Exercises: OBQA basics

Exercise 1: Chasing

Let us consider the ruleset $\mathcal{R}$ containing the following rules:

1. $\forall x \forall y \ r(x, y) \rightarrow \exists z \ s(y, z)$
2. $\forall x \forall y \ s(x, y) \rightarrow \exists z \ p(x, y, z)$
3. $\forall x \forall y \forall z \ r(x, y) \land p(y, z, t) \rightarrow s(y, z) \land q(z)$

Let $D = \{r(a, b), r(b, c), s(c, d)\}$.

1. Compute the chase of $D$ with respect to $\mathcal{R}$
2. Does it holds that for any instance $D$, the chase of $D$ with respect to $\mathcal{R}$ is finite?

Exercise 2: Atomic Head

Let $\mathcal{R}$ be a set of existential rules. Show that there exists a set of rules $\mathcal{R}'$ such that:

1. All the rules in $\mathcal{R}'$ have exactly one atom in the head;
2. For any Boolean conjunctive query $q$ whose atoms have predicates appearing in $\mathcal{R}$, for any database $D$, it holds that $D, \mathcal{R} \models q$ if and only if $D, \mathcal{R}' \models q$.

Exercise 2: Weak Acyclicity

With each set of existential rules $\mathcal{R}$ can be associated a graph, called the predicate graph of $\mathcal{R}$, and here denoted by $\text{PG}(\mathcal{R})$. It is defined as follows:

1. The vertices of $\text{PG}(\mathcal{R})$ are the pairs $(p, i)$, where $p$ is a predicate appearing in $\mathcal{R}$ and $i$ is an integer between 1 and the arity of $p$;
2. There is a normal edge from $(p, i)$ to $(q, j)$ if there is a rule $\sigma$ in which there is a variable $y$ appearing at position $(p, i)$ in the body of $\sigma$ and at position $(q, j)$ in the head of $\sigma$;
3. There is a special edge from $(p, i)$ to $(q, j)$ if there is a rule $\sigma$ in which there is a variable $y$ appearing at position $(p, i)$ in the body of $\sigma$ and at some position in the head of $\sigma$, and an existential variable $z$ appearing at position $(q, j)$ in the head of $\sigma$.

1. Draw the predicate graph of the ruleset containing:
   - $\forall x \forall y \ r(x, y) \rightarrow \exists z \ r(y, z)$;
   - $\forall x \forall y \ s(x, y) \rightarrow r(y, x)$;
   - $\forall x \forall y \forall z \ r(x, y) \land r(y, z) \rightarrow r(x, z)$.
2. A critical cycle is a cycle containing a special edge. Show that if $\text{PG}(\mathcal{R})$ does not contain any critical cycle (we say that $\mathcal{R}$ is weakly-acyclic, w.a. for short), then the chase w.r.t. $\mathcal{R}$ always terminates.
3. Provide an example of ruleset whose predicate graph contains a critical cycle and for which the chase always terminates.