interpretation $\mathcal{I}$, $\mathsf{nnf}(\neg C)^{\mathcal{I}} = (\neg C)^{\mathcal{I}}$.

In the base case, C is an atomic concept A or is of the form $\neg A$ for an atomic concept A. In this case, $\mathsf{nnf}(C) = C$ is in NNF, and for every interpretation $\mathcal{I}$, $C^{\mathcal{I}} = \mathsf{nnf}(C)^{\mathcal{I}}$ holds trivially. Moreover, if $C = A$, $\mathsf{nnf}(\neg C) = \neg A$ and if $C = \neg A$, $\mathsf{nnf}(\neg C) = \mathsf{nnf}(\neg(\neg A)) = \mathsf{nnf}(A) = A$ so in both cases, $\mathsf{nnf}(\neg C)$ is in NNF and for every interpretation $\mathcal{I}$, $\mathsf{nnf}(\neg C)^{\mathcal{I}} = (\neg C)^{\mathcal{I}}$.

- If C is of the form $C_1 \sqcap C_2$ with C_1 and C_2 two $\mathcal{ALC}$ concepts such that $\mathsf{nnf}(C_1)$, $\mathsf{nnf}(C_2)$, $\mathsf{nnf}(\neg C_1)$, and $\mathsf{nnf}(\neg C_2)$ are in NNF and for every interpretation $\mathcal{I}$, $C_i^{\mathcal{I}} = \mathsf{nnf}(C_i)^{\mathcal{I}}$ and $\mathsf{nnf}(\neg C_i)^{\mathcal{I}} = (\neg C_i)^{\mathcal{I}}$ $(1 \leq i \leq 2)$, then
    1. $\mathsf{nnf}(C) = \mathsf{nnf}(C_1 \sqcap C_2) = \mathsf{nnf}(C_1) \sqcap \mathsf{nnf}(C_2)$ is in NNF (since negation appears only in front of atomic concepts in $\mathsf{nnf}(C_1)$ and $\mathsf{nnf}(C_2)$);
    2. for every interpretation $\mathcal{I}$, $\mathsf{nnf}(C)^{\mathcal{I}} = (\mathsf{nnf}(C_1) \sqcap \mathsf{nnf}(C_2))^{\mathcal{I}} = \mathsf{nnf}(C_1)^{\mathcal{I}} \cap \mathsf{nnf}(C_2)^{\mathcal{I}} = C_1^{\mathcal{I}} \cap C_2^{\mathcal{I}} = (C_1 \sqcap C_2)^{\mathcal{I}} = C^{\mathcal{I}}$;
    3. $\mathsf{nnf}(\neg C) = \mathsf{nnf}(\neg(C_1 \sqcap C_2)) = \mathsf{nnf}(\neg C_1) \sqcup \mathsf{nnf}(\neg C_2)$ is in NNF (since negation appears only in front of atomic concepts in $\mathsf{nnf}(\neg C_1)$ and $\mathsf{nnf}(\neg C_2)$);
    4. for every interpretation $\mathcal{I}$, $\mathsf{nnf}(\neg C)^{\mathcal{I}} = (\mathsf{nnf}(\neg C_1) \sqcup \mathsf{nnf}(\neg C_2))^{\mathcal{I}} = \mathsf{nnf}(\neg C_1)^{\mathcal{I}} \cup \mathsf{nnf}(\neg C_2)^{\mathcal{I}} = (\neg C_1^{\mathcal{I}}) \cup (\neg C_2^{\mathcal{I}}) = (\Delta^{\mathcal{I}} \setminus C_1^{\mathcal{I}}) \cup (\Delta^{\mathcal{I}} \setminus C_2^{\mathcal{I}}) = \Delta^{\mathcal{I}} \setminus (C_1^{\mathcal{I}} \cap C_2^{\mathcal{I}}) = (\neg(C_1 \cap C_2))^{\mathcal{I}} = (\neg C)^{\mathcal{I}}$.
- The case where C is of the form $C_1 \sqcup C_2$ is similar.
- If C is of the form $\exists R.C'$ with C' an $\mathcal{ALC}$ concept such that $\mathsf{nnf}(C')$ and $\mathsf{nnf}(\neg C')$ are in NNF and for every interpretation $\mathcal{I}$, $C'^{\mathcal{I}} = \mathsf{nnf}(C')^{\mathcal{I}}$ and $\mathsf{nnf}(\neg C')^{\mathcal{I}} = (\neg C')^{\mathcal{I}}$, then
    1. $\mathsf{nnf}(C) = \mathsf{nnf}(\exists R.C') = \exists R.\mathsf{nnf}(C')$ is in NNF (since negation appears only in front of atomic concepts in $\mathsf{nnf}(C')$);
    2. for every interpretation $\mathcal{I}$, $\mathsf{nnf}(C)^{\mathcal{I}} = (\exists R.\mathsf{nnf}(C'))^{\mathcal{I}} = \{u \mid (u,v) \in R^{\mathcal{I}}, v \in \mathsf{nnf}(C')^{\mathcal{I}}\} = \{u \mid (u,v) \in R^{\mathcal{I}}, v \in C'^{\mathcal{I}}\} = (\exists R.C')^{\mathcal{I}} = C^{\mathcal{I}}$;
    3. $\mathsf{nnf}(\neg C) = \mathsf{nnf}(\neg(\exists R.C')) = \forall R.(\mathsf{nnf}(\neg C'))$ is in NNF (since negation appears only in front of atomic concepts in $\mathsf{nnf}(\neg C')$);
    4. for every interpretation $\mathcal{I}$, $\mathsf{nnf}(\neg C)^{\mathcal{I}} = (\forall R.(\mathsf{nnf}(\neg C')))^{\mathcal{I}} = \{u \mid (u,v) \in R^{\mathcal{I}} \implies v \in \mathsf{nnf}(\neg C')^{\mathcal{I}}\} = \{u \mid (u,v) \in R^{\mathcal{I}} \implies v \in (\neg C')^{\mathcal{I}}\} = (\neg(\exists R.C'))^{\mathcal{I}} = (\neg C)^{\mathcal{I}}$.
- The case where C is of the form $\forall R.C'$ is similar.
- If C is of the form $\neg C'$ with C' an $\mathcal{ALC}$ concept, such that $\mathsf{nnf}(C')$ is in NNF and for every interpretation $\mathcal{I}$, $C'^{\mathcal{I}} = \mathsf{nnf}(C')^{\mathcal{I}}$ and $\mathsf{nnf}(\neg C')^{\mathcal{I}} = (\neg C')^{\mathcal{I}}$, then
    1. $\mathsf{nnf}(C) = \mathsf{nnf}(\neg C')$ is in NNF by assumption;
    2. for every interpretation $\mathcal{I}$, $\mathsf{nnf}(C)^{\mathcal{I}} = \mathsf{nnf}(\neg C')^{\mathcal{I}} = (\neg C')^{\mathcal{I}} = C^{\mathcal{I}}$;
    3. $\mathsf{nnf}(\neg C) = \mathsf{nnf}(\neg(\neg C')) = \mathsf{nnf}(C')$ is in NNF by assumption;
    4. for every interpretation $\mathcal{I}$, $\mathsf{nnf}(\neg C)^{\mathcal{I}} = \mathsf{nnf}(C')^{\mathcal{I}} = C'^{\mathcal{I}} = (\neg C)^{\mathcal{I}}$.

