Exercises: Description Logics Basics

Exercise 1
Consider the following interpretation $I$ with $\Delta^I = \{a, b, c, d\}$.

- $A^I = \{a, b\}$
- $B^I = \{b\}$
- $C^I = \{c, d\}$
- $R^I = \{(a, b), (a, c)\}$
- $S^I = \{(a, b), (b, c)\}$

For each of the following concepts $D$, list all elements $x$ of $\Delta^I$ such that $x \in D^I$:

1. $A \sqcap \exists S.C$
2. $B \sqcup (C \sqcap \exists S. \top)$
3. $\forall R.C$
4. $\forall S.C$
5. $A \sqcap \neg \exists R. \top$
6. $\exists R. \exists S. \top$

Correction

1. $A \sqcap \exists S.C = \{b\}$
2. $B \sqcup (C \sqcap \exists S. \top) = \{b, c\}$
3. $\forall R.C = \{b, c, d\}$
4. $\forall S.C = \{b, c, d\}$
5. $A \sqcap \neg \exists R. \top = \{b\}$
6. $\exists R. \exists S. \top = \{a\}$

For each of the following TBox statements, say whether it is satisfied in $I$.

1. $A \sqsubseteq B \sqcup C$
2. $A \sqsubseteq \exists S. \top$
3. $\exists S. \sqsubseteq B \sqsubseteq C$
4. $A \sqsubseteq \neg C$

Correction

1. No: $I \not\models A \sqsubseteq B \sqcup C$ because $\{a, b\} \not\subseteq \{b, c, d\}$
2. Yes: $I \models A \sqsubseteq \exists S. \top$ because $\{a, b\} \subseteq \{a\}$
3. Yes: $I \models \exists S. \sqsubseteq B \sqsubseteq C$ because $\{c\} \subseteq \{c, d\}$
4. Yes: $I \models A \sqsubseteq \neg C$ because $\{a, b\} \subseteq \{a, b\}$

Exercise 2
Consider the following TBox and ABoxes.

$T = \{A \sqsubseteq \forall R.B, \exists R.B \sqsubseteq C, B \sqsubseteq \neg C\}$

- $A_1 = \{A(a)\}$
- $A_2 = \{A(a), R(a, b)\}$
- $A_3 = \{A(a), R(a, b), C(b)\}$

1. Does $T \models A \sqsubseteq C$ ?
2. Does $T \models A \sqcap \exists R. \top \sqsubseteq \neg B$ ?
3. Is $B \sqcap \exists R.B$ satisfiable w.r.t. $T$ ? If so give a model of $T$ where $B \sqcap \exists R.B$ is non-empty.
4. Is $A \sqcap \forall R.C$ satisfiable w.r.t. $T$ ? If so give a model of $T$ where $A \sqcap \forall R.C$ is non-empty.
5. Is the knowledge base $\langle T, A_1 \rangle$ satisfiable ? If so, give a model of $\langle T, A_1 \rangle$.
6. Is the knowledge base $\langle T, A_2 \rangle$ satisfiable ? If so, give a model of $\langle T, A_2 \rangle$. 
7. Is the knowledge base $\langle T, A_3 \rangle$ satisfiable? If so, give a model of $\langle T, A_3 \rangle$.

8. Does $\langle T, A_1 \rangle \models C(a)$?

9. Does $\langle T, A_2 \rangle \models C(a)$?

10. Does $\langle T, A_3 \rangle \models C(a)$?

**Correction**

1. No. Consider the following interpretation $I$ on domain $\Delta^I = \{a\}$: $A^I = \{a\}$, $B^I = \emptyset$, $C^I = \emptyset$, $R^I = \emptyset$. $I$ is a model of $T$ and $A^I \not\subseteq C^I$ so $T \not\models A \subseteq C$.

