Correction of Exercise Sheet
Description Logics

Correction of Exercise 1: Modelisation

1. PhD students are students and researchers.
   \[ \text{PhDStudent} \sqsubseteq \text{Student} \sqcap \text{Researcher} : \text{in } ALC \]

2. Professors are not PhD students.
   \[ \text{Professor} \sqsubseteq \neg \text{PhDStudent} : \text{in } ALC \]

3. PhD students are employed by some university.
   \[ \text{PhDStudent} \sqsubseteq \exists \text{employedBy.University} : \text{in } ALC \]

4. Those who are employed by some university are researchers, professors, administrative staff workers or technical staff workers.
   \[ \exists \text{employedBy.University} \sqsubseteq \text{Researcher} \sqcup \text{Professor} \sqcup \text{AdminStaff} \sqcup \text{TechnicalStaff} : \text{in } ALC \]

5. Teachers are exactly the persons that teach some course.
   \[ \text{Teacher} \equiv \text{Person} \sqcap \exists \text{teach.Course} : \text{in } ALC \]

6. Professors teach at least two courses.
   \[ \text{Professor} \sqsubseteq \exists 2 \text{teach.Course} : \text{not in } ALC \text{ (number restriction)} \]

7. PhD students are supervised by a researcher.
   \[ \text{PhDStudent} \sqsubseteq \exists \text{supervise} \neg \text{Researcher} : \text{not in } ALC \text{ (inverse role)} \]

8. PhD students teach only tutorials or hands-on-sessions.
   \[ \text{PhDStudent} \sqsubseteq \forall \text{teach.(Tutorial} \sqcup \text{HandsOnSession)} : \text{in } ALC \]

9. Administrative staff workers do not supervise PhD students.
   \[ \text{AdminStaff} \sqsubseteq \forall \text{supervise.}(\neg \text{PhDStudent}) : \text{in } ALC \]

10. Researchers are members of a department which is part of a university.
    \[ \text{Researcher} \sqsubseteq \exists \text{memberOf.(Department} \sqcap \exists \text{partOf.University}) : \text{in } ALC \]

11. Students that are not PhD students are not employed by a university.
    \[ \text{Student} \sqcap \neg \text{PhDStudent} \sqsubseteq \neg \exists \text{employedBy.University} : \text{in } ALC \]

12. Things that are taught are courses.
    \[ \top \sqsubseteq \forall \text{teach.Course} : \text{in } ALC \text{ (equivalent to } \exists \text{teach.}\top \sqsubseteq \text{Course which is not in } ALC \text{)} \]

13. Courses are attended by at most 50 students.
    \[ \text{Course} \sqsubseteq \leq 50 \text{attend.}\top \sqsubseteq \text{Student} : \text{not in } ALC \text{ (number restriction, inverse role)} \]

14. Courses taught by Ana are not hands-on-sessions.
    \[ \text{Course} \sqcap \exists \text{teach.}\{\text{ana}\} \sqsubseteq \neg \text{HandsOnSession} : \text{not in } ALC \text{ (nominals, inverse role)} \]

15. Ana is a researcher.
    \[ \text{Researcher}(\text{ana}) \]

16. John is a PhD student who teaches logic and is supervised by Ana.
    \[ \text{PhDStudent}(\text{john}), \text{teach}(\text{john}, \text{logic}), \text{supervise}(\text{ana}, \text{john}) \]
Can you express that PhD students are employed by the same university that the one the department they are member of is part of?
No. Best try: PhDStudent ⊆ EmployeeBy.(University ⊓∃partOf−.(Department ⊓∃memberOf−.PhDStudent)) but no way to say that the PhD student is the same.

Correction of Exercise 2: Interpretations

1. \((A \cap ∃S.C)^I = \{b\}\)  
2. \((B \cup (C \cap ∃S−\cdot.T))^I = \{b, c\}\)  
3. \((∀R.C)^I = \{b, c, d\}\)  
4. \((∃S.C)^I = \{b, c, d\}\)  
5. \((A \cap ¬∃R.T)^I = \{b\}\)

1. No: \(I \nvDash A \subseteq B \cup C\) because \(\{a, b\} \nsubseteq \{b, c, d\}\)
2. Yes: \(I \vdash A \subseteq ∃S.T\) because \(\{a, b\} \subseteq \{a, b\}\)
3. Yes: \(I \vdash ∃S−\cdot.B \subseteq C\) because \(\{c\} \subseteq \{c, d\}\)
4. Yes: \(I \vdash A \subseteq ¬C\) because \(\{a, b\} \subseteq \{a, b\}\)