Hence, for every $\mathcal{ALC}$ concept C, $\mathsf{nnf}(C)$ is in NNF and for every interpretation $\mathcal{I}$, $C^{\mathcal{I}} = \mathsf{nnf}(C)^{\mathcal{I}}$.

Correction of Exercise 11: Adapting tableau algorithm for another DL

Take as input $\langle \mathcal{T}, \mathcal{A} \rangle$ where $\mathcal{T}$ is a TBox that contains only role inclusions of the form $R \sqsubseteq S$ or $R \sqsubseteq \neg S$.

- Start with $\mathcal{A}_c = \mathcal{A}$.
- At each stage, apply to $\mathcal{A}_c$ one of the following rules that extends $\mathcal{A}_c$ with new assertions:
  - If $R(a,b) \in \mathcal{A}_c$, $R \sqsubseteq S \in \mathcal{T}$, and $S(a,b) \notin \mathcal{A}_c$, adds $S(a,b)$ to $\mathcal{A}_c$.
  - If $R(a,b) \in \mathcal{A}_c$, $R \sqsubseteq \neg S \in \mathcal{T}$, and $\neg S(a,b) \notin \mathcal{A}_c$, adds $\neg S(a,b)$ to $\mathcal{A}_c$.
- Stop applying rules when either:
  1. $\mathcal{A}_c$ contains a clash, that is, a pair $\{R(a,b), \neg R(a,b)\}$.
  2. $\mathcal{A}_c$ is clash-free and complete, meaning that no rule can be applied to $\mathcal{A}_c$.
- Return "yes" if $\mathcal{A}_c$ is clash-free, "no" otherwise.