2. Yes. Let $I$ be a model of $T$ and $e$ be an element of $\Delta^I$ such that $e \in (A \cap \exists R.T)^I = A^I \cap (\exists R.T)^I$. Since $e \in (\exists R.T)^I$, there exists $d \in \Delta^I$ such that $(e, d) \in R^I$. Since $I$ is a model of $T$, $I \models A \subseteq \forall R.B$, so $e \in A^I$ and $(e, d) \in R^I$ implies that $d \in B^I$. Hence $e \in (\exists R.B)^I$. Since $I |\models \exists R.B \subseteq C$, it follows that $e \in C^I$. Finally, since $I \models B \subseteq \neg C$, $B^I \subseteq C^I \setminus \Delta^I$ so $e \notin B^I$. We have shown that for every model $I$ of $T$, $(A \cap \exists R.T)^I \subseteq \Delta^I \setminus B^I$, i.e. $I \models A \cap \exists R.T \subseteq \neg B$. This is exactly the definition of $\models T \models A \cap \exists R.T \subseteq \neg B$.

3. No. Assume for a contradiction that there exists a model $I$ of $T$ such that $(B \cap \exists R.B)^I$ is non-empty and let $e \in (B \cap \exists R.B)^I$. Since $I \models \exists R.B \subseteq C$ and $e \in (\exists R.B)^I$, then $e \in C^I$. It follows that $e$ belongs to $B^I$ and to $C^I$, so $B^I \subseteq \Delta^I \setminus C^I$, which contradicts $T \models B \subseteq \neg C$.

4. Yes. Consider the model $I$ of $T$ given in the correction of question 1. $(A \cap \forall R.C)^I = \{a\}$ is non-empty.

5. Yes. We just need to extend the interpretation given in the correction of question 1 by setting $a^I = a$ to obtain a model of $\langle T, A_1 \rangle$.

6. Yes. Consider the following interpretation $I$ on domain $\Delta^I = \{a, b\}$: $a^I = a$, $b^I = b$, $A^I = \{a\}$, $B^I = \{b\}$, $C^I = \{a\}$, $R^I = \{(a, b)\}$. $I$ is a model of $\langle T, A_2 \rangle$.

7. No. Assume for a contradiction that $\langle T, A_3 \rangle$ has a model $I$. We must have $a^I \in A^I$ and $(a^I, b^I) \in R^I$ so since $I \models A \subseteq \forall R.B$, it follows that $b \in B^I$. However, we also must have $b \in C^I$, which contradicts $I \models B \subseteq \neg C$.

8. No. The model of $\langle T, A_1 \rangle$ given in question 5 does not satisfy $C(a)$.

9. Yes. Let $I$ be a model of $\langle T, A_2 \rangle$. Since $I \models A(a)$ and $I \models R(a, b)$, then $a^I \in A^I$ and $(a^I, b^I) \in R^I$. Since $I \models A \subseteq \forall R.B$, it follows that $b \in B^I$. Hence $a^I \in (\exists R.B)^I$, so since $I \models B \subseteq \neg C$, $a^I \in C^I$. We have shown that for every model $I$ of $\langle T, A_2 \rangle$, $a^I \in C^I$. This is exactly the definition of $\models T \models A(a)$.

10. Yes. Since $\langle T, A_3 \rangle$ has no model, it is true that $a^I \in C^I$ in every model of $\langle T, A_3 \rangle$. An unsatisfiable knowledge base entails every logical axiom.

**Exercise 3**

Translate the following TBox statements into first-order logic.

1. $\exists R.\exists S.T \subseteq B \cup C$
2. $A \cap \neg B \subseteq \forall R.C$
3. $\exists R.\neg A \subseteq \neg C$
4. $A \cup \exists R.B \subseteq \exists S.T$

**Correction**

1. $\forall x (\exists y (R(x, y) \land \exists z S(y, z)) \Rightarrow B(x) \lor C(x))$
2. $\forall x (A(x) \land \neg B(x) \Rightarrow \forall y (R(x, y) \Rightarrow C(y)))$
3. $\forall x (\exists y (R(y, x) \land A(y)) \Rightarrow \neg C(x))$
4. $\forall x (A(x) \lor \exists y (R(x, y) \land B(y)) \Rightarrow \exists z S(x, z))$