Correction of Exercise 3: Basic reasoning

1. No. Consider the following interpretation \(I\) on domain \(Δ^I = \{a\}\): \(A^I = \{a\}, B^I = \emptyset, C^I = \emptyset, R^I = \emptyset\). \(I\) is a model of \(T\) and \(A^I \nsubseteq C^I\) so \(T \nvDash A \subseteq C\).
2. Yes. Let \(I\) be a model of \(T\) and \(e\) be an element of \(Δ^I\) such that \(e \in (A \cap ∃R.T)^I = A^I \cap (∃R.T)^I\). Since \(e \in (∃R.T)^I\), there exists \(d \in Δ^I\) such that \(e, d) \in R^I\). Since \(I\) is a model of \(T\), \(I \vdash A \subseteq ∀R.B\), so \(e \in A^I\) and \(e, d) \in R^I\) implies that \(d \in B^I\). Hence \(e \in (∃R.B)^I\). Since \(I \vdash ∃R.B \subseteq C\), it follows that \(e \in C^I\). Finally, since \(I \vdash B \subseteq ¬C\), \(B^I \subseteq Δ^I \setminus C^I\) so \(e \notin B^I\). We have shown that for every model \(I\) of \(T\), \((A \cap ∃R.T)^I \subseteq Δ^I \setminus B^I\), i.e. \(I \vdash A \cap ∃R.T \subseteq ¬B\). This is exactly the definition of \(T \vdash A \cap ∃R.T \subseteq ¬B\).
3. No. Assume for a contradiction that there exists a model \(I\) of \(T\) such that \((B \cap ∃R.B)^I\) is non-empty and let \(e \in (B \cap ∃R.B)^I\). Since \(I \vdash ∃R.B \subseteq C\) and \(e \in (∃R.B)^I\), then \(e \in C^I\). It follows that \(e\) belongs to \(B^I\) and to \(C^I\), so \(B^I \nsubseteq Δ^I \setminus C^I\), which contradicts \(T \vdash B \subseteq ¬C\).
4. Yes. Consider the model \(I\) of \(T\) given in the correction of question 1. \((A \cap ∀R.C)^I = \{a\}\) is non-empty.
5. Yes. We just need to extend the interpretation given in the correction of question 1 by setting \(a^I = a\) to obtain a model of \(⟨T, A_1⟩\).
6. Yes. Consider the following interpretation \(I\) on domain \(Δ^I = \{a, b\}\): \(A^I = \{a\}, B^I = \{b\}, C^I = \{a\}, R^I = \{(a, b)\}\). \(I\) is a model of \(⟨T, A_2⟩\).
7. No. Assume for a contradiction that \(⟨T, A_3⟩\) has a model \(I\). We must have \(a^I \in A^I\) and \((a^I, b^I) \in R^I\) so since \(I \vdash A \subseteq ∀R.B\), it follows that \(b \in B^I\). However, we also must have \(b \in C^I\), which contradicts \(I \vdash B \subseteq ¬C\).
8. No. The model of \(⟨T, A_1⟩\) given in question 5 does not satisfy \(C(a)\).
9. Yes. Let \(I\) be a model of \(⟨T, A_2⟩\). Since \(I \vdash A(a)\) and \(I \vdash R(a, b)\), then \(a^I \in A^I\) and \((a^I, b^I) \in R^I\). Since \(I \vdash A \subseteq ∀R.B\), it follows that \(b \in B^I\). Hence \(a^I \in (∃R.B)^I\), so since \(I \vdash ∃R.B \subseteq C\), \(a^I \in C^I\). We have shown that for every model \(I\) of \(⟨T, A_2⟩\), \(a^I \in C^I\). This is exactly the definition of \(⟨T, A_2⟩ \vdash C(a)\).
10. Yes. Since \(⟨T, A_3⟩\) has no model, it is true that \(a^I \in C^I\) in every model of \(⟨T, A_3⟩\). An unsatisfiable knowledge base entails every logical axiom.
Correction of Exercise 4: DL fragments

Minimal fragments of $\mathcal{ALC}$: $\{\land, \neg, \exists\}$, $\{\land, \neg, \forall\}$, $\{\lor, \neg, \exists\}$, $\{\lor, \neg, \forall\}$.

Proof for the $\{\land, \neg, \exists\}$ fragment:
Let $C$ be an $\mathcal{ALC}$ concept. We first show by induction on the structure of $C$ that there exists $C'$ in the $\{\land, \neg, \exists\}$ fragment that is equivalent to $C$.

Base case: If $C$ is an atomic concept, then $C$ is in the $\{\land, \neg, \exists\}$ fragment.

- If $C = C_1 \cap C_2$, and $C_1$, $C_2$ are $\mathcal{ALC}$ concepts equivalent to $C_1'$ and $C_2'$ in the $\{\land, \neg, \exists\}$ fragment, then $C$ is equivalent to $C' = C_1' \cap C_2'$ which belongs to the fragment.
- If $C = C_1 \cup C_2$, and $C_1$, $C_2$ are $\mathcal{ALC}$ concepts equivalent to $C_1'$ and $C_2'$ in the $\{\land, \neg, \exists\}$ fragment, then $C$ is equivalent to $C' = (\neg(C_1' \cap \neg C_2'))$ which belongs to the fragment.
- If $C = \neg C_1$ and $C_1$ is an $\mathcal{ALC}$ concept equivalent to $C_1'$ in the $\{\land, \neg, \exists\}$ fragment, then $C$ is equivalent to $C' = \neg C_1'$ which belongs to the fragment.
- If $C = \exists R.C_1$ and $C_1$ is an $\mathcal{ALC}$ concept equivalent to $C_1'$ in the $\{\land, \neg, \exists\}$ fragment, then $C$ is equivalent to $C' = \exists R.C_1'$ which belongs to the fragment.
- If $C = \forall R.C_1$ and $C_1$ is an $\mathcal{ALC}$ concept equivalent to $C_1'$ in the $\{\land, \neg, \exists\}$ fragment, then $C$ is equivalent to $C' = \neg(\exists R.\neg C_1')$ which belongs to the fragment.

We now show that every sub-fragment of the $\{\land, \neg, \exists\}$ fragment does not capture $\mathcal{ALC}$. Let $A$ and $B$ be atomic concepts.