The algorithm adds exactly one assertion of the form $S(a,b)$ or $\neg S(a,b)$ at each step and the number of such assertions is bounded by $2 \times r \times i^2$ where r is the number of role names in $\mathcal{T}$ and i is the number of individual names in $\mathcal{A}$. Hence, $\mathcal{A}_c$ will contain a clash or be complete before $2 \times r \times i^2$ steps and the algorithm terminates.

If the algorithm return "yes", we define $\mathcal{I}$ by $\Delta^{\mathcal{I}} = \{a \mid a \text{ individual in } \mathcal{A}\}$, $A^{\mathcal{I}} = \{a \mid A(a) \in \mathcal{A}\}$ for every concept name A, $R^{\mathcal{I}} = \{(a,b) \mid R(a,b) \in \mathcal{A}_c\}$ for every role name R. It is clear that $\mathcal{I}$ is a model of $\mathcal{A}$. We show that it is a model of $\mathcal{T}$:

- Let $R \sqsubseteq S \in \mathcal{T}$ and $(a,b) \in R^{\mathcal{I}}$. By construction of $\mathcal{I}$, $R(a,b) \in \mathcal{A}_c$. Since $\mathcal{A}_c$ is complete, $S(a,b) \in \mathcal{A}_c$ (otherwise the rule that adds it is applicable). It follows that $(a,b) \in S^{\mathcal{I}}$. Hence $\mathcal{I} \models R \sqsubseteq S$.
- Let $R \sqsubseteq S \in \mathcal{T}$ and $(a,b) \in R^{\mathcal{I}}$. By construction of $\mathcal{I}$, $R(a,b) \in \mathcal{A}_c$. Since $\mathcal{A}_c$ is complete, $\neg S(a,b) \in \mathcal{A}_c$ (otherwise the rule that adds it is applicable). Since $\mathcal{A}_c$ is clash-free, $S(a,b) \notin \mathcal{A}_c$. It follows that $(a,b) \notin S^{\mathcal{I}}$. Hence $\mathcal{I} \models R \sqsubseteq \neg S$.

It follows that $\mathcal{I} \models \langle \mathcal{T}, \mathcal{A} \rangle$, i.e., $\langle \mathcal{T}, \mathcal{A} \rangle$ is satisfiable. Hence the algorithm is sound.

To show completeness, we show that the rules preserve the satisfiability of $\langle \mathcal{T}, \mathcal{A}_c \rangle$. Assume that $\langle \mathcal{T}, \mathcal{A}_c \rangle$ is satisfiable.

- If $\langle \mathcal{T}, \mathcal{A}_c \cup \{S(a,b)\} \rangle$ is obtained by applying the first rule, there is $R(a,b) \in \mathcal{A}_c$ and $R \sqsubseteq S \in \mathcal{T}$. Since $\langle \mathcal{T}, \mathcal{A}_c \rangle$ is satisfiable, there is a model $\mathcal{I}$ of $\langle \mathcal{T}, \mathcal{A}_c \rangle$. Since $\mathcal{I} \models R(a,b)$, then $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in R^{\mathcal{I}}$, so since $\mathcal{I} \models R \sqsubseteq S$, then $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in S^{\mathcal{I}}$. Hence $\mathcal{I} \models \langle \mathcal{T}, \mathcal{A}_c \cup \{S(a,b)\} \rangle$, i.e., $\langle \mathcal{T}, \mathcal{A}_c \cup \{S(a,b)\} \rangle$ is satisfiable.
- If $\langle \mathcal{T}, \mathcal{A}_c \cup \{\neg S(a,b)\} \rangle$ is obtained by applying the first rule, there is $R(a,b) \in \mathcal{A}_c$ and $R \sqsubseteq \neg S \in \mathcal{T}$. Since $\langle \mathcal{T}, \mathcal{A}_c \rangle$ is satisfiable, there is a model $\mathcal{I}$ of $\langle \mathcal{T}, \mathcal{A}_c \rangle$. Since $\mathcal{I} \models R(a,b)$, then $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in R^{\mathcal{I}}$, so since $\mathcal{I} \models R \sqsubseteq \neg S$, then $(a^{\mathcal{I}}, b^{\mathcal{I}}) \notin S^{\mathcal{I}}$. Hence $\mathcal{I} \models \langle \mathcal{T}, \mathcal{A}_c \cup \{\neg S(a,b)\} \rangle$, i.e., $\langle \mathcal{T}, \mathcal{A}_c \cup \{\neg S(a,b)\} \rangle$ is satisfiable.

