We show that $A \subseteq B$ if and only if $A \cap \neg B$ is not satisfiable. If $A \subseteq B$, then in any model $I$ of the TBox, it holds that the $A^I \subseteq B^I$. Hence $A^I \cap (\Delta^I \setminus B^I) = \emptyset$. As $(A \cap \neg B)^I = A^I \cap (\Delta^I \setminus B^I)$, the interpretation of $A \cap \neg B$ is empty in any model of the TBox, hence $A \cap \neg B$ is not satisfiable.

Conversely, if $A$ is not a subconcept of $B$ there exists a model $I$ of the TBox in which there exists $e \in A^I$ such that $e \notin B^I$. Hence, $e \notin \neg B^I$, and by definition of conjunction, $e \in (A \land \neg B)^I$. We have exhibited a model in which $(A \land \neg B)$ has a non empty interpretation, and $A \land \neg B$ is thus satisfiable.

Thus, in order to decide whether $A$ is a subconcept of $B$, one can check whether $A \land \neg B$ is satisfiable. As satisfiability in $\mathcal{EL}$ is trivial (every concept is satisfiable) and subsumption is not, there cannot be a reduction from subsumption to satisfiability.

Exercise 2: Negation Normal Form

We prove the result by induction on the concepts.

- Base case: the concepts are atomic names, negation of atomic names, $\top$ or $\bot$: these concepts are already in NNF, $\neg \top$ is equivalent to $\bot$, and $\neg \bot$ is equivalent to $\top$
- The concept is the conjunction of two (smaller) concepts: $C_1 \land C_2$: by induction, there exists $C'_1$ in NNF equivalent to $C_1$, and $C'_2$ in NNF equivalent to $C_2$: hence $C'_1 \land C'_2$ is in NNF and equivalent to $C_1 \land C_2$
- Same reasoning apply for disjunction, universal and existential restriction;
- If a concept is of the shape $\neg C$, we distinguish according to the shape of $C$:
  - if $C$ is an atomic name, $\top$, or $\bot$, this has already been taken care of in the base case;
  - if $C$ is a conjunction, say $C_1 \land C_2$, then $\neg C$ is equivalent to $\neg C_1 \lor \neg C_2$, and we transform both subconcepts by induction;
  - similarly, if $\neg(C_1 \lor C_2)$ is equivalent to $\neg C_1 \land \neg C_2$, $\neg \forall R.C$ is equivalent to $\exists R.\neg C$, and $\neg \exists R.C$ is equivalent to $\forall R.\neg C$.

Exercise 3: Tableau Algorithm

1. We apply the tableau algorithm to $\exists R.(A \cap B) \cap \forall R.(C \cup \neg A) \cap \forall R.(\neg C \cap \exists R.A)$
   
   1. $S_0 = \{ \exists R.(A \cap B) \cap \forall R.(C \cup \neg A) \cap \forall R.(\neg C \cap \exists R.A) (a) \}$
   2. $S_0^1 = S_0 \cup \{ \exists R.(A \cap B) (a), \forall R.(C \cup \neg A) (a), \forall R.(\neg C \cap \exists R.A) (a) \}$ by application of the $\cap$-rule;
   3. $S_0^2 = S_0^1 \cup \{ R(a,b), (A \cap B)(b) \}$ by application of the $\exists$-rule;
   4. $S_0^3 = S_0^2 \cup \{ A(b), B(b) \}$ by application of the $\cap$-rule;
   5. $S_0^4 = S_0^3 \cup \{ (C \cup \neg (A))(b) \}$ by application of the $\forall$-rule;
   6. $S_0^5 = S_0^4 \cup \{ C(b) \}$ by application of the $\cup$-rule; note that a second ABox should be created, namely $S_0^4 \cup \{ \neg A(b) \}$. However, this ABox containing a clash, we safely ignore it;
The only remaining ABox contains a clash: the concept is not satisfiable.