- $A \cap B$ cannot be expressed on $\{\neg, \exists\}$
- $\neg A$ cannot be expressed on $\{\land, \exists\}$
- $\exists R.A$ cannot be expressed on $\{\land, \neg\}$

Correction of Exercise 5: Translation to FOL

1. $\forall x (\exists y (R(x, y) \land \exists z S(y, z)) \Rightarrow B(x) \lor C(x))$
2. $\forall x (A(x) \land \neg B(x) \Rightarrow \forall y (R(x, y) \Rightarrow C(y)))$
3. $\forall x (\exists y (R(y, x) \land A(y)) \Rightarrow \neg C(x))$
4. $\forall x (A(x) \lor \exists y (R(x, y) \land B(y)) \Rightarrow \exists z S(x, z))$

Correction of Exercise 6: Negation normal form

1. $\text{nff}((\neg A \cup \forall R.((\neg(B \cap \neg C)))))) = \text{nff}(\neg A) \cap \text{nff}(\neg(\forall R.((\neg(B \cap \neg C))))))$
   $= \text{nff}(A) \cap \exists R.\text{nff}(\neg((\neg(B \cap \neg C))))$
   $= A \cap \exists R.\text{nff}(B \cap \neg C)$
   $= A \cap \exists R.((\neg(B) \cap \text{nff}(\neg C))$
   $= A \cap \exists R.(B \cap \neg C)$

2. $\text{nff}((\neg(\exists R.((\neg(S.A))) \cap \neg(\forall R.B))) = \text{nff}(\neg(\exists R.((\neg(S.A))) \cap \neg(\forall R.B))$
   $= \forall R.\text{nff}(\neg(\neg(S.A))) \cap \exists R.\text{nff}(\neg B)$
   $= \forall R.\text{nff}(S.A) \cap \exists R.(\neg B)$
   $= \forall R.\exists S.\text{nff}(A) \cap \exists R.(\neg B)$
   $= \forall R.S.A \cap \exists R.(\neg B)$
Correction of Exercise 7: Tableau algorithm for concept satisfiability

1. ∃R ∃S.A ∨ ∀R ∀S.¬A is not satisfiable. Indeed, every ABox generated by the tableau algorithm contains a clash:

\[(∃R ∃S.A ∨ ∀R ∀S.¬A)(a₀)\]
\[\quad | (∃R ∃S.A)(a₀)\]
\[\quad | (∀R ∀S.¬A)(a₀)\]
\[\quad | R(a₀, a₁)\]
\[\quad | (∃S.A)(a₁)\]
\[\quad | (∀S.¬A)(a₁)\]
\[\quad | S(a₁, a₂)\]
\[\quad | A(a₂)\]
\[\quad | ¬A(a₂)\]
\[✗\]

2. ∃R.B ∨ ∀R ∀S.R.A ∨ ∀R.¬A is satisfiable. The interpretation I defined by $B^I = \{a₁\}$, $A^I = ∅$ and $R^I = \{(a₀, a₁)\}$ is such that $(∃R.B ∨ ∀R ∀S.R.A ∨ ∀R.¬A)^I$ is non-empty.

\[(∃R.B ∨ ∀R ∀S.R.A ∨ ∀R.¬A)(a₀)\]
\[\quad | (∃R.B)(a₀)\]
\[\quad | (∀R ∀S.R.A ∨ ∀R.¬A)(a₀)\]
\[\quad | (∀R ∀R.A)(a₀)\]
\[\quad | (∀R.¬A)(a₀)\]
\[\quad | R(a₀, a₁)\]
\[\quad | B(a₁)\]
\[\quad | ∀R.A(a₁)\]
\[\quad | ¬A(a₁)\]
\[✓\]

Correction of Exercise 8: Tableau algorithm for KB satisfiability

To decide whether $\mathcal{T} \models A \sqsubseteq C$ with the tableau algorithm, we need to check whether $\{A \sqcap ¬C\}$ is satisfiable w.r.t. $\mathcal{T}$, i.e., whether $⟨\mathcal{T}, \{A \sqcap ¬C\}(a)⟩$ is satisfiable.

$⟨\mathcal{T}, \{A \sqcap ¬C\}(a)⟩$ is satisfiable so $\mathcal{T} \not\models A \sqsubseteq C$. A model of $\mathcal{T}$ that shows it is:

$\Delta^I = \{a, a₁\}$
$A^I = \{a, a₁\}$
$B^I = \{a₁\}$
$C^I = ∅$
$R^I = \{(a₁, (a₁, a₁))\}$
Correction of Exercise 9: Tableau algorithm for KB satisfiability – Optimization

\[ \mathcal{T} = \{ A \sqsubseteq \forall R.B, B \sqsubseteq \neg F, E \sqsubseteq G, A \sqsubseteq D \sqcup E, D \sqsubseteq \exists R.F, \exists R.\neg B \sqsubseteq G \} \]

All axioms in \( \mathcal{T} \) are inclusions with atomic left- or right-hand side.

- For inclusions \( A \sqsubseteq D \) with atomic left-hand side, replace the TBox-rule by
  TBox-atomic-left-rule: if \( A(a) \in \mathcal{A} \), a is not blocked, \( A \sqsubseteq D \in \mathcal{T} \) (\( A \) atomic), and \( D(a) \notin \mathcal{A} \), replace \( \mathcal{A} \) with \( \mathcal{A} \cup \{ D(a) \} \).