If $\langle \mathcal{T}, \mathcal{A} \rangle$ is satisfiable, since applying the rules preserve satisfiability, the ABox obtained when the algorithm terminates is clash-free, and the algorithm returns "yes". Hence the algorithm is complete.

Correction of Exercise 12: Normal form of $\mathcal{EL}$ TBoxes

Normalize the following $\mathcal{EL}$ TBox.

$$\mathcal{T} = \{A \sqsubseteq \exists R.\exists S.C, \quad A \sqcap \exists R.\exists S.C \sqsubseteq B \sqcap C, \quad \exists R.\top \sqcap B \sqsubseteq \exists S.\exists R.D\}$$

The normalization step generates the following axioms:

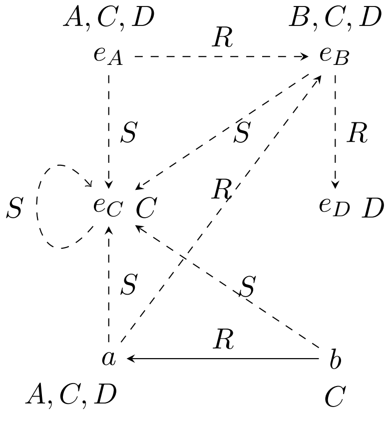
- $A \sqsubseteq \exists R.A_1$
- $A_1 \sqsubseteq \exists S.C$
- $A \sqcap \exists R.\exists S.C \sqsubseteq A_2$
- $A_2 \sqsubseteq B \sqcap C$
- $A \sqcap A_3 \sqsubseteq A_2$
- $\exists R.\exists S.C \sqsubseteq A_3$
- $\exists R.A_4 \sqsubseteq A_3$
- $\exists S.C \sqsubseteq A_4$
- $A_2 \sqsubseteq B$
- $A_2 \sqsubseteq C$
- $\exists R.\top \sqcap B \sqsubseteq A_5$
- $A_5 \sqsubseteq \exists S.\exists R.D$
- $\exists R.\top \sqsubseteq A_6$
- $A_6 \sqcap B \sqsubseteq A_5$
- $A_5 \sqsubseteq \exists S.A_7$
- $A_7 \sqsubseteq \exists R.D$

out of which only the axioms being in normal form are kept:

- $A \sqsubseteq \exists R.A_1$
- $A_1 \sqsubseteq \exists S.C$
- $A \sqcap A_3 \sqsubseteq A_2$
- $\exists R.A_4 \sqsubseteq A_3$
- $\exists S.C \sqsubseteq A_4$
- $A_2 \sqsubseteq B$
- $A_2 \sqsubseteq C$
- $\exists R.\top \sqsubseteq A_6$
- $A_6 \sqcap B \sqsubseteq A_5$
- $A_5 \sqsubseteq \exists S.A_7$
- $A_7 \sqsubseteq \exists R.D$

Correction of Exercise 13: Compact canonical model

$$\begin{aligned}&\mathcal{T} = \{\ A \sqsubseteq \exists R.B, \quad B \sqsubseteq \exists R.D, \quad C \sqsubseteq \exists S.C, \quad A \sqcap C \sqsubseteq D, \quad B \sqcap C \sqsubseteq D, \quad \exists R.\top \sqsubseteq C\ \}\\ &\mathcal{A} = \{\ A(a), \quad R(b,a)\ \}\end{aligned}$$



It follows that $\mathcal{T}$ entails the following atomic concept inclusions (besides those that belong to $\mathcal{T}$ and the trivial ones of the form $X \sqsubseteq X$): $A \sqsubseteq C$, $A \sqsubseteq D$, $B \sqsubseteq C$, $B \sqsubseteq D$, and the following assertions (besides those that belong to $\mathcal{A}$): $C(a)$, $D(a)$ and $C(b)$.

Correction of Exercise 14: Saturation algorithm

$$\begin{aligned}\mathcal{T} = \{\ & A \sqsubseteq B, \quad \exists R.\top \sqsubseteq D, \quad H \sqsubseteq \exists P.A, \quad D \sqsubseteq M,\\ & B \sqsubseteq \exists R.E, \quad D \sqcap M \sqsubseteq H, \quad A \sqsubseteq \exists S.B, \quad \exists S.M \sqsubseteq G\ \}\\ \mathcal{A} = \{\ & D(a), \quad S(a,b), \quad R(b,a)\ \}\end{aligned}$$