2. We apply the tableau algorithm to $\exists R.(A \sqcap B) \sqcap \forall R.(-A \sqcup C) \sqcap \forall R.(-B \sqcup -C)$

1. $S_0 = \{A \sqcap -C(a)\}$
2. $S_1^0 = S_0 \cup \{A(a), -C(a)\}$ by application of the $\sqcap$-rule
3. $S_2^0 = S_1^0 \cup \{-A \sqcup \exists R.(A \sqcup -B)(a)\}$ by application of the TBox-rule, for the first axiom on $a$;
4. $S_3^0 = S_2^0 \cup \exists R.(A \sqcup -B)(a)$ by application of the $\sqcup$-rule (the other ABox contains a clash);
5. $S_4^0 = S_3^0 \cup \{R(a, b), A \sqcup -B(b)\}$ by application of the $\exists$-rule;
6. $S_5^0 = S_4^0 \cup \{\forall R.B \sqcup C(a)\}$ by application of the TBox-rule, for the second axiom on $a$;
7. $S_6^0 = S_5^0 \cup \forall R.B(a)$ by application of the $\sqcup$-rule (the other ABox contain a clash);
8. $S_7^0 = S_6^0 \cup \{B(b)\}$ by application of the $\forall$-rule;
9. $S_8^0 = S_7^0 \cup \{A(b)\}$, by application of the $\sqcup$-rule, on the atom added in $S_4^0$ (the other ABox contains a clash)
10. $S_9^0 = S_8^0 \cup \forall R.-C \sqcup C(a)$ by application of the TBox-rule for the third axiom on $a$;
11. $S_10^0 = S_9^0 \cup \forall R.-C(a)$ by application of the $\sqcup$-rule (the other ABox contains a clash)
12. $S_11^0 = S_7^0 \cup \{-C(b)\}$ by application of the $\forall$-rule
13. $S_12^0 = S_11^0 \cup \{-A \sqcup \exists R.(A \sqcup -B)(b)\}$ by application of the TBox-rule, for the first axiom on $b$;
14. $S_13^0 = S_12^0 \cup \exists R.(A \sqcup -B)(b)$ by application of the $\sqcup$-rule (the other ABox contains a clash);
15. $S_14^0 = S_13^0 \cup \{R(b, c), A \sqcup -B(c)\}$ by application of the $\exists$-rule; note that $c$ is blocked by $b$
16. $S_15^0 = S_14^0 \cup \forall R.B \sqcup C(b)$ by application of the TBox-rule, for the second axiom on $b$;
17. $S_16^0 = S_15^0 \cup \forall R.B(b)$ by application of the $\sqcup$-rule (the other ABox contain a clash);
18. $S_17^0 = S_16^0 \cup \{B(c)\}$ by application of the $\forall$-rule;
19. $S_18^0 = S_17^0 \cup \forall R.-C \sqcup C(b)$ by application of the TBox-rule for the third axiom on $b$;
20. $S_0^{19} = S_0^{18} \cup \{\forall R.\neg C(b)\}$ by application of the $\cup$-rule (the other ABox contains a lash);

21. $S_0^{20} = S_0^{19} \cup \{\neg C(c)\}$, by application of the $\forall$-rule.

At this step, $S_0^{20}$ is complete (as $c$ is blocked by $b$), and clash-free. Hence, $A \sqcap \neg C$ is satisfiable and thus $A$ is not subsumed by $C$.

An example of model witnessing this non subsumption is an infinite chain of individuals $a_i$, such that $R(a_i, a_{i+1})$ holds (and no other $R$ atoms exist), $A(a_i)$ holds for any $i$, and $B(a_i)$ holds for any $i$.

**Exercise 4: Conservative Extensions**

For the first property, we have to consider each rule of the normalization algorithm. Let us consider the first rule:

$$\hat{C} \sqsubseteq \hat{D} \rightarrow \hat{C} \sqsubseteq A, A \sqsubseteq \hat{D}$$

and let us consider $T_2$ obtained from $T_1$ by applying this rule.