- For inclusions \( D \sqsubseteq A \) with atomic right-hand side, replace the TBox-rule by
  TBox-atomic-right-rule: if \( \neg A(a) \in \mathcal{A} \), a is not blocked, \( D \sqsubseteq A \in \mathcal{T} \) (\( A \) atomic), and \( \neg D(a) \notin \mathcal{A} \), replace \( \mathcal{A} \) with \( \mathcal{A} \cup \{ \neg D(a) \} \).

1. \( \mathcal{T} \models A \sqsubseteq E \)

2. \( \mathcal{T} \not\models E \sqsubseteq F \)

3. \( \mathcal{T} \models A \sqsubseteq G \)

4. \( \mathcal{T} \models D \sqsubseteq G \)

5. \( \mathcal{T} \not\models G \sqsubseteq F \)
Correction of Exercise 10: Negation normal form algorithm

Let $C$ be an $\mathcal{ALC}$ concept. We show by structural induction that

1. $\text{nnf}(C)$ is in NNF;
2. for every interpretation $\mathcal{I}$, $C^\mathcal{I} = \text{nnf}(C)^\mathcal{I}$;
3. $\text{nnf}(\neg C)$ is in NNF;
4. for every interpretation $\mathcal{I}$, $\text{nnf}(\neg C)^\mathcal{I} = (\neg C)^\mathcal{I}$.

In the base case, $C$ is an atomic concept $A$ or is of the form $\neg A$ for an atomic concept $A$. In this case, $\text{nnf}(C) = C$ is in NNF, and for every interpretation $\mathcal{I}$, $C^\mathcal{I} = \text{nnf}(C)^\mathcal{I}$ holds trivially. Moreover, if $C = A$, $\text{nnf}(\neg C) = \neg A$ and if $C = \neg A$, $\text{nnf}(\neg C) = \neg(\neg(\neg A)) = \neg A$ so in both cases, $\text{nnf}(\neg C)$ is in NNF and for every interpretation $\mathcal{I}$, $\text{nnf}(\neg C)^\mathcal{I} = (\neg C)^\mathcal{I}$.

- If $C$ is of the form $C_1 \sqcap C_2$ with $C_1$ and $C_2$ two $\mathcal{ALC}$ concepts such that $\text{nnf}(C_1)$, $\text{nnf}(C_2)$, $\text{nnf}(\neg C_1)$, and $\text{nnf}(\neg C_2)$ are in NNF and for every interpretation $\mathcal{I}$, $C^\mathcal{I}_1 = \text{nnf}(C_1)^\mathcal{I}$ and $\text{nnf}(\neg C^\mathcal{I}_1) = (\neg C_1)^\mathcal{I}$ (1 ≤ $i$ ≤ 2), then
  1. $\text{nnf}(C) = \text{nnf}(C_1 \sqcap C_2) = \text{nnf}(C_1) \sqcap \text{nnf}(C_2)$ is in NNF (since negation appears only in front of atomic concepts in $\text{nnf}(C_1)$ and $\text{nnf}(C_2)$);
  2. for every interpretation $\mathcal{I}$, $\text{nnf}(C)^\mathcal{I} = (\text{nnf}(C_1) \sqcap \text{nnf}(C_2))^\mathcal{I} = \text{nnf}(C_1)^\mathcal{I} \sqcap \text{nnf}(C_2)^\mathcal{I} = (C_1 \sqcap C_2)^\mathcal{I} = C^\mathcal{I}$;
  3. $\text{nnf}(\neg C) = \text{nnf}(\neg(C_1 \sqcap C_2)) = \text{nnf}(\neg C_1) \sqcup \text{nnf}(\neg C_2)$ is in NNF (since negation appears only in front of atomic concepts in $\text{nnf}(\neg C_1)$ and $\text{nnf}(\neg C_2)$);
  4. for every interpretation $\mathcal{I}$, $\text{nnf}(\neg C)^\mathcal{I} = (\text{nnf}(\neg C_1) \sqcup \text{nnf}(\neg C_2))^\mathcal{I} = \text{nnf}(\neg C_1)^\mathcal{I} \sqcup \text{nnf}(\neg C_2)^\mathcal{I} = (\neg C^\mathcal{I}_1) \sqcup (\neg C^\mathcal{I}_2) = (\Delta^\mathcal{I} \setminus C_1^\mathcal{I}) \cup (\Delta^\mathcal{I} \setminus C_2^\mathcal{I}) = (\Delta^\mathcal{I} \setminus (C_1^\mathcal{I} \cap C_2^\mathcal{I})) = (\neg (C_1 \sqcap C_2))^\mathcal{I} = (\neg C)^\mathcal{I}$.

- The case where $C$ is of the form $C_1 \sqcup C_2$ is similar.