1. We start by classifying $\mathcal{T}$:

$$\overline{A \sqsubseteq A} \quad \overline{B \sqsubseteq B} \quad \overline{D \sqsubseteq D} \quad \overline{E \sqsubseteq E} \quad \overline{M \sqsubseteq M} \quad \overline{G \sqsubseteq G} \quad \overline{H \sqsubseteq H}$$

$$\overline{A \sqsubseteq \top} \quad \overline{B \sqsubseteq \top} \quad \overline{D \sqsubseteq \top} \quad \overline{E \sqsubseteq \top} \quad \overline{M \sqsubseteq \top} \quad \overline{G \sqsubseteq \top} \quad \overline{H \sqsubseteq \top}$$

$$\frac{B \sqsubseteq \exists R.E \quad E \sqsubseteq \top \quad \exists R.\top \sqsubseteq D}{B \sqsubseteq D} \qquad \frac{A \sqsubseteq B \quad B \sqsubseteq D}{A \sqsubseteq D}$$

$$\frac{B \sqsubseteq D \quad D \sqsubseteq M}{B \sqsubseteq M} \qquad \frac{A \sqsubseteq B \quad B \sqsubseteq M}{A \sqsubseteq M}$$

$$\frac{D \sqsubseteq M \quad D \sqsubseteq D \quad D \sqcap M \sqsubseteq H}{D \sqsubseteq H} \qquad \frac{B \sqsubseteq D \quad D \sqsubseteq H}{B \sqsubseteq H} \qquad \frac{A \sqsubseteq B \quad B \sqsubseteq H}{A \sqsubseteq H}$$

$$\frac{A \sqsubseteq \exists S.B \quad B \sqsubseteq M \quad \exists S.M \sqsubseteq G}{A \sqsubseteq G}$$

We next find all assertions entailed by $\langle \mathcal{T}, \mathcal{A} \rangle$:

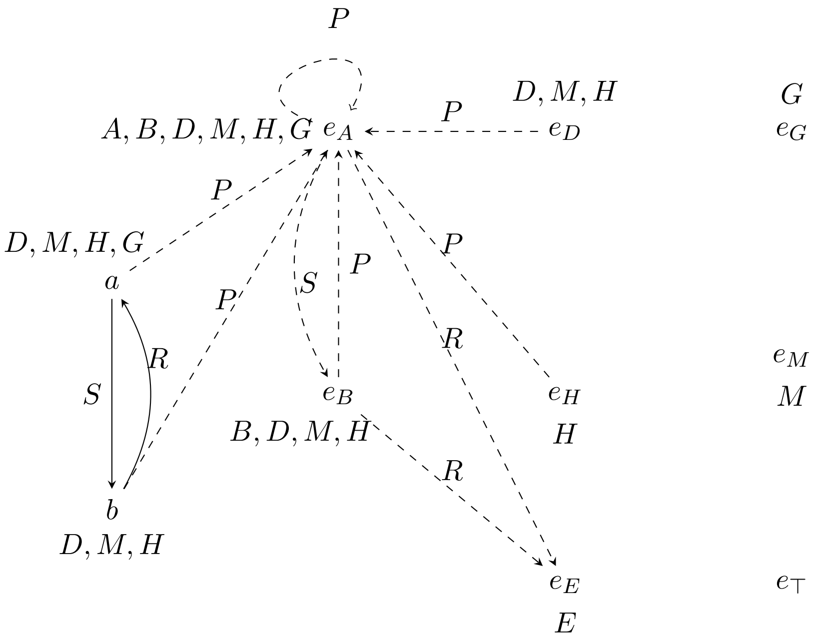
$$\frac{}{\top(a)} \qquad \frac{}{\top(b)}$$

$$\frac{D(a) \quad D \sqsubseteq M}{M(a)} \qquad \frac{D(a) \quad D \sqsubseteq H}{H(a)}$$

$$\frac{R(b,a) \quad \top(a) \quad \exists R.\top \sqsubseteq D}{D(b)} \qquad \frac{D(b) \quad D \sqsubseteq M}{M(b)} \qquad \frac{D(b) \quad D \sqsubseteq H}{H(b)}$$

$$\frac{S(a,b) \quad M(b) \quad \exists S.M \sqsubseteq G}{G(a)}$$

2. Compact canonical model:



Correction of Exercise 15: Properties of conservative extensions

1. *If $\mathcal{T}_2$ is a conservative extension of $\mathcal{T}_1$ and $\mathcal{T}_3$ is a conservative extension of $\mathcal{T}_2$, then $\mathcal{T}_3$ is a conservative extension of $\mathcal{T}_1$.*

   Let $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ be three TBoxes such that $\mathcal{T}_2$ is a conservative extension of $\mathcal{T}_1$ and $\mathcal{T}_3$ is a conservative extension of $\mathcal{T}_2$.