- applying it add the symbol $A$ to the signature, hence the signature of $T_1$ is included in the signature of $T_2$
- as all the axioms of $T_1$ except the one on which the rule has been applied belong to $T_2$, they are all satisfied in any model of $T_2$. As the model fulfills that $\hat{C} \sqsubseteq A$ and that $A \sqsubseteq \hat{D}$, it holds that $\hat{C} \sqsubseteq \hat{D}$, hence this axiom as well is satisfied in any model of $T_2$, which shows that any model of $T_2$ is a model of $T_1$.
- starting from a model $I_1$ of $T_1$, the only thing that remains to define is an interpretation for $A$. Any interpretation $A^{T_2}$ such that $\hat{C}^{I_1} \subseteq A^{T_2} \subseteq \hat{D}^{I_2}$ would work, and one can among other choose $A^{T_2} = \hat{C}^{I_1}$.

The other rules can be treated similarly.

For the second property, let us consider $T_1$, $T_2$ and $T_3$ such that $T_2$ is a conversative extension of $T_1$ and $T_3$ is a conservative extension of $T_2$. Let us show that $T_3$ is a conservative extension of $T_1$:

- the signature of $T_1$ is included in the signature of $T_2$ which is included in the signature of $T_3$, hence the signature of $T_1$ is included in the signature of $T_3$
- any model of $T_3$ is a model of $T_2$, and is hence a model of $T_1$
- any model of $T_1$ can be extented in a model of $T_2$ (by interpreting the concepts/relations that are not already interpreted in $T_1$), which can itself be extended in a model of $T_3$ (by interpreting the concepts/relations that are not already interpreted in $T_2$).

Hence $T_3$ is a conservative extension of $T_1$.

For the last property, let us assume that $T_1 \models C \subseteq D$. As any model of $T_2$ is a model of $T_1$, it holds that in any model $I$ of $T_2$, $C^I \subseteq D^I$. Hence $T_2 \models C \subseteq D$. Conversely, if $T_2 \models C \subseteq D$, then let us consider an arbitrary model $I_1$ of $T_1$. There exists $I_2$ such that is a model of $T_2$ which coincide with $I_1$ on the original concepts/roles, hence $C^{I_2} = D^{I_2} \subseteq D^{I_1}$, and thus $T_1 \models C \subseteq D$, which concludes the proof.

**Exercise 5: Subsumption Algorithm**

Consider the following TBox:

- $A \sqsubseteq \exists R.\exists S.C$
- $A \sqcap \exists R.\exists S.C \sqsubseteq B \sqcap C$
- $\exists R.\top \sqcap B \sqsubseteq \exists S.\exists R.D$
For any pair of concepts \((A, B)\) on the TBox vocabulary, the algorithm first normalizes the TBox, then applies the classification rules. The normalization step generates the following axioms:

- \(A \sqsubseteq \exists R.A_1\)
- \(A_1 \sqsubseteq \exists S.C\)
- \(A \sqcap \exists R.\exists S.C \sqsubseteq A_2\)
- \(A_2 \sqsubseteq B \sqcap C\)
- \(A \sqcap A_3 \sqsubseteq A_2\)
- \(\exists R.\exists S.C \sqsubseteq A_3\)
- \(\exists R.A_4 \sqsubseteq A_3\)
- \(\exists S.C \sqsubseteq A_4\)
- \(A_2 \sqsubseteq B\)
- \(A_2 \sqsubseteq C\)
- \(\exists R.\top \sqcap B \sqsubseteq A_5\)
- \(A_5 \sqsubseteq \exists S.\exists R.D\)
- \(\exists R.\top \sqsubseteq A_6\)
- \(A_6 \sqcap B \sqsubseteq A_5\)
- \(A_5 \sqsubseteq \exists S.A_7\)
- \(A_7 \sqsubseteq \exists R.D\)

out of which only the axioms being sequents are kept. The second step saturate this normalized TBox by applying the classification rules. One can easily check that during this saturation, all the classification rules are used.