- If $C$ is of the form $\exists R.C'$ with $C'$ an $\mathcal{ALC}$ concept such that $\text{nnf}(C')$ and $\text{nnf}(\neg C')$ are in NNF and for every interpretation $\mathcal{I}$, $C^\mathcal{I} = \text{nnf}(C')^\mathcal{I}$ and $\text{nnf}(\neg C^\mathcal{I}) = (\neg C')^\mathcal{I}$, then
  1. $\text{nnf}(C) = \text{nnf}(\exists R.C') = \exists R.\text{nnf}(C')$ is in NNF (since negation appears only in front of atomic concepts in $\text{nnf}(C')$);
  2. for every interpretation $\mathcal{I}$, $\text{nnf}(C)^\mathcal{I} = (\exists R.\text{nnf}(C'))^\mathcal{I} = \{u \mid (u, v) \in R^\mathcal{I}, v \in \text{nnf}(C')^\mathcal{I}\} = \{u \mid (u, v) \in R^\mathcal{I}, v \in C^\mathcal{I}\} = (\exists R.C')^\mathcal{I} = C^\mathcal{I}$;
  3. $\text{nnf}(\neg C) = \text{nnf}(\neg(\exists R.C')) = \forall R.(\text{nnf}(\neg C'))$ is in NNF (since negation appears only in front of atomic concepts in $\text{nnf}(\neg C')$);
  4. for every interpretation $\mathcal{I}$, $\text{nnf}(\neg C)^\mathcal{I} = (\forall R.(\text{nnf}(\neg C')))^\mathcal{I} = \{u \mid (u, v) \in R^\mathcal{I} \implies v \in \text{nnf}(\neg C)^\mathcal{I}\} = \{u \mid (u, v) \in R^\mathcal{I} \implies v \in (\neg C)^\mathcal{I}\} = (\neg(\exists R.C'))^\mathcal{I} = (\neg C)^\mathcal{I}$.

- The case where $C$ is of the form $\forall R.C'$ is similar.

- If $C$ is of the form $\neg C'$ with $C'$ an $\mathcal{ALC}$ concept, such that $\text{nnf}(C')$ is in NNF and for every interpretation $\mathcal{I}$, $C^\mathcal{I} = \text{nnf}(C')^\mathcal{I}$ and $\text{nnf}(\neg C^\mathcal{I}) = (\neg C')^\mathcal{I}$, then
  1. $\text{nnf}(C) = \text{nnf}(\neg C')$ is in NNF by assumption;
  2. for every interpretation $\mathcal{I}$, $\text{nnf}(C)^\mathcal{I} = \text{nnf}(\neg C')^\mathcal{I} = (\neg C')^\mathcal{I} = C^\mathcal{I}$;
  3. $\text{nnf}(\neg C) = \text{nnf}(\neg(\neg C')) = \text{nnf}(C')$ is in NNF by assumption;
  4. for every interpretation $\mathcal{I}$, $\text{nnf}(\neg C)^\mathcal{I} = \text{nnf}(C')^\mathcal{I} = C^\mathcal{I} = (\neg C)^\mathcal{I}$.

Hence, for every $\mathcal{ALC}$ concept $C$, $\text{nnf}(C)$ is in NNF and for every interpretation $\mathcal{I}$, $C^\mathcal{I} = \text{nnf}(C)^\mathcal{I}$.
Correction of Exercise 11: Adapting tableau algorithm for another DL

Take as input \( \langle T, A \rangle \) where \( T \) is a TBox that contains only role inclusions of the form \( R \sqsubseteq S \) or \( R \sqsubseteq \neg S \).

- Start with \( A_c = A \).
- At each stage, apply to \( A_c \) one of the following rules that extends \( A_c \) with new assertions:
  - If \( R(a,b) \in A_c \), \( R \sqsubseteq S \in T \), and \( S(a,b) \notin A_c \), adds \( S(a,b) \) to \( A_c \).
  - If \( R(a,b) \in A_c \), \( R \sqsubseteq \neg S \in T \), and \( \neg S(a,b) \notin A_c \), adds \( \neg S(a,b) \) to \( A_c \).
- Stop applying rules when either:
  1. \( A_c \) contains a clash, that is, a pair \( \{ R(a,b), \neg R(a,b) \} \).
  2. \( A_c \) is clash-free and complete, meaning that no rule can be applied to \( A_c \).
- Return “yes” if \( A_c \) is clash-free, “no” otherwise.

The algorithm adds exactly one assertion of the form \( S(a,b) \) or \( \neg S(a,b) \) at each step and the number of such assertions is bounded by \( 2 \times r \times i^2 \) where \( r \) is the number of role names in \( T \) and \( i \) is the number of individual names in \( A \). Hence, \( A_c \) will contain a clash or be complete before \( 2 \times r \times i^2 \) steps and the algorithm terminates.

If the algorithm returns “yes”, we define \( I \) by \( \Delta^I = \{ a \mid a \text{ individual in } A \} \), \( A^I = \{ a \mid A(a) \in A \} \) for every concept name \( A \), \( R^I = \{ (a,b) \mid R(a,b) \in A_c \} \) for every role name \( R \). It is clear that \( I \) is a model of \( A \).

- Let \( R \sqsubseteq S \in T \) and \( (a,b) \in R^I \). By construction of \( I \), \( R(a,b) \in A_c \). Since \( A_c \) is complete, \( S(a,b) \in A_c \) (otherwise the rule that adds it is applicable). It follows that \( (a,b) \in S^I \). Hence \( I \models R \sqsubseteq S \).
- Let \( R \sqsubseteq S \in T \) and \( (a,b) \in R^I \). By construction of \( I \), \( R(a,b) \in A_c \). Since \( A_c \) is complete, \( \neg S(a,b) \in A_c \) (otherwise the rule that adds it is applicable). Since \( A_c \) is clash-free, \( S(a,b) \notin A_c \). It follows that \( (a,b) \notin S^I \). Hence \( I \models R \sqsubseteq \neg S \).