   - Since $\mathcal{T}_2$ is a conservative extension of $\mathcal{T}_1$, then the signature of $\mathcal{T}_1$ is included in the signature of $\mathcal{T}_2$. Since $\mathcal{T}_3$ is a conservative extension of $\mathcal{T}_2$, then the signature of $\mathcal{T}_2$ is included in the signature of $\mathcal{T}_3$. Hence the signature of $\mathcal{T}_1$ is included in the signature of $\mathcal{T}_3$.
   - Let $\mathcal{I}$ be a model of $\mathcal{T}_3$. Since $\mathcal{T}_3$ is a conservative extension of $\mathcal{T}_2$, then $\mathcal{I}$ is a model of $\mathcal{T}_2$. Since $\mathcal{T}_2$ is a conservative extension of $\mathcal{T}_1$, it follows that $\mathcal{I}$ is a model of $\mathcal{T}_1$. Hence every model of $\mathcal{T}_3$ is a model of $\mathcal{T}_1$.
   - Let $\mathcal{I}_1$ be a model of $\mathcal{T}_1$. Since $\mathcal{T}_2$ is a conservative extension of $\mathcal{T}_1$, then there is a model $\mathcal{I}_2$ of $\mathcal{T}_2$ such that

  - $\Delta^{\mathcal{I}_1} = \Delta^{\mathcal{I}_2}$
  - $A^{\mathcal{I}_1} = A^{\mathcal{I}_2}$ for every atomic concept in the signature of $\mathcal{T}_1$
  - $R^{\mathcal{I}_1} = R^{\mathcal{I}_2}$ for every role in the signature of $\mathcal{T}_1$

  Since $\mathcal{T}_3$ is a conservative extension of $\mathcal{T}_2$ and $\mathcal{I}_2$ is a model of $\mathcal{T}_2$, then there exists a model $\mathcal{I}_3$ of $\mathcal{T}_3$ such that

  - $\Delta^{\mathcal{I}_2} = \Delta^{\mathcal{I}_3}$
  - $A^{\mathcal{I}_2} = A^{\mathcal{I}_3}$ for every atomic concept in the signature of $\mathcal{T}_2$
  - $R^{\mathcal{I}_2} = R^{\mathcal{I}_3}$ for every role in the signature of $\mathcal{T}_2$

  Since the signature of $\mathcal{T}_1$ is included in the signature of $\mathcal{T}_2$, it follows that $\mathcal{I}_3$ is a model of $\mathcal{T}_3$ such that

  - $\Delta^{\mathcal{I}_1} = \Delta^{\mathcal{I}_3}$
  - $A^{\mathcal{I}_1} = A^{\mathcal{I}_3}$ for every atomic concept in the signature of $\mathcal{T}_1$
  - $R^{\mathcal{I}_1} = R^{\mathcal{I}_3}$ for every role in the signature of $\mathcal{T}_1$

Hence $\mathcal{T}_3$ is a conservative extension of $\mathcal{T}_1$.

2. *If $\mathcal{T}_2$ is a conservative extension of $\mathcal{T}_1$ and C and D are concepts containing only concept and role names from $\mathcal{T}_1$, then it holds that $\mathcal{T}_1 \models C \sqsubseteq D$ if and only if $\mathcal{T}_2 \models C \sqsubseteq D$.*

   Let $\mathcal{T}_2$ be a conservative extension of $\mathcal{T}_1$.

   - Assume that $\mathcal{T}_1 \models C \sqsubseteq D$. Let $\mathcal{I}$ be a model of $\mathcal{T}_2$. Since $\mathcal{T}_2$ is a conservative extension of $\mathcal{T}_1$, then $\mathcal{I}$ is a model of $\mathcal{T}_1$. Hence, since $\mathcal{T}_1 \models C \sqsubseteq D$, $\mathcal{I} \models C \sqsubseteq D$. Since this holds for every model of $\mathcal{T}_2$, it follows that $\mathcal{T}_2 \models C \sqsubseteq D$.
   - Conversely, assume that $\mathcal{T}_2 \models C \sqsubseteq D$. Let $\mathcal{I}_1$ be a model of $\mathcal{T}_1$. Since $\mathcal{T}_2$ is a conservative extension of $\mathcal{T}_1$, there exists a model $\mathcal{I}_2$ of $\mathcal{T}_2$ such that
     - $\Delta^{\mathcal{I}_1} = \Delta^{\mathcal{I}_2}$
     - $A^{\mathcal{I}_1} = A^{\mathcal{I}_2}$ for every atomic concept in the signature of $\mathcal{T}_1$
     - $R^{\mathcal{I}_1} = R^{\mathcal{I}_2}$ for every role in the signature of $\mathcal{T}_1$

     We show by structural induction that for every $\mathcal{EL}$ concept E such that E contains only concept and role names from $\mathcal{T}_1$, $E^{\mathcal{I}_1} = E^{\mathcal{I}_2}$.

     - Base case: E is an atomic concept in the signature of $\mathcal{T}_1$ so $E^{\mathcal{I}_1} = E^{\mathcal{I}_2}$.
     - Induction step:
       - Case $E = \neg F$, F contains only concept and role names from $\mathcal{T}_1$ and we assume by induction that $F^{\mathcal{I}_1} = F^{\mathcal{I}_2}$. Thus $E^{\mathcal{I}_1} = \Delta^{\mathcal{I}_1} \setminus F^{\mathcal{I}_1} = \Delta^{\mathcal{I}_2} \setminus F^{\mathcal{I}_2} = E^{\mathcal{I}_2}$.
       - Case $E = F_1 \sqcap F_2$, F_1 and F_2 contain only concept and role names from $\mathcal{T}_1$ and we assume by induction that $F_1^{\mathcal{I}_1} = F_1^{\mathcal{I}_2}$ and $F_2^{\mathcal{I}_1} = F_2^{\mathcal{I}_2}$. Thus $E^{\mathcal{I}_1} = F_1^{\mathcal{I}_1} \cap F_2^{\mathcal{I}_1} = F_1^{\mathcal{I}_2} \cap F_2^{\mathcal{I}_2} = E^{\mathcal{I}_2}$.
       - Case $E = \exists R.F$ with R in the signature of $\mathcal{T}_1$, F contains only concept and role names from $\mathcal{T}_1$ and we assume by induction that $F^{\mathcal{I}_1} = F^{\mathcal{I}_2}$. It holds that $R^{\mathcal{I}_1} = R^{\mathcal{I}_2}$ so $E^{\mathcal{I}_1} = \{u \mid (u,v) \in R^{\mathcal{I}_1}, v \in F^{\mathcal{I}_1}\} = \{u \mid (u,v) \in R^{\mathcal{I}_2}, v \in F^{\mathcal{I}_2}\} = E^{\mathcal{I}_2}$.

     Since $\mathcal{T}_2 \models C \sqsubseteq D$, then $C^{\mathcal{I}_2} \subseteq D^{\mathcal{I}_2}$. Since C and D are concepts containing only concept and role names from $\mathcal{T}_1$, it follows that $C^{\mathcal{I}_1} \subseteq D^{\mathcal{I}_1}$, i.e., $\mathcal{I}_1 \models C \sqsubseteq D$. Since this holds for every model of $\mathcal{T}_1$, it follows that $\mathcal{T}_1 \models C \sqsubseteq D$.

3. *If $\mathcal{T}_2$ is a conservative extension of $\mathcal{T}_1$, then for every ABox $\mathcal{A}$ and assertion α that use only atomic concepts and roles from $\mathcal{T}_1$, $\langle \mathcal{T}_1, \mathcal{A} \rangle \models \alpha$ iff $\langle \mathcal{T}_2, \mathcal{A} \rangle \models \alpha$.*

   Let $\mathcal{T}_2$ be a conservative extension of $\mathcal{T}_1$ and $\mathcal{A}$ and α be an ABox and an assertion that use only atomic concepts and roles from $\mathcal{T}_1$.

   - Assume that $\langle \mathcal{T}_1, \mathcal{A} \rangle \models \alpha$. Let $\mathcal{I}$ be a model of $\langle \mathcal{T}_2, \mathcal{A} \rangle$. Since $\mathcal{T}_2$ is a conservative extension of $\mathcal{T}_1$ and $\mathcal{I}$ is a model of $\mathcal{T}_2$, then $\mathcal{I}$ is a model of $\mathcal{T}_1$. Since $\langle \mathcal{T}_1, \mathcal{A} \rangle \models \alpha$ and $\mathcal{I}$ is a model of $\mathcal{A}$ and $\mathcal{T}_1$, then $\mathcal{I} \models \alpha$. Since this holds for every model of $\langle \mathcal{T}_2, \mathcal{A} \rangle$, it follows that $\langle \mathcal{T}_2, \mathcal{A} \rangle \models \alpha$.