It follows that \( I \models \langle T, A \rangle \), i.e., \( \langle T, A \rangle \) is satisfiable. Hence the algorithm is sound.

To show completeness, we show that the rules preserve the satisfiability of \( \langle T, A_c \rangle \). Assume that \( \langle T, A_c \rangle \) is satisfiable.

- If \( \langle T, A_c \cup \{ S(a,b) \} \rangle \) is obtained by applying the first rule, there is \( R(a,b) \in A_c \) and \( R \sqsubseteq S \in T \). Since \( \langle T, A_c \rangle \) is satisfiable, there is a model \( I \) of \( \langle T, A_c \rangle \). Since \( I \models R(a,b) \), then \( (a^I, b^I) \in R^I \), so since \( I \models R \sqsubseteq S \), then \( (a^I, b^I) \in S^I \). Hence \( I \models \langle T, A_c \cup \{ S(a,b) \} \rangle \), i.e., \( \langle T, A_c \cup \{ S(a,b) \} \rangle \) is satisfiable.
- If \( \langle T, A_c \cup \{ \neg S(a,b) \} \rangle \) is obtained by applying the first rule, there is \( R(a,b) \in A_c \) and \( R \sqsubseteq \neg S \in T \). Since \( \langle T, A_c \rangle \) is satisfiable, there is a model \( I \) of \( \langle T, A_c \rangle \). Since \( I \models R(a,b) \), then \( (a^I, b^I) \in R^I \), so since \( I \models R \sqsubseteq \neg S \), then \( (a^I, b^I) \notin S^I \). Hence \( I \models \langle T, A_c \cup \{ \neg S(a,b) \} \rangle \), i.e., \( \langle T, A_c \cup \{ \neg S(a,b) \} \rangle \) is satisfiable.

If \( \langle T, A \rangle \) is satisfiable, since applying the rules preserve satisfiability, the ABox obtained when the algorithm terminates is clash-free, and the algorithm returns “yes”. Hence the algorithm is complete.

Correction of Exercise 12: Normal form of \( \mathcal{EL} \) TBoxes

Normalize the following \( \mathcal{EL} \) TBox.

\[
T = \{ A \sqsubseteq \exists R.\exists S.C, \quad A \sqcap \exists R.\exists S.C \sqsubseteq B \sqcap C, \quad \exists R.T \sqcap B \sqsubseteq \exists S.\exists R.D \}
\]

The normalization step generates the following axioms:
Correction of Exercise 13: Compact canonical model

\[ \mathcal{T} = \{ A \sqsubseteq \exists R.B, B \sqsubseteq \exists R.D, C \sqsubseteq \exists S.C, A \cap C \sqsubseteq D, B \cap C \sqsubseteq D, \exists R.\top \sqsubseteq C \} \]
\[ \mathcal{A} = \{ A(a), R(b,a) \} \]

\begin{align*}
A, C, D & \quad R \quad B, C, D \\
S & \quad R \quad e_A \quad e_B \\
S & \quad R \\
A, C, D & \quad R \quad b \\
A, C, D & \quad R \quad b \\
A, C, D & \quad R \quad b
\end{align*}

It follows that \( \mathcal{T} \) entails the following atomic concept inclusions (besides those that belong to \( \mathcal{T} \) and the trivial ones of the form \( X \sqsubseteq X \)): \( A \sqsubseteq C, A \sqsubseteq D, B \sqsubseteq C, B \sqsubseteq D, \) and the following assertions (besides those that belong to \( \mathcal{A} \)): \( C(a), D(a) \) and \( C(b) \).

Correction of Exercise 14: Saturation algorithm

\[ \mathcal{T} = \{ A \sqsubseteq B, \exists R.\top \sqsubseteq D, H \sqsubseteq \exists P.A, D \sqsubseteq M, B \sqsubseteq \exists R.E, D \cap M \sqsubseteq H, A \sqsubseteq \exists S.B, \exists S.M \sqsubseteq G \} \]
\[ \mathcal{A} = \{ D(a), S(a,b), R(b,a) \} \]

1. We start by classifying \( \mathcal{T} \):

\[ \begin{array}{cccccccc}
A \sqsubseteq A & B \sqsubseteq B & D \sqsubseteq D & E \sqsubseteq E & M \sqsubseteq M & G \sqsubseteq G & H \sqsubseteq H \\
A \sqsubseteq T & B \sqsubseteq T & D \sqsubseteq T & E \sqsubseteq T & M \sqsubseteq T & G \sqsubseteq T & H \sqsubseteq T
\end{array} \]
We next find all assertions entailed by \( \langle T, A \rangle \):

\[
\begin{align*}
\top(a) & \quad \top(b) \\
D(a) & \quad D(b) \\
M(a) & \quad M(b) \\
H(a) & \quad H(b) \\
\end{align*}
\]

2. Compact canonical model:

Correction of Exercise 15: Properties of conservative extensions

1. If \( T_2 \) is a conservative extension of \( T_1 \) and \( T_3 \) is a conservative extension of \( T_2 \), then \( T_3 \) is a conservative extension of \( T_1 \).

Let \( T_1, T_2 \) and \( T_3 \) be three TBoxes such that \( T_2 \) is a conservative extension of \( T_1 \) and \( T_3 \) is a conservative extension of \( T_2 \).

- Since \( T_2 \) is a conservative extension of \( T_1 \), then the signature of \( T_1 \) is included in the signature of \( T_2 \). Since \( T_3 \) is a conservative extension of \( T_2 \), then the signature of \( T_2 \) is included in the signature of \( T_3 \). Hence the signature of \( T_1 \) is included in the signature of \( T_3 \).
- Let \( I \) be a model of \( T_3 \). Since \( T_3 \) is a conservative extension of \( T_2 \), then \( I \) is a model of \( T_2 \). Since \( T_2 \) is a conservative extension of \( T_1 \), it follows that \( I \) is a model of \( T_1 \). Hence every model of \( T_3 \) is a model of \( T_1 \).
- Let \( I_1 \) be a model of \( T_1 \). Since \( T_2 \) is a conservative extension of \( T_1 \), then there is a model \( I_2 \) of \( T_2 \) such that
3. Since $\mathcal{I}_3$ is a conservative extension of $\mathcal{I}_2$ and $\mathcal{I}_2$ is a model of $\mathcal{T}_2$, then there exists a model $\mathcal{I}_3$ of $\mathcal{T}_3$ such that

- $\Delta^{\mathcal{I}_1} = \Delta^{\mathcal{I}_2}$
- $A^{\mathcal{I}_1} = A^{\mathcal{I}_2}$ for every atomic concept in the signature of $\mathcal{T}_1$
- $R^{\mathcal{I}_1} = R^{\mathcal{I}_2}$ for every role in the signature of $\mathcal{T}_1$

Hence $\mathcal{T}_3$ is a conservative extension of $\mathcal{T}_1$.

2. If $\mathcal{T}_2$ is a conservative extension of $\mathcal{T}_1$ and $C$ and $D$ are concepts containing only concept and role names from $\mathcal{T}_1$, then it holds that $\mathcal{T}_1 \models C \subseteq D$ if and only if $\mathcal{T}_2 \models C \subseteq D$.

Let $\mathcal{T}_2$ be a conservative extension of $\mathcal{T}_1$.

- Assume that $\mathcal{T}_1 \models C \subseteq D$. Let $\mathcal{I}$ be a model of $\mathcal{T}_2$. Since $\mathcal{T}_2$ is a conservative extension of $\mathcal{T}_1$, then $\mathcal{I}$ is a model of $\mathcal{T}_1$. Hence, since $\mathcal{T}_1 \models C \subseteq D$, $\mathcal{I} \models C \subseteq D$. Since this holds for every model of $\mathcal{T}_2$, it follows that $\mathcal{T}_2 \models C \subseteq D$.

- Conversely, assume that $\mathcal{T}_2 \models C \subseteq D$. Let $\mathcal{I}_1$ be a model of $\mathcal{T}_1$. Since $\mathcal{T}_2$ is a conservative extension of $\mathcal{T}_1$, there exists a model $\mathcal{I}_2$ of $\mathcal{T}_2$ such that

- $\Delta^{\mathcal{I}_1} = \Delta^{\mathcal{I}_2}$
- $A^{\mathcal{I}_1} = A^{\mathcal{I}_2}$ for every atomic concept in the signature of $\mathcal{T}_1$
- $R^{\mathcal{I}_1} = R^{\mathcal{I}_2}$ for every role in the signature of $\mathcal{T}_1$

We show by structural induction that for every $\mathcal{EL}$ concept $E$ such that $E$ contains only concept and role names from $\mathcal{T}_1$, $E^{\mathcal{I}_1} = E^{\mathcal{I}_2}$.

- Base case: $E$ is an atomic concept in the signature of $\mathcal{T}_1$ so $E^{\mathcal{I}_1} = E^{\mathcal{I}_2}$.

- Induction step:
  * Case $E = \neg F$, $F$ contains only concept and role names from $\mathcal{T}_1$ and we assume by induction that $F^{\mathcal{I}_1} = F^{\mathcal{I}_2}$. Thus $E^{\mathcal{I}_1} = \Delta^{\mathcal{I}_1} \setminus F^{\mathcal{I}_1} = \Delta^{\mathcal{I}_2} \setminus F^{\mathcal{I}_2} = E^{\mathcal{I}_2}$.
  * Case $E = F_1 \sqcap F_2$, $F_1$ and $F_2$ contain only concept and role names from $\mathcal{T}_1$ and we assume by induction that $F_1^{\mathcal{I}_1} = F_1^{\mathcal{I}_2}$ and $F_2^{\mathcal{I}_1} = F_2^{\mathcal{I}_2}$. Thus $E^{\mathcal{I}_1} = F_1^{\mathcal{I}_1} \sqcap F_2^{\mathcal{I}_1} = F_1^{\mathcal{I}_2} \sqcap F_2^{\mathcal{I}_2} = E^{\mathcal{I}_2}$.
  * Case $E = \exists R.F$ with $R$ in the signature of $\mathcal{T}_1$, $F$ contains only concept and role names from $\mathcal{T}_1$ and we assume by induction that $F^{\mathcal{I}_1} = F^{\mathcal{I}_2}$. $R^{\mathcal{I}_1} = R^{\mathcal{I}_2}$ so $E^{\mathcal{I}_1} = \{ u \mid (u,v) \in R^{\mathcal{I}_1}, v \in F^{\mathcal{I}_1} \} = \{ u \mid (u,v) \in R^{\mathcal{I}_2}, v \in F^{\mathcal{I}_2} \} = E^{\mathcal{I}_2}$.