- Conversely, assume that $\langle \mathcal{T}_2, \mathcal{A} \rangle \models \alpha$. Let $\mathcal{I}_1$ be a model of $\langle \mathcal{T}_1, \mathcal{A} \rangle$. Since $\mathcal{T}_2$ is a conservative extension of $\mathcal{T}_1$ and $\mathcal{I}_1$ is a model of $\mathcal{T}_1$, there exists a model $\mathcal{I}_2$ of $\mathcal{T}_2$ such that
  - $\Delta^{\mathcal{I}_1} = \Delta^{\mathcal{I}_2}$
  - $A^{\mathcal{I}_1} = A^{\mathcal{I}_2}$ for every atomic concept in the signature of $\mathcal{T}_1$
  - $R^{\mathcal{I}_1} = R^{\mathcal{I}_2}$ for every role in the signature of $\mathcal{T}_1$

  Since $\mathcal{I}_1 \models \mathcal{A}$ and concepts and roles used in $\mathcal{A}$ are in the signature of $\mathcal{T}_1$, then $\mathcal{I}_2 \models \mathcal{A}$. It follows that $\mathcal{I}_2$ is a model of $\langle \mathcal{T}_2, \mathcal{A} \rangle$, so $\mathcal{I}_2 \models \alpha$. Since α is of the form $A(a)$ or $R(a, b)$ with A, R in the signature of $\mathcal{T}_1$, it follows that $\mathcal{I}_1 \models \alpha$. Since this holds for every model of $\langle \mathcal{T}_1, \mathcal{A} \rangle$, it follows that $\langle \mathcal{T}_1, \mathcal{A} \rangle \models \alpha$.

Correction of Exercise 16: Conservative extensions

$\mathcal{T}_2 = \mathcal{T}_1 \cup \{A \sqsubseteq C,\ D \sqsubseteq B\}$

1. $\mathcal{T}_2$ is a conservative extension of $\mathcal{T}_1$:
   - Since $\mathcal{T}_1 \subseteq \mathcal{T}_2$, the signature of $\mathcal{T}_1$ is included in the signature of $\mathcal{T}_2$.
   - Since $\mathcal{T}_1 \subseteq \mathcal{T}_2$, every model of $\mathcal{T}_2$ is a model of $\mathcal{T}_1$.
   - Let $\mathcal{I}_1$ be a model of $\mathcal{T}_1$. We define an interpretation $\mathcal{I}_2$ by
     - $\Delta^{\mathcal{I}_2} = \Delta^{\mathcal{I}_1}$
     - $E^{\mathcal{I}_2} = E^{\mathcal{I}_1}$ for every atomic concept in the signature of $\mathcal{T}_1$
     - $R^{\mathcal{I}_2} = R^{\mathcal{I}_1}$ for every role in the signature of $\mathcal{T}_1$
     - $A^{\mathcal{I}_2} = C^{\mathcal{I}_1}$
     - $B^{\mathcal{I}_2} = D^{\mathcal{I}_1}$

     $\mathcal{I}_2$ is a model of $\mathcal{T}_1$ since it coincides with $\mathcal{I}_1$ on the signature of $\mathcal{T}_1$ and $\mathcal{I}_2 \models A \sqsubseteq C$ and $\mathcal{I}_2 \models D \sqsubseteq B$ by construction of $\mathcal{I}_2$. Hence $\mathcal{I}_2$ is a model of $\mathcal{T}_2$.
2. $\mathcal{T}_2 \cup \{A \sqsubseteq B\}$ is a conservative extension of $\mathcal{T}_1$: The proof is similar to the previous question except that we define $B^{\mathcal{I}_2} = D^{\mathcal{I}_1} \cup C^{\mathcal{I}_1}$: It still holds that $\mathcal{I}_2 \models A \sqsubseteq C$ and $\mathcal{I}_2 \models D \sqsubseteq B$ (since $D^{\mathcal{I}_1} \subseteq D^{\mathcal{I}_1} \cup C^{\mathcal{I}_1}$) and $\mathcal{I}_2 \models A \sqsubseteq B$ since $C^{\mathcal{I}_1} \subseteq D^{\mathcal{I}_1} \cup C^{\mathcal{I}_1}$.
3. If $\mathcal{T}_1 \not\models D \sqsubseteq C$, then $\mathcal{T}_2 \cup \{B \sqsubseteq A\}$ is not a conservative extension of $\mathcal{T}_1$ because $\mathcal{T}_2 \cup \{B \sqsubseteq A\} \models D \sqsubseteq C$.