Since $\mathcal{T}_2 \models C \subseteq D$, then $C^{\mathcal{I}_2} \subseteq D^{\mathcal{I}_2}$. Since $C$ and $D$ are concepts containing only concept and role names from $\mathcal{T}_1$, it follows that $C^{\mathcal{I}_1} \subseteq D^{\mathcal{I}_1}$, i.e., $\mathcal{T}_1 \models C \subseteq D$. Since this holds for every model of $\mathcal{T}_1$, it follows that $\mathcal{T}_1 \models C \subseteq D$.

3. If $\mathcal{T}_2$ is a conservative extension of $\mathcal{T}_1$, then for every ABox $\mathcal{A}$ and assertion $\alpha$ that use only atomic concepts and roles from $\mathcal{T}_1$, $\langle \mathcal{T}_1, \mathcal{A} \rangle \models \alpha$ if and only if $\langle \mathcal{T}_2, \mathcal{A} \rangle \models \alpha$.

Let $\mathcal{T}_2$ be a conservative extension of $\mathcal{T}_1$ and $\mathcal{A}$ and $\alpha$ be an ABox and an assertion that use only atomic concepts and roles from $\mathcal{T}_1$.

- Assume that $\langle \mathcal{T}_1, \mathcal{A} \rangle \models \alpha$. Let $\mathcal{I}$ be a model of $\langle \mathcal{T}_2, \mathcal{A} \rangle$. Since $\mathcal{T}_2$ is a conservative extension of $\mathcal{T}_1$ and $\mathcal{I}$ is a model of $\mathcal{T}_2$, then $\mathcal{I}$ is a model of $\mathcal{T}_1$. Since $\langle \mathcal{T}_1, \mathcal{A} \rangle \models \alpha$ and $\mathcal{I}$ is a model of $\mathcal{A}$ and $\mathcal{T}_1$, then $\mathcal{I} \models \alpha$. Since this holds for every model of $\langle \mathcal{T}_2, \mathcal{A} \rangle$, it follows that $\langle \mathcal{T}_2, \mathcal{A} \rangle \models \alpha$. 

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Conversely, assume that \( \langle T_2, A \rangle \models \alpha \). Let \( I_1 \) be a model of \( \langle T_1, A \rangle \). Since \( T_2 \) is a conservative extension of \( T_1 \) and \( I_1 \) is a model of \( T_1 \), there exists a model \( I_2 \) of \( T_2 \) such that
- \( \Delta^{I_2} = \Delta^{I_1} \)
- \( A^{I_2} = A^{I_1} \) for every atomic concept in the signature of \( T_1 \)
- \( R^{I_2} = R^{I_1} \) for every role in the signature of \( T_1 \)

Since \( I_1 \models A \) and concepts and roles used in \( A \) are in the signature of \( T_1 \), then \( I_2 \models A \). It follows that \( I_2 \) is a model of \( \langle T_2, A \rangle \), so \( I_2 \models \alpha \). Since \( \alpha \) is of the form \( A(a) \) or \( R(a, b) \) with \( A, R \) in the signature of \( T_1 \), it follows that \( I_1 \models \alpha \). Since this holds for every model of \( \langle T_1, A \rangle \), it follows that \( \langle T_1, A \rangle \models \alpha \).

**Correction of Exercise 16: Conservative extensions**

\( T_2 = T_1 \cup \{ A \sqsubseteq C, D \sqsubseteq B \} \)

1. \( T_2 \) is a conservative extension of \( T_1 \):
   - Since \( T_1 \subseteq T_2 \), the signature of \( T_1 \) is included in the signature of \( T_2 \).
   - Since \( T_1 \subseteq T_2 \), every model of \( T_2 \) is a model of \( T_1 \).
   - Let \( I_1 \) be a model of \( T_1 \). We define an interpretation \( I_2 \) by
     - \( \Delta^{I_2} = \Delta^{I_1} \)
     - \( E^{I_2} = E^{I_1} \) for every atomic concept in the signature of \( T_1 \)
     - \( R^{I_2} = R^{I_1} \) for every role in the signature of \( T_1 \)
     - \( A^{I_2} = C^{I_1} \)
     - \( B^{I_2} = D^{I_1} \)

   \( I_2 \) is a model of \( T_1 \) since it coincides with \( I_1 \) on the signature of \( T_1 \) and \( I_2 \models A \sqsubseteq C \) and \( I_2 \models D \sqsubseteq B \) by construction of \( I_2 \). Hence \( I_2 \) is a model of \( T_2 \).

2. \( T_2 \cup \{ A \sqsubseteq B \} \) is a conservative extension of \( T_1 \): The proof is similar to the previous question except that we define \( B^{I_2} = D^{I_1} \cup C^{I_1} \): It still holds that \( I_2 \models A \sqsubseteq C \) and \( I_2 \models D \sqsubseteq B \) (since \( D^{I_1} \subseteq D^{I_1} \cup C^{I_1} \)) and \( I_2 \models A \sqsubseteq B \) since \( C^{I_1} \subseteq D^{I_1} \cup C^{I_1} \).

3. If \( T_1 \not\models D \sqsubseteq C \), then \( T_2 \cup \{ B \sqsubseteq A \} \) is not a conservative extension of \( T_1 \) because \( T_2 \cup \{ B \sqsubseteq A \} \models D \sqsubseteq